LONG-TIME STABILITY AND CONVERGENCE OF THE UNIAXIAL PERFECTLY MATCHED LAYER METHOD FOR TIME-DOMAIN ACOUSTIC SCATTERING PROBLEMS

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Abstract. The uniaxial perfectly matched layer (PML) method uses rectangular domain to define the PML problem and thus provides greater flexibility and efficiency in dealing with problems involving anisotropic scatterers. In this paper we first derive the uniaxial PML method for solving the time-domain scattering problem based on the Laplace transform and the complex coordinate stretching in the frequency domain. We prove the long-time stability of the initial-boundary value problem of the uniaxial PML system for piecewise constant medium property and show the exponential convergence of the time-domain uniaxial PML method. Our analysis shows that for fixed PML absorbing medium property, any error of the time-domain PML method can be achieved by enlarging the thickness of the PML layer as $\ln T$ for large $T > 0$. Numerical experiments are included to illustrate the efficiency of the PML method.

Key words. Uniaxial perfectly matched layer, time-domain scattering, convergence, stability.

1. Introduction. We propose and study a uniaxial perfectly matched layer (PML) method for solving the acoustic scattering problem with the sound-hard boundary condition on the obstacle:

$$\frac{\partial u}{\partial t} = -\mathrm{div}\mathbf{p} + f(x,t), \quad \frac{\partial \mathbf{p}}{\partial t} = -\nabla u \qquad \text{in} \ \ (\mathbb{R}^2 \backslash \bar{D}) \times (0,T), \tag{1.1}$$

$$\mathbf{p} \cdot \mathbf{n}_D = 0 \qquad \text{on} \ \ \Gamma_D \times (0,T), \tag{1.2}$$

$$\sqrt{r}(u - \mathbf{p} \cdot \hat{x}) \to 0, \qquad \text{as} \ \ r = |x| \to \infty, \ \ \text{a.e.} \ \ t \in (0,T), \tag{1.3}$$

$$u|_{t=0} = u_0, \quad \mathbf{p}|_{t=0} = \mathbf{p}_0. \tag{1.4}$$

Here u is the pressure and $\mathbf{p}$ is the velocity field of the wave. $D \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary Γ_D and $\mathbf{n}_D$ is the unit outer normal to Γ_D. $f, u_0, \mathbf{p}_0$ are assumed to be supported in the circle $B_R = \{x \in \mathbb{R}^2 : |x| < R\}$ for some $R > 0$. (1.3) is the radiation condition which corresponds to the well-known Sommerfeld radiation condition in the frequency domain. We remark that the results in this paper can be easily extended to solve scattering problems with other boundary conditions such as the sound-soft or the impedance boundary conditions on Γ_D.

Since the work of Bérenger [5] which proposed a PML technique for solving Maxwell equations, various constructions of PML absorbing layers have been proposed and studied in the literature (cf. e.g. Turkel and Yefet [27], Teixeira and Chew [26] for the reviews). The basic idea of the PML technique is to surround the computational domain by a layer of finite thickness with specially designed model medium that absorbs all the waves that propagate from inside the computational domain.

The initial Bérenger's PML method is based on the splitting of the electromagnectic fields, which is later proved by Abarbanel and Gottlieb [1] that it is only weakly well-posed and thus may suffer instability in practical applications. The other

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so-called "unsplit-field" PML formulations are strongly well-posed, including the uniaxial PML method developed in Sacks *et al* [25], Zhao-Cangellaris [29], and Gedney [16] for the Maxwell equations and a second order PML formulation in Grote and Sim [17] for wave equations. In curvilinear coordinates, the split-field PML method is introduced in Collino and Monk [13] and the unsplit-field PML methods are introduced in Petropoulous [24] and [26] for Maxwell equations.

Although the PML approach has been proved very successful in the practical applications, there are few mathematical results on the convergence of the PML methods. In the frequency domain, the convergence is studied in Lassas and Somersalo [21], Hohage *et al* [19] for the acoustic scattering problems with the circular layers and in Chen and Wu [10], Kim and Pasciak [20], Chen and Zheng [11] with the uniaxial PML layers. It is proved in [21, 19, 10, 11] that the PML solution converges exponentially to the solution of the original scattering problem as the thickness of the PML layer tends to infinite. In Chen and Wu [9], Chen and Liu [8], an adaptive PML technique is proposed and studied in which a posteriori error estimate is used to determine the PML parameters. For the time-domain PML method, the planar PML method in one space direction is considered in Hagstrom [18] for the wave equation. In de Hoop *et al* [14], Diaz and Joly [15], the PML system with point source is analyzed based on the Cagniard - de Hoop method. In Chen [7], the convergence of the time-domain PML method with circular layer is proved by using the exponential decay esitmate of the modified Bessel functions.

The long-time stability of the PML method is also a much studied topic in the literature (see e.g. Bécache and Joly [3], Bécache *et al* [4], Appelö *et al* [2]). For a PML method to be practically useful, it must be stable in time, that is, the solution should not grow exponentially in time. We remark that the well-posedness of the PML system which follows from the theory of symmetric hyperbolic systems allows the exponential growth of the solutions. In [3, 4, 2, 17] the stability of the Cauchy problem of the PML systems is considered separately in each part of the domain where the PML medium property is assumed to be constant.

In this paper we first prove the long-time stability of the initial-boundary value problem of the uniaxial PML system in the PML layer for piecewise constant PML medium properties. The proof is based on an energy argument which is inspired by the method in [3] and the stability analysis of uniaxial PML method in the frequency domain in [11]. We also remark that the stability analysis of the PML method in [20] for the Helmholtz equation is difficult to be used for the time domain analysis as the dependence of the constant in the inf-sup condition on the wave number is not explicit. The second purpose of the paper is to prove the convergence of the time-domain uniaxial PML method. Our technique to prove the PML convergence is different from that for circular PML layer [7]. It is based on the stability estimate for the initial-boundary value problem of the PML system in the first part of the paper and an exponential decay estimate of the PML extension in the time domain. This estimate is derived by using the method of Laplace transform, the integral representation of the exterior Dirichlet problem for the Helmholtz equation, and the idea of the complex coordinate stretching.

The layout of the paper is as follows. In section 2 we derive the uniaxial PML formulation for (1.1)-(1.4) by using the method of complex coordinate stretching of Chew and Weedon [12] in the frequency domain, and returning to the time domain by using the inverse Laplace transform. In section 3 we show the stability of the initial boundary value problem of the PML system in the PML layer. In section 4 we prove

the exponential decay of the PML extension based on the property of the modified Bessel function and the potential theory. The results in Sections 3 and 4 are used to prove the convergence of the PML method in Section 5. In section 6 we present two examples to show the performance of the PML method.

2. The PML equation. For any $s \in \mathbb{C}$ such that $\mathrm{Re}\,(s) > 0$, let $u_{\mathrm{L}} = \mathscr{L}(u)$ and $\mathbf{p}_{\mathrm{L}} = \mathscr{L}(\mathbf{p})$ be the Laplace transform of u and $\mathbf{p}$ in time

$$u_{\mathrm{L}}(x,s) = \int_0^\infty e^{-st} u(x,t)\,\mathrm{d}t, \quad \mathbf{p}_{\mathrm{L}}(x,s) = \int_0^\infty e^{-st}\mathbf{p}(x,t)\,\mathrm{d}t.$$

Since $\mathscr{L}(\partial_t u) = s u_{\mathrm{L}} - u_0$ and $\mathscr{L}(\partial_t \mathbf{p}) = s\mathbf{p}_{\mathrm{L}} - \mathbf{p}_0$, by taking the Laplace transform of (1.1) we get

$$s u_{\mathrm{L}} - u_0 = -\mathrm{div}\mathbf{p}_{\mathrm{L}} + f_{\mathrm{L}}, \quad s\mathbf{p}_{\mathrm{L}} - \mathbf{p}_0 = -\nabla u_{\mathrm{L}} \quad \text{in} \quad \mathbb{R}^2\backslash\bar{D}, \tag{2.1}$$

where $f_{\mathrm{L}} = \mathscr{L}(f)$.

Let D be contained in the interior of the rectangle $B_1 = \{x \in \mathbb{R}^2 : |x_1| < L_1/2, |x_2| < L_2/2\}$. Let $\Gamma_1 = \partial B_1$ and $\mathbf{n}_1$ the unit outer normal to Γ_1. Because $f, u_0, \mathbf{p}_0$ are all supported inside B_1, we know that u_{L} satisfies the Helmholtz equation outside B_1

$$-\Delta u_{\mathrm{L}} + s^2 u_{\mathrm{L}} = 0 \qquad \text{in} \quad \mathbb{R}^2\backslash\bar{B}_1. \tag{2.2}$$

Moreover, (1.3) implies that u_{L} satisfies the radiation condition

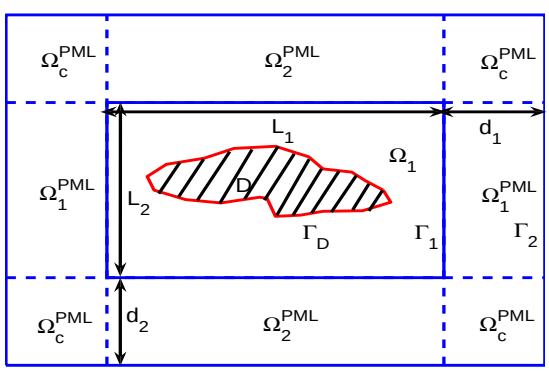

FIG. 2.1. *Setting of the scattering problem with the PML layer.*

$$\sqrt{r}\left(\frac{\partial u_{\mathrm{L}}}{\partial r} + s u_{\mathrm{L}}\right) \to 0, \qquad \text{as} \quad r \to \infty.$$

Let $T_s : H^{1/2}(\Gamma_1) \to H^{-1/2}(\Gamma_1)$ be the Dirichlet-to-Neumann operator for the Helmholtz equation (2.2) with the complex wave number s. For any $\mu \in H^{1/2}(\Gamma_1)$, $T_s(\mu) = \dfrac{1}{s}\dfrac{\partial w}{\partial \mathbf{n}_1}$ on Γ_1, where $\mathbf{n}_1$ is the unit outer normal to Γ_1 and w is the solution

of the following exterior Dirichlet problem of the Helmholtz equation

$$-\Delta w + s^2 w = 0 \qquad \text{in } \mathbb{R}^2\backslash\bar{B}_1, \tag{2.3}$$

$$w = \mu \qquad \text{on } \Gamma_1, \tag{2.4}$$

$$\sqrt{r}\left(\frac{\partial w}{\partial r} + sw\right) \to 0 \qquad \text{as } r = |x| \to \infty. \tag{2.5}$$

For the complex wave number s with $\mathrm{Re}\,(s) > 0$, it is known by Lax-Milgram lemma that (2.3)-(2.5) has a unique solution $w \in H^1(\mathbb{R}^2\backslash\bar{B}_1)$ (cf. e.g. Nedelec [23, Lemma 2.6.6]). Thus $T_s : H^{1/2}(\Gamma_1) \to H^{-1/2}(\Gamma_1)$ is well-defined and is a continuous linear operator. We also remark that the well-posedness of (2.3)-(2.5) is also known for the case of real wave number when $\mathrm{Re}\,(s) = 0$ (cf. e.g. McLean [22, Theorem 9.11]).

Since $\mathbf{p}_0$ is supported in B_1, (2.1) implies,

$$\mathbf{p}_{\mathrm{L}} \cdot \mathbf{n}_1 + T_s(u_{\mathrm{L}}) = 0 \qquad \text{on } \Gamma_1.$$

Taking the inverse Laplace transform, we get the Dirichlet-to-Neumann boundary condition

$$\mathbf{p} \cdot \mathbf{n}_1 + \mathcal{T}(u) = 0 \qquad \text{on } \Gamma_1 \times (0, T), \tag{2.6}$$

where $\mathcal{T} = \mathscr{L}^{-1} \circ T_s \circ \mathscr{L}$. The acoustic scattering problem (1.1)-(1.4) is reduced to the following problem in the bounded domain $\Omega_1 \times (0, T)$ with $\Omega_1 = B_1\backslash\bar{D}$,

$$\frac{\partial u}{\partial t} = -\mathrm{div}\mathbf{p} + f(x,t), \quad \frac{\partial \mathbf{p}}{\partial t} = -\nabla u \qquad \text{in } \Omega_1 \times (0, T), \tag{2.7}$$

$$\mathbf{p} \cdot \mathbf{n}_D = 0 \qquad \text{on } \Gamma_D \times (0, T), \tag{2.8}$$

$$\mathbf{p} \cdot \mathbf{n}_1 + \mathcal{T}(u) = 0 \qquad \text{on } \Gamma_1 \times (0, T), \tag{2.9}$$

$$u|_{t=0} = u_0, \quad \mathbf{p}|_{t=0} = \mathbf{p}_0. \tag{2.10}$$

The well-posedness and stability of the reduced problem follows directly from Theorem 2.1 in [7]. In particular, $\mathcal{T}(u) \in L^2(0, T; H^{-1/2}(\Gamma_1))$.

Now we turn to the introduction of the absorbing PML layer. Let $B_2 = \{x \in \mathbb{R}^2 : |x_1| < L_1/2 + d_1, |x_2| < L_2/2 + d_2\}$ be the rectangle which contains B_1. Let

$$\alpha_1(x_1) = 1 + s^{-1}\sigma_1(x_1), \quad \alpha_2(x_2) = 1 + s^{-1}\sigma_2(x_2)$$

be the model medium property, where $\sigma_j(x_j) = 0$ if $|x_j| \le L_j/2$ and

$$\sigma_j(x_j) = \sigma_0 \ \text{ if } \ |x_j| \ge L_j/2, \quad j = 1, 2. \tag{2.11}$$

Here $\sigma_0 > 0$ is a constant. Denote by $\tilde{x}_j$ the complex coordinate defined by

$$\tilde{x}_j = \begin{cases} x_j & \text{if } |x_j| < L_j/2, \\ \int_0^{x_j} \alpha_j(t)\,\mathrm{d}t & \text{if } |x_j| \ge L_j/2. \end{cases} \tag{2.12}$$

To derive the PML equation, we first notice that by the third Green formula, the solution of the exterior Dirichlet problem (2.3)-(2.5) satisfies

$$w = -\Psi^s_{\mathrm{SL}}(\lambda) + \Psi^s_{\mathrm{DL}}(\mu) \qquad \text{in } \mathbb{R}^2\backslash\bar{B}_1, \tag{2.13}$$

where $\lambda = \frac{\partial w}{\partial \mathbf{n}_1}\Big|_{\Gamma_1} = sT_s(\mu) \in H^{-1/2}(\Gamma_1)$ is the Neumann trace of w on Γ_1. $\Psi^s_{\rm SL}, \Psi^s_{\rm DL}$ are respectively the single and double layer potentials [22],

$$\Psi^s_{\rm SL}(\lambda)(x) = \int_{\Gamma_1} G_s(x,y)\lambda(y)\,{\rm d}s(y), \quad \forall\, \lambda \in H^{-1/2}(\Gamma_1), \tag{2.14}$$

$$\Psi^s_{\rm DL}(\mu)(x) = \int_{\Gamma_1} \frac{\partial G_s(x,y)}{\partial \mathbf{n}_1(y)}\mu(y)\,{\rm d}s(y), \quad \forall\, \mu \in H^{1/2}(\Gamma_1). \tag{2.15}$$

Here G_s is the fundamental solution of the Helmholtz equation satisfying the Sommerfeld radiation condition [28]

$$G_s(x,y) = \frac{1}{2\pi}K_0(s|x-y|), \tag{2.16}$$

where $K_0(z)$ is the modified Bessel function of order zero.

We follow the method of complex stretching [12] to introduce the PML equation in the frequency domain. For any $z \in \mathbb{C}$, denote by $z^{1/2}$ as the analytic branch of $\sqrt{z}$ such that ${\rm Re}\,(z^{1/2}) > 0$ for any $z \in \mathbb{C}\backslash(-\infty, 0]$. Let

$$\rho_s(\tilde{x}, y) = \big(s^2((\tilde{x}_1 - y_1)^2 + (\tilde{x}_2 - y_2)^2)\big)^{1/2} \tag{2.17}$$

be the complex distance and define

$$\tilde{G}_s(x,y) = \frac{1}{2\pi}K_0(\rho_s(\tilde{x}, y)). \tag{2.18}$$

It is easy to see that $\tilde{G}_s$ is smooth for $x \in \mathbb{R}^2\backslash\bar{B}_1$ and $y \in \bar{B}_1$. Now we can define the modified single and double layer potentials

$$\tilde{\Psi}^s_{\rm SL}(\lambda)(x) = \int_{\Gamma_1} \tilde{G}_s(x,y)\lambda(y)\,{\rm d}s(y), \quad \forall\, \lambda \in H^{-1/2}(\Gamma_1),$$

$$\tilde{\Psi}^s_{\rm DL}(\mu)(x) = \int_{\Gamma_1} \frac{\partial \tilde{G}_s(x,y)}{\partial \mathbf{n}_1(y)}\mu(y)\,{\rm d}s(y), \quad \forall\, \mu \in H^{1/2}(\Gamma_1).$$

It is clear that $\tilde{\Psi}^s_{\rm SL}(\lambda), \tilde{\Psi}^s_{\rm DL}(\mu)$ are smooth in $\mathbb{R}^2\backslash\bar{B}_1$, and for any $\lambda \in H^{-1/2}(\Gamma_1), \mu \in H^{1/2}(\Gamma_1)$,

$$\gamma_D^+\tilde{\Psi}^s_{\rm SL}(\lambda) = \gamma_D^+\Psi^s_{\rm SL}(\lambda), \quad \gamma_D^+\tilde{\Psi}^s_{\rm DL}(\mu) = \gamma_D^+\Psi^s_{\rm DL}(\mu) \tag{2.19}$$

where $\gamma_D^+ : H^1_{loc}(\mathbb{R}^2\backslash\bar{B}_1) \to H^{1/2}(\Gamma_1)$ is the trace operator.

For any $\lambda \in H^{-1/2}(\Gamma_1), \mu \in H^{1/2}(\Gamma_1)$, let $\mathbb{E}(\lambda, \mu)$ be the PML extension given by

$$\mathbb{E}(\lambda, \mu) = -\tilde{\Psi}^s_{\rm SL}(\lambda) + \tilde{\Psi}^k_{\rm DL}(\mu) \qquad \text{for} \;\; x \in \mathbb{R}^2\backslash\bar{B}_1. \tag{2.20}$$

By (2.19) and (2.13) we know that

$$\gamma_D^+\mathbb{E}(\lambda, \mu) = -\gamma_D^+\Psi^k_{\rm SL}(\lambda) + \gamma_D^+\Psi^k_{\rm DL}(\mu) = \gamma_D^+ w = \mu \qquad \text{on} \;\; \Gamma_1$$

for any $\mu \in H^{1/2}(\Gamma_1)$ with $\lambda = sT_s(\mu) \in H^{-1/2}(\Gamma_1)$. Let

$$\tilde{u}_{\rm L} = \mathbb{E}(\frac{\partial u_{\rm L}}{\partial \mathbf{n}_1}\Big|_{\Gamma_1}, u_{\rm L}|_{\Gamma_1}) \tag{2.21}$$

be the PML extension of $u_{\rm L}|_{\Gamma_1}$ which satisfies $\gamma_D^+\tilde{u}_{\rm L} = u_{\rm L}|_{\Gamma_1}$ on Γ_1. Since $K_0(z)$ decays exponentially on the right half complex plane [8], heuristically $\tilde{u}_{\rm L}(x)$ will decay exponentially when x is away from Γ_1.

It is obvious that $\tilde{u}_{\rm L}$ satisfies

$$-\tilde{\Delta}\tilde{u}_{\rm L} + s^2\tilde{u}_{\rm L} = 0 \qquad \text{in } \mathbb{R}^2\backslash\bar{B}_1,$$

where $\tilde{\Delta} = \widetilde{\rm div}\tilde{\nabla}$ and $\tilde{\nabla}v = (\frac{1}{\alpha_1}\frac{\partial v}{\partial x_1}, \frac{1}{\alpha_2}\frac{\partial v}{\partial x_2})^T$, $\widetilde{\rm div}\mathbf{q} = \frac{1}{\alpha_1}\frac{\partial \mathbf{q}_1}{\partial x_1} + \frac{1}{\alpha_2}\frac{\partial \mathbf{q}_2}{\partial x_2}$. Thus we get the PML equation in the frequency domain

$$-{\rm div}(A\nabla\tilde{u}_{\rm L}) + s^2\alpha_1\alpha_2\tilde{u}_{\rm L} = 0, \tag{2.22}$$

where $A = {\rm diag}(\alpha_2/\alpha_1, \alpha_1/\alpha_2)$ is a diagonal matrix.

The desired time-domain PML system will be obtained by taking the inverse Laplace transform of (2.21). To that purpose, we introduce new variables

$$s\tilde{\mathbf{p}}_{\rm L}^* = -\nabla\tilde{u}_{\rm L}, \quad s\tilde{\mathbf{p}}_{\rm L} = -A\nabla\tilde{u}_{\rm L}, \quad s\tilde{u}_{\rm L}^* = \sigma_1\sigma_2\tilde{u}_{\rm L}, \quad \forall x \in \mathbb{R}^2\backslash\bar{B}_1. \tag{2.23}$$

The PML equation (2.22) becomes

$${\rm div}\tilde{\mathbf{p}}_{\rm L} + s\alpha_1\alpha_2\tilde{u}_{\rm L} = 0. \tag{2.24}$$

Let, for $x \in \mathbb{R}^2\backslash\bar{B}_1$,

$$\tilde{u} = \mathscr{L}^{-1}(\tilde{u}_{\rm L}),\ \tilde{u}^* = \mathscr{L}^{-1}(\tilde{u}_{\rm L}^*),\ \tilde{\mathbf{p}} = \mathscr{L}^{-1}(\tilde{\mathbf{p}}_{\rm L}),\ \tilde{\mathbf{p}}^* = \mathscr{L}^{-1}(\tilde{\mathbf{p}}_{\rm L}^*), \tag{2.25}$$

with $\tilde{u}|_{t=0} = 0$, $\tilde{u}^*|_{t=0} = 0$, $\tilde{\mathbf{p}}|_{t=0} = 0$, and $\tilde{\mathbf{p}}^*|_{t=0} = 0$. The first two equations in (2.23) imply

$$s\alpha_1\tilde{\mathbf{p}}_{{\rm L},1} = s\alpha_2\tilde{\mathbf{p}}_{{\rm L},1}^*, \quad s\alpha_2\tilde{\mathbf{p}}_{{\rm L},2} = s\alpha_1\tilde{\mathbf{p}}_{{\rm L},2}^*,$$

which, since $s\alpha_1 = s + \sigma_1$ and $s\alpha_2 = s + \sigma_2$, by taking the inverse Laplace transform yield

$$\frac{\partial\tilde{\mathbf{p}}_1}{\partial t} + \sigma_1\tilde{\mathbf{p}}_1 = \frac{\partial\tilde{\mathbf{p}}_1^*}{\partial t} + \sigma_2\tilde{\mathbf{p}}_1^*, \quad \frac{\partial\tilde{\mathbf{p}}_2}{\partial t} + \sigma_2\tilde{\mathbf{p}}_2 = \frac{\partial\tilde{\mathbf{p}}_2^*}{\partial t} + \sigma_1\tilde{\mathbf{p}}_2^*.$$

Introduce the matrices $\Lambda_1(x) = {\rm diag}(\sigma_1(x_1), \sigma_2(x_2)), \Lambda_2(x) = {\rm diag}(\sigma_2(x_2), \sigma_1(x_1))$, we can rewrite the above equations in the compact form

$$\frac{\partial\tilde{\mathbf{p}}}{\partial t} + \Lambda_1\tilde{\mathbf{p}} = \frac{\partial\tilde{\mathbf{p}}^*}{\partial t} + \Lambda_2\tilde{\mathbf{p}}_2^*. \tag{2.26}$$

By taking the inverse Laplace transform in the first and third equation in (2.23) we have

$$\frac{\partial\tilde{\mathbf{p}}^*}{\partial t} + \nabla\tilde{u} = 0, \quad \frac{\partial\tilde{u}^*}{\partial t} = \sigma_1\sigma_2\tilde{u}. \tag{2.27}$$

Finally by taking the inverse Laplace transform in (2.24) we get

$$\frac{\partial\tilde{u}}{\partial t} + {\rm div}\tilde{\mathbf{p}} + (\sigma_1 + \sigma_2)\tilde{u} + \tilde{u}^* = 0. \tag{2.28}$$

Since $\tilde{u} = u$, $\tilde{\mathbf{p}} = \mathbf{p}$ on Γ_1, $\tilde{u}, \tilde{\mathbf{p}}$ can be viewed as the extension of the solution $(u, \mathbf{p})$ of the problem (1.1)-(1.4). Moreover, since $\sigma_1 = \sigma_2 = 0$ inside the rectangle B_1,

if we set $\tilde{u} = u, \tilde{\mathbf{p}} = \mathbf{p}, \tilde{u}^* = 0, \tilde{\mathbf{p}}^* = \mathbf{p}$ in $[B_1 \backslash \bar{D}] \times (0,T)$, then $(\tilde{u}, \tilde{\mathbf{p}}, \tilde{u}^*, \tilde{\mathbf{p}}^*)$ satisfies (2.26), (2.27), and, instead of (2.28),

$$\frac{\partial \tilde{u}}{\partial t} + \mathrm{div}\tilde{\mathbf{p}} + (\sigma_1 + \sigma_2)\tilde{u} + \tilde{u}^* = f. \tag{2.29}$$

We summarize the above consideration in the following lemma.

LEMMA 2.1. *Let $(u, \mathbf{p})$ be the solution of the problem* (1.1)-(1.4) *which is extended to be $(\tilde{u}, \tilde{\mathbf{p}})$ outside B_1 according to* (2.25). *Let $(\tilde{u}^*, \tilde{\mathbf{p}}^*)$ be defined in* (2.25) *for $x \in \mathbb{R}^2 \backslash \bar{B}_1$ and $(\tilde{u}^*, \tilde{\mathbf{p}}^*) = (0, \mathbf{p})$ for $x \in B_1$. Then $(\tilde{u}, \tilde{\mathbf{p}}, \tilde{u}^*, \tilde{\mathbf{p}}^*)$ satisfies the PML system* (2.29), (2.26)-(2.27) *outside $\bar{D}$ and satisfies the initial and boundary conditions $\tilde{u}|_{t=0} = u_0, \tilde{\mathbf{p}}|_{t=0} = \mathbf{p}_0, \tilde{u}^*|_{t=0} = 0, \tilde{\mathbf{p}}^*|_{t=0} = \mathbf{p}_0$ in $\mathbb{R}^2 \backslash \bar{D}$ and $\tilde{\mathbf{p}} \cdot \mathbf{n}_D = 0$ on $\Gamma_D \times (0,T)$.*

We define the following initial-boundary value problem for $(\hat{u}, \hat{\mathbf{p}}, \hat{u}^*, \hat{\mathbf{p}}^*)$ which is referred as the PML problem in the rest of this paper, where $\Omega_2 = B_2 \backslash \bar{D}$,

$$\frac{\partial \hat{u}}{\partial t} + \mathrm{div}\hat{\mathbf{p}} + (\sigma_1 + \sigma_2)\hat{u} + \hat{u}^* = f, \quad \frac{\partial \hat{u}^*}{\partial t} = \sigma_1\sigma_2\hat{u} \quad \text{in } \Omega_2 \times (0,T), \tag{2.30}$$

$$\frac{\partial \hat{\mathbf{p}}^*}{\partial t} + \nabla \hat{u} = 0, \quad \frac{\partial \hat{\mathbf{p}}}{\partial t} + \Lambda_1 \hat{\mathbf{p}} = \frac{\partial \hat{\mathbf{p}}^*}{\partial t} + \Lambda_2 \hat{\mathbf{p}}^* \quad \text{in } \Omega_2 \times (0,T), \tag{2.31}$$

$$\hat{\mathbf{p}} \cdot \mathbf{n}_D = 0 \quad \text{on } \Gamma_D \times (0,T), \qquad \hat{u} = 0 \quad \text{on } \Gamma_2 \times (0,T), \tag{2.32}$$

$$\hat{u}|_{t=0} = u_0, \quad \hat{\mathbf{p}}|_{t=0} = \mathbf{p}_0, \quad \hat{u}^*|_{t=0} = 0, \quad \hat{\mathbf{p}}^*|_{t=0} = \mathbf{p}_0 \quad \text{in } \Omega_2. \tag{2.33}$$

By the construction of the PML problem, $(\hat{u}, \hat{\mathbf{p}})$ is designed to approximate the solution $(u, \mathbf{p})$ of the original scattering problem (1.1)-(1.4) in the domain $\Omega_1 \times (0,T)$.

Notice that (2.30)-(2.33) is a first order symmetric hyperbolic system whose well-posedness follows from the standard theory (see e.g. Chen [6]). Here we state the well-posedness of the PML problem (2.30)-(2.33) and omit the proof.

THEOREM 2.2. *Let $f \in L^\infty(0,T; L^2(\Omega_1))$. Assume that $u_0 \in H^1(\Omega_2)$, $\mathbf{p}_0 \in H(\mathrm{div}; \Omega_2)$ are supported in B_1 and satisfy the compatibility conditions $\mathbf{p}_0 \cdot \mathbf{n}_D = 0$, $\nabla u_0 \cdot \mathbf{n}_D = 0$ on Γ_D. Then the PML problem* (2.30)-(2.33) *has a unique strong solution $(\hat{u}, \hat{\mathbf{p}}, \hat{u}^*, \hat{\mathbf{p}}^*)$ satisfying*

$$\hat{u} \in L^\infty(0,T; H^1(\Omega_2)) \cup W^{1,\infty}(0,T; L^2(\Omega_2)), \quad \hat{u}^* \in W^{1,\infty}(0,T; L^2(\Omega_2)),$$
$$\hat{\mathbf{p}} \in L^\infty(0,T; H(\mathrm{div}; \Omega_2)) \cup W^{1,\infty}(0,T; L^2(\Omega_2)), \quad \hat{\mathbf{p}}^* \in W^{1,\infty}(0,T; L^2(\Omega_2)).$$

3. The stability of the PML system. In this section we consider the stability of the initial-boundary value problem of the following PML system in $\Omega^{\mathrm{PML}} \times (0,T)$, where $\Omega^{\mathrm{PML}} = \Omega_2 \backslash \bar{\Omega}_1$,

$$\frac{\partial \phi}{\partial t} + \mathrm{div}\Phi + (\sigma_1 + \sigma_2)\phi + \phi^* = f_1, \quad \frac{\partial \phi^*}{\partial t} = \sigma_1\sigma_2\phi \quad \text{in } \Omega^{\mathrm{PML}} \times (0,T), \tag{3.1}$$

$$\frac{\partial \Phi^*}{\partial t} + \nabla \phi = f_2, \quad \frac{\partial \Phi}{\partial t} + \Lambda_1 \Phi = \frac{\partial \Phi^*}{\partial t} + \Lambda_2 \Phi^* \quad \text{in } \Omega^{\mathrm{PML}} \times (0,T), \tag{3.2}$$

$$\phi = 0 \quad \text{on } \partial\Omega^{\mathrm{PML}} \times (0,T), \tag{3.3}$$

$$\phi|_{t=0} = 0, \quad \Phi|_{t=0} = 0, \quad \phi^*|_{t=0} = 0, \quad \Phi^*|_{t=0} = 0 \quad \text{in } \Omega^{\mathrm{PML}}. \tag{3.4}$$

Here we assume that $f_1, f_2 \in H^1(0,T; L^2(\Omega_2))$. We denote (see Figure 2.1)

$$\Omega_1^{\mathrm{PML}} = \{x \in \Omega^{\mathrm{PML}} : |x_1| > L_1/2, \quad |x_2| < L_2/2\},$$
$$\Omega_2^{\mathrm{PML}} = \{x \in \Omega^{\mathrm{PML}} : |x_1| < L_1/2, \quad |x_2| > L_2/2\},$$
$$\Omega_c^{\mathrm{PML}} = \{x \in \Omega^{\mathrm{PML}} : |x_1| > L_1/2, \quad |x_2| > L_2/2\}.$$

We first show the following estimate for the stability of the initial-boundary value problem of PML system.

THEOREM 3.1. *Let* $(\phi, \Phi, \phi^*, \Phi^*)$ *be the solution of the PML problem* (3.1)-(3.4), *we have the following stability estimate*

$$\begin{aligned}
&\max_{0\le t\le T}\left(\|\partial_t\phi+\sigma_0\phi\|_{L^2(\Omega^{\rm PML})}+\|\partial_t\Phi\|_{L^2(\Omega^{\rm PML})}+\sum_{j=1}^{2}\|\sigma_0\Phi_j\|_{L^2(\Omega_j^{\rm PML})}\right)\\
&\le C\left(\|f_1(\cdot,0)\|_{L^2(\Omega^{\rm PML})}+\|f_2(\cdot,0)\|_{L^2(\Omega^{\rm PML})}\right)\\
&+C\int_0^T\left(\|\partial_t f_1\|_{L^2(\Omega^{\rm PML})}+\|\partial_t f_2+\sigma_0 f_2\|_{L^2(\Omega^{\rm PML})}\right)\,{\rm d}t,
\end{aligned}$$

where the constant C *is independent of* σ_0 *and* T.

Proof. We differentiate the first equation in (3.1) in time t and use the second equation in (3.1) to obtain

$$\frac{\partial^2\phi}{\partial t^2}+{\rm div}\frac{\partial\Phi}{\partial t}+(\sigma_1+\sigma_2)\frac{\partial\phi}{\partial t}+\sigma_1\sigma_2\phi=\partial_t f_1\qquad\text{in }\ \Omega^{\rm PML}\times(0,T).$$

Multiplying the above equation by $\partial_t\phi+\sigma_0\phi$ and integrating over $\Omega^{\rm PML}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\rm d}{{\rm d}t}\|\partial_t\phi+\sigma_0\phi\|^2_{L^2(\Omega^{\rm PML})}+\int_{\Omega^{\rm PML}}{\rm div}\frac{\partial\Phi}{\partial t}\left(\partial_t\phi+\sigma_0\phi\right)\,{\rm d}x\\
&+\int_{\Omega^{\rm PML}}\left[(\sigma_1+\sigma_2-\sigma_0)\partial_t\phi+\sigma_1\sigma_2\phi\right]\left(\partial_t\phi+\sigma_0\phi\right)\,{\rm d}x=\int_{\Omega^{\rm PML}}\partial_t f_1\left(\partial_t\phi+\sigma_0\phi\right)\,{\rm d}x.
\end{aligned}$$

Since $\sigma_1+\sigma_2-\sigma_0\ge 0$ in $\Omega^{\rm PML}$, the integration in time from 0 to t of the third term on the left-hand of above equation is non-negative. Thus

$$\begin{aligned}
&\frac{1}{2}\|\partial_t\phi+\sigma_0\phi\|^2_{L^2(\Omega^{\rm PML})}+\int_0^t\int_{\Omega^{\rm PML}}{\rm div}\frac{\partial\Phi}{\partial t}\left(\partial_t\phi+\sigma_0\phi\right)\,{\rm d}x\,{\rm d}t\\
&\le\frac{1}{2}\|\partial_t\phi|_{t=0}\|^2_{L^2(\Omega^{\rm PML})}+\int_0^t\int_{\Omega^{\rm PML}}\partial_t f_1\left(\partial_t\phi+\sigma_0\phi\right)\,{\rm d}x\,{\rm d}t.
\end{aligned}\tag{3.5}$$

Here we have used that fact that $\phi(\cdot,0)=0$. Next we differentiate the first equation in (3.2) in time, multiply the first equation in (3.2) by σ_0, and add the two obtained equations

$$\frac{\partial^2\Phi^*}{\partial t^2}+\sigma_0\frac{\partial\Phi^*}{\partial t}+\nabla\left(\partial_t\phi+\sigma_0\phi\right)=\partial_t f_2+\sigma_0 f_2\qquad\text{in }\ \Omega^{\rm PML}\times(0,T).\tag{3.6}$$

On the other hand, by the second equation in (3.2) we have

$$\begin{aligned}
\frac{\partial^2\Phi^*}{\partial t^2}+\sigma_0\frac{\partial\Phi^*}{\partial t}&=\frac{\partial^2\Phi}{\partial t^2}+\Lambda_1\frac{\partial\Phi}{\partial t}+(\sigma_0 I-\Lambda_2)\frac{\partial\Phi^*}{\partial t}\\
&=\frac{\partial^2\Phi}{\partial t^2}+(\sigma_0 I+\Lambda_1-\Lambda_2)\frac{\partial\Phi}{\partial t}+(\sigma_0 I-\Lambda_2)(\Lambda_1\Phi-\Lambda_2\Phi^*).
\end{aligned}$$

Insert the above identity to (3.6) and multiply the obtained equation by $\partial_t\Phi$ and integrate over $\Omega^{\rm PML}$ we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\rm d}{{\rm d}t}\|\partial_t\Phi\|^2_{L^2(\Omega^{\rm PML})}+\int_{\Omega^{\rm PML}}\nabla\left(\partial_t\phi+\sigma_0\phi\right)\partial_t\Phi+\int_{\Omega^{\rm PML}}\left(\sigma_0 I+\Lambda_1-\Lambda_2\right)|\partial_t\Phi|^2\\
&+\int_{\Omega^{\rm PML}}(\sigma_0 I-\Lambda_2)(\Lambda_1\Phi-\Lambda_2\Phi^*)\partial_t\Phi\,{\rm d}x=\int_{\Omega^{\rm PML}}(\partial_t f_2+\sigma_0 f_2)\partial_t\Phi\,{\rm d}x.
\end{aligned}\tag{3.7}$$

Since $\sigma_0 I + \Lambda_1 - \Lambda_2 \geq 0$ in Ω^{PML}, we have

$$\int_{\Omega^{\mathrm{PML}}} (\sigma_0 I + \Lambda_1 - \Lambda_2) \left|\partial_t \Phi\right|^2 \, \mathrm{d}x \geq 0.$$

Notice that $\sigma_0 I - \Lambda_2 = 0$ in Ω_c^{PML}, $\sigma_0 I - \Lambda_2 = \mathrm{diag}(\sigma_0, 0)$ in Ω_1^{PML}, and $\sigma_0 I - \Lambda_2 = \mathrm{diag}(0, \sigma_0)$ in Ω_2^{PML}, we obtain that

$$\int_{\Omega^{\mathrm{PML}}} (\sigma I - \Lambda_2)(\Lambda_1 \Phi - \Lambda_2 \Phi^*) \partial_t \Phi \, \mathrm{d}x = \frac{\sigma_0^2}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\|\Phi_1\|^2_{L^2(\Omega_1^{\mathrm{PML}})} + \|\Phi_2\|^2_{L^2(\Omega_2^{\mathrm{PML}})} \right),$$

Thus it follows from (3.7) that

$$\begin{aligned} &\frac{1}{2} \left(\|\partial_t \Phi\|^2_{L^2(\Omega^{\mathrm{PML}})} + \sum_{j=1}^{2} \|\sigma_0 \Phi_j\|^2_{L^2(\Omega_j^{\mathrm{PML}})} \right) + \int_0^t \int_{\Omega^{\mathrm{PML}}} \nabla \left(\partial_t \phi + \sigma_0 \phi \right) \partial_t \Phi \, \mathrm{d}x \, \mathrm{d}t \\ &\leq \frac{1}{2} \|\partial_t \Phi|_{t=0}\|^2_{L^2(\Omega^{\mathrm{PML}})} + \int_0^t \int_{\Omega^{\mathrm{PML}}} \left(\partial_t f_2 + \sigma_0 f_2 \right) \partial_t \Phi \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{3.8}$$

By adding (3.5) and (3.8) we have

$$\begin{aligned} &\frac{1}{2} \left(\|\partial_t \phi + \sigma_0 \phi\|^2_{L^2(\Omega^{\mathrm{PML}})} + \|\partial_t \Phi\|^2_{L^2(\Omega^{\mathrm{PML}})} + \sum_{j=1}^{2} \|\sigma_0 \Phi_j\|^2_{L^2(\Omega_j^{\mathrm{PML}})} \right) \\ &\leq \frac{1}{2} \left(\|\partial_t \phi|_{t=0}\|^2_{L^2(\Omega^{\mathrm{PML}})} + \|\partial_t \Phi|_{t=0}\|^2_{L^2(\Omega^{\mathrm{PML}})} \right) \\ &+ \int_0^t \int_{\Omega^{\mathrm{PML}}} \partial_t f_1 \left(\partial_t \phi + \sigma_0 \phi \right) \, \mathrm{d}x \, \mathrm{d}t + \int_0^t \int_{\Omega^{\mathrm{PML}}} (\partial_t f_2 + \sigma_0 f_2) \partial_t \Phi \, \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

This completes the proof by using the Cauchy-Schwarz inequality and the following compatibility conditions

$$\partial_t \phi|_{t=0} = f_1(x, 0), \quad \partial_t \Phi|_{t=0} = \partial_t \Phi^*|_{t=0} = f_2(x, 0),$$

which follows from (3.1)-(3.2) and the fact that $\phi(\cdot, 0) = 0, \Phi(\cdot, 0) = 0$. ∎

LEMMA 3.2. *Let $\psi = \partial_t \phi + \sigma_0 \phi$. We have*

$$\max_{0 \leq t \leq T} \|\sigma_0 \phi\|_{L^2(\Omega^{\mathrm{PML}})} \leq \max_{0 \leq t \leq T} \|\psi\|_{L^2(\Omega^{\mathrm{PML}})}.$$

Proof. We first notice that since $\phi(\cdot, 0) = 0$,

$$\phi(x, t) = \int_0^t e^{\sigma_0 (s-t)} \psi(x, s) \, \mathrm{d}s.$$

Now

$$\begin{aligned} |\sigma_0 \phi|^2 = \sigma_0^2 \left| \int_0^t e^{\sigma_0 (s-t)} \psi(x, s) \, \mathrm{d}s \right|^2 &\leq \int_0^t \sigma_0 e^{\sigma_0 (s-t)} \, \mathrm{d}s \cdot \int_0^t \sigma_0 e^{\sigma_0 (s-t)} |\psi|^2 \, \mathrm{d}s \\ &\leq \int_0^t \sigma_0 e^{\sigma_0 (s-t)} |\psi|^2 \, \mathrm{d}s. \end{aligned}$$

Thus

$$\int_{\Omega^{\rm PML}} |\sigma_0\phi|^2 \,\mathrm{d}x \le \max_{0\le t\le T} \|\psi\|^2_{L^2(\Omega^{\rm PML})} \int_0^t \sigma_0 e^{\sigma_0(s-t)}\,\mathrm{d}s \le \max_{0\le t\le T} \|\psi\|^2_{L^2(\Omega^{\rm PML})}.$$

This completes the proof. ∎

The following theorem shows the stability of the PML system in the PML layer.

THEOREM 3.3. *Let* $(\phi, \Phi, \phi^*, \Phi^*)$ *be the solution of the PML system* (3.1)-(3.4), *we have the following stability estimate*

$$\begin{aligned}
&\max_{0\le t\le T} \left(\|\partial_t\phi\|_{L^2(\Omega^{\rm PML})} + \|\partial_t\Phi\|_{L^2(\Omega^{\rm PML})} + \sigma_0^{-1}\|\partial_t\phi^*\|_{L^2(\Omega^{\rm PML})} + \|\partial_t\Phi^*\|_{L^2(\Omega^{\rm PML})}\right)\\
&\le C\left(\|f_1(\cdot,0)\|_{L^2(\Omega^{\rm PML})} + \|f_2(\cdot,0)\|_{L^2(\Omega^{\rm PML})}\right)\\
&+ C\int_0^T \left(\|\partial_t f_1\|_{L^2(\Omega^{\rm PML})} + \|\partial_t f_2 + \sigma_0 f_2\|_{L^2(\Omega^{\rm PML})}\right)\,\mathrm{d}t,
\end{aligned}$$

where the constant C *is independent of* σ_0 *and* T.

Proof. The estimate for $\partial_t\phi$ follows from Lemma 3.2 and Theorem 3.1. The estimate for $\partial_t\phi^*$ follows easily from the second equation in (3.1), Lemma 3.2 and Theorem 3.1. To estimate $\partial_t\Phi^*$ we first notice that $\Phi^* = \Phi$ in $\Omega_c^{\rm PML}$. In $\Omega_1^{\rm PML}$, we have

$$\frac{\partial\Phi_1^*}{\partial t} = \frac{\partial\Phi_1}{\partial t} + \sigma_0\Phi_1, \quad \frac{\partial\Phi_2^*}{\partial t} + \sigma_0\Phi_2^* = \frac{\partial\Phi_2}{\partial t}.$$

Thus the estimate of $\|\partial_t\Phi_1^*\|_{L^2(\Omega_1^{\rm PML})}$ can be proved by using Theorem 3.1. The estimate of $\|\partial_t\Phi_2^*\|_{L^2(\Omega_1^{\rm PML})}$ follows from Theorem 3.1 and the augument in Lemma 3.2. Similarly we can bound $\|\partial_t\Phi^*\|_{L^2(\Omega_2^{\rm PML})}$. This completes the proof. ∎

We remark that our stability analysis is inspired by the energy analysis in [3] in which the stability is proved separately for the PML system in the unbounded domain parallel to the axises and in the unbounded corner domain. Here we show the stability of the PML system in the whole truncated PML layer which will be useful for the convergence analysis of the time domain PML method in this paper.

4. Exponential decay of the PML extension. In this section we show the exponential decay of the PML extension in the time domain which is the inverse Laplace transform of $\tilde{u}_{\rm L}$ in (2.21)

$$\tilde{u} = \mathscr{L}^{-1}\left(\mathbb{E}(\frac{\partial u_{\rm L}}{\partial \mathbf{n}_1}\Big|_{\Gamma_1}, u_{\rm L}|_{\Gamma_1})\right).$$

We start with the following elementary lemma.

LEMMA 4.1. *For any* $z_1 = a_1 + \mathbf{i}b_1, z_2 = a_2 + \mathbf{i}b_2$ *with* $a_1, b_1, a_2, b_2 \in \mathbb{R}$ *such that* $b_1^2 + b_2^2 > 0$, *we have*

$$\mathrm{Re}\,[(z_1^2 + z_2^2)^{1/2}] \ge \frac{|a_1b_1 + a_2b_2|}{\sqrt{b_1^2 + b_2^2}}.$$

Proof. For any $a, b \in \mathbb{R}$ we know that

$$\mathrm{Re}\,[(a + \mathbf{i}b)^{1/2}] = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}.$$

Here we used the convention that $z^{1/2}$ is the analytic branch of $\sqrt{z}$ such that $\mathrm{Re}\,(z^{1/2}) > 0$ for any $z \in \mathbb{C}\backslash(-\infty, 0]$. It is easy to check that $\mathrm{Re}\,[(a+\mathbf{i}b)^{1/2}]$ is a increasing function in $a \in \mathbb{R}$. Let $z_1^2 + z_2^2 = a + \mathbf{i}b$, then

$$a + \mathbf{i}b = \left(\frac{a_1b_1 + a_2b_2}{\sqrt{b_1^2+b_2^2}} + \mathbf{i}\sqrt{b_1^2+b_2^2}\right)^2 + \frac{(a_2b_1 - a_1b_2)^2}{b_1^2+b_2^2}.$$

Let $a' = a - \frac{(a_2b_1-a_1b_2)^2}{b_1^2+b_2^2}$, we have

$$\mathrm{Re}\,[(a'+\mathbf{i}b)^{1/2}] = \frac{|a_1b_1 + a_2b_2|}{\sqrt{b_1^2+b_2^2}}.$$

On the other hand, since $a' \le a$, we know that $\mathrm{Re}\,[(a+\mathbf{i}b)^{1/2}] \ge \mathrm{Re}\,[(a'+\mathbf{i}b)^{1/2}]$. This completes the proof. □

In the following we will always make the following assumption on the thickness of the PML layer which is rather mild in the practical applications.

(H1) $d_1 = d_2 = d$ and $L = \max(L_1, L_2) \le C_0 d$ for some constant $C_0 > 0$.

We remark that in this paper the aspect ratio of the domain B_1 is held fixed which indicates that C_0 is bounded. When the aspect ratio grows, then C_0 will grow which implies our error estimate in Theorem 5.2 may break down. One possible way to overcome the difficulty is to use a PML layer not of the same thickness in x_1 and x_2 directions as required in (H1). In particular, if we require $\frac{d_1}{L_1} = \frac{d_2}{L_2}$ which is denoted as θ, then the exponential decay of the PML extension can still be achieved as $\theta \to \infty$.

By (2.11) and (H1) we know that

$$\bar{\sigma} := \int_0^{L_1/2+d_1} \sigma_1(t)\,\mathrm{d}t = \int_0^{L_2/2+d_2} \sigma_2(t)\,\mathrm{d}t = \sigma_0 d.$$

The following lemma will be used in proving the exponential decay of the modified Bessel function.

LEMMA 4.2. *Let $s = s_1 + \mathbf{i}s_2$, $s_1 > 0, s_2 \in \mathbb{R}$. Then for any $x \in \Gamma_2, y \in \Gamma_1$, the complex distance in* (2.17) *satisfies*

$$|\mathrm{Re}\,[\rho_s(\tilde{x}, y)/s]| \ge d/\sqrt{2}, \qquad \mathrm{Re}\,[\rho_s(\tilde{x}, y)] \ge \gamma\bar{\sigma},$$

where $\gamma = 1/\sqrt{1+(1+L/d)^2}$.

Proof. Let $z_j = \tilde{x}_j - y_j = (x_j - y_j) + s^{-1}x_j\hat{\sigma}_j(x_j)$, where

$$\hat{\sigma}_j = \frac{1}{x_j}\int_0^{x_j} \sigma_j(t)\mathrm{d}t.$$

Then by Lemma 4.1,

$$\begin{aligned}
&|\mathrm{Re}\,[\rho_s(\tilde{x}, y)/s]| \\
&= \mathrm{Re}\,[((\tilde{x}_1 - y_1)^2 + (\tilde{x}_2 - y_2)^2)^{1/2}] \\
&= \mathrm{Re}\left[\left(\left(x_1 - y_1 + \frac{s_1x_1\hat{\sigma}_1}{s_1^2+s_2^2} - \mathbf{i}\frac{s_2x_1\hat{\sigma}_1}{s_1^2+s_2^2}\right)^2 + \left(x_2 - y_2 + \frac{s_1x_2\hat{\sigma}_2}{s_1^2+s_2^2} - \mathbf{i}\frac{s_2x_2\hat{\sigma}_2}{s_1^2+s_2^2}\right)^2\right)^{1/2}\right] \\
&\ge \frac{\left|\left(x_1 - y_1 + \frac{s_1x_1\hat{\sigma}_1}{s_1^2+s_2^2}\right)\frac{s_2x_1\hat{\sigma}_1}{s_1^2+s_2^2} + \left(x_2 - y_2 + \frac{s_1x_2\hat{\sigma}_2}{s_1^2+s_2^2}\right)\frac{s_2x_2\hat{\sigma}_2}{s_1^2+s_2^2}\right|}{\sqrt{\frac{s_2^2(x_1\hat{\sigma}_1)^2}{(s_1^2+s_2^2)^2} + \frac{s_2^2(x_2\hat{\sigma}_2)^2}{(s_1^2+s_2^2)^2}}}.
\end{aligned}$$

For any $x \in \Gamma_2, y \in \Gamma_1$, it is easy to see that $x_j\hat{\sigma}_j(x_j - y_j) \geq 0$. Thus for any $x \in \Gamma_2, y \in \Gamma_1$,

$$|\mathrm{Re}\,[\rho_s(\tilde{x}, y)/s]| \geq \frac{|x_1 - y_1|\,|x_1\hat{\sigma}_1| + |x_2 - y_2|\,|x_2\hat{\sigma}_2|}{\sqrt{(x_1\hat{\sigma}_1)^2 + (x_2\hat{\sigma}_2)^2}} \geq \frac{d \cdot \bar{\sigma}}{\sqrt{2}\bar{\sigma}} = \frac{d}{\sqrt{2}}.$$

This shows the first estimate. To show the second estimate, let $z_j = s(\tilde{x}_j - y_j) = s(x_j - y_j) + x_j\hat{\sigma}_j(x_j)$, Again by Lemma 4.1,

$$\begin{aligned}
&\mathrm{Re}\,[\rho_s(\tilde{x}, y)] \\
&= \mathrm{Re}\,[(s^2((\tilde{x}_1 - y_1)^2 + (\tilde{x}_2 - y_2)^2))^{1/2}] \\
&\geq \frac{|(s_1(x_1 - y_1) + x_1\hat{\sigma}_1)(s_2(x_1 - y_1)) + (s_1(x_2 - y_2) + x_2\hat{\sigma}_2)(s_2(x_2 - y_2))|}{\sqrt{s_2^2(x_1 - y_1)^2 + s_2^2(x_2 - y_2)^2}}.
\end{aligned}$$

Thus for any $x \in \Gamma_2, y \in \Gamma_1$,

$$\mathrm{Re}\,[\rho_s(\tilde{x}, y)] \geq \frac{|x_1\hat{\sigma}_1|\,|x_1 - y_1| + |x_2\hat{\sigma}_2|\,|x_2 - y_2|}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}}.$$

If $x_1 = \pm(L_1/2 + d) \in \Gamma_2$, we have $|x_1\hat{\sigma}_1| = \bar{\sigma}$ and $\dfrac{|x_2 - y_2|}{|x_1 - y_1|} \leq \dfrac{L_2 + d}{d} \leq 1 + \dfrac{L}{d}$. Consequently we deduce

$$\mathrm{Re}\,[\rho_s(\tilde{x}, y)] \geq \frac{\bar{\sigma}}{\sqrt{1 + (1 + L/d)^2}} = \gamma\bar{\sigma}.$$

Similarly we can show the same estimate for $x_2 = \pm(L_2/2 + d) \in \Gamma_2$. This completes the proof. ∎

We need the modified Bessel function $K_\nu(z)$ of order ν, $\nu \in \mathbb{C}$, which is the solution of the differential equation

$$z^2\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + z\frac{\mathrm{d}w}{\mathrm{d}z} - (z^2 + \nu^2)w = 0$$

satisfying the asymptotic behavior $K_\nu(z) \sim \mathbf{i}(\frac{\pi}{2z})^{1/2}e^{-z-\frac{\pi}{4}\mathbf{i}}$ as $|z| \to \infty$. We refer to the treatise Watson [28] for extensive studies on the special function $K_\nu(z)$. We recall the Schläfli integral representation formula [28, P. 181], for $z \in \mathbb{C}$ such that $|\arg z| < \pi/2$,

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh \nu t\,\mathrm{d}t. \tag{4.1}$$

This implies easily the following lemma.

Lemma 4.3. *For any $\nu \in \mathbb{R}$, $\theta_2 \geq \theta_1 > 0$, we have*

$$K_\nu(\theta_2) \leq e^{-(\theta_2 - \theta_1)}K_\nu(\theta_1).$$

The following lemma on the estimates of the fundamental solution $\tilde{G}_s$ of the PML equation will play an important role in the analysis in this paper.

Lemma 4.4. *Let γ, σ_0, d be so chosen that*

$$\gamma\bar{\sigma} \geq 1. \tag{4.2}$$

Then there exists a constant $C > 0$ independent of s, σ_0, d but may depend on the constant C_0 in (H1) such that for any $x \in \Gamma_2, y \in \Gamma_1$,
(i) $|\tilde{G}_s(x,y)| \le C\, e^{-\gamma\bar{\sigma}}$;
(ii) $\dfrac{\partial \tilde{G}_s}{\partial x_j} = sP^s_{1,j} + P^s_{0,j}$ *and* $|P^s_{l,j}| \le C\sigma_0(1+\sigma_0 s_1^{-1})\, e^{-\gamma\bar{\sigma}}, \quad l = 0,1,\ j = 1,2$;
(iii) $\dfrac{\partial \tilde{G}_s}{\partial y_j} = sQ^s_{1,j}$ *and* $|Q^s_{1,j}| \le C(1+\sigma_0 s_1^{-1})\, e^{-\gamma\bar{\sigma}}, \quad j = 1,2$;
(iv) $\dfrac{\partial^2 \tilde{G}_s}{\partial x_i \partial y_j} = s^2 R^s_{2,ij} + sR^s_{1,ij} + R^s_{0,ij}$ *and* $|R^s_{l,ij}| \le C\sigma_0^2(1+\sigma_0 s_1^{-1})^2\, e^{-\gamma\bar{\sigma}}, \quad l = 0,1,2,\ i,j = 1,2$.

Proof. By the Schläfli integral representation formula (4.1), it is easy to see that $|K_0(z)| < K_0(\mathrm{Re}\,(z))$ for any $z \in \mathbb{C}$ such that $\mathrm{Re}\,(z) > 0$. Thus, by (2.18) and Lemma 4.3, if $\mathrm{Re}\,[\rho_s(\tilde{x},y)] \ge 1$,

$$|\tilde{G}_s| \le \frac{1}{2\pi}|K_0(\rho_s(\tilde{x},y))| \le \frac{1}{2\pi}K_0(\mathrm{Re}\,[\rho_s(\tilde{x},y)]) \le \frac{1}{2\pi}K_0(1)e^{-(\mathrm{Re}\,[\rho_s(\tilde{x},y)]-1)}.$$

This proves (i) by Lemma 4.2 and (4.2).

To show (ii) we first notice that

$$\frac{\partial \tilde{G}_s}{\partial x_j} = \frac{1}{2\pi}K_0'(\rho_s(\tilde{x},y))\frac{(\tilde{x}_j - y_j)(s+\sigma_j)}{\rho_s(\tilde{x},y)/s} = sP^s_{1,j} + P^s_{0,j}.$$

From the identity $K_0'(z) = -K_1(z)$ and Lemma 4.2, $P^s_{1,j}$ has the following estimate

$$\begin{aligned}|P^s_{1,j}| &= \frac{1}{2\pi}|K_0'(\rho_s(\tilde{x},y))|\left|\frac{\tilde{x}_j - y_j}{\rho_s(\tilde{x},y)/s}\right| \le \frac{1}{2\pi}|K_1(\rho_s(\tilde{x},y))|\frac{L_j + d + s_1^{-1}\bar{\sigma}}{|\mathrm{Re}\,[\rho_s(\tilde{x},y)/s]|}\\ &\le \frac{1}{2\pi}K_1(1)e^{-(\gamma\bar{\sigma}-1)}\frac{L_j + d + s_1^{-1}\bar{\sigma}}{(d/\sqrt{2})} \le C(1+\sigma_0 s_1^{-1})\, e^{-\gamma\bar{\sigma}}.\end{aligned}$$

The estimate for $P^s_{0,j}$ follows from the similar argument. Similarly, we can prove (iii) by

$$\frac{\partial \tilde{G}_s}{\partial y_j} = \frac{1}{2\pi}K_0'(\rho_s(\tilde{x},y))\frac{(\tilde{x}_j - y_j)(-s)}{\rho_s(\tilde{x},y)/s} = sQ^s_{1,j}.$$

To prove (iv) we note that

$$\begin{aligned}\frac{\partial^2 \tilde{G}_s}{\partial x_i \partial y_j} =& -\frac{1}{2\pi}K_1'(\rho_s(\tilde{x},y))\frac{-(\tilde{x}_i - y_i)(\tilde{x}_j - y_j)(s+\sigma_i)s}{\rho_s^2(\tilde{x},y)/s^2}\\ &-\frac{1}{2\pi}K_1(\rho_s(\tilde{x},y))\frac{-(\rho_s/s)^2\delta_{ij}(s+\sigma_i) + (\tilde{x}_i - y_i)(\tilde{x}_j - y_j)(s+\sigma_i)}{\rho_s^3(\tilde{x},y)/s^3}.\end{aligned}$$

By using the identity $K_1'(z) = -\frac{1}{2}(K_0(z) + K_2(z))$, we have

$$\begin{aligned}|K_1'(\rho_s(\tilde{x},y))| &\le \frac{1}{2}(\,|K_0(\rho_s(\tilde{x},y))| + |K_2(\rho_s(\tilde{x},y))|\,)\\ &\le e^{-(\gamma\bar{\sigma}-1)}\,\frac{1}{2}(K_0(1) + K_2(1)) = e^{-(\gamma\bar{\sigma}-1)}|K_1'(1)|.\end{aligned}$$

On the other hand, by Lemma 4.2, $|\rho_s/s|^{-1} \le Cd^{-1} \le C\sigma_0$, where we have used the assumption (4.2) in the last estimate. Therefore,

$$\frac{\partial^2 \tilde{G}_s}{\partial x_i \partial y_j} = s^2 R^s_{2,ij} + sR^s_{1,ij} + R^s_{0,ij},$$

where $R_{l,ij}$ has the following estimate

$$|R^s_{l,ij}| \le C\sigma_0^2 (1+\sigma_0 s_1^{-1})^2\, e^{-\gamma\bar{\sigma}}, \quad l = 0,1,2,\ i,j = 1,2.$$

This completes the proof. ∎

Now we are in the position to estimate the modified single and double layer potentials $\tilde{\Psi}^k_{\mathrm{SL}}, \tilde{\Psi}^k_{\mathrm{DL}}$. Throughout the paper we shall use the $H^{1/2}(\Gamma_j)$ norm, $j = 1,2$,

$$\|v\|_{H^{1/2}(\Gamma_j)} = \left(|\Gamma_j|^{-1}\|v\|^2_{L^2(\Gamma_j)} + |v|^2_{\frac{1}{2},\Gamma_j}\right)^{1/2},$$

where

$$|v|^2_{\frac{1}{2},\Gamma_j} = \int_{\Gamma_j}\int_{\Gamma_j} \frac{|v(x)-v(x')|^2}{|x-x'|^2}\,\mathrm{d}s(x)\,\mathrm{d}s(x').$$

To estimate the PML solution in time domain, we need the following Lemma.

Lemma 4.5. *For any $s_1 > 0$ and $f, g \in L^2(0,T;L^2(\Gamma))$, we have*

$$\left\|\int_\Gamma f * g\,\mathrm{d}s_x\right\|_{L^2(0,T)} \le e^{s_1 T}\left(\max_{-\infty<s_2<+\infty}\|(\mathscr{L}f)(s_1+\mathbf{i}s_2)\|_{L^2(\Gamma)}\right)\|\,g\,\|_{L^2(0,T;L^2(\Gamma))}.$$

Proof. Let $\tilde{f}, \tilde{g}$ be the extension of f, g in $\mathbb{R}$ such that $\tilde{f} = 0, \tilde{g} = 0$ outside the interval $(0,T)$. Notice that

$$\mathscr{L}\tilde{f}\cdot\mathscr{L}\tilde{g} = \mathscr{L}(\tilde{f}*\tilde{g}) = \mathscr{F}(e^{-s_1 t}(\tilde{f}*\tilde{g})), \quad \mathscr{L}\tilde{g} = \mathscr{F}(e^{-s_1 t}\tilde{g}),$$

where $\mathscr{F}$ is the Fourier transform in t. By using Parseval equality we obtain

$$\begin{aligned}
\int_0^T e^{-2s_1 t}\left|\int_\Gamma \tilde{f}*\tilde{g}\,\mathrm{d}s_x\right|^2 \mathrm{d}t &= 2\pi\int_{-\infty}^{+\infty}\left|\int_\Gamma \mathscr{L}\tilde{f}\cdot\mathscr{L}\tilde{g}\mathrm{d}s_x\right|^2 \mathrm{d}s_2 \\
&\le 2\pi\int_{-\infty}^{+\infty}\|\mathscr{L}\tilde{f}\|^2_{L^2(\Gamma)}\|\mathscr{L}\tilde{g}\|^2_{L^2(\Gamma)}\mathrm{d}s_2 \\
&\le 2\pi\max_{-\infty<s_2<+\infty}\|\mathscr{L}\tilde{f}\|^2_{L^2(\Gamma)}\int_{-\infty}^{+\infty}\|\mathscr{L}\tilde{g}\|^2_{L^2(\Gamma)}\mathrm{d}s_2 \\
&= \max_{-\infty<s_2<+\infty}\|\mathscr{L}\tilde{f}\|^2_{L^2(\Gamma)}\int_{-\infty}^{+\infty}\|e^{-s_1 t}\tilde{g}\|^2_{L^2(\Gamma)}\mathrm{d}t \\
&\le \max_{-\infty<s_2<+\infty}\|\mathscr{L}f\|^2_{L^2(\Gamma)}\|g\|^2_{L^2(0,T;L^2(\Gamma))}.
\end{aligned}$$

From the definition of convolution, we have

$$(\tilde{f}*\tilde{g})(t) = \int_0^t \tilde{f}(t-s)\tilde{g}(s)\,\mathrm{d}s = \int_0^t f(t-s)g(s)\,\mathrm{d}s = (f*g)(t).$$

This completes the proof. ∎

LEMMA 4.6. *For any $\mu(x,t) \in H^2(0,T;L^2(\Gamma_1))$ satisfying $\mu|_{t=0} = 0, \partial_t\mu|_{t=0} = 0$ on Γ_1, let $\mu_{\text{L}} = \mathscr{L}\mu$ and*

$$v(x,t) = \mathscr{L}^{-1}\left(\tilde{\Psi}^s_{\text{DL}}(\mu_{\text{L}})\right) = \int_{\Gamma_1} \mathscr{L}^{-1}\left(\frac{\partial \tilde{G}_s(x,y)}{\partial \mathbf{n}_1(y)}\mu_{\text{L}}(y)\right) \mathrm{d}s_y$$

be the double layer potential. Then

$$\|v\|_{L^2(0,T;H^{1/2}(\Gamma_2))} \le C\sigma_0^2 d(1+\sigma_0 T)^2\, e^{-\gamma\bar{\sigma}}\|\mu\|_{H^2(0,T;L^2(\Gamma_1))}.$$

where the constant C is independent of σ_0, d and T.

Proof. From the definition of norm $H^{1/2}(\Gamma_2)$, it is easy to see that

$$\|v\|_{L^2(0,T;H^{1/2}(\Gamma_2))} \le C\|v\|_{L^2(0,T;L^\infty(\Gamma_2))} + Cd\|\nabla_x v\|_{L^2(0,T;L^\infty(\Gamma_2))}.$$

For any $x \in \Gamma_2$, we have

$$v(x,t) = \int_{\Gamma_1} \mathscr{L}^{-1}\left(\mathbf{Q}_1^s(x,y)\cdot\mathbf{n}_1(y)\right) * \partial_t\mu(y)\,\mathrm{d}s_y,$$

$$\nabla_x v(x,t) = \int_{\Gamma_1} \sum_{l=0}^{2} \mathscr{L}^{-1}(R_l^s(x,y)\cdot\mathbf{n}_1(y)) * \partial_t^l\mu(y)\,\mathrm{d}s_y,$$

where $\mathbf{Q}_1^s = (Q_{1,1}^s, Q_{1,2}^s)^T$ and the matrix $R_l^s = (R_{l,ij}^s)_{i,j=1}^2$ with elements $Q_{1,j}^s$, $R_{l,ij}^s$ are given in Lemma 4.4.

Then Lemma 4.5 implies that

$$\|v(x,t)\|_{L^2(0,T)} \le Ce^{s_1 T}\max_{s_2\in\mathbb{R}}\|\mathbf{Q}_1^s(x,\cdot)\|_{L^2(\Gamma_1)}\|\mu\|_{H^1(0,T;L^2(\Gamma_1))},$$

$$\|\nabla_x v(x,t)\|_{L^2(0,T)} \le Ce^{s_1 T}\sum_{l=0}^{2}\max_{s_2\in\mathbb{R}}\|R_l^s(x,\cdot)\|_{L^2(\Gamma_1)}\|\mu\|_{H^2(0,T;L^2(\Gamma_1))}.$$

By Lemma 4.4 (iii),

$$\|\mathbf{Q}_1^s(x,\cdot)\|_{L^2(\Gamma_1)} \le C(1+\sigma_0 s_1^{-1})\, e^{-\gamma\bar{\sigma}}.$$

By Lemma 4.4 (iv),

$$\sum_{l=0}^{2}\|R_l^s(x,\cdot)\|_{L^2(\Gamma_1)} \le C\sigma_0^2(1+\sigma_0 s_1^{-1})^2\, e^{-\gamma\bar{\sigma}}.$$

This completes the proof by letting $s_1 = T^{-1}$. ∎

LEMMA 4.7. *For any $\lambda(x,t) \in H^1(0,T;L^2(\Gamma_1))$ satisfying $\lambda|_{t=0} = 0$ on Γ_1, let $\lambda_{\text{L}} = \mathscr{L}\lambda$ and*

$$v(x,t) = \mathscr{L}^{-1}\left(\tilde{\Psi}^s_{\text{SL}}(\lambda_{\text{L}})\right) = \int_{\Gamma_1} \mathscr{L}^{-1}\left(\tilde{G}_s(x,y)\lambda_{\text{L}}(y)\right) \mathrm{d}s_y$$

be the single layer potential. Then

$$\|v\|_{L^2(0,T;H^{1/2}(\Gamma_2))} \le C\sigma_0 d(1+\sigma_0 T)\, e^{-\gamma\bar{\sigma}}\|\lambda\|_{H^1(0,T;L^2(\Gamma_1))},$$

where the constant C is independent of σ_0, d and T.

Proof. From the definition of norm $H^{1/2}(\Gamma_2)$, it is easy to see that

$$\|v\|_{L^2(0,T;H^{1/2}(\Gamma_2))} \le C\|v\|_{L^2(0,T;L^\infty(\Gamma_2))} + Cd\|\nabla_x v\|_{L^2(0,T;L^\infty(\Gamma_2))}.$$

For any $x \in \Gamma_2$, we have

$$v(x,t) = \int_{\Gamma_1} \mathscr{L}^{-1}(\tilde{G}_s(x,y)) * \lambda(y)\,\mathrm{d}s_y,$$
$$\nabla_x v(x,t) = \int_{\Gamma_1} \sum_{l=0}^{1} \mathscr{L}^{-1}(\mathbf{P}_l^s(x,y)) * \partial_t^l \lambda(y)\,\mathrm{d}s_y,$$

where $\mathbf{P}_l^s = (P_{l,1}^s, P_{l,2}^s)^T$, $l = 0, 1$. Then Lemma 4.5 implies that

$$\|v(x,t)\|_{L^2(0,T)} \le Ce^{s_1 T} \max_{s_2\in\mathbb{R}} \|\tilde{G}_s(x,\cdot)\|_{L^2(\Gamma_1)} \|\lambda\|_{L^2(0,T;L^2(\Gamma_1))},$$
$$\|\nabla_x v(x,t)\|_{L^2(0,T)} \le Ce^{s_1 T} \sum_{l=0}^{1} \max_{s_2\in\mathbb{R}} \|\mathbf{P}_l^s(x,\cdot)\|_{L^2(\Gamma_1)} \|\lambda\|_{H^1(0,T;L^2(\Gamma_1))}.$$

By Lemma 4.4 (i) and (ii), for any $x \in \Gamma_2$,

$$\|\tilde{G}_s(x,\cdot)\|_{L^2(\Gamma_1)} \le C\|\tilde{G}_s(x,\cdot)\|_{L^\infty(\Gamma_1)} \le C\, e^{-\gamma\bar{\sigma}},$$
$$\sum_{l=0}^{1} \|\mathbf{P}_l^s(x,\cdot)\|_{L^2(\Gamma_1)} \le C \sum_{l=0}^{1} \|\mathbf{P}_l^s(x,\cdot)\|_{L^\infty(\Gamma_1)} \le C\sigma_0(1+\sigma_0 s_1^{-1})\, e^{-\gamma\bar{\sigma}}.$$

This completes the proof by letting $s_1 = T^{-1}$. ∎

The following theorem which is a direct consequence of Lemmas 4.6-4.7 is the main result of this section.

THEOREM 4.8. *For any $\lambda \in H^1(0,T;L^2(\Gamma_1))$, $\mu \in H^2(0,T;L^2(\Gamma_1))$ satisfying $\lambda|_{t=0} = 0$, $\mu|_{t=0} = 0$, $\partial_t\mu|_{t=0} = 0$ on Γ_1, let $\lambda_{\text{L}} = \mathscr{L}\lambda$, $\mu_{\text{L}} = \mathscr{L}\mu$ and $\mathbb{E}(\lambda_{\text{L}}, \mu_{\text{L}})$ be the PML extension defined in* (2.20)*. Then there exists a constant $C > 0$ independent of σ_0, d, T but may depend on C_0 in (H1), such that*

$$\begin{aligned}\|\mathscr{L}^{-1}\mathbb{E}(\lambda_{\text{L}}, \mu_{\text{L}})\|_{L^2(0,T;H^{1/2}(\Gamma_2))} &\le C\sigma_0 d(1+\sigma_0 T)\, e^{-\gamma\bar{\sigma}} \|\lambda\|_{H^1(0,T;L^2(\Gamma_1))} \\ &\quad + C\sigma_0^2 d(1+\sigma_0 T)^2\, e^{-\gamma\bar{\sigma}} \|\mu\|_{H^2(0,T;L^2(\Gamma_1))}.\end{aligned}$$

5. Convergence analysis. In this section, we consider the convergence of the PML method. We start with following lemma.

LEMMA 5.1. *For any $w \in L^2(0,T;H^{1/2}(\Gamma_1))$, we have*

$$-\mathrm{Re}\int_0^T e^{-2s_1 t}\langle \mathcal{T}(w), w\rangle_{\Gamma_1}\,\mathrm{d}t \ge 0,$$

where $\langle\cdot,\cdot\rangle_{\Gamma_1}$ is the duality pairing between $H^{-1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$ which is the extension of the inner product of $L^2(\Gamma_1)$.

Proof. Let $w_{\text{L}} = \mathscr{L}(w)$, from the definition of Dirichlet-to-Neumann operator $\mathcal{T}$, we know that $\mathcal{T}(w) = \mathscr{L}^{-1}(T_s(w_{\text{L}}))$, where w_{L} satisfies (2.3)-(2.5). By applying the

first Green identity over the bounded domain $\Omega_R = B_R \backslash \bar{B}_1$, where $R > 0$ such that $\bar{B}_1 \subset B_R = \{x \in \mathbb{R}^2 : \ |x| < R\}$, we see that

$$\left(\frac{1}{s}\nabla w_{\rm L}, \nabla \bar{w}_{\rm L}\right) + (s w_{\rm L}, \bar{w}_{\rm L}) + \langle T_s(w_{\rm L}), w_{\rm L}\rangle_{\Gamma_1} - \langle \frac{1}{s}\frac{\partial w_{\rm L}}{\partial r}, w_{\rm L}\rangle_{\partial B_R} = 0. \quad (5.1)$$

Take the real part of (5.1) and notice that

$$\left|\frac{1}{s}\frac{\partial w}{\partial r} + w\right|^2 = \left|\frac{1}{s}\frac{\partial w}{\partial r}\right|^2 + |w|^2 + 2\mathrm{Re}\left(\frac{1}{s}\frac{\partial w}{\partial r}\bar{w}\right),$$

we have

$$\begin{aligned}\mathrm{Re}\left(\frac{1}{s}\nabla w_{\rm L}, \nabla \bar{w}_{\rm L}\right) + \mathrm{Re}\,(s w_{\rm L}, \bar{w}_{\rm L}) + \mathrm{Re}\,\langle T_s(w_{\rm L}), w_{\rm L}\rangle_{\Gamma_1} \\ + \frac{1}{2}\left\|\frac{1}{s}\frac{\partial w_{\rm L}}{\partial r}\right\|^2_{L^2(\partial B_R)} + \frac{1}{2}\|w_{\rm L}\|^2_{L^2(\partial B_R)} = \frac{1}{2}\left\|\frac{1}{s}\frac{\partial w_{\rm L}}{\partial r} + w_{\rm L}\right\|^2_{L^2(\partial B_R)}. \quad (5.2)\end{aligned}$$

Since $s_1 > 0$, the first two terms in (5.2) are positive. By the radiation condition (2.5), the right hand side of (5.2) tends to zero when $R \to \infty$. This yields that $\mathrm{Re}\,\langle T_s(w_{\rm L}), w_{\rm L}\rangle_{\Gamma_1} \le 0$, then by Parseval equality, we obtain

$$\begin{aligned}-\mathrm{Re}\int_0^T e^{-2s_1 t}\langle \mathcal{T}(w), w\rangle_{\Gamma_1}\mathrm{d}t &= -\mathrm{Re}\int_{-\infty}^{\infty} e^{-2s_1 t}\langle \mathscr{L}^{-1}(T_s(w_{\rm L})), \mathscr{L}^{-1}(w_{\rm L})\rangle_{\Gamma_1}\mathrm{d}t \\ &= -2\pi\mathrm{Re}\int_{-\infty}^{\infty}\langle T_s(w_{\rm L}), w_{\rm L}\rangle_{\Gamma_1}\mathrm{d}s_2 \ge 0.\end{aligned}$$

This completes the proof. □

For $\hat{u}_{\rm L} = \mathscr{L}(\hat{u})$ we define its PML extension as in (2.21),

$$\tilde{\hat{u}}_{\rm L} = \mathbb{E}(\frac{\partial \hat{u}_{\rm L}}{\partial \mathbf{n}_1}\Big|_{\Gamma_1}, \hat{u}_{\rm L}|_{\Gamma_1}),$$

and define the corresponding variables $(\tilde{\hat{u}}, \tilde{\hat{\mathbf{p}}}, \tilde{\hat{u}}^*, \tilde{\hat{\mathbf{p}}}^*)$ as in (2.25) which satisfies the PML system in $\mathbb{R}^2\backslash\bar{D}$. Let $(\hat{u}, \hat{\mathbf{p}}, \hat{u}^*, \hat{\mathbf{p}}^*)$ be the solution of the PML system (2.30)-(2.33) in $\Omega^{\rm PML}$. Take $\phi = \tilde{\hat{u}} - \hat{u}$, $\Phi = \tilde{\hat{\mathbf{p}}} - \hat{\mathbf{p}}$, $\phi^* = \tilde{\hat{u}}^* - \hat{u}^*$, $\Phi^* = \tilde{\hat{\mathbf{p}}}^* - \hat{\mathbf{p}}^*$, then $(\phi, \Phi, \phi^*, \Phi^*)$ satisfies the following system

$$\frac{\partial \phi}{\partial t} + \mathrm{div}\Phi + (\sigma_1 + \sigma_2)\phi + \phi^* = 0, \quad \frac{\partial \phi^*}{\partial t} = \sigma_1\sigma_2\phi \quad \text{in } \Omega^{\rm PML}\times(0,T), \quad (5.3)$$

$$\frac{\partial \Phi^*}{\partial t} + \nabla\phi = 0, \quad \frac{\partial \Phi}{\partial t} + \Lambda_1\Phi = \frac{\partial \Phi^*}{\partial t} + \Lambda_2\Phi^* \quad \text{in } \Omega^{\rm PML}\times(0,T), \quad (5.4)$$

$$\phi = 0 \quad \text{on } \Gamma_1\times(0,T), \qquad \phi = \tilde{\hat{u}}|_{\Gamma_2} \quad \text{on } \Gamma_2\times(0,T), \quad (5.5)$$

$$\phi|_{t=0} = 0, \quad \Phi|_{t=0} = 0, \quad \phi^*|_{t=0} = 0, \quad \Phi^*|_{t=0} = 0 \quad \text{in } \Omega^{\rm PML}. \quad (5.6)$$

Let ζ be a lifting function in $\Omega^{\rm PML}\times(0,T)$ such that $\zeta = 0$ on Γ_1 and $\zeta = \tilde{\hat{u}}|_{\Gamma_2}$ on Γ_2. It is easy to see that we can choose $\zeta(\cdot, 0) = 0$. Then $(\tilde{\phi}, \tilde{\phi}^*, \Phi, \Phi^*)$, where $\tilde{\phi} = \phi - \zeta$, $\tilde{\phi}^* = \phi^* - \int_0^t \sigma_1\sigma_2\zeta$ satisfies the PML system (3.1)-(3.4) with

$$f_1 = -\partial_t\zeta - (\sigma_1 + \sigma_2)\zeta - \int_0^t \sigma_1\sigma_2\zeta, \qquad f_2 = -\nabla\zeta \quad \text{in } \Omega^{\rm PML}\times(0,T),$$

and zero initial and boundary values. From Theorem 3.3 we have

$$\max_{0\le t\le T}\left(\|\partial_t\phi\|_{L^2(\Omega^{\rm PML})}+\|\partial_t\Phi\|_{L^2(\Omega^{\rm PML})}\right)$$
$$\le C\int_0^T\left(\|\partial_t f_1\|_{L^2(\Omega^{\rm PML})}+\|\partial_t f_2+\sigma_0 f_2\|_{L^2(\Omega^{\rm PML})}\right)\,{\rm d}t.$$

By (5.3) we have

$$\|{\rm div}\Phi\|_{L^2(\Omega^{\rm PML})}\le\|\partial_t\phi\|_{L^2(\Omega^{\rm PML})}+2\|\sigma_0\phi\|_{L^2(\Omega^{\rm PML})}+\|\phi^*\|_{L^2(\Omega^{\rm PML})}.$$

Thus by Lemma 3.2 we have

$$\max_{0\le t\le T}\left(\|{\rm div}\Phi\|_{L^2(\Omega^{\rm PML})}+\|\partial_t\Phi\|_{L^2(\Omega^{\rm PML})}\right)$$
$$\le C(1+\sigma_0T)\int_0^T\left(\|\partial_t f_1\|_{L^2(\Omega^{\rm PML})}+\|\partial_t f_2+\sigma f_2\|_{L^2(\Omega^{\rm PML})}\right)\,{\rm d}t$$
$$\le C(1+\sigma_0T)\sigma_0^2T^{1/2}\left(\|\zeta\|_{H^2(0,T;L^2(\Omega^{\rm PML}))}+\|\zeta\|_{H^1(0,T;H^1(\Omega^{\rm PML}))}\right).$$

The above estimate is valid for any ζ implies that

$$\max_{0\le t\le T}\left(\|{\rm div}\Phi\|_{L^2(\Omega^{\rm PML})}+\|\partial_t\Phi\|_{L^2(\Omega^{\rm PML})}\right)$$
$$\le C(1+\sigma_0T)\sigma_0^2T^{1/2}\left(\|\tilde{\hat{u}}\|_{H^2(0,T;H^{1/2}(\Gamma_2))}\right). \tag{5.7}$$

Now we are in the position to show the convergence of the time domain PML method in the following theorem.

THEOREM 5.2. *Let* $(u,\mathbf{p})$ *be the solution of the problem* (1.1)-(1.4) *and* $(\hat{u},\hat{\mathbf{p}},\hat{u}^*,\hat{\mathbf{p}}^*)$ *be the solution of the PML problem* (2.30)-(2.33). *Then there exists a constant* $C>0$ *independently of* σ_0,d,T *but may depend on* C_0 *in (H1) such that*

$$\max_{0\le t\le T}\left(\|u-\hat{u}\|_{L^2(\Omega_1)}+\|\mathbf{p}-\hat{\mathbf{p}}\|_{L^2(\Omega_1)}\right)$$
$$\le CT^{3/2}\sigma_0^3d(1+\sigma_0T)^3\,e^{-\gamma\bar{\sigma}}\left(\|\partial\hat{u}/\partial\mathbf{n}_1\|_{H^3(0,T;L^2(\Gamma_1))}+\sigma_0\|\hat{u}\|_{H^4(0,T;L^2(\Gamma_1))}\right).$$

Proof. From (2.7) and (2.30)-(2.31), we have

$$\frac{\partial(u-\hat{u})}{\partial t}+{\rm div}(\mathbf{p}-\hat{\mathbf{p}})=0,\quad\text{in}\ \ \Omega_1\times(0,T), \tag{5.8}$$
$$\frac{\partial(\mathbf{p}-\hat{\mathbf{p}})}{\partial t}+\nabla(u-\hat{u})=0,\quad\text{in}\ \ \Omega_1\times(0,T). \tag{5.9}$$

By testing (5.8) with $v\in H^1(\Omega_1)$ and using (2.6) we get

$$\left(\frac{\partial(u-\hat{u})}{\partial t},v\right)-(\mathbf{p}-\hat{\mathbf{p}},\nabla v)-\langle\mathcal{T}(u-\hat{u}),v\rangle_{\Gamma_1}=\langle\hat{\mathbf{p}}\cdot\mathbf{n}_1+\mathcal{T}(\hat{u}),v\rangle_{\Gamma_1}. \tag{5.10}$$

Let $w=u-\hat{u}$ and $w^*=\int_0^t(u-\hat{u}){\rm d}t$. From (5.9) we have $\mathbf{p}-\hat{\mathbf{p}}=-\nabla w^*$. Thus by taking $v=w$ we have

$$\frac{1}{2}\frac{\rm d}{{\rm d}t}\left(\|w\|^2_{L^2(\Omega_1)}+\|\nabla w^*\|^2_{L^2(\Omega_1)}\right)-\langle\mathcal{T}(w),w\rangle_{\Gamma_1}=\langle\hat{\mathbf{p}}\cdot\mathbf{n}_1+\mathcal{T}(\hat{u}),w\rangle_{\Gamma_1}. \tag{5.11}$$

Denote

$$X(0,T;\Omega_1)=\left\{v\in L^\infty(0,T;L^2(\Omega_1)),\ v^*=\int_0^t v\mathrm{d}t\in L^\infty(0,T;H^1(\Omega_1))\right\},$$

which is a Banach space with the norm

$$\|v\|_{X(0,T;\Omega_1)}=\sup_{0\le t\le T}\left(\|v\|^2_{L^2(\Omega_1)}+\|\nabla v^*\|^2_{L^2(\Omega_1)}\right)^{1/2}.$$

Define

$$Y(0,T;\Gamma_1)=\left\{\varphi:\int_0^T\langle\varphi,v\rangle_{\Gamma_1}\mathrm{d}t<\infty,\ \forall v\in X(0,T;\Omega_1)\right\}.$$

It is easy to see that $Y(0,T;\Gamma_1)$ is also a Banach space with the norm

$$\|\varphi\|_{Y(0,T;\Gamma_1)}=\sup_{v\in X(0,T;\Omega_1)}\frac{\left|\int_0^T\langle\varphi,v\rangle_{\Gamma_1}\mathrm{d}t\right|}{\|v\|_{X(0,T;\Omega_1)}}.$$

From (5.11) and Lemma 5.1 we obtain

$$\|e^{-s_1t}w\|^2_{X(0,T;\Omega_1)}\le C\|e^{-s_1t}(\hat{\mathbf{p}}\cdot\mathbf{n}_1+\mathcal{T}(\hat{u}))\|_{Y(0,T;\Gamma_1)}\|e^{-s_1t}w\|_{X(0,T;\Omega_1)}.$$

By letting $s_1\to 0$, we get

$$\sup_{0\le t\le T}\left(\|w\|_{L^2(\Omega_1)}+\|\nabla w^*\|_{L^2(\Omega_1)}\right)\le C\|\hat{\mathbf{p}}\cdot\mathbf{n}_1+\mathcal{T}(\hat{u})\|_{Y(0,T;\Gamma_1)}. \tag{5.12}$$

It is clear that $\mathcal{T}(\hat{u})=-\tilde{\hat{\mathbf{p}}}\cdot\mathbf{n}_1$ on $\Gamma_1\times(0,T)$. Therefore we need to estimate $\|(\hat{\mathbf{p}}-\tilde{\hat{\mathbf{p}}})\cdot\mathbf{n}_1\|_{Y(0,T;\Gamma_1)}=\|\Phi\cdot\mathbf{n}_1\|_{Y(0,T;\Gamma_1)}$. Notice that any function $v\in X(0,T;\Omega_1)$ can be extended to $\Omega^{\mathrm{PML}}\times(0,T)$, such that $v=0$ on $\Gamma_2\times(0,T)$ and $\|v\|_{X(0,T;\Omega^{\mathrm{PML}})}\le C\|v\|_{X(0,T;\Omega_1)}$. Thus

$$\|\Phi\cdot\mathbf{n}_1\|_{Y(0,T;\Gamma_1)}\le C\sup_{v\in X(0,T;\Omega^{\mathrm{PML}})}\frac{\left|\int_0^T\langle\Phi\cdot\mathbf{n}_1,v\rangle_{\Gamma_1}\mathrm{d}t\right|}{\|v\|_{X(0,T;\Omega^{\mathrm{PML}})}}.$$

On the other hand, since $v=0$ on Γ_2, we have

$$\int_0^T\langle\Phi\cdot\mathbf{n}_1,v\rangle_{\Gamma_1}\mathrm{d}t=\int_0^T\left[(\mathrm{div}\Phi,v)_{\Omega^{\mathrm{PML}}}+(\Phi,\nabla v)_{\Omega^{\mathrm{PML}}}\right]\mathrm{d}t.$$

Integrating by parts we obtain

$$\begin{aligned}\int_0^T(\Phi,\nabla v)_{\Omega^{\mathrm{PML}}}\mathrm{d}t&=(\Phi(\cdot,T),\nabla v^*(\cdot,T))_{\Omega^{\mathrm{PML}}}-\int_0^T(\partial_t\Phi,\nabla v^*)_{\Omega^{\mathrm{PML}}}\mathrm{d}t\\&\le\max_{0\le t\le T}\|\nabla v^*\|_{L^2(\Omega^{\mathrm{PML}})}\int_0^T\|\partial_t\Phi\|_{L^2(\Omega^{\mathrm{PML}})}\mathrm{d}t,\end{aligned}$$

where we have used the fact that $\Phi|_{t=0}=0$. Therefore,

$$\|\Phi\cdot\mathbf{n}_1\|_{Y(0,T;\Gamma_1)}\le C\int_0^T\left(\|\mathrm{div}\Phi\|_{L^2(\Omega^{\mathrm{PML}})}+\|\partial_t\Phi\|_{L^2(\Omega^{\mathrm{PML}})}\right)\mathrm{d}t,$$

and consequently, by (5.12) and (5.7),

$$\sup_{0\le t\le T}\left(\|w\|_{L^2(\Omega_1)}+\|\nabla w^*\|_{L^2(\Omega_1)}\right)\le C\int_0^T\left(\|\mathrm{div}\Phi\|_{L^2(\Omega^{\mathrm{PML}})}+\|\partial_t\Phi\|_{L^2(\Omega^{\mathrm{PML}})}\right)\mathrm{d}t$$
$$\le C(1+\sigma_0T)\sigma_0^2T^{3/2}\left(\|\tilde{\hat{u}}\|_{H^2(0,T;H^{1/2}(\Gamma_2))}\right).$$

This completes the proof by using Theorem 4.8. □

The theorem indicates that for the fixed PML absorbing coefficient σ_0, any error of the time domain PML method can be achieved by enlarging the thickness of the PML layer as $\ln T$ for large $T>0$.

6. Numerical Implementation. In this section, we present three numerical examples to illustrate the performance of the PML method. The computations are carried out in Matlab programming environment. We first briefly describe the numerical method used in our computations.

Denote by $H^1_{\Gamma_2}(\Omega_2)=\{v\in H^1(\Omega_2):\ v=0 \text{ on } \Gamma_2\}$. The equivalent weak formulation of (2.30)-(2.33) is: Find $u\in L^2(0,T;H^1_{\Gamma_2}(\Omega_2))$, $\mathbf{p}\in L^2(0,T;L^2(\Omega_2))$, $u^*\in L^2(0,T;L^2(\Omega_2))$, $\mathbf{p}^*\in L^2(0,T;L^2(\Omega_2))$ such that

$$\left(\frac{\partial u}{\partial t},v\right)-(\mathbf{p},\nabla v)+((\sigma_1+\sigma_2)u,v)+(u^*,v)=(f,v),\quad\forall\, v\in H^1_{\Gamma_2}(\Omega_2),\tag{6.1}$$
$$\left(\frac{\partial \mathbf{p}^*}{\partial t},\mathbf{q}^*\right)+(\nabla u,\mathbf{q}^*)=0,\quad\forall\,\mathbf{q}^*\in L^2(\Omega_2),\tag{6.2}$$
$$\left(\frac{\partial u^*}{\partial t},v^*\right)-(\sigma_1\sigma_2u,v^*)=0,\quad\forall\, v^*\in H^1(\Omega_2),\tag{6.3}$$
$$\left(\frac{\partial \mathbf{p}}{\partial t},\mathbf{q}\right)+(\Lambda_1\mathbf{p},\mathbf{q})=\left(\frac{\partial \mathbf{p}^*}{\partial t},\mathbf{q}\right)+(\Lambda_2\mathbf{p}^*,\mathbf{q}),\quad\forall\,\mathbf{q}\in L^2(\Omega_2),\tag{6.4}$$
$$u|_{t=0}=u_0,\quad \mathbf{p}|_{t=0}=\mathbf{p}_0,\quad u^*|_{t=0}=0,\quad \mathbf{p}^*|_{t=0}=\mathbf{p}_0\quad\text{ in }\Omega_2.\tag{6.5}$$

Let $\{t_0,\cdots,t_N\}$ be a partition of the time interval $[0,T]$ and $\tau_n=t_n-t_{n-1}$ be the n-th timestep size. Let $\mathcal{M}_h$ be a regular triangulation of Ω_2. We assume the elements $K\in\mathcal{M}_h$ may have one curved edge align with Γ_D so that $\Omega_2=\cup_{K\in\mathcal{M}_h}K$. Let $V_h\subset H^1(\Omega_2)$ be the conforming linear finite element space over $\mathcal{M}_h$, and $\overset{\circ}{V}_h=\{v_h\in V_h:\ v_h=0 \text{ on } \Gamma_2\}$. Let $W_h\subset L^2(\Omega_2)$ be the piecewise constant element space over $\mathcal{M}_h$.

We use the Crank-Nicolson scheme in time. Denote by

$$\partial v^n=\frac{v^n-v^{n-1}}{\tau_n},\quad \bar{v}^n=\frac{v^n+v^{n-1}}{2},$$

then the fully discrete scheme of (6.1)-(6.5) reads as follows: Given $(U^{n-1},\mathbf{P}^{n-1},U^{*,n-1},\mathbf{P}^{*,n-1})\in\overset{\circ}{V}_h\times(W_h)^2\times V_h\times(W_h)^2$, find $(U^n,\mathbf{P}^n,U^{*,n},\mathbf{P}^{*,n})\in\overset{\circ}{V}_h\times(W_h)^2\times V_h\times(W_h)^2$ such that

$$(\partial U^n,V)-(\bar{\mathbf{P}}^n,\nabla V)+((\sigma_1+\sigma_2)\bar{U}^n,V)+(\bar{U}^{*,n},V)=(\hat{f}_n,V),\quad\forall\, V\in\overset{\circ}{V}_h,$$
$$(\partial\mathbf{P}^{*,n},\mathbf{Q}^*)+(\nabla\bar{U}^n,\mathbf{Q}^*)=0,\quad\forall\,\mathbf{Q}^*\in(W_h)^2,$$
$$(\partial U^{*,n},V^*)-(\sigma_1\sigma_2\bar{U}^n,V^*)=0,\quad\forall\, V^*\in V_h,$$
$$(\partial\mathbf{P}^n,\mathbf{Q})+(\Lambda_1\bar{\mathbf{P}}^n,\mathbf{Q})=(\partial\mathbf{P}^{*,n},\mathbf{Q})+(\Lambda_2\bar{\mathbf{P}}^{*,n},\mathbf{Q}),\quad\forall\,\mathbf{Q}\in(W_h)^2,$$

where $\hat{f}_n := \frac{1}{\tau_n}\int_{t_{n-1}}^{t_n} f(t)\,\mathrm{d}t$ is the mean value of f over $[t_{n-1}, t_n]$.

Example 1. Let $B_1 = [-0.5, 0.5]^2$, the source is located at origin with the source term

$$f(x,t) = \delta(x)\,t, \tag{6.6}$$

which exists for all time. The initial values u_0 and $\mathbf{p}_0$ are taken as zeros. By using the time-domain Green's function, we can get the exact solution excited by this source

$$u(x,t) = \int_\Omega \int_0^t G_{t-s}(x,y)\partial_t f(y,s)\,\mathrm{d}s\,\mathrm{d}y. \tag{6.7}$$

The delta function is approximated by the Gaussian function and numerical integration is also used to compute the above convolution.

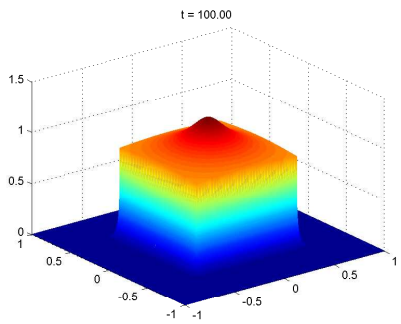

FIG. 6.1. *Example 1: numerical solution at final step $T = 100.0$ with a point source at origin.*

First, we take $L_1 = L_2 = 1.0$, fix $d_1 = d_2 = 0.5$ and choose $\sigma_0 = 25$. In the computation, a mesh of 227533 nodes is used which is refined around the boundaries where the coefficients of the PML system are discontinuous and the final time is set to be $T = 100.0$. In Figure 6.1, we show the numerical solution at the final step. It is observed that the waves are attenuated in the PML layer without spurious reflection.

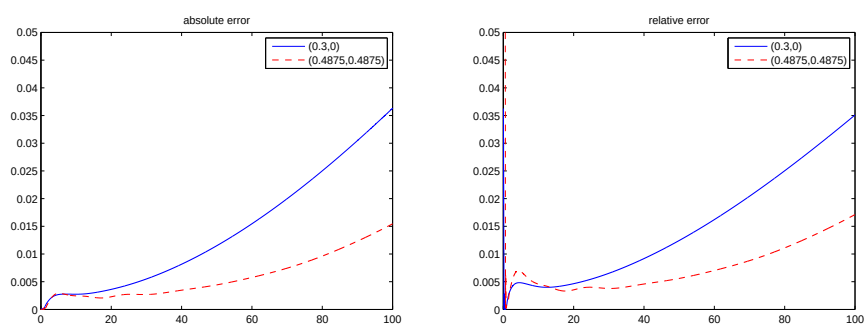

FIG. 6.2. *Example 1: The absolute and relative errors of the numerical solutions at points $(0.3, 0)$ and $(0.4875, 0.4875)$ for different times ($d_1 = d_2 = 0.5$).*

In Figure 6.2, we show the absolute and relative errors of the numerical solution at points $(0.3, 0)$ and $(0.4875, 0.4875)$ for different times. It is observed that the long time stability and convergence hold even for the point very close to the PML interface.

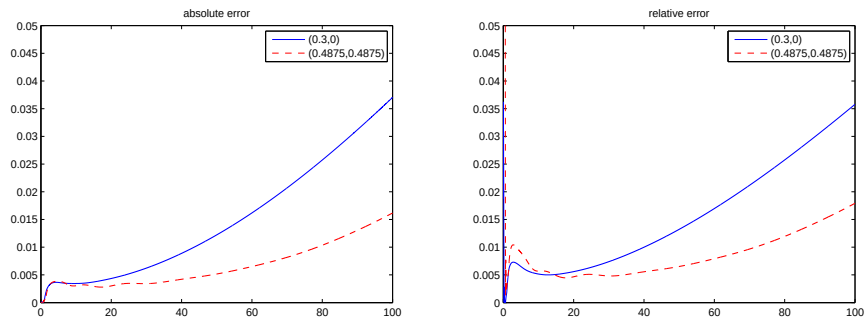

FIG. 6.3. *Example 1: The absolute and relative errors of the numerical solutions at points* $(0.3, 0)$ *and* $(0.4875, 0.4875)$ *for different times* ($d_1 = d_2 = 0.25$).

Then, we fix $d_1 = d_2 = 0.25$, choose $\sigma_0 = 50$ and do the numerical experiment again. The mesh of 298113 nodes is used and the final time is also set to be $T = 100.0$. Figure 6.3 shows the computational errors at the above points. The long time stability and convergence are also observed.

Example 2. Let $B_1 = [-0.5, 0.5]^2$, the source is located at origin with the source term

$$f(x, t) = \delta(x) \sin(\omega t), \tag{6.8}$$

which oscillates with $\omega = 1$ for all time. The initial values u_0 and $\mathbf{p}_0$ are taken as zeros. The exact solution can also be computed by using (6.7).

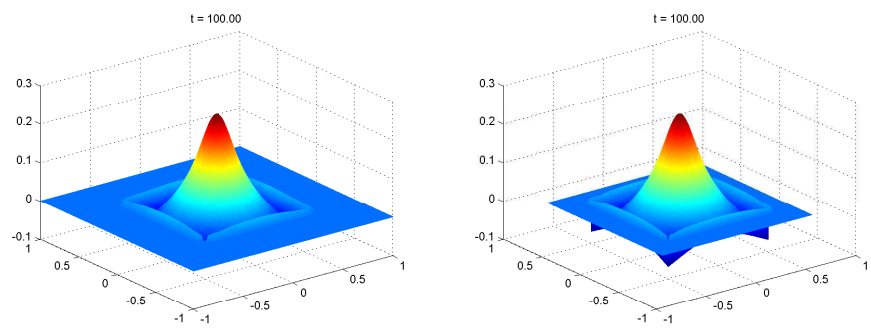

FIG. 6.4. *Example 2: The numerical solution at final step* $T = 100.0$ *with an oscillating source at origin (left:* $d_1 = d_2 = 0.5$*, right:* $d_1 = d_2 = 0.25$*).*

In this example, we take $L_1 = L_2 = 1.0$ and choose two sets of PML parameters the same as those in Example 1. We also use the same meshes and set the final time $T = 100.0$. Figure 6.4 displays the numerical solutions at the final step. It shows that the waves are absorbed rapidly as they propagate through the PML layer in each case.

In Figure 6.5, we compare the exact solution to the numerical solution at point $(0.4875, 0.4875)$ for different times. The solid curve represents the exact solution and the dashed curve the numerical one. It shows a very good agreement between the numerical solutions and the exact ones, even for a long time computation.

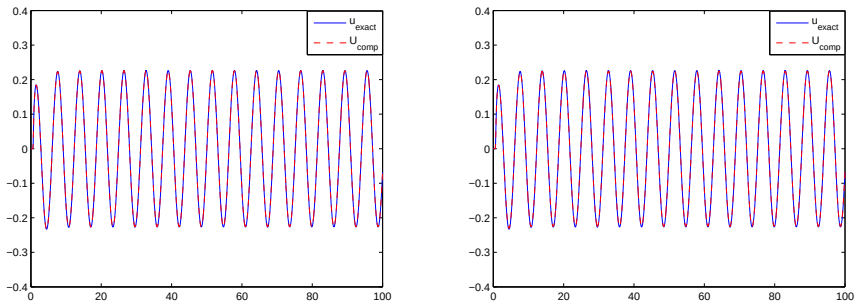

FIG. 6.5. *Example 2: The exact and numerical solution at point* $(0.4875, 0.4875)$ *for different times (left:* $d_1 = d_2 = 0.5$*, right:* $d_1 = d_2 = 0.25$*).*

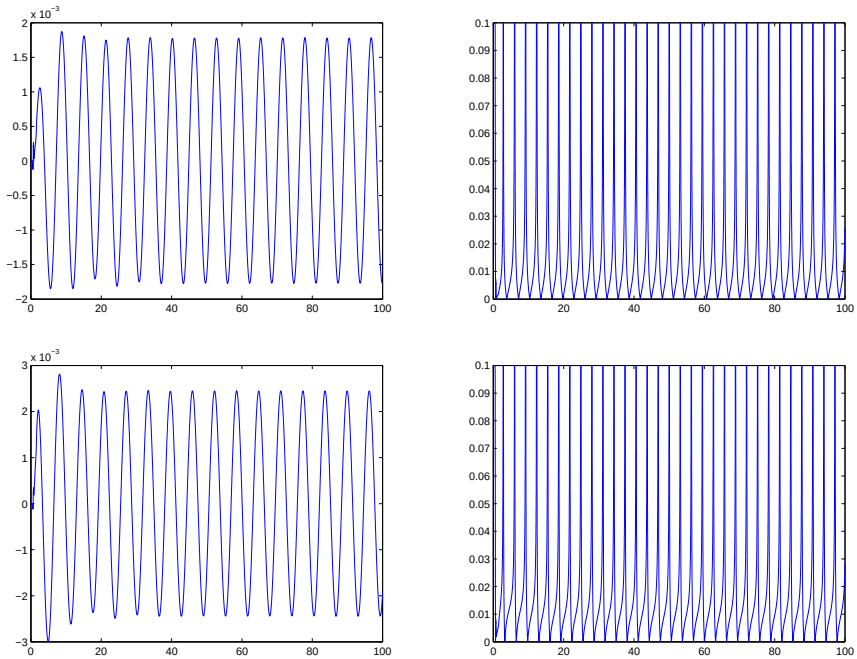

FIG. 6.6. *Example 2: The absolute and relative errors of the numerical solution at point* $(0.4875, 0.4875)$ *for different times (top:* $d_1 = d_2 = 0.5$*, bottom:* $d_1 = d_2 = 0.25$*).*

In Figure 6.6, we show the absolute and relative errors of the numerical solution at the above point. It is observed that the long time stability and convergence hold for each choice of the PML parameters.

Finally, we remark that the computational errors in Example 1 increase in time due to the presence of the source for all time and the computational errors oscillate in Example 2 due to the oscillation of the source. They both grow no faster than the polynomial function of T in the error bound in Theorem 5.2. We also remark that the computational errors shown in the numerical examples include both the errors due to the PML method that are analyzed in this paper and the errors of the discretization

of the PML system which deserve further studies.

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