AN ADAPTIVE FINITE ELEMENT METHOD FOR THE $\mathbf{H} - \psi$ FORMULATION OF TIME-DEPENDENT EDDY CURRENT PROBLEMS

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Abstract. In this paper, we develop an adaptive finite element method based on reliable and efficient *a posteriori* error estimates for the $\mathbf{H} - \psi$ formulation of eddy current problems with multiply connected conductors. Multiply connected domains are considered by making "cuts". The competitive performance of the method is demonstrated by an engineering benchmark problem, Team Workshop Problem 7, and a singular problem with analytic solution.

Key words. Maxwell's equations, eddy current problem, adaptive finite element method, multiply connected conductor, Team Workshop Problem 7

AMS subject classifications. 65M60, 65M50, 78A25

1. Introduction. Eddy currents appear in almost all electromagnetic devices. They cause energy loss and may reduce lifespan of devices. Three dimensional eddy current problems describe very low-frequency electromagnetic phenomena by quasi-static Maxwell's equations. In this case, displacement currents may be neglected (see [1] and [7, Ch.8]), thus Maxwell's equations become

$$\begin{cases} \mathbf{curl}\,\mathbf{H} = \mathbf{J} & \text{in} \quad \mathbb{R}^3, & \text{(Ampere's law)} \\ \mu \dfrac{\partial \mathbf{H}}{\partial t} + \mathbf{curl}\,\mathbf{E} = 0 & \text{in} \quad \mathbb{R}^3, & \text{(Farady's law)} \\ \operatorname{div}(\mu \mathbf{H}) = 0 & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field, and $\mathbf{J}$ is the total current defined by:

$$\mathbf{J} = \begin{cases} \sigma\,\mathbf{E} & \text{in} \quad \Omega_c, & \text{(conducting region)} \\ \mathbf{J}_s & \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega_c}. & \text{(nonconducting region)} \end{cases} \tag{1.2}$$

In (1.1) and (1.2), μ is the magnetic permeability, σ is the electric conductivity, $\mathbf{J}_s$ is the solenoidal source current carried by some coils in the air, and Ω_c is the conducting region which carries eddy currents. To avoid extra complicated constraints on $\mathbf{J}_s$, we assume $\operatorname{supp}(\mathbf{J}_s) \cap \bar{\Omega}_c = \emptyset$.

(1.1) – (1.2) may be simplified into different forms by virtue of various field variables (see [19] and references therein). Generally speaking, each of these simplified formulations contains at least an unknown vector function defined in the conducting region, plus an unknown vector function or an unknown scalar function defined in the nonconducting region. From the point of view of numerical computation, the

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latter case needs less degrees of freedom and thus is more favorable. In this paper, we adopt a formulation based on the magnetic field $\mathbf{H}$ in the conducting region and the magnetic scalar potential ψ in the nonconducting region. When all connected components of the conducting region are simply connected, the scalar potential ψ belongs to $H^1(\mathbb{R}^3 \setminus \overline{\Omega_c})$ and the problem is relatively easy to be dealt with in the framework of finite element method. Otherwise, in the case of multiply connected conductors, ψ is discontinuous somewhere in the nonconducting region [2] and thus the problem becomes more difficult. We focus on this case and treat the discontinuities of ψ by making "cuts" in the nonconducting region.

Eddy current problems involve discontinuous coefficients, reentrant corners of material interfaces, and skin effect. Thus local singularities and internal layers of the solution arise. We refer to [17] for the descriptions of the eddy current limit and the singularities of the solutions. Among various numerical methods for eddy current problem, the finite element method is most popular (see [7, Ch.8], [8], and references therein). We also refer to [16] for the finite integration method. It is well known that the adaptive finite element method is very efficient for problems with local singularities since it produces "quasi-optimal" mesh by using reliable and efficient error estimates [11] [33]. *A posteriori* error estimates are computable quantities in terms of the discrete solution and known datum that measure the actual discretization errors without the knowledge of the exact solution. They are essential in designing algorithms for mesh modification which equidistribute the error and optimize the computation. Ever since the pioneering work of Babuška and Rheinboldt [3], the adaptive finite element methods based on *a posteriori* error estimates have become a central theme in scientific and engineering computing. The ability of error control and the asymptotically optimal approximation property (see e.g. [9], [25], and [14]) make the adaptive finite element method attractive for complicated physical and industrial processes (cf. e.g. [10] and [12]).

A posteriori error estimates for Nédélec $\mathbf{H}(\mathbf{curl})$-conforming edge elements are obtained in [24] for Maxwell scattering problems, in [4] for the electric field-based formulation of eddy current problems, in [5] for higher order edge element approximation of eddy current problems, in [31] and [32] for the fem-bem coupling scheme of eddy current problems, and also in [13] for time-harmonic Maxwell equations with singularities. The key ingredient in the analysis is the orthogonal Helmholtz decomposition $\mathbf{v} = \nabla\varphi + \Psi$, where for any $\mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega)$, $\varphi \in H^1(\Omega)$, and $\Psi \in \mathbf{H}(\mathbf{curl}; \Omega)$. Since a stable edge element interpolation operator is not available for functions in $\mathbf{H}(\mathbf{curl}; \Omega)$, some kind of regularity result for $\Psi \in \mathbf{H}(\mathbf{curl}; \Omega)$ is required. This regularity result is proved in [24] for domains with smooth boundary and in [4] for convex polyhedral domains. If one removes the orthogonality requirement in the Helmholtz decomposition, the regularity $\Psi \in \mathbf{H}^1(\Omega)$ can be proved for a large class of non-convex polygonal domains or domains having screens [13] [18] [27]. In this paper, the proof of the Helmholtz decomposition becomes more complicated due to the presence of the discontinuities of the magnetic potential in the nonconducting region. We treat this difficulty by introducing some finite element functions into the decomposition.

In this paper, we develop an adaptive finite element method based on reliable and efficient *a posteriori* error estimates for the $\mathbf{H} - \psi$ formulation of eddy current problems with multiply connected conductors (see [Chapter 8, 7] for the formulation of time-harmonic problems). We compute two challenging problems to demonstrate the competitive performance of our method. One is an engineering benchmark problem, the Team Workshop Problem 7, and another is a singular problem with analytic solution. The results indicate that our adaptive method has the following very desirable quasi-optimality property:

$$\eta_{\text{total}} \approx C\, N_{\text{total}}^{-1/4}$$

is valid asymptotically, where η_{total} is the total error estimate (see Theorem 4.1), and $N_{\text{total}} := \sum_{n=1}^{M} N_n$ with M being the number of time steps and N_n being the number of elements of the mesh $\mathcal{T}_n$ at the n-th timestep.

The rest of the paper is arranged as follows: In section 2, we derive the $\mathbf{H} - \psi$ based formulation of time-dependent eddy current problems. The equivalent weak formulation and its well-posedness are also given in this section. In section 3, we introduce a coupled conforming finite element approximation to the $\mathbf{H} - \psi$ based formulation and prove the Helmholtz decomposition of the variational space. In section 4, we derive reliable and efficient residual-based *a posteriori* error estimates. In section 5, we report the numerical results for a singular solution and the Team Workshop Problem 7, and compare them with experimental results to show the competitive performance of the method proposed in this paper.

2. Magnetic field and magnetic scalar potential based formulation. Let $\Omega \subset \mathbb{R}^3$ be a sufficiently large convex polyhedral domain containing all conductors and coils (see Fig. 2.1 for a typical model with one conductor and one coil). Denote the conducting domain by Ω_c which consists of all conductors. We assume that Ω_c is bounded and each of its connected components is a Lipschitz domain.

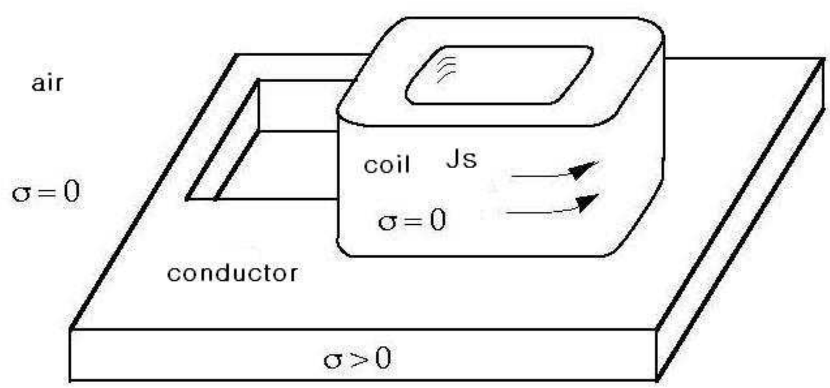

FIG. 2.1. *Setting of the eddy current problems: A conductor with a hole and a coil.*

We assume that μ and σ are real valued $L^\infty(\Omega)$ functions and there exist two positive constants $\mu_{\min}$ and $\sigma_{\min}$ such that $\mu \geq \mu_{\min}$ in Ω and $\sigma \geq \sigma_{\min}$ in Ω_c. Furthermore, we assume $\sigma \equiv 0$ outside of Ω_c.

Let $L^2(\Omega)$ be the usual Hilbert space of square integrable functions equipped with the following inner product and norm:

$$(u,\, v) := \int_\Omega u(\mathbf{x})\, v(\mathbf{x})\, d\,\mathbf{x} \quad \text{and} \quad \|u\|_{0,\Omega} := (u,\, v)^{1/2}.$$

$H^m(\Omega) := \{v \in L^2(\Omega) \,:\, D^\xi v \in L^2(\Omega),\ |\xi| \le m\}$ equipped with the following norm and semi-norm

$$\|u\|_{m,\Omega} := \left(\sum_{|\xi|\le m} \|D^\xi u\|_{0,\Omega}^2\right)^{1/2} \quad \text{and} \quad |u|_{m,\Omega} := \left(\sum_{|\xi|= m} \|D^\xi u\|_{0,\Omega}^2\right)^{1/2},$$

where ξ represents non-negative triple index. $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ whose functions have zero traces on $\partial\Omega$. Throughout the paper we denote vector-valued quantities by boldface notation, such as $\mathbf{L}^2(\Omega) := (L^2(\Omega))^3$. Define

$$\mathbf{H}(\mathbf{curl};\Omega) := \{\mathbf{v} \in \mathbf{L}^2(\Omega) \,:\, \mathbf{curl}\,\mathbf{v} \in \mathbf{L}^2(\Omega)\},$$
$$\mathbf{H}_0(\mathbf{curl};\Omega) := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl};\Omega) \,:\, \mathbf{n}\times\mathbf{v} = 0 \ \text{ on } \partial\Omega\}.$$

$\mathbf{H}(\mathbf{curl};\Omega)$ is equipped with the following norm:

$$\|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} := \left(\|\mathbf{v}\|_{0,\Omega}^2 + \|\mathbf{curl}\,\mathbf{v}\|_{0,\Omega}^2\right)^{1/2}.$$

Since $\operatorname{div}\mathbf{J}_s \equiv 0$, there exists a source magnetic field $\mathbf{H}_s$ such that

$$\mathbf{J}_s = \mathbf{curl}\,\mathbf{H}_s \quad \text{in } \mathbb{R}^3. \tag{2.1}$$

The field $\mathbf{H}_s$ can be written explicitly by the Biot-Savart Law for general coils:

$$\mathbf{H}_s := \mathbf{curl}\,\mathbf{A}_s \quad \text{where} \quad \mathbf{A}_s(\mathbf{x}) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\mathbf{J}_s(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y}.$$

In the following we are going to find the residual $\mathbf{H}_0 := \mathbf{H} - \mathbf{H}_s$ which is called the reaction field in [21]. Clearly, by (1.1), (1.2), and (2.1), we have

$$\mathbf{curl}\,\mathbf{H}_0 = 0 \quad \text{in } \Omega\setminus\overline{\Omega_c}.$$

Our goal is to write $\mathbf{H}_0$ as $\nabla\psi$ for some scalar potential ψ. Since $\Omega\setminus\overline{\Omega_c}$ may not be simply connected, ψ may not be unique. To deal with this difficulty, we introduce the following assumption (see [2, Hypothesis 3.3]):

Hypothesis 2.1. *There exist I connected open surfaces $\Sigma_0, \cdots, \Sigma_I$, called "cuts", contained in $\Omega\setminus\overline{\Omega_c}$, such that*

- (*i*) *each cut Σ_i is an open part of some smooth two-dimensional manifold with Lipschitz-continuous boundary, $i = 1, \cdots, I$;*
- (*ii*) *the boundary of Σ_i is contained in $\partial\Omega_c$ and $\overline{\Sigma}_i \cap \overline{\Sigma}_j = \emptyset$ for $i \ne j$;*
- (*iii*) *the open set $\Omega^\circ := (\Omega\setminus\overline{\Omega_c})\setminus(\cup_{i=1}^I \Sigma_i)$ is simply connected and pseudo-Lipschitz (see [2, Definition 3.1] for the definition of pseudo-Lipschitz domain).*

For each Σ_i, we fix its unit normal vector $\mathbf{n}$ pointing to one side (see Fig. 2.2 for $I = 1$). Define

$$\Theta := \{\varphi \in H^1(\Omega^\circ) \,:\, [\varphi]_{\Sigma_j} = \text{const.},\ \ 1 \le j \le I\},$$

where $[\varphi]_{\Sigma_j}$ is the jump of φ across the cut Σ_j. For any $\varphi \in \Theta$, we can extend $\nabla\varphi \in \mathbf{L}^2(\Omega^\circ)$ continuously to a function $\tilde{\nabla}\varphi \in \mathbf{L}^2(\Omega\setminus\overline{\Omega_c})$ such that

$$\tilde{\nabla}\varphi = \nabla\varphi \quad \text{in } \Omega^\circ.$$

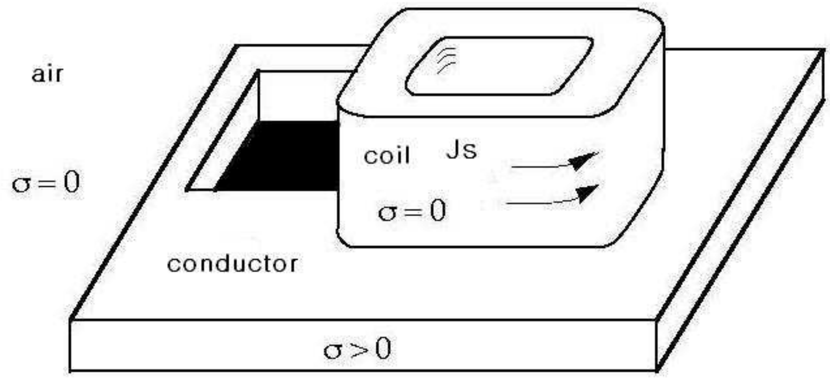

FIG. 2.2. *Setting of the eddy current problems: Stem the hole by making "cut".*

LEMMA 2.2. [2, Lemma 3.11] *Let $\varphi \in H^1(\Omega^\circ)$. Then $\varphi \in \Theta$ if and only if*

$$\mathbf{curl}\,(\tilde{\nabla}\varphi) = 0 \quad \text{in } \Omega \setminus \overline{\Omega_c}\,.$$

Since Ω° is simply connected, by Lemma 2.2, there exists a unique potential $\psi \in \Theta/\mathbb{R}^1$ such that

$$\mathbf{H}_0 = \nabla\psi \quad \text{in } \Omega^\circ.$$

Thus the second equation in (1.1) becomes

$$\begin{cases} \mu \dfrac{\partial\,(\mathbf{H}_s + \nabla\psi)}{\partial t} + \mathbf{curl}\,\mathbf{E} = 0 \quad \text{in } \Omega^\circ, \\ \mu \dfrac{\partial\,(\mathbf{H}_s + \mathbf{H}_0)}{\partial t} + \mathbf{curl}\,\mathbf{E} = 0 \quad \text{in } \Omega_c\,. \end{cases} \tag{2.2}$$

For the initial conditions, we set

$$\psi(\cdot, 0) = 0, \quad \mathbf{H}_0(\cdot, 0) = \mathbf{0}. \tag{2.3}$$

Since the total electro-magnetic energy is finite, we may assume $\mathbf{H} \in \mathbf{L}^2(\mathbb{R}^3)$ which implies $\mathbf{curl}\,\mathbf{E} \in \mathbf{L}^2(\mathbb{R}^3)$. Assuming Ω large enough, we set the following boundary condition on $\partial\Omega$:

$$\nabla\psi \cdot \mathbf{n} = -\mathbf{H}_s \cdot \mathbf{n} \quad \text{on } \partial\Omega\,. \tag{2.4}$$

Our next goal is going to derive a weak formula for (1.1), starting from (2.2). Similar development can be found in [7, Chapter 8]).

Since the tangential field $\mathbf{H}_0 \times \mathbf{n}$ is continuous through $\partial\Omega_c$, we add this constraint to the test functions and define

$$\mathbf{X} = \Big\{ \mathbf{v} \;:\; \mathbf{v} = \tilde{\nabla}\varphi \text{ in } \Omega \setminus \overline{\Omega_c} \text{ for some } \varphi \in \Theta/\mathbb{R}^1 \text{ and } \mathbf{v} = \mathbf{w} \text{ in } \Omega_c$$
$$\text{for some } \mathbf{w} \in \mathbf{H}(\mathbf{curl};\, \Omega_c) \text{ such that } \tilde{\nabla}\varphi \times \mathbf{n} = \mathbf{w} \times \mathbf{n} \text{ on } \partial\Omega_c \Big\}.$$

It is clear that $\mathbf{X} \subset \mathbf{H}(\mathbf{curl};\, \Omega)$. For any $\varphi \in \Theta/\mathbb{R}^1$, we multiply the first equation of (2.2) by $\nabla\varphi$, integrate by part to obtain

$$\frac{\partial}{\partial t} \int_{\Omega^\circ} \mu\, (\nabla\psi + \mathbf{H}_s) \cdot \nabla\varphi = - \int_{\Omega^\circ} \mathbf{curl}\, \mathbf{E} \cdot \nabla\varphi = - \int_{\partial\Omega^\circ} \mathbf{curl}\, \mathbf{E} \cdot \mathbf{n}\, \varphi\, .$$

Note that $\partial\Omega^\circ = \partial\Omega \cup \partial\Omega_c \cup (\cup_{j=1}^{I} \Sigma_j)$. By (2.2) and (2.4) we have $\mathbf{curl}\, \mathbf{E} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Thus

$$\text{(2.5)} \quad \frac{\partial}{\partial t} \int_{\Omega^\circ} \mu\, (\nabla\psi + \mathbf{H}_s) \cdot \nabla\varphi = \sum_{j=1}^{I} \int_{\Sigma_j} \mathbf{E} \cdot [\mathbf{n} \times \nabla\varphi]_{\Sigma_j} + \int_{\partial\Omega_c} \mathbf{E} \cdot (\mathbf{n} \times \tilde{\nabla}\varphi)$$
$$= \int_{\partial\Omega_c} \mathbf{E} \cdot (\mathbf{n} \times \tilde{\nabla}\varphi),$$

where $\mathbf{n}$ is the unit outer normal to $\partial\Omega_c$, and we have used the fact that $[\nabla\varphi \times \mathbf{n}]_{\Sigma_j} = \mathbf{0}$ on Σ_j because of $\varphi \in \Theta$. For any $\mathbf{w} \in \mathbf{H}(\mathbf{curl};\, \Omega_c)$, we multiply the second equation of (2.2) by $\mathbf{w}$ and integrate by part to obtain

$$\frac{\partial}{\partial t} \int_{\Omega_c} \mu\, (\mathbf{H}_s + \mathbf{H}_0) \cdot \mathbf{w} = - \int_{\Omega_c} \mathbf{curl}\, \mathbf{E} \cdot \mathbf{w} = \int_{\partial\Omega_c} \mathbf{E} \cdot (\mathbf{n} \times \mathbf{w}) - \int_{\Omega_c} \mathbf{E} \cdot \mathbf{curl}\, \mathbf{w}\, .$$

By (1.2) and the first equation of (1.1), we have

$$\text{(2.6)} \quad \frac{\partial}{\partial t} \int_{\Omega_c} \mu\, (\mathbf{H}_s + \mathbf{H}_0) \cdot \mathbf{w} + \int_{\Omega_c} \sigma^{-1} \mathbf{curl}\, \mathbf{H}_0 \cdot \mathbf{curl}\, \mathbf{w} = \int_{\partial\Omega_c} \mathbf{E} \cdot (\mathbf{n} \times \mathbf{w}),$$

where $\mathbf{n}$ is the unit normal on $\partial\Omega_c$ pointing to the exterior of Ω_c, and we have used (2.1) and the fact that $\mathbf{J}_s \equiv \mathbf{0}$ in Ω_c. By the tangential continuity of the electric field $\mathbf{E}$, we add (2.5) to (2.6) and obtain, for any $\mathbf{v} \in \mathbf{X}$ such that $\mathbf{v} = \tilde{\nabla}\varphi$ in $\Omega \setminus \overline{\Omega_c}$ and $\mathbf{v} = \mathbf{w}$ in Ω_c,

$$\frac{\partial}{\partial t} \int_{\Omega^\circ} \mu\, \nabla\psi \cdot \nabla\varphi + \frac{\partial}{\partial t} \int_{\Omega_c} \mu\, \mathbf{H}_0 \cdot \mathbf{w} + \int_{\Omega_c} \sigma^{-1} \mathbf{curl}\, \mathbf{H}_0 \cdot \mathbf{curl}\, \mathbf{w} = - \frac{\partial}{\partial t} \int_{\Omega} \mu\, \mathbf{H}_s \cdot \mathbf{v}.$$

For the convenience in notation, we drop the subscript of $\mathbf{H}_0$ and denote the reaction field by $\mathbf{H}$ in the rest of this paper. Thus we are led to the following variational problem based on the magnetic reaction field and magnetic scalar potential: Find $\mathbf{H} \in \mathbf{L}^2((0,T); \mathbf{X})$ such that $\mathbf{H}(\cdot, 0) \equiv 0$ and

$$\text{(2.7)} \qquad \frac{\partial}{\partial t} \int_{\Omega} \mu\, \mathbf{H} \cdot \mathbf{v} + \int_{\Omega_c} \sigma^{-1} \mathbf{curl}\, \mathbf{H} \cdot \mathbf{curl}\, \mathbf{v} = - \frac{\partial}{\partial t} \int_{\Omega} \mu \mathbf{H}_s \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{X}.$$

It is easy to prove the following theorem by the Galerkin method. Here we omit the details.

THEOREM 2.3. *Assume that $\partial(\mu\mathbf{H}_s)/\partial t \in \mathbf{L}^2(\Omega)$ is Lipschitz-continuous with respect to t. Then the initial problem (2.7) has a unique solution $\mathbf{H} \in \mathbf{L}^2((0,T); \mathbf{X})$.*

3. Finite element approximations. We use a fully discrete scheme to approximate (2.7). Let $\{t_0, \cdots, t_M\}$ form a partition of the time interval [0, T] and $\tau_n = t_n - t_{n-1}$ be the n-th timestep. Let $\mathcal{T}_n$ be a regular tetrahedral triangulation of Ω such that $\mathcal{T}_n^c := \mathcal{T}_n|_{\Omega_c}$ and $\mathcal{T}_n^\circ := \mathcal{T}_n|_{\Omega^\circ}$ are triangulations of Ω_c and Ω° respectively. Let $\mathcal{T}_{\rm init}$ be the initial regular triangulation of Ω such that each $\mathcal{T}_n$, $n = 0, \cdots, M$, is a refinement of $\mathcal{T}_{\rm init}$.

Let $V_n \subset H^1(\Omega)$ and $V_n^\circ \subset H^1(\Omega^\circ)$ be the conforming linear Lagrangian finite element spaces over $\mathcal{T}_n$ and $\mathcal{T}_n^\circ$ respectively, and $\mathbf{V}_n^c \subset \mathbf{H}(\mathbf{curl}; \Omega_c)$ be the Nédélec edge element space of the lowest order over $\mathcal{T}_n^c$ [26]. We introduce the finite element space $\mathbf{X}_n \subset \mathbf{X}$ by

$$\mathbf{X}_n = \Big\{\mathbf{v} \,:\, \mathbf{v} = \tilde{\nabla}\varphi_n \text{ in } \Omega \setminus \overline{\Omega_c} \text{ for some } \varphi_n \in \Theta \cap V_n^\circ/\mathbb{R}^1 \text{ and } \mathbf{v} = \mathbf{w}_n \text{ in } \Omega_c$$
$$\text{for some } \mathbf{w}_n \in \mathbf{V}_n^c \text{ such that } \tilde{\nabla}\varphi_n \times \mathbf{n} = \mathbf{w}_n \times \mathbf{n} \text{ on } \partial\Omega_c\Big\}.$$

Thus a fully discrete scheme of (2.7) is: Find $\mathbf{H}_n \in \mathbf{X}_n$ such that $\mathbf{H}_0 \equiv \mathbf{0}$ and

$$\int_\Omega \mu \frac{\mathbf{H}_n - \mathbf{H}_{n-1}}{\tau_n} \cdot \mathbf{v} + \int_{\Omega_c} \sigma^{-1} \mathbf{curl}\,\mathbf{H}_n \cdot \mathbf{curl}\,\mathbf{v} = \int_\Omega \bar{\mathbf{f}}_n \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{X}_n, \tag{3.1}$$

where $\mathbf{f} := -\mu\, \partial \mathbf{H}_s/\partial t$ and $\bar{\mathbf{f}}_n := \frac{1}{\tau_n}\int_{t_{n-1}}^{t_n} \mathbf{f}$ is the mean value of $\mathbf{f}$ over $[t_{n-1}, t_n]$. The uniqueness and existence of solutions to (3.1) follows directly from the Lax-Milgram Lemma.

For each "cut" Σ_i, let q_i be the $H^1(\Omega^\circ)$-conforming linear finite element function satisfying

$$[q_i]_{\Sigma_j} = \delta_{ij},\ 1 \le j \le I, \text{ and } q_i(A) = 0, \text{ for any node } A \text{ not on } \overline{\Sigma}_i. \tag{3.2}$$

Denote the edges in $\bar{\Omega} \setminus \Omega_c$ by $\mathcal{E}_{\rm init}^{\bar{\Omega}\setminus\Omega_c}$. For any $E \in \mathcal{E}_{\rm init}^{\bar{\Omega}\setminus\Omega_c}$, let A_1 and A_2 be its two endpoints. We define $\mathbf{w}_E$ by $\mathrm{supp}(\mathbf{w}_E) = \bigcup_{E \subset \partial T, T \in \mathcal{T}_{\rm init}} \overline{T}$ and for any $T \subset \mathrm{supp}(\mathbf{w}_E)$

$$\mathbf{w}_E = \lambda_2^T \nabla \lambda_1^T - \lambda_2^T \nabla \lambda_2^T \quad \text{in } T,$$

where λ_1^T and λ_2^T are the barycentric coordinates of T with respect to A_1 and A_2 respectively. In fact, $\mathbf{w}_E$ is the corresponding canonical basis function of the lowest order Nédélec edge element space over $\mathcal{T}_{\rm init}$ [26]. We define

$$\mathbf{q}_i := \sum_{E \in \mathcal{E}_{\rm init}^{\bar{\Omega}\setminus\Omega_c}} \int_E \nabla q_i \cdot dE\ \mathbf{w}_E. \tag{3.3}$$

The key ingredient in the analysis of *a posteriori* error estimates for Maxwell's equations is Helmholtz-type decompositions for functions in $\mathbf{H}(\mathbf{curl}; \Omega)$. In the next, we will introduce an $\mathbf{H}(\mathbf{curl})$-stable decomposition for $\mathbf{X}$. Since both Ω_c and $\Omega\setminus\overline{\Omega_c}$ are multiply connected, it is difficult to find a scalar function ψ with constant jumps across all "cuts" to define the irrotational part. Instead, we represent these discontinuities by the help of some finite element function.

THEOREM 3.1. *Let $\mathbf{X}_{\rm init}$ be the finite element space over $\mathcal{T}_{\rm init}$. For any $\mathbf{v} \in \mathbf{X}$, there exists a $\varphi \in H^1(\Omega)/\mathbb{R}^1$, a $\mathbf{v}_{\rm init} \in \mathbf{X}_{\rm init}$, and a $\mathbf{v}_s \in \mathbf{H}(\mathbf{curl}; \Omega) \cap \mathbf{H}^1(\Omega_c)$ such that $\mathbf{v}_s = \mathbf{0}$ in $\Omega \setminus \overline{\Omega_c}$ and*

$$\mathbf{v} = \nabla\varphi + \mathbf{v}_{\rm init} + \mathbf{v}_s. \tag{3.4}$$

Furthermore, there exists a positive C depending only on Ω and $\mathcal{T}_{\rm init}$ such that

$$\|\varphi\|_{1,\Omega} + \|\mathbf{v}_s\|_{1,\Omega_c} + \|\mathbf{v}_{\rm init}\|_{\mathbf{H}(\mathbf{curl};\,\Omega)} \le C\|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl};\,\Omega)}\,. \tag{3.5}$$

Proof. For any $\mathbf{v} \in \mathbf{X}$ such that $\mathbf{v} = \tilde{\nabla} f$ in $\Omega \setminus \overline{\Omega_c}$ for some $f \in \Theta/\mathbb{R}^1$, define

$$\varphi_{\rm init} := \sum_{i=1}^{I} [f]_{\Sigma_i}\, q_i \quad \text{ and } \quad \mathbf{v}_{\rm init} := \sum_{i=1}^{I} [f]_{\Sigma_i}\, \mathbf{q}_i. \tag{3.6}$$

where q_i and $\mathbf{q}_i$ are defined in (3.2) and (3.3) respectively. Clearly, by the definition of $\mathbf{q}_i$, $\mathbf{v}_{\rm init} \in \mathbf{X}_{\rm init}$ is an $\mathbf{H}(\mathbf{curl})$-continuous extension of $\tilde{\nabla}\varphi_{\rm init}$ to Ω_c. We also have $f - \varphi_{\rm init} \in H^1(\Omega \setminus \overline{\Omega_c})$. By the trace theorem and Schwartz's inequality,

$$|[f]_{\Sigma_i}| = \frac{1}{|\Sigma_i|}\int_{\Sigma_i} |[f]_{\Sigma_i}| \le \left(\frac{1}{|\Sigma_i|}\int_{\Sigma_i} |[f]_{\Sigma_i}|^2\right)^{1/2} \le C\|f\|_{1,\Omega^\circ},$$

for all $i = 1, \cdots, I$. Thus there exists a constant C depending only on $\Sigma_1, \cdots, \Sigma_I$ and $\mathcal{T}_{\rm init}$ such that

$$\|\varphi_{\rm init}\|_{1,\Omega^\circ} \le C\|f\|_{1,\Omega^\circ} \quad \text{and} \quad \|\mathbf{v}_{\rm init}\|_{\mathbf{H}(\mathbf{curl};\,\Omega)} \le C\|f\|_{1,\Omega^\circ}. \tag{3.7}$$

By Stein's extension theorem [30, Theorem 5, p.181] and (3.7), there exists an extension of $f - \varphi_{\rm init}$ denoted by $\varphi_0 \in H^1(\Omega)/\mathbb{R}^1$ such that

$$\varphi_0 = f - \varphi_{\rm init}, \quad \text{in } \Omega \setminus \overline{\Omega_c}, \tag{3.8}$$

$$\|\varphi_0\|_{1,\Omega} \le C\|f - \varphi_{\rm init}\|_{1,\Omega\setminus\overline{\Omega_c}} \le C\|f\|_{1,\Omega^\circ} \le C\|\nabla f\|_{0,\Omega^\circ}. \tag{3.9}$$

By (3.6) and (3.8), we have $\mathbf{v} - \nabla\varphi_0 - \mathbf{v}_{\rm init} \in \mathbf{H}_0(\mathbf{curl};\Omega_c)$. In view of (3.4), we only need to decompose $\mathbf{v} - \nabla\varphi_0 - \mathbf{v}_{\rm init}$ into a gradient part and an $\mathbf{H}^1$-smooth part.

Denote $\mathbf{w} := \mathbf{v} - \nabla\varphi_0 - \mathbf{v}_{\rm init}$ and extend $\mathbf{w}$ by zero to $\Omega\setminus\overline{\Omega_c}$. Clearly the extension $\tilde{\mathbf{w}} \in \mathbf{H}_0(\mathbf{curl};\Omega)$. By Theorem 3.4 of [20, p. 45] and Theorem 3.7 of [20, p. 52], there exists a $\Psi \in \mathbf{H}^1(\Omega)$ such that

$$\mathbf{curl}\,\Psi = \mathbf{curl}\,\tilde{\mathbf{w}}, \quad \operatorname{div}\Psi = 0, \quad \text{in } \Omega, \tag{3.10}$$

$$\|\Psi\|_{1,\Omega} \le C(\|\mathbf{curl}\,\Psi\|_{0,\Omega} + \|\operatorname{div}\Psi\|_{0,\Omega}) = C\|\mathbf{curl}\,\mathbf{w}\|_{0,\Omega_c}. \tag{3.11}$$

Moreover, by (3.10) and Theorem 2.9 of [20, p. 31], there exists a $\xi \in H_0^1(\Omega)$ such that

$$\tilde{\mathbf{w}} = \Psi + \nabla\xi, \quad \text{in } \Omega, \tag{3.12}$$

$$\|\xi\|_{1,\Omega} \le C|\xi|_{1,\Omega} \le C\|\mathbf{w}\|_{\mathbf{H}(\mathbf{curl};\,\Omega_c)}, \tag{3.13}$$

$$|\xi|_{2,\Omega\setminus\overline{\Omega_c}} \le \|\Psi\|_{1,\Omega} \le C\|\mathbf{curl}\,\mathbf{w}\|_{0,\Omega_c}. \tag{3.14}$$

Since $\Omega \setminus \overline{\Omega_c}$ is a Lipschitz domain, by Stein's extension theorem [30, Theorem 5, p. 181], there exists an extension of $\xi|_{\Omega\setminus\overline{\Omega_c}}$ denoted by $\tilde{\xi} \in H^2(\mathbb{R}^3)$ such that

$$\tilde{\xi} = \xi \quad \text{in } \Omega \setminus \overline{\Omega_c} \quad \text{and} \quad \|\tilde{\xi}\|_{2,\mathbb{R}^3} \le C\|\xi\|_{2,\Omega\setminus\overline{\Omega_c}} \le C\|\mathbf{w}\|_{\mathbf{H}(\mathbf{curl};\,\Omega_c)}. \tag{3.15}$$

Define $p := \xi - \tilde{\xi} \in H_0^1(\Omega_c)$ and $\mathbf{v}_s := \Psi + \nabla\tilde{\xi} \in \mathbf{H}^1(\Omega_c) \cap \mathbf{H}_0(\mathbf{curl};\,\Omega_c)$. Combining (3.10)–(3.15) yields

$$\mathbf{v} - \nabla\varphi_0 - \mathbf{v}_{\rm init} = \nabla p + \mathbf{v}_s, \quad \text{in } \Omega_c, \tag{3.16}$$

$$\|p\,\|_{1,\Omega_c} + \|\mathbf{v}_s\|_{1,\Omega_c} \le C\|\mathbf{v} - \nabla\varphi_0 - \mathbf{v}_{\rm init}\|_{\mathbf{H}(\mathbf{curl};\,\Omega_c)} \le C\|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl};\,\Omega)}. \tag{3.17}$$

Extend p and $\mathbf{v}_s$ by zero to the exterior of Ω_c and denote the extensions by the same notations. Define $\varphi := \varphi_0 + p$. Then (3.16) yields (3.4) and (3.7), (3.9), (3.17) yield (3.5). □

REMARK 3.2. *If $\Omega\backslash\overline{\Omega_c}$ in Theorem 3.1 is simply connected, the finite element term $\mathbf{v}_{init}$ in (3.4) and (3.5) will disappear. The decomposition in Theorem 3.1 extends the so-called Birman-Solomyak decomposition of $H_0(\mathbf{curl};\, D)$ for non-convex $D \in \mathbb{R}^3$ in [6], [18], and [27].*

To derive our error estimates, we introduce the Scott-Zhang Operator [29] $\mathcal{I}_n : H^1(\Omega) \to V_n$ and the Beck-Hiptmair-Hoppe-Wohlmuth Operator [4] $\Pi_n :\ \mathbf{H}^1(\Omega_c) \cap \mathbf{H}_0(\mathbf{curl};\, \Omega_c) \to \mathbf{V}_n^c$. $\mathcal{I}_n$ and Π_n satisfy the following approximation and stability properties respectively: For any $\phi_h \in V_n,\ \phi \in H^1(\Omega),\ \mathbf{w}_h \in \mathbf{X}_n$, and $\mathbf{w} \in \mathbf{H}^1(\Omega_c) \cap \mathbf{H}_0(\mathbf{curl};\, \Omega_c)$

$$
\tag{3.18} \left\{ \begin{aligned} &\mathcal{I}_n\phi_h = \phi_h, \\ &\|\nabla \mathcal{I}_n \phi\|_{0,T} \le C\, |\phi|_{1,D_T}, \\ &\|\phi - \mathcal{I}_n\phi\|_{0,T} \le C\, h_T\, |\phi|_{1,D_T}, \\ &\|\phi - \mathcal{I}_n\phi\|_{0,F} \le C\, h_F^{1/2}\, |\phi|_{1,D_F}, \end{aligned} \right.
$$

$$
\tag{3.19} \left\{ \begin{aligned} &\Pi_n\mathbf{w}_h = \mathbf{w}_h, \\ &\|\Pi_n\mathbf{w}\|_{\mathbf{H}(\mathbf{curl};\,T)} \le C\, \|\mathbf{w}\|_{1,D_T}, \\ &\|\mathbf{w} - \Pi_n\mathbf{w}\|_{0,T} \le C\, h_T\, |\mathbf{w}|_{1,D_T}, \\ &\|\mathbf{w} - \Pi_n\mathbf{w}\|_{0,F} \le C\, h_F^{1/2}\, |\mathbf{w}|_{1,D_F}, \end{aligned} \right.
$$

where D_A is the union of elements in $\mathcal{T}_k$ with non-empty intersection with A, $A = T$ or F.

4. Residual based *a posteriori* error estimates. For the sake of convenience, we neglect iterative errors in the solution of linear algebraic systems. Let $\mathbf{H}$ and $\mathbf{H}_n$ be the solutions of (2.7) and (3.1) respectively. For $t \in [t_{n-1}, t_n]$, let $l(t) = (t - t_{n-1})/\tau_n$, and define

$$
\tag{4.1} \mathbf{H}_h(t) := l(t)\, \mathbf{H}_n + (1 - l(t))\, \mathbf{H}_{n-1}.
$$

Hereafter we define the error function by $\mathbf{e}(t) := \mathbf{H}(t) - \mathbf{H}_h(t)$. Combining (2.7) and (3.1), we have

$$
\tag{4.2} \left(\mu\frac{\partial \mathbf{e}}{\partial t},\, \mathbf{v}\right) + (\mathbf{curl}\,(\mathbf{H} - \mathbf{H}_n),\, \mathbf{curl}\,\mathbf{v}) = (\mathbf{f} - \bar{\mathbf{f}}_n,\, \mathbf{v}) + r_n(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{X},
$$

where $\mathbf{f}$ and $\bar{\mathbf{f}}_n$ are defined in the beginning of Section 3, and

$$
r_n(\mathbf{v}) := \left(\bar{\mathbf{f}}_n - \mu\,\frac{\partial \mathbf{H}_h}{\partial t},\, \mathbf{v}\right) - (\mathbf{curl}\,\mathbf{H}_n,\, \mathbf{curl}\,\mathbf{v}).
$$

THEOREM 4.1. *There exists a positive constant C depending only on Ω, μ, and σ such that for any $0 \le m \le M$,*

$$
\tag{4.3} \|\sqrt{\mu}\,\mathbf{e}(t_m)\|_{0,\Omega}^2 + \|\mathbf{curl}\,\mathbf{e}\|^2_{\mathbf{L}^2((0,T);\,\mathbf{L}^2(\Omega))} \le C \sum_{n=1}^{m} \tau_n \left\{ \left(\eta^n_{\text{time}}\right)^2 + \left(\eta^n_{\text{space}}\right)^2 \right\},
$$

where the a posteriori error estimates are given by

$$\begin{aligned}\left(\eta_{\text{time}}^n\right)^2 &= \|\mathbf{curl}(\mathbf{H}_n - \mathbf{H}_{n-1})\|_{0,\Omega_c}^2 + \tau_n^{-1}\|\mathbf{f} - \bar{\mathbf{f}}_n\|_{\mathbf{L}^2((t_{n-1},t_n);\,\mathbf{L}^2(\Omega))}^2,\\ \left(\eta_{\text{space}}^n\right)^2 &= \sum_{T\in\mathcal{T}_n}\left(\eta_{0,T}^n\right)^2 + \sum_{T\in\mathcal{T}_n^c}\left(\eta_{1,T}^n\right)^2 + \sum_{F\in\mathcal{F}_n^{\Omega}}\left(\eta_{0,F}^n\right)^2\\ &\quad + \sum_{F\in\mathcal{F}_n^{\Omega_c}}\left(\eta_{1,F}^n\right)^2 + \sum_{F\in\mathcal{F}_n^{\partial\Omega}}\left(\eta_{0,B,F}^n\right)^2,\end{aligned}$$

with the local error indicators defined by

$$\begin{aligned}\eta_{0,T}^n &:= h_T\left\|\operatorname{div}\left(\bar{\mathbf{f}}_n - \mu\frac{\partial\mathbf{H}_h}{\partial t}\right)\right\|_{0,T},\\ \eta_{1,T}^n &:= h_T\left\|\bar{\mathbf{f}}_n - \mu\frac{\partial\mathbf{H}_h}{\partial t} - \mathbf{curl}(\sigma^{-1}\,\mathbf{curl}\,\mathbf{H}_n)\right\|_{0,T},\\ \eta_{0,F}^n &:= \sqrt{h_F}\left\|\left[\left(\bar{\mathbf{f}}_n - \mu\frac{\partial\mathbf{H}_h}{\partial t}\right)\cdot\mathbf{n}\right]_F\right\|_{0,F},\\ \eta_{1,F}^n &:= \sqrt{h_F}\left\|\left[\sigma^{-1}\mathbf{curl}\,\mathbf{H}_n\times\mathbf{n}\right]_{J,F}\right\|_{0,F},\\ \eta_{0,B,F}^n &:= \sqrt{h_F}\left\|\left(\bar{\mathbf{f}}_n - \mu\frac{\partial\mathbf{H}_h}{\partial t}\right)\cdot\mathbf{n}\right\|_{0,F}.\end{aligned}$$

Here $\mathcal{F}_n^{\Omega}$, $\mathcal{F}_n^{\Omega_c}$, and $\mathcal{F}_n^{\partial\Omega}$ denote the edges in Ω, in Ω_c, and on $\partial\Omega$ respectively.

Proof. It is easy to show the following identity:

$$\begin{aligned}&2\left(\mathbf{curl}\,(\mathbf{H} - \mathbf{H}_n),\,\mathbf{curl}\,\mathbf{e}\right)\\ &= \|\mathbf{curl}\,(\mathbf{H} - \mathbf{H}_n)\|_{0,\Omega_c}^2 + \|\mathbf{curl}\,\mathbf{e}\|_{0,\Omega_c}^2 - \|\mathbf{curl}\,(\mathbf{H}_h - \mathbf{H}_n)\|_{0,\Omega_c}^2.\end{aligned}$$

Taking $\mathbf{v} = \mathbf{e}$ in (4.2), we deduce that

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\mu}\mathbf{e}\|_{0,\Omega}^2 + \|\mathbf{curl}\,(\mathbf{H} - \mathbf{H}_n)\|_{0,\Omega_c}^2 + \|\mathbf{curl}\,\mathbf{e}\|_{0,\Omega_c}^2\\ &= \|\mathbf{curl}\,(\mathbf{H}_h - \mathbf{H}_n)\|_{0,\Omega_c}^2 + 2\,(\mathbf{f} - \bar{\mathbf{f}}_n,\,\mathbf{e}) + 2\,r_n(\mathbf{e}).\end{aligned}\tag{4.4}$$

Integrating (4.4) in time from 0 to $t^* \in (0,\, t_m]$, $m \geq 1$, and using the initial condition, we have

$$\begin{aligned}&\|\sqrt{\mu}\mathbf{e}(t^*)\|_{0,\Omega}^2 + \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n\wedge t^*}\left\{\|\mathbf{curl}\,(\mathbf{H} - \mathbf{H}_n)\|_{0,\Omega_c}^2 + \|\mathbf{curl}\,\mathbf{e}\|_{0,\Omega_c}^2\right\}\\ &= \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n\wedge t^*}\left\{\|\mathbf{curl}\,(\mathbf{H}_h - \mathbf{H}_n)\|_{0,\Omega_c}^2 + 2\,|(\mathbf{f} - \bar{\mathbf{f}}_n,\,\mathbf{e})| + 2\,|r_n(\mathbf{e})|\right\},\end{aligned}\tag{4.5}$$

where $t_n \wedge t^* = \min(t_n,\, t^*)$. By (4.1) and direct calculations, we have

$$\int_{t_{n-1}}^{t_n}\|\mathbf{curl}\,(\mathbf{H}_h - \mathbf{H}_n)\|_{0,\Omega_c}^2 = \frac{\tau_n}{3}\,\|\mathbf{curl}(\mathbf{H}_n - \mathbf{H}_{n-1})\|_{0,\Omega_c}^2\,.\tag{4.6}$$

Let $\mathbf{X}_{\text{init}}$ be the finite element space with respect to the initial partition $\mathcal{T}_{\text{init}}$ of Ω. According to Theorem 3.1, we can decompose $\mathbf{e}$ into

$$\mathbf{e} = \nabla\varphi + \mathbf{e}_{\text{init}} + \mathbf{e}_s,$$

where $\varphi \in H^1(\Omega)/\mathbb{R}^1$, $\mathbf{e}_{\rm init} \in \mathbf{X}_{\rm init}$, and $\mathbf{e}_s \in \mathbf{H}_0(\mathbf{curl};\,\Omega_c) \cap \mathbf{H}^1(\Omega_c)$ satisfy

$$(4.7)\quad \mathbf{e}_s = 0 \ \text{ in } \Omega\backslash\overline{\Omega_c} \quad \text{and} \quad \|\varphi\|_{1,\Omega} + \|\mathbf{e}_s\|_{1,\Omega_c} + \|\mathbf{e}_{\rm init}\|_{\mathbf{H}(\mathbf{curl};\,\Omega)} \le C\,\|\mathbf{e}\|_{\mathbf{H}(\mathbf{curl};\,\Omega)}\,.$$

By virtue of (3.1) and $\mathbf{X}_{\rm init} \subset \mathbf{X}_n$, we have

$$r_n(\mathbf{e}) = r_n(\nabla(\varphi - \mathcal{I}_n\varphi)) + r_n(\mathbf{e}_s - \Pi_n \mathbf{e}_s)\,. \tag{4.8}$$

By the Galerkin orthogonality, the formula of integration by part, (3.18), (3.19), Schwartz's inequality, and (4.7), we have

$$\begin{aligned}
(4.9)\ |r_n(\nabla\varphi - \nabla\mathcal{I}_n\varphi)| &= \left|\int_\Omega \left(\bar{\mathbf{f}}_n - \mu\frac{\partial \mathbf{H}_h}{\partial t}\right)\cdot\nabla(\varphi - \mathcal{I}_n\varphi)\right| \\
\le \sum_{T\in\mathcal{T}_n}\left|\int_T \operatorname{div}\left(\bar{\mathbf{f}}_n - \mu\frac{\partial \mathbf{H}_h}{\partial t}\right)(\mathcal{I}_n\varphi - \varphi)\right| &+ \sum_{F\in\mathcal{F}_n^\Omega}\left|\int_F \left[\bar{\mathbf{f}}_n - \mu\frac{\partial \mathbf{H}_h}{\partial t}\right]_F\cdot\mathbf{n}\,(\varphi - \mathcal{I}_n\varphi)\right| \\
+ \sum_{F\in\mathcal{F}_n^{\partial\Omega}}\left|\int_F \left(\bar{\mathbf{f}}_n - \mu\frac{\partial \mathbf{H}_h}{\partial t}\right)\cdot\mathbf{n}\,(\varphi - \mathcal{I}_n\varphi)\right|, \\
\le C\left\{\sum_{T\in\mathcal{T}_n}(\eta_{0,T}^n)^2 + \sum_{F\in\mathcal{F}_n^\Omega}(\eta_{0,F}^n)^2 + \sum_{F\in\mathcal{F}_n^{\partial\Omega}}(\eta_{0,B,F}^n)^2\right\}^{1/2} &\|\sqrt{\mu}\,\mathbf{e}\|_{\mathbf{H}(\mathbf{curl};\,\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
(4.10)\qquad & |r_n(\mathbf{e}_s - \Pi_n\mathbf{e}_s)| \\
&= \left|\int_{\Omega_c}\left\{\left(\bar{\mathbf{f}}_n - \mu\frac{\partial \mathbf{H}_h}{\partial t}\right)\cdot(\mathbf{e}_s - \Pi_n\mathbf{e}_s) - \mathbf{curl}\,\mathbf{H}_n\cdot\mathbf{curl}\,(\mathbf{e}_s - \Pi_n\mathbf{e}_s)\right\}\right| \\
&\le \sum_{T\in\mathcal{T}_n^c}\left|\int_T\left\{\bar{\mathbf{f}}_n - \mu\frac{\partial \mathbf{H}_h}{\partial t} - \mathbf{curl}(\sigma^{-1}\,\mathbf{curl}\,\mathbf{H}_n)\right\}\cdot(\mathbf{e}_s - \Pi_n\mathbf{e}_s)\right| \\
&\quad + \sum_{F\in\mathcal{F}_n^{\Omega_c}}\left|\int_F\left[\sigma^{-1}\,\mathbf{curl}\,\mathbf{H}_n\times\mathbf{n}\right]_F\cdot(\mathbf{e}_s - \Pi_n\mathbf{e}_s)\right|, \\
&\le C\left\{\sum_{T\in\mathcal{T}_n^c}\left(\eta_{1,T}^n\right)^2 + \sum_{F\in\mathcal{F}_n^{\Omega_c}}\left(\eta_{1,F}^n\right)^2\right\}^{1/2}\|\sqrt{\mu}\,\mathbf{e}\|_{\mathbf{H}(\mathbf{curl};\,\Omega)}.
\end{aligned}$$

Combing (4.8)–(4.10) yields

$$|r_n(\mathbf{e})| \le C\,\eta_{\rm space}^n\,\|\sqrt{\mu}\,\mathbf{e}\|_{\mathbf{H}(\mathbf{curl};\,\Omega)}.$$

Thus we have

$$(4.11)\int_{t_{n-1}}^{t_n}\left\{|(\mathbf{f} - \bar{\mathbf{f}}_n,\,\mathbf{e})| + |r_n(\mathbf{e})|\right\} \le \int_{t_{n-1}}^{t_n}\left(\|\mathbf{f} - \bar{\mathbf{f}}_n\|_{0,\Omega} + \eta_{\rm space}^n\right)\|\sqrt{\mu}\,\mathbf{e}\|_{\mathbf{H}(\mathbf{curl};\,\Omega)}.$$

Substituting (4.6) and (4.11) into (4.5), for any $t^* \in (t_{m-1},\, t_m]$, we have

$$
\begin{aligned}
(4.12)\quad & \|\sqrt{\mu}\mathbf{e}(t^*)\|_{0,\Omega}^2 + \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n \wedge t^*} \left\{ \|\mathbf{curl}\,(\mathbf{H}-\mathbf{H}_n)\|_{0,\Omega_c}^2 + \|\mathbf{curl}\,\mathbf{e}\|_{0,\Omega_c}^2 \right\} \\
& \le \frac{1}{3} \sum_{n=1}^{m} \tau_n \,\|\mathbf{curl}(\mathbf{H}_n - \mathbf{H}_{n-1})\|_{0,\Omega_c}^2 + C \sum_{n=1}^{m} \tau_n \left(\eta_{\rm space}^n\right)^2 \\
& \quad + C \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n \wedge t^*} \|\mathbf{f} - \bar{\mathbf{f}}_n\|_{0,\Omega}^2 + \frac{1}{2} \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n \wedge t^*} \|\mathbf{curl}\,\mathbf{e}\|_{0,\Omega_c}^2 + \frac{1}{2} \max_{0 \le t \le t^*} \|\sqrt{\mu}\mathbf{e}(t)\|_{0,\Omega}^2.
\end{aligned}
$$

For any m, we may choose $t^* \in [0,\, t_m]$ such that $\|\sqrt{\mu}\mathbf{e}(t^*)\|_{0,\Omega}^2 = \max_{0 \le t \le t_m} \|\sqrt{\mu}\mathbf{e}(t)\|_{0,\Omega}^2$ first to get the estimate for $\|\sqrt{\mu}\mathbf{e}(t^*)\|_{0,\Omega}^2$, then use (4.12) again to obtain (4.3). $\square$

REMARK 4.2. *$\eta_{1,T}^n$ and $\eta_{0,T}^n$ are the residuals of the second and third equations of (1.1) respectively. $\eta_{1,F}^n$ and $\eta_{0,F}^n$ reflect the continuity conditions of the electromagnetic fields. $\eta_{0,B,F}^n$ is the boundary residual of the last equation of (2.2).*

To show the sharpness of the spatial error estimator, we need to bound $\eta_{\rm space}^n$ by the error between the discrete solution and the continuous solution. In view of (3.1), we observe that for fixed t_n and time-step size τ_n, we are essentially controlling the error between $\mathbf{H}_n$ and $\mathbf{H}_n^*$ by adapting the current mesh $\mathcal{T}_n$. Here $\mathbf{H}_n^*$ is the solution of the following continuous problem: Given the discrete solution $\mathbf{H}_{n-1} \in \mathbf{X}_{n-1}$ at t_{n-1}, find $\mathbf{H}_n^* \in \mathbf{X}$ such that

$$
\left(\mu \frac{\mathbf{H}_n^* - \mathbf{H}_{n-1}}{\tau_n},\, \mathbf{v} \right) + \int_{\Omega_c} \mathbf{curl}\,\mathbf{H}_n^* \cdot \mathbf{curl}\,\mathbf{v} = (\bar{\mathbf{f}}_n,\, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{X}.
$$

Similar to the arguments in [4, Section 5] and [11], we obtain the following theorem of lower bound estimates in terms of $\mathbf{H}_n$ and $\mathbf{H}_n^*$. The proof is omitted.

THEOREM 4.3. *Let μ and σ be piecewise constants. Then there exists a constant C independent of the mesh $\mathcal{T}_n$ such that*

$$
\begin{aligned}
(4.13)\quad \left(\eta_{\rm space}^n\right)^2 \le\; & C\tau_n^{-2} \|\mathbf{H}_n^* - \mathbf{H}_n\|_{0,\Omega}^2 + C \|\mathbf{curl}(\mathbf{H}_n^* - \mathbf{H}_n)\|_{0,\Omega_c}^2 \\
& + C \sum_{T \in \mathcal{T}_n^c} h_T^2 \|\mathbf{g}_n - \mathcal{Q}_T \mathbf{g}_n\|_{0,T}^2 + C \sum_{F \in \mathcal{F}_n^{\Omega}} h_F \,\|(I - \mathcal{Q}_F)[\mathbf{g}_n \cdot \mathbf{n}]_F\|_{0,F}^2 \\
& + C \sum_{T \in \mathcal{T}_n} h_T^2 \|\mathrm{div}\,(\mathbf{g}_n - \mathcal{P}_T \mathbf{g}_n)\|_{0,T}^2,
\end{aligned}
$$

where $\mathbf{g}_n := \bar{\mathbf{f}}_n - \mu\, \partial \mathbf{H}_h / \partial t$, $\mathcal{Q}_T$: $\mathbf{L}^2(T) \to \mathbf{P}_2(T)$ and $\mathcal{Q}_F$: $\mathbf{L}^2(F) \to \mathbf{P}_2(F)$ are $\mathbf{L}^2$-projections, $\mathcal{P}_T$: $\mathbf{H}(\mathrm{div};\, T) \to \mathbf{P}_1(T)$ is the $\mathbf{H}(\mathrm{div})$-projection. $\mathbf{P}_k(D)$ is the space of vector polynomials of maximal degree k defined on D for $D = T$ or F.

REMARK 4.4. *The last three terms in the righthand side of (4.13) are higher order than $\tau_n^{-1} \|\mathbf{H}_n^* - \mathbf{H}_n\|_{0,\Omega} + \|\mathbf{curl}(\mathbf{H}_n^* - \mathbf{H}_n)\|_{0,\Omega_c}$, supposing that $\mathcal{T}_n = \mathcal{T}_{n-1}$ and the source field $\mathbf{H}_s$ is smooth enough in time. It reflects the coarseness of known datum $\mathbf{H}_{n-1}$ and $\mathbf{H}_s$ on the current mesh $\mathcal{T}_n$.*

5. Adaptive algorithm and numerical results. The implementation of our adaptive algorithm is based on the adaptive finite element package ALBERT [28] and is carried out on Origin 3800.

We define the local *a posteriori* error estimator over an element $T \in \mathcal{T}_n$ by

$$\eta_T := \left\{ C_0 \left(\eta_{0,T}^n\right)^2 + C_0 \left(\eta_{1,T}^n\right)^2 + \frac{C_1}{2} \sum_{F \subset \partial T} \left[\left(\eta_{0,F}^n\right)^2 + \left(\eta_{1,F}^n\right) \right]^2 \right\}^{1/2},$$

where $\eta_{0,F} = \eta_{0,B,F}$ if $F \subset \partial\Omega$. Define the time error estimate, the global spacial error estimate, the maximal element error estimate over $\mathcal{T}_n$ respectively by

$$\eta_{\text{time}}^n := \left\{ \|\mathbf{curl}(\mathbf{H}_n - \mathbf{H}_{n-1})\|_{0,\Omega_c}^2 + \tau_n^{-1} \|\mathbf{f} - \bar{\mathbf{f}}_n\|_{\mathbf{L}^2((t_{n-1},t_n);\,\mathbf{L}^2(\Omega))}^2 \right\}^{1/2},$$

$$\eta_{\text{space}}^n := \left(\sum_{T \in \mathcal{T}_n} \eta_T^2 \right)^{1/2}, \qquad \eta_{\max}^n = \max_{T \in \mathcal{T}_n} \eta_T.$$

In real computations, we choose $C_0 = 100$ and $C_1 = 1$. Denote the interpolation operator of Nédélec's lowest order edge element over the mesh $\mathcal{T}_n$ by Υ_n. Now we describe the adaptive algorithm used in this paper. For similar time and space adaptive strategies, we refer to the documentation of ALBERT [28] and references therein.

Algorithm 5.1. (Time and space adaptive algorithm)

Given the end time $t_{\mathsf{end}} > 0$, an initial coarse triangulation of Ω denoted by $\mathcal{T}_{\mathsf{init}}$, positive tolerances $\mathtt{TOL}_{\mathsf{init}}$, $\mathtt{TOL}_{\mathsf{time}}$, $\mathtt{TOL}_{\mathsf{space}}$, parameters $\delta_1 \in (0,1)$, $\delta_2 > 1$, and $\theta_{\mathsf{time}} \in (0,\ 1)$.

1. Mesh refinements at t_0:
   - set the initial solution by $\mathbf{H}_0 := \Upsilon_0 \mathbf{H}(0,\cdot)$ over the mesh $\mathcal{T}_0 := \mathcal{T}_{\mathsf{init}}$
   - While $\|\mathbf{H}(0,\cdot) - \mathbf{H}_0\|_{\mathbf{H}(\mathbf{curl};\,\Omega)} > \mathtt{TOL}_{\mathsf{init}}$ do

     Refine each element $T \in \mathcal{T}_0$ satisfying

     $$\|\mathbf{H}(0,\cdot) - \mathbf{H}_0\|_{\mathbf{H}(\mathbf{curl};\,T)} > 0.6 \max_{T \in \mathcal{T}_0} \|\mathbf{H}(0,\cdot) - \mathbf{H}_0\|_{\mathbf{H}(\mathbf{curl};\,T)}$$

   end while
2. While $t_n \le t_{\mathsf{end}}$ do
   (I) given $\mathbf{H}_{n-1}$, $\mathcal{T}_{n-1}$, and the timestep size τ_{n-1} from the previous time step
   - $\mathcal{T}_n := \mathcal{T}_{n-1}$, $\tau_n := \tau_{n-1}$, $t_n := t_{n-1} + \tau_n$
   - solve the discrete problem (3.1) on $\mathcal{T}_n$ using known data $\mathbf{H}_{n-1}$
   - compute the time error estimate η_{time}^n, the local error estimator η_T on each $T \in \mathcal{T}_n$, the global error estimate η_{space}^n, and the maximal element error estimate η_{max}^n

   (II) while $\eta_{\mathsf{time}}^n > \mathtt{TOL}_{\mathsf{time}}/\sqrt{t_{\mathsf{end}}}$ do
   - $\tau_n := \delta_1 \tau_n$, $t_n := t_{n-1} + \tau_n$
   - solve the discrete problem (3.1) on $\mathcal{T}_n$
   - compute η_{time}^n, η_{space}^n, η_{max}^n, and η_T on each $T \in \mathcal{T}_n$

   end while

   (III) while $\eta_{\mathsf{space}}^n > \mathtt{TOL}_{\mathsf{space}}/\sqrt{t_{\mathsf{end}}}$ do
   - mark each element $T \in \mathcal{T}_n$ for refinement if $\eta_T > 0.6\,\eta_{\mathsf{max}}^n$ and for coarsening if $\eta_T < 0.1\,\eta_{\mathsf{max}}^n$
   - if elements are marked then
     - adapt mesh $\mathcal{T}_n$ to produce a modified $\mathcal{T}_n$
     - solve the discrete problem (3.1) on $\mathcal{T}_n$

– compute η^n_{time}, η^n_{space}, η^n_{max}, and η_T on each $T \in \mathcal{T}_n$
end if

- while $\eta^n_{\mathsf{time}} > \mathtt{TOL}_{\mathsf{time}}/\sqrt{t_{\mathsf{end}}}$ do
  – $\tau_n := \delta_1 \tau_n, \quad t_n := t_{n-1} + \tau_n$
  – solve the discrete problem (3.1) on $\mathcal{T}_n$
  – compute η^n_{time}, η^n_{space}, η^n_{max}, and η_T on each $T \in \mathcal{T}_n$

  end while

end while

(IV) if $\eta^n_{\mathsf{time}} < \theta_{\mathsf{time}} \cdot \mathtt{TOL}_{\mathsf{time}}/\sqrt{t_{\mathsf{end}}}$ then $\tau_n := \delta_2\, \tau_n$

end while.

In the following, we report two numerical experiments to demonstrate the competitive behavior of the proposed method.

Example 5.1. We consider the problem (2.7) defined on the three-dimensional domain $\Omega = (-1,\, 1)^3$. The conducting region is an “L-shaped” domain

$$\Omega_c = (-0.5,\, 0.5)^3 \setminus \{(0.0,\, 0.5) \times (0.0,\, 0.5) \times (-0.5,\, 0.5)\}.$$

Let $\mu \equiv 1.0$ and $\sigma \equiv 100.0$. The righthand side $\mathbf{H}_s$ is so chosen that the exact solution of (2.7) is $\mathbf{H}(\mathbf{x},\, t) = s(t)\, \nabla\psi(\mathbf{x})$, where

$$s(t) = \mathrm{sign}(0.5 - t) \times \left[1 - e^{-10000\times(t-0.5)^2}\right],$$

$$\psi(\mathbf{x}) = \begin{cases} 0 & \text{if} \quad |x_2| \geq 0.5, \\ (1-x_1^2)^2\,(1-x_2^2)^2\,(0.25-x_3^2)^2\,\sqrt{\frac{r-x_1}{2}} & \text{elsewhere,} \end{cases}$$

and $r^2 = x_1^2 + x_2^2$ in cylindrical coordinates. Fig.5.1 shows the graph of the function $s(t)$ which varies very rapidly near $t = 0.5$.

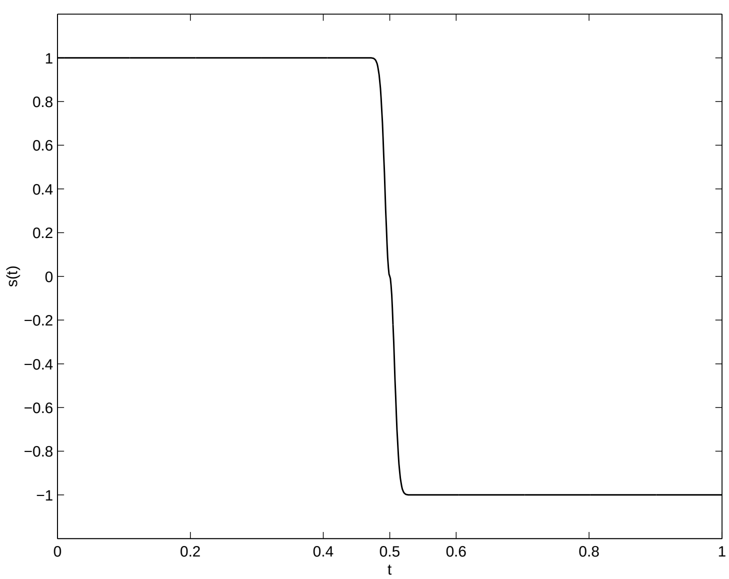

Fig. 5.1. *The graph of function $s(t)$.*

Fig. 5.2 shows the time step sizes at different time. We observe that the time step sizes are very small near $t = 0.5$ where the solution varies very rapidly in time.

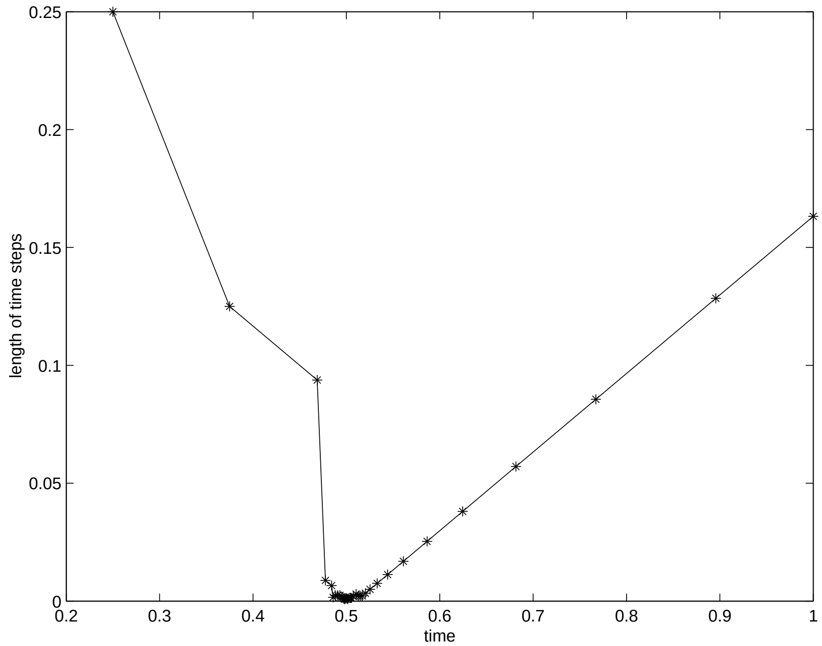

FIG. 5.2. *The time step sizes at different time.*

Fig. 5.3 shows the curve of $\log \|\mathbf{H}(t_M) - \mathbf{H}_M\|_{\mathbf{H}(\mathbf{curl};\,\Omega)}$ versus $\log N_{\text{total}}$, where $t_M = 1.0$ is the final time, M is the number of time steps, $N_{\text{total}} = \sum_{n=1}^{M} N_n$ is the total number of elements of all time steps, and N_n is the number of elements of $\mathcal{T}_n$. Fig. 5.4 shows the curve of $\log E_{\text{total}}$ versus $\log N_{\text{total}}$, where E_{total} is the total energy error defined by

$$E_{\text{total}}^2 = \sum_{n=1}^{M} \tau_n \|\mathbf{H}(t_n) - \mathbf{H}_n\|^2_{\mathbf{H}(\mathbf{curl};\,\Omega)},$$

and $\mathbf{H}_n$ is the solution of (3.1). Fig.5.5 shows the curve of $\log \eta_{\text{total}}$ versus $\log N_{\text{total}}$, where η_{total} is the total error estimate defined by

$$\eta_{\text{total}}^2 = \sum_{n=1}^{M} \tau_n \left\{ \left(\eta_{\text{time}}^n\right)^2 + \left(\eta_{\text{space}}^n\right)^2 \right\}.$$

They indicate that the adaptive meshes and the associated numerical complexity are quasi-optimal, i.e.

$$\|\mathbf{H}(t_M) - \mathbf{H}_M\|_{\mathbf{H}(\mathbf{curl};\,\Omega)} \approx C\, N_{\text{total}}^{-1/4}, \qquad E_{\text{total}} \approx C\, N_{\text{total}}^{-1/4}, \qquad \eta_{\text{total}} \approx C\, N_{\text{total}}^{-1/4}.$$

Fig. 5.6 shows an adaptive mesh of 773,736 elements on Ω at the final time t_M after 61 adaptive iterations in time and space. We observe that the mesh is locally refined near the segment $\{x_1 = x_2 = 0,\, -0.5 < x_3 < 0.5\}$ where the solution is singular.

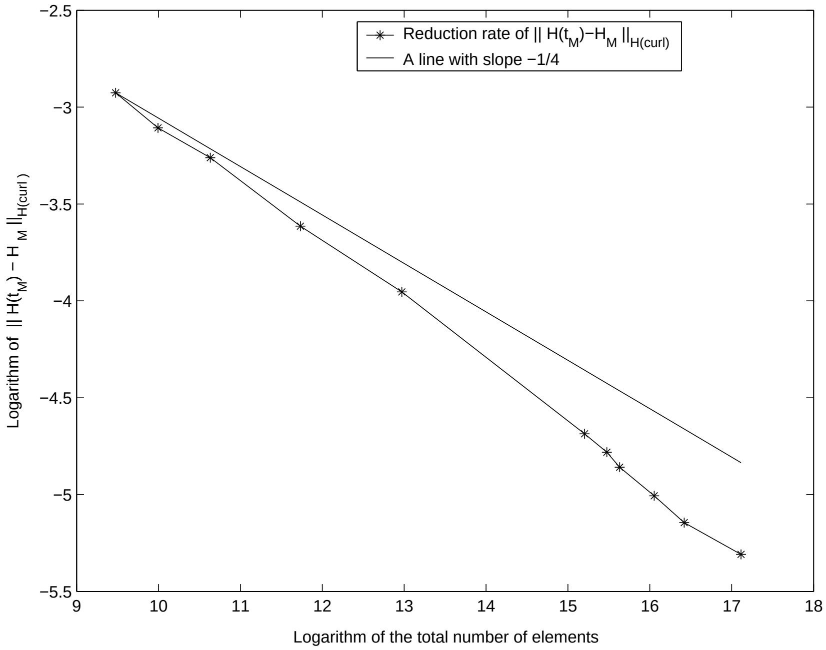

FIG. 5.3. *Quasi-optimality of the adaptive mesh refinements of the error at the final time* $\|\mathbf{H}(t_M) - \mathbf{H}_M\|_{\mathbf{H}(\mathbf{curl};\,\Omega)}$ *(Example 5.1).*

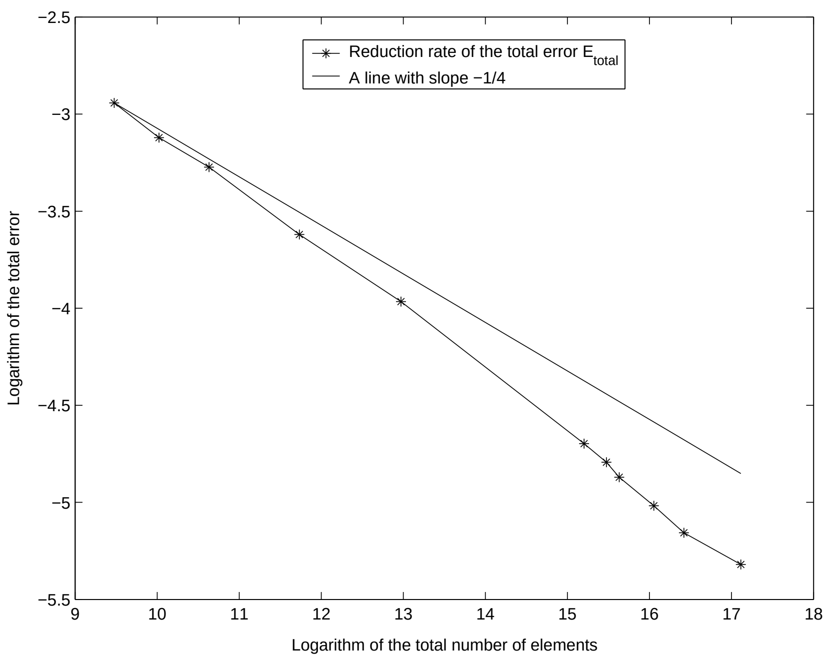

FIG. 5.4. *Quasi-optimality of the adaptive mesh refinements of the total error* E_{total} *(Example 5.1).*

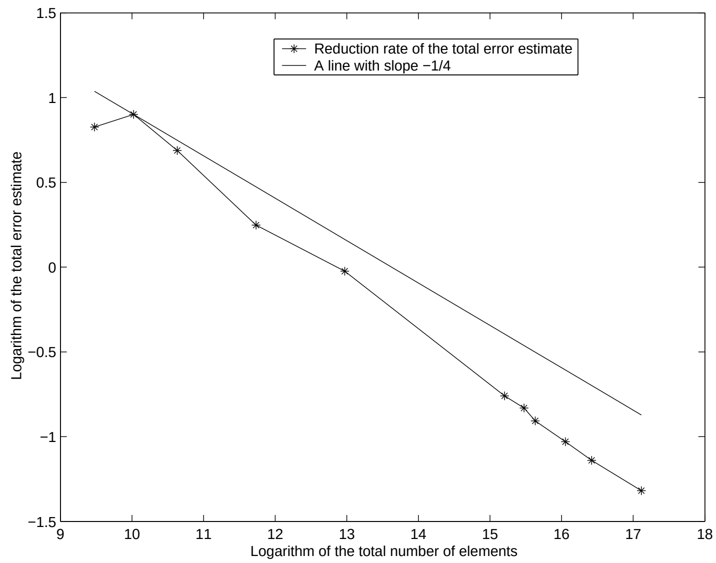

FIG. 5.5. *Quasi-optimality of the adaptive mesh refinements of the total error estimate* η_{total} *(Example 5.1).*

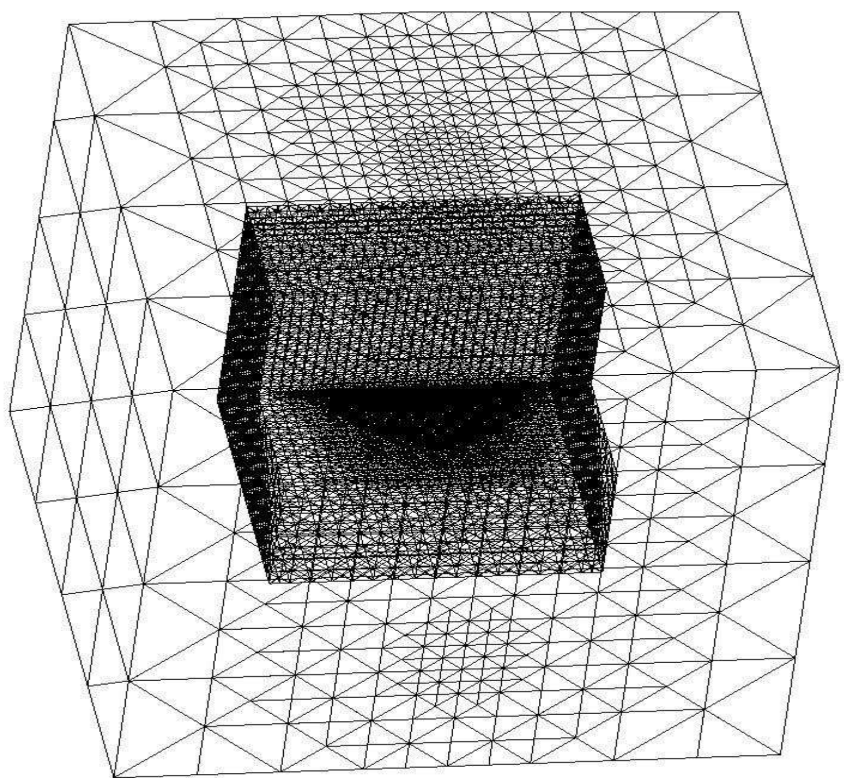

FIG. 5.6. *An adaptively refined mesh of 773,736 elements at the final time* t_M *after 61 adaptive iterations in time and space (Example 5.1).*

Example 5.2. We compute the Team Workshop Problem 7. This problem consists of an aluminum plate with a hole above which a racetrack shaped coil is placed (see Fig.5.7). The aluminum plate has a conductivity of 3.526×10^7 Siemens/Metre and the sinal driving current of the coil is 2742 Ampere/Turn. The frequency of the driving current is $\omega = 50$ Hertz.

Since the driving current is time-harmonic, most numerical methods developed for this problem are frequency domain methods. We remark that our method is applicable to many time-dependent electromagnetic problems with three-dimensional multiply connected geometry. We set Ω to be a cubic domain with one-meter edges and start the computation with zero initial value. The result becomes steady after one period. We compare the peak values of the vertical magnetic flux μH_z with measured values on some points. These points are located at $y = 72\,\text{mm}$, $z = 34\,\text{mm}$, and $x = (18 \times i)\,\text{mm}$ where $i = 0, \cdots, 16$ (see Fig. 5.7).

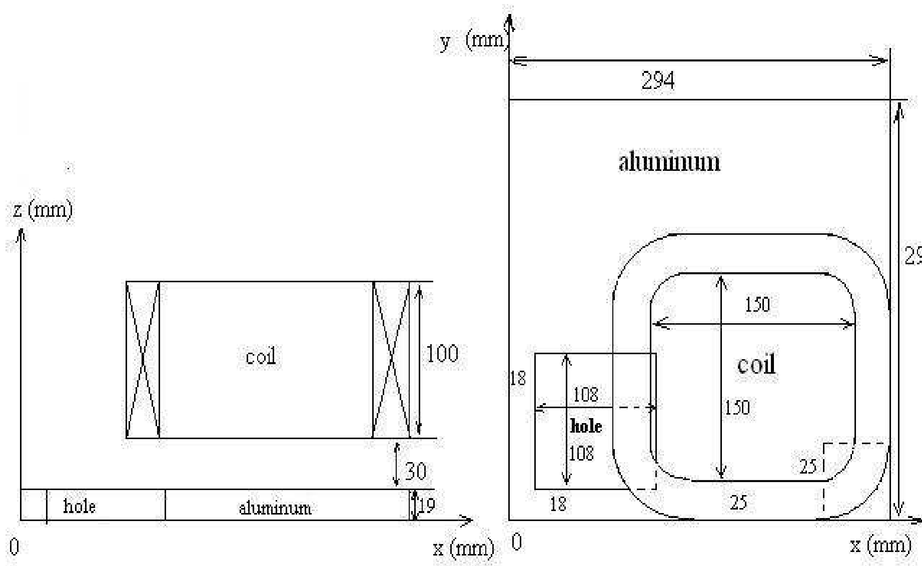

Fig. 5.7. *The geometry of Team Workshop Problem 7 in frontal view with specified positions. All geometry dimensions are given in mm.*

Fig.5.8–5.11 show the numerical values of μH_z on measured points at time $t_M = \mathrm{T} = 2.75\,periods$, which are obtained on different adaptive meshes. With the number of degrees of freedom increasing, they coincide with the experimental values better and better. Fig.5.11 shows a very good agreement with the measured values.

Fig.5.12 shows the curve of $\log \eta_{\text{total}}$ versus $\log N_{\text{total}}$. It indicates that the adaptive method based on our *a posteriori* error estimates has the very desirable quasi-optimality property:

$$\eta_{\text{total}} \approx C\, N_{\text{total}}^{-1/4}$$

is valid asymptotically.

Fig. 5.13 shows an adaptively refined mesh of 2,263,668 elements after 18 adaptive iterations from 77,760 initial elements. We observe that the mesh is locally refined on the surface of the conductor.

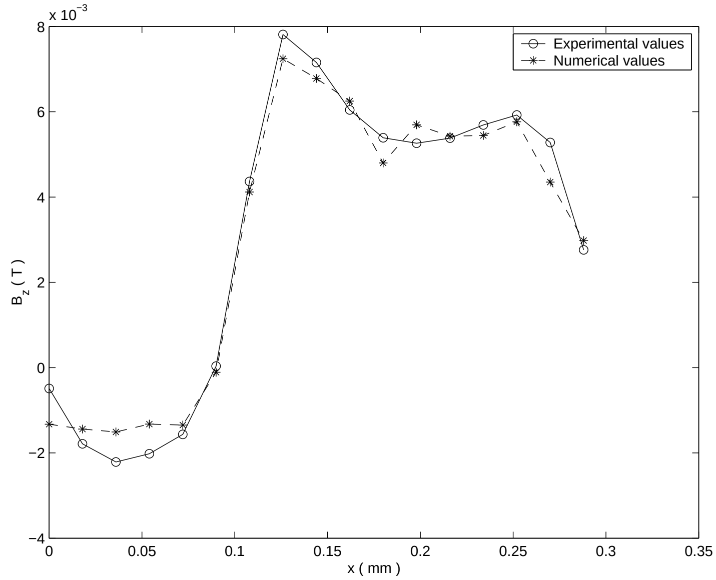

FIG. 5.8. *Numerical values of* μH_z *with* $M = 55$, $N_{total} = 5,555,550$, $\eta_{total} = 0.0126$, *the number of degrees of freedom on* $\mathcal{T}_M$ *is 36,714.*

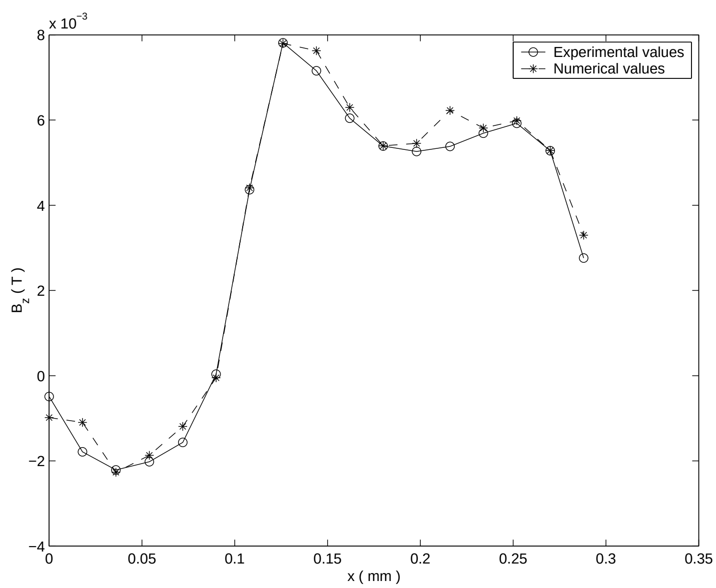

FIG. 5.9. *Numerical values of* μH_z *with* $M = 55$, $N_{total} = 16,152,290$, $\eta_{total} = 0.0105$, *the number of degrees of freedom on* $\mathcal{T}_M$ *is 120,558.*

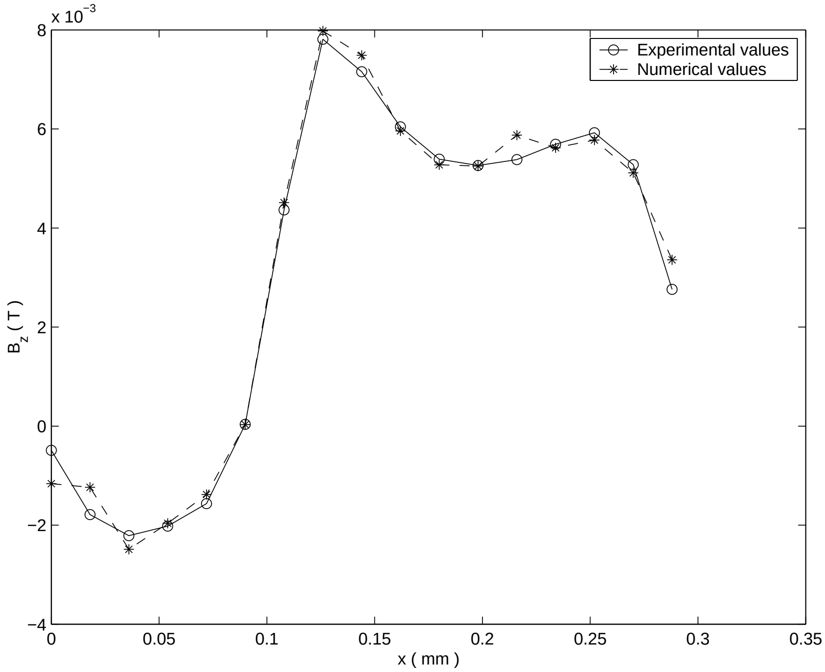

FIG. 5.10. *Numerical values of* μH_z *with* $M = 110$, $N_{total} = 73,068,160$, $\eta_{total} = 0.0065$, *the number of degrees of freedom on* $\mathcal{T}_M$ *is 277,883.*

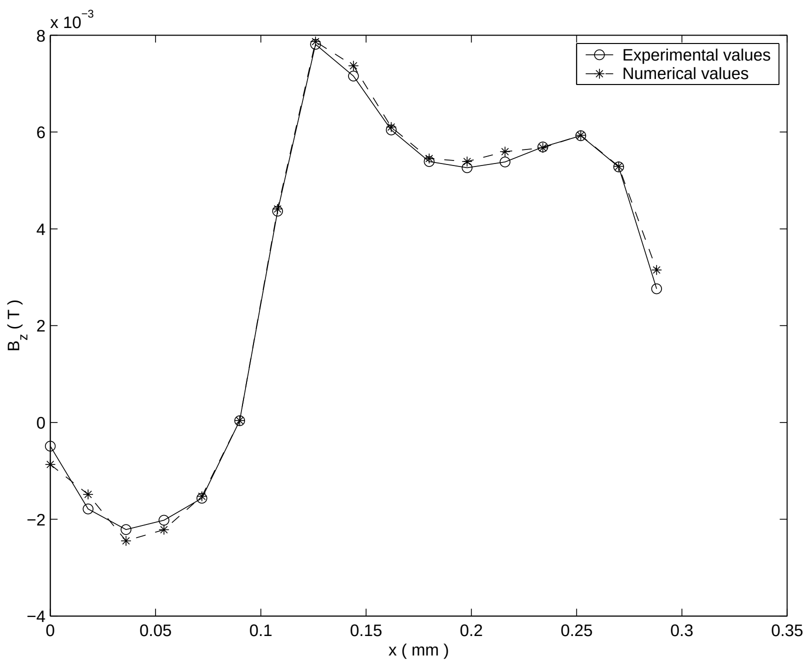

FIG. 5.11. *Numerical values of* μH_z *with* $M = 110$, $N_{total} = 249,003,480$, $\eta_{total} = 0.0047$, *the number of degrees of freedom on* $\mathcal{T}_M$ *is 873,971.*

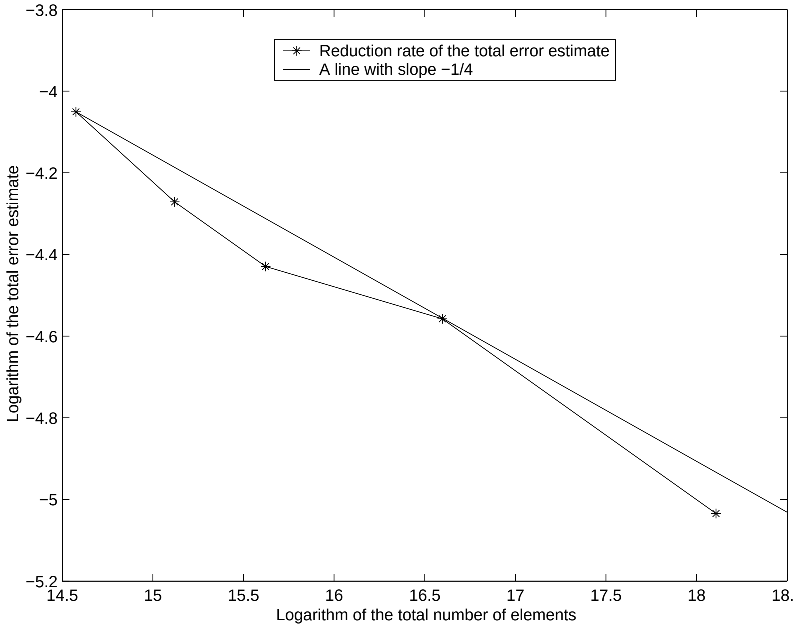

Fig. 5.12. *Quasi-optimality of the adaptive mesh refinements of the total a posteriori error estimate.*

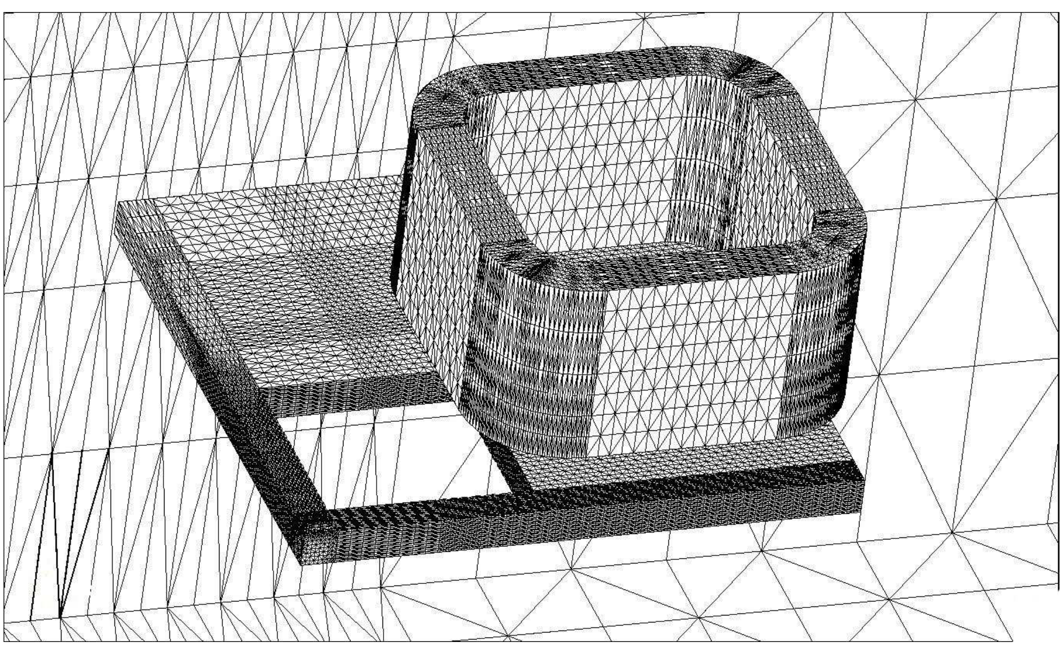

Fig. 5.13. *An adaptively refined mesh of 2,263,668 elements after 18 adaptive iterations from 77,760 initial elements.*

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