PML METHOD FOR ELECTROMAGNETIC SCATTERING PROBLEM IN A TWO-LAYER MEDIUM*

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Abstract. The perfectly matched layer (PML) method is well-studied for acoustic scattering problems, electromagnetic scattering problems, and more recently, elastic scattering problems, with homogeneous background media. The purpose of this paper is to present the stability and exponential convergence of the PML method for three-dimensional electromagnetic scattering problem in a two-layer medium. The main contributions of this paper are threefold. Firstly, we establish the well-posedness of the original scattering problem for any Dirichlet boundary value in $\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)$ where Γ_D stands for the boundary of the scatterer. Secondly, we propose a new weak formulation for the original problem where the Dirichlet-to-Neumann operator is proposed on a truncation boundary inside PML. This argument is favorable to the analysis for the PML Dirichlet-to-Neumann operator. The inf-sup condition is proved for the sesquilinear form. Thirdly, we establish the well-posedness of the PML problem and prove that the approximate solution converges to the original scattering solution exponentially as either the PML absorbing coefficient or the thickness of the PML increases.

Key words. Perfectly matched layer, electromagnetic scattering problem, dyadic Green's function, Maxwell's equation, two-layer medium

AMS subject classifications. 35Q60, 65N30

1. Introduction. We propose and study the perfectly matched layer (PML) method for solving the electromagnetic scattering problem in a two-layer medium:

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} - k^2\boldsymbol{E} = 0 \quad \text{in } \mathbb{R}^3_\pm \setminus \bar{D}, \tag{1.1a}$$

$$\boldsymbol{n}\times\boldsymbol{E} = \boldsymbol{g} \quad \text{on } \Gamma_D, \tag{1.1b}$$

$$[\boldsymbol{n}\times\mathbf{curl}\,\boldsymbol{E}] = [\boldsymbol{n}\times\boldsymbol{E}] = 0 \qquad \text{on } \Sigma, \tag{1.1c}$$

$$\lim_{\rho\to\infty}\int_{\partial B(\rho)} |\mathbf{curl}\,\boldsymbol{E}\times\boldsymbol{n} - \mathbf{i}k\boldsymbol{E}|^2 = 0, \tag{1.1d}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{g}$ is determined by the incoming wave, $D\subset\mathbb{R}^3$ is a bounded domain with Lipschitz-continuous boundary Γ_D, $B(\rho)=\{\boldsymbol{x}\in\mathbb{R}^3:\ |\boldsymbol{x}|<\rho\}$ is the ball of radius ρ and centering at the origin, and $\boldsymbol{n}$ stands for the unit outer normal to D and $B(\rho)$ on their respective boundaries. We assume the wave number k is positive and piecewise constant, defined by

$$k(\boldsymbol{x}) = \begin{cases} k_+, & \text{if } \boldsymbol{x}\in\mathbb{R}^3_+, \\ k_-, & \text{if } \boldsymbol{x}\in\mathbb{R}^3_-, \end{cases} \tag{1.2}$$

where $\mathbb{R}^3_\pm = \{(x_1,x_2)\in\mathbb{R}^3:\ \pm x_2>0\}$. Without loss of generality we assume in this paper that $k_->k_+>0$. We remark that the boundary condition (1.1b) is not essential for our results. In fact, (1.1b) can be replaced by other boundary conditions such as Neumann or impedance boundary conditions on Γ_D. Furthermore, we can scale the system such that the diameter of the scatterer satisfies $\mathrm{diam}(D)\geq 1$.

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The basic idea of the PML method is to surround the computational domain by a layer of specially designed model medium that absorbs all waves propagating from inside the computational domain [1]. The convergence of the PML method for homogeneous background materials has drawn considerable attentions in the literature. Lassas and Somersalo [27,28] and Hohage, Schmidt, and Zschiedrich [25] studied the acoustic scattering problems for circular and smooth PMLs. It is proved that the PML solution converges exponentially to the solution of the original scattering problem as the thickness of the PML tends to infinity. We also refer to the work of Collino and Monk for PML in curvilinear coordinates [21]. In 2003, Chen and Wu proposed the adaptive PML finite element method for acoustic grating problems [15]. The adaptive PML method provides a complete numerical strategy to solve the scattering problems in the framework of finite element which produces automatically a coarse mesh size away from the fixed domain and thus makes the total computational costs insensitive to the thickness of the absorbing PMLs. Later on, the adaptive PML finite element method was extended to accoustic scattering problems in [14,16], to electromagnetic scattering problems in [12,13], to multiple scattering problems in [26,31], and to grating problems [2,3]. In 2005, Bao and Wu first proved the exponential convergence of PML method for Maxwell's equations [4]. Bramble and Pasciak also studied the stability and exponential convergence of PML method for acoustic and electromagnetic scattering problems in a series of papers [5–8]. They use both circular coordinates and Cartesian coordinates in constructing wave-absorbing materials. We also refer to the recent papers on PML methods for elastic scattering problems [9,18] and to [11,17] for exponential convergence of time-domain PML.

The studies mentioned above assume homogeneous background materials, that is, the wave number is constant away from the scatterer. The analysis for scattering problem is very challenging for layered media since the scattering waves usually consist of both propagating modes and evanescent modes. For two-layer media, Chen and Zheng first proved the stability and exponential convergence of the uniaxial PML method for two-dimensional acoustic scattering problems [19]. Their proof is very technical and relies on the Cagniard-de Hoop transform for the Green's function. Electromagnetic scattering problems in two-layer media have broad applications in both scientific and engineering areas, such as, near-field imaging, detection of buried objects, and so on. But the convergence of the PML method is an open issue. In 1998, Cutzach and Hazard proved the existence and uniqueness for electromagnetic scattering problem in a two-layer medium with incident plane waves or incident point source [23] (see also Monk's book [30, Chapter 12]). We also refer to Coyle and Monk [22] and Monk [30, Chapter 12] for the finite element approximation using transparent boundary condition and to [29] for the coupling of finite element method and boundary element method.

For scattering problems in layered media, the scattered waves become much more complicated and high-accuracy approximation of the radiation boundary condition becomes much difficult [24]. It is well-known that the numerical method using PML have two advantages compared with that using the Dirichlet-to-Neumann (DtN) operator. Firstly, it does not compute the Green's function which is very complicated for layered medium, particularly, in three-dimensional case. Secondly, the numerical method using PML usually yields an algebraic system with sparse matrix. It is favorable in designing effective precondtioners for the discrete problem. The purpose of this paper is to investigate the theoretical aspect of the PML method for electromagnetic scattering problem in a two-layer medium. The main theme is threefold.

- We prove the well-posedness of the scattering problem for any Dirichlet boundary data $\boldsymbol{g} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)$.
- We propose a new weak formulation for the scattering problem where the DtN operator is defined on the truncation boundary inside PML. This formulation is favorite in proving the stability of both the original DtN operator and the PML DtN operator. We also prove the inf-sup condition of the sesquilinear form which plays the key role in the convergence of the PML method.
- We introduce the Cagniard-de Hoop transform to the dyadic Green's function and prove that the Green's function decays exponentially in PML. This is the major novelty of this paper. We prove the well-posedness of the PML problem and the exponential convergence of the PML solution as either the absorbing coefficient or the thickness of the PML tends to infinity.

The layout of this paper is organized as follows. In section 2, we present the Fourier integral form of the dyadic Green's function for the scattering problem. The uniqueness and existence of the scattering solution are proved for any Dirichlet boundary data $\boldsymbol{g} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)$. In section 3, we derive the Cagniard-de Hoop representation of the Green's function. In section 4, we introduce the PML by means of complex coordinate stretching and prove the exponential decay of the modified Green's function. In section 5, we study an equivalent exterior problem of the scattering problem in the wave-absorbing material. Different from traditional ways in PML analysis, we propose a new weak formulation for the exterior problem where the truncation boundary is located inside PML. In section 6, we prove the inf-sup condition for the sesquilinear form of the weak formulation. In Section 7, we propose the PML approximation to the exterior problem on the truncated domain. The well-posedness and exponential convergence of the PML problem are also proved.

2. The well-posedness of the scattering problem. The purpose of this section is to study the weak solution of (1.1). First we introduce some Sobolev spaces.

2.1. Sobolev spaces. For a domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary $\Gamma = \partial\Omega$, let $L^2(\Omega)$ be the space of square-integrable functions, $H^1(\Omega) \subset L^2(\Omega)$ be the subspace whose functions have square-integrable gradients, and $\boldsymbol{H}(\mathbf{curl}, \Omega) \subset \boldsymbol{L}^2(\Omega)$ be the subspace whose functions have square-integrable curls. Throughout the paper we denote vector-valued quantities by boldface notation, such as $\boldsymbol{L}^2(\Omega) := L^2(\Omega)^3$. From [10], we have the surjective mappings

$$\begin{aligned}
\gamma :\ & H^1(\Omega) \to H^{1/2}(\Gamma), && \gamma\varphi = \varphi \quad \text{on } \Gamma,\\
\gamma_t :\ & \boldsymbol{H}(\mathbf{curl}, \Omega) \to \boldsymbol{H}^{-1/2}(\mathrm{Div}; \Gamma), && \gamma_t \boldsymbol{u} = \boldsymbol{n} \times \boldsymbol{u} \quad \text{on } \Gamma,\\
\gamma_T :\ & \boldsymbol{H}(\mathbf{curl}, \Omega) \to \boldsymbol{H}^{-1/2}(\mathrm{Curl}; \Gamma), && \gamma_T \boldsymbol{u} = \boldsymbol{n} \times (\boldsymbol{u} \times \boldsymbol{n}) \quad \text{on } \Gamma,
\end{aligned}$$

where Div, Curl stand for the surface divergence and surface scalar curl operators. For convenience, we define equivalent norms on the respective surface Sobolev spaces

$$\|\lambda\|_{H^{1/2}(\Gamma)} = \inf_{\substack{v \in H^1(\Omega)\\ \gamma v = \lambda}} \|v\|_{H^1(\Omega)} \qquad \forall\, \lambda \in H^{1/2}(\Gamma), \tag{2.1}$$

$$\|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div};\Gamma)} = \inf_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl},\Omega)\\ \gamma_t \boldsymbol{v} = \boldsymbol{\lambda}}} \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \qquad \forall\, \boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}; \Gamma), \tag{2.2}$$

$$\|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Curl};\Gamma)} = \inf_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl},\Omega)\\ \gamma_T \boldsymbol{v} = \boldsymbol{\lambda}}} \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \qquad \forall\, \boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathrm{Curl}; \Gamma). \tag{2.3}$$

For any $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$, it holds that

$$\mathrm{Div}(\gamma_t \boldsymbol{u}) = -\,\mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{n}, \quad \mathrm{Curl}(\gamma_T \boldsymbol{u}) = \mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{n} \quad \text{on } \Gamma.$$

It is known that $\boldsymbol{H}^{-1/2}(\mathrm{Div};\Gamma)$ and $\boldsymbol{H}^{-1/2}(\mathrm{Curl};\Gamma)$ are dual spaces. For any $\boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathrm{Div};\Gamma)$ and $\boldsymbol{\xi} \in \boldsymbol{H}^{-1/2}(\mathrm{Curl};\Gamma)$, the duality paring is defined by

$$\langle \boldsymbol{\lambda}, \boldsymbol{\xi} \rangle_\Gamma := \int_\Omega (\boldsymbol{u}_{\boldsymbol{\lambda}} \cdot \mathbf{curl}\, \boldsymbol{u}_{\boldsymbol{\xi}} - \mathbf{curl}\, \boldsymbol{u}_{\boldsymbol{\lambda}} \cdot \boldsymbol{u}_{\boldsymbol{\xi}}) \tag{2.4}$$

where $\boldsymbol{u}_{\boldsymbol{\lambda}}, \boldsymbol{u}_{\boldsymbol{\xi}} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ satisfy $\boldsymbol{\lambda} = \gamma_t \boldsymbol{u}_{\boldsymbol{\lambda}}$ and $\boldsymbol{\xi} = \gamma_T \boldsymbol{u}_{\boldsymbol{\xi}}$ on Γ.

For any $S \subset \Gamma$, the subspaces with zero trace and zero tangential trace on S are denoted respectively by

$$H_S^1(\Omega) := \{v \in H^1(\Omega) : \gamma v = 0 \ \text{ on } S\},$$
$$\boldsymbol{H}_S(\mathbf{curl}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega) : \gamma_t \boldsymbol{v} = 0 \ \text{ on } S\}.$$

In particular, we use the conventional notations

$$H_0^1(\Omega) := H_\Gamma^1(\Omega), \qquad \boldsymbol{H}_0(\mathbf{curl}, \Omega) := \boldsymbol{H}_\Gamma(\mathbf{curl}, \Omega).$$

2.2. Dyadic Green's function for the layered medium. The dyadic Green's function is the main tool in our analysis for the well-posedness of the scattering problem and the exponential convergence of the PML method. We follow arguments similar to those in [30, Section 12.4] to derive the dyadic Green's function $\mathbb{G}(k; \boldsymbol{x}, \boldsymbol{y})$ for the layered medium. The details are omitted here. Throughout the paper, we shall use the convention that for any $z \in \mathbb{C}$, $z^{1/2}$ is the branch of the square root $\sqrt{z}$ such that $\mathrm{Re}(z^{1/2}) \geq 0$. This corresponds to the left half real axis as the branch cut in the complex plane. Then we have, for $z = z_1 + \mathbf{i} z_2$, $z_1, z_2 \in \mathbb{R}$,

$$z^{1/2} = \sqrt{\frac{|z| + z_1}{2}} + \mathbf{i}\, \mathrm{sgn}(z_2) \sqrt{\frac{|z| - z_1}{2}}\ . \tag{2.5}$$

Let $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ be the column vectors of the tensor $\mathbb{G}$. Each $\mathbf{g}_j$ satisfies an electromagnetic scattering problem with polarized dipole source

$$\mathbf{curl}\,\mathbf{curl}\, \mathbf{g}_j(k; \boldsymbol{x}, \cdot) - k^2 \mathbf{g}_j(k; \boldsymbol{x}, \cdot) = \delta_{\boldsymbol{x}} \boldsymbol{e}_j \qquad \text{in } \mathbb{R}^3_\pm, \tag{2.6a}$$
$$[\boldsymbol{n} \times \mathbf{curl}\, \mathbf{g}_j(k; \boldsymbol{x}, \cdot)] = [\boldsymbol{n} \times \mathbf{g}_j(k; \boldsymbol{x}, \cdot)] = 0 \qquad \text{on } \Sigma, \tag{2.6b}$$
$$\lim_{\rho \to \infty} \int_{\partial B(\rho)} |\mathbf{curl}\, \mathbf{g}_j(k; \boldsymbol{x}, \cdot) \times \boldsymbol{n} - \mathbf{i} k\, \mathbf{g}_j(k; \boldsymbol{x}, \cdot)|^2 = 0, \tag{2.6c}$$

where $\delta_{\boldsymbol{x}}(\boldsymbol{y}) = \delta(|x_1 - y_1|)\delta(|x_2 - y_2|)\delta(|x_3 - y_3|)$ stands for the Dirac source at $\boldsymbol{x} \in \mathbb{R}^3$ and $\boldsymbol{e}_j$ is the unit vector along the positive direction of the x_j-axis, $j = 1, 2, 3$. Similar to the scattering problem in free space, we write $\mathbb{G}$ as

$$\mathbb{G}(k; \boldsymbol{x}, \cdot) = \mathbb{H}(k; \boldsymbol{x}, \cdot) + k_\pm^{-2} \nabla\, \mathrm{div}\, \mathbb{H}(k; \boldsymbol{x}, \cdot) \quad \text{in } \mathbb{R}^3_\pm, \tag{2.7}$$

where $\mathbb{H}$ is the Hertz tensor and can be split into $\mathbb{H} = \mathbb{S} - \mathbb{P}$, $\mathbb{S}$ is the double source tensor standing for the singular part, and $\mathbb{P}$ is the perturbation tensor standing for the regular part.

Let $\Phi(\omega; \boldsymbol{x}, \boldsymbol{y}) = \frac{e^{\mathbf{i}\omega|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}$ be the fundamental solution of the Helmholtz equation with constant number $\omega > 0$, namely,

$$\Delta \Phi(\omega; \boldsymbol{x}, \cdot) + \omega^2 \Phi(\omega; \boldsymbol{x}, \cdot) = -\delta_{\boldsymbol{x}} \qquad \text{in } \mathbb{R}^3,$$
$$\lim_{\rho \to \infty} \int_{\partial B(\rho)} \left| \frac{\partial \Phi}{\partial \boldsymbol{n}}(\omega; \boldsymbol{x}, \cdot) - \mathbf{i}\omega \Phi(\omega; \boldsymbol{x}, \cdot) \right|^2 = 0.$$

Let $\mathbb{I}$ denote the identity matrix. Then the double source tensor is given by

$$\mathbb{S}(k;\boldsymbol{x},\boldsymbol{y}) = \mathbb{I}\times\begin{cases}\Phi(k_+;\boldsymbol{x},\boldsymbol{y})-\Phi(k_+;\boldsymbol{x}',\boldsymbol{y}) & \text{if } x_3>0,\ y_3>0,\\ \Phi(k_-;\boldsymbol{x},\boldsymbol{y})-\Phi(k_-;\boldsymbol{x}',\boldsymbol{y}) & \text{if } x_3<0,\ y_3<0,\\ 0 & \text{elsewhere,}\end{cases} \tag{2.8}$$

where $\boldsymbol{x}'=(x_1,x_2,-x_3)$ is the image of $\boldsymbol{x}=(x_1,x_2,x_3)$ with respect to Σ.

The perturbation tensor $\mathbb{P}$ satisfy the matrix Helmholtz equation

$$\Delta\mathbb{P}(k;\boldsymbol{x},\cdot)+k_\pm^2\mathbb{P}(k;\boldsymbol{x},\cdot)=0 \quad \text{in } \mathbb{R}^3_\pm. \tag{2.9}$$

From [30, Chapter 12], the perturbation tensor has the form

$$\mathbb{P}=\begin{pmatrix}\mathrm{P}_{11} & 0 & 0\\ 0 & \mathrm{P}_{22} & 0\\ \mathrm{P}_{13} & \mathrm{P}_{23} & \mathrm{P}_{33}\end{pmatrix}. \tag{2.10}$$

The entries can be written into Fourier integrals. For any function f, define

$$J(f;\boldsymbol{x},\boldsymbol{y}) := \frac{\mathbf{i}}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\lambda_1,\lambda_2)e^{\mathbf{i}[(x_1-y_1)\lambda_1+(x_2-y_2)\lambda_2+(|x_3|+|y_3|)\mu_+]}\mathrm{d}\lambda_1\mathrm{d}\lambda_2,$$

where $\mu_\pm$ are square roots defined by the limiting absorption principle

$$\mu_\pm(\lambda_1,\lambda_2)=\lim_{\varepsilon\to0^+}\left[(k_\pm+\mathbf{i}\varepsilon)^2-\lambda_1^2-\lambda_2^2\right]^{1/2} \qquad \forall\,(\lambda_1,\lambda_2)\in\mathbb{R}^2. \tag{2.11}$$

From (2.5), this implies that

$$\operatorname{Im}\mu_\pm(\lambda_1,\lambda_2)=\left(\lambda_1^2+\lambda_2^2-k_\pm^2\right)^{1/2} \quad \text{for } \sqrt{\lambda_1^2+\lambda_2^2}\ge k_->k_+.$$

Clearly the integral in $J(f;\boldsymbol{x},\boldsymbol{y})$ is convergent absolutely for any function satisfying

$$|f(\lambda_1,\lambda_2)|\le C(1+\lambda_1^2+\lambda_2^2)^m \qquad \forall\, m\in\mathbb{R}.$$

For convenience, we write

$$h_1=\frac{1}{\mu_++\mu_-}, \qquad h_2=\frac{1}{k_-^2\mu_++k_+^2\mu_-}, \qquad h_3=\frac{k_-^2-k_+^2}{k_-^2\mu_++k_+^2\mu_-}h_1. \tag{2.12}$$

The entries of $\mathbb{P}$ are defined respectively as follows (cf. [30, Section 12.4]): for $j=1,2$,

$$\mathrm{P}_{jj}(k;\boldsymbol{x},\boldsymbol{y})=\begin{cases}J(h_1;\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_+,\\ J(h_1e^{\mathbf{i}(\mu_+-\mu_-)y_3};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x}\in\mathbb{R}^3_+,\ \boldsymbol{y}\in\mathbb{R}^3_-,\\ J(h_1e^{\mathbf{i}(\mu_+-\mu_-)x_3};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x}\in\mathbb{R}^3_-,\ \boldsymbol{y}\in\mathbb{R}^3_+,\\ J(h_1e^{\mathbf{i}(\mu_+-\mu_-)(x_3+y_3)};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_-,\end{cases} \tag{2.13}$$

$$\mathrm{P}_{j3}(k;\boldsymbol{x},\boldsymbol{y}) = \begin{cases} J(\lambda_j h_3;\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_+,\\ J(\lambda_j h_3 e^{\mathbf{i}(\mu_+-\mu_-)y_3};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x}\in\mathbb{R}^3_+,\ \boldsymbol{y}\in\mathbb{R}^3_-,\\ J(\lambda_j h_3 e^{\mathbf{i}(\mu_+-\mu_-)x_3};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x}\in\mathbb{R}^3_-,\ \boldsymbol{y}\in\mathbb{R}^3_+,\\ J(\lambda_j h_3 e^{\mathbf{i}(\mu_+-\mu_-)(x_3+y_3)};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_-, \end{cases} \tag{2.14}$$

$$\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y}) = \begin{cases} J(k_-^2 h_2;\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_+,\\ J(k_-^2 h_2 e^{\mathbf{i}(\mu_+-\mu_-)y_3};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x}\in\mathbb{R}^3_+,\ \boldsymbol{y}\in\mathbb{R}^3_-,\\ J(k_+^2 h_2 e^{\mathbf{i}(\mu_+-\mu_-)x_3};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x}\in\mathbb{R}^3_-,\ \boldsymbol{y}\in\mathbb{R}^3_+,\\ J(k_+^2 h_2 e^{\mathbf{i}(\mu_+-\mu_-)(x_3+y_3)};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_-. \end{cases} \tag{2.15}$$

To end this subsection, we study the singularity of the perturbation tensor.

LEMMA 2.1. *There exists a constant $C > 0$ depending only on k such that, for any $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_\pm$, $i = 0, 1$, and $j = 1, 2, 3$,*

$$\left|\frac{\partial^i}{\partial x_j}\mathbb{P}(k;\boldsymbol{x},\boldsymbol{y})\right| + \left|\frac{\partial^i}{\partial y_j}\mathbb{P}(k;\boldsymbol{x},\boldsymbol{y})\right| \le C\left(1+|\boldsymbol{x}-\boldsymbol{y}|^{-i-1}\right) \quad \text{if} \quad x_3y_3<0, \tag{2.16}$$

$$\left|\frac{\partial^i}{\partial x_j}\mathbb{P}(k;\boldsymbol{x},\boldsymbol{y})\right| + \left|\frac{\partial^i}{\partial y_j}\mathbb{P}(k;\boldsymbol{x},\boldsymbol{y})\right| \le C\left(1+|\boldsymbol{x}-\boldsymbol{y}'|^{-i-1}\right) \quad \text{if} \quad x_3y_3>0. \tag{2.17}$$

Proof. Without loss of generality, we only consider $\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y})$ for $\boldsymbol{x}\in\mathbb{R}^3_+$ and $\boldsymbol{y}\in\mathbb{R}^3_-$. The proofs for other cases are similar.

Write $\xi = \sqrt{\lambda_1^2+\lambda_2^2}$ for convenience. Then (2.11) indicates $\operatorname{Im}\mu_\pm = \sqrt{\xi^2-k_\pm^2}$ for $\xi \ge k_\pm$ and $\operatorname{Im}\mu_\pm = 0$ otherwise. We have $\operatorname{Im}\mu_+ \ge \operatorname{Im}\mu_-$, $\left|e^{\mathbf{i}(\mu_+-\mu_-)y_3}\right| \le 1$, and

$$|\mu_+-\mu_-| = \frac{k_-^2-k_+^2}{|\mu_-+\mu_+|} = O\left(\xi^{-1}\right) \qquad \text{as } \xi\to\infty. \tag{2.18}$$

Write $z = x_3 - y_3$ for convenience. From (2.15), we deduce that

$$\left|\frac{\partial^i}{\partial x_j}\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y})\right| \le C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\mu_+|^{i-1}e^{-z\operatorname{Im}\mu_+}\mathrm{d}\lambda_1\mathrm{d}\lambda_2 \le C(1+z^{-1-i}), \tag{2.19}$$

for any $i = 0, 1$ and $j = 1, 2, 3$. It suffices to prove the lemma for $z < 1$.

Recall that $\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y}) = k_-^2 J(h_2e^{\mathbf{i}(\mu_+-\mu_-)y_3};\boldsymbol{x},\boldsymbol{y})$. By Taylor's expansion,

$$e^{\mathbf{i}(\mu_+-\mu_-)y_3} = 1+\mathbf{i}(\mu_+-\mu_-)y_3-\frac{1}{2}(\mu_+-\mu_-)^2y_3^2+O\left(\xi^{-3}\right),$$
$$h_2 = \frac{1}{(k_+^2+k_-^2)\mu_+}\left[1+\frac{k_+^2(\mu_+-\mu_-)}{(k_+^2+k_-^2)\mu_+}\right]+O(\xi^{-4}) \qquad \text{as } \xi\to\infty.$$

We can write $h_2e^{\mathbf{i}(\mu_+-\mu_-)y_3} = \sum_{j=1}^3 R_j$ where

$$R_1 = \frac{1}{(k_+^2+k_-^2)\mu_+}, \qquad R_2 = \frac{\mathbf{i}(k_+^2-k_-^2)y_3}{2(k_+^2+k_-^2)}\frac{1}{\mu_+^2},$$
$$R_3 = \frac{k_+^2-k_-^2}{(k_+^2+k_-^2)^2}\left[\frac{k_+^2}{2}+\frac{y_3^2}{8}(k_-^4-k_+^4)\right]\frac{1}{\mu_+^3}+O\left(\xi^{-4}\right).$$

Then we have $\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y}) = k_-^2 \sum_{j=1}^3 J(R_j;\boldsymbol{x},\boldsymbol{y})$.

From [30, Chapter 12], the fundamental solution can be written as

$$\Phi(k_+;\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2} J(\mu_+^{-1};\boldsymbol{x},\boldsymbol{y}).$$

This shows that

$$J(R_1;\boldsymbol{x},\boldsymbol{y}) = \frac{2}{k_+^2+k_-^2}\Phi(k_+;\boldsymbol{x},\boldsymbol{y}), \qquad |J(R_1;\boldsymbol{x},\boldsymbol{y})| \le C\rho^{-1}. \tag{2.20}$$

For the second term, since

$$\frac{\partial}{\partial x_3} J(R_2;\boldsymbol{x},\boldsymbol{y}) = \frac{(k_-^2-k_+^2)y_3}{k_+^2+k_-^2}\Phi(k_+;\boldsymbol{x},\boldsymbol{y}), \tag{2.21}$$

similar to (2.19), we have $J(R_1;\boldsymbol{x},\boldsymbol{y}) \le C(1+z^{-1})$. This yields

$$\begin{aligned} |J(R_2;\boldsymbol{x},\boldsymbol{y})| &= \left| J(R_1;(x_1,x_2,\rho),\boldsymbol{y}) - \frac{(k_-^2-k_+^2)y_3}{k_+^2+k_-^2}\int_{x_3}^{\rho} \Phi(k_+;(x_1,x_2,t),\boldsymbol{y})\mathrm{d}t \right| \\ &\le C\left[1+(\rho-y_3)^{-1}\right] + C(\rho-x_3)\max_{x_3\le t\le\rho} |\Phi(k_+;(x_1,x_2,t),\boldsymbol{y})| \\ &\le C\left(1+\rho^{-1}\right). \end{aligned} \tag{2.22}$$

The third term is easy to be estimated as follows

$$|J(R_3;\boldsymbol{x},\boldsymbol{y})| \le C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\lambda_1^2+\lambda_2^2)^{-3/2}\mathrm{d}\lambda_1\mathrm{d}\lambda_2 \le C.$$

Combining the above inequalities, we get

$$|\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y})| \le C\left(1+\rho^{-1}\right).$$

The derivatives of P_{33} can be estimated by using (2.20)–(2.21) and similar arguments as in the proof of (2.22). We do not elaborate on the details here. ☐

2.3. Existence and uniqueness of the scattering solution. Now we study the well-posedness of the scattering problem. The idea is inspired by [23] and [30, Chapter 12] where incident point sources and incident plane waves are considered. Let $B_0 = B(R_0)$, $R_0 \ge 1$, be the ball containing D and where the scattering field is interested. Write $\Omega_0 = B_0\backslash\bar{D}$ and $\Gamma_0 = \partial B_0$ for convenience. Let $\chi \in C_0^\infty(\mathbb{R}^3)$ be the cut-off function satisfying $\mathrm{supp}(\chi) \subset B_0$ and $\chi \equiv 1$ in $\bar{D}$.

We introduce the modified Green's function $\mathbb{G}_\chi(\boldsymbol{x},\boldsymbol{y}) = \chi(\boldsymbol{y})\mathbb{G}(k;\boldsymbol{x},\boldsymbol{y})$ and define the wave propagation operator by, for any $\boldsymbol{u} \in \boldsymbol{L}^1(\Omega_0)$,

$$\mathcal{P}(\boldsymbol{u}) := \int_{\Omega_0} \left[\mathbf{curl}_{\boldsymbol{y}}\,\mathbf{curl}_{\boldsymbol{y}}\,\mathbb{G}_\chi(\cdot,\boldsymbol{y}) - k^2\mathbb{G}_\chi(\cdot,\boldsymbol{y})\right]^\top \boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\,. \tag{2.23}$$

From [30, Section 12.4.3], the scattering solution $\boldsymbol{E}$ of (1.1) satisfies

$$\boldsymbol{E}(\boldsymbol{x}) = \mathcal{P}(\boldsymbol{E})(\boldsymbol{x}) \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^3_\pm\backslash\bar{B}_0\,.$$

THEOREM 2.2. *For any $\boldsymbol{g} \in \boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)$, (1.1) has a unique solution $\boldsymbol{E}$. Moreover, for any bounded domain $\Omega \subset \mathbb{R}^3\backslash\bar{D}$, there exists a constant $C > 0$ depending only on k and Ω such that*

$$\|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \le C\,\|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}\,. \tag{2.24}$$

Proof. The uniqueness of the solution is proved by Cutzach and Hazard in [23]. It is left to prove the existence. Let $B_1 = B(R_1)$ be a ball containing B_0 and write $\Omega_1 = B_1 \backslash \bar{D}$, $\Gamma_1 = \partial B_1$. Define

$$\boldsymbol{U} = \left\{ \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega_1) : \gamma_T \boldsymbol{v} \in \boldsymbol{L}^2(\Gamma_1) \right\}, \quad \boldsymbol{U}_0 = \{ \boldsymbol{v} \in \boldsymbol{U} : \gamma_t \boldsymbol{v} = 0 \text{ on } \Gamma_D \}.$$

By [30, Theorem 4.1], $\boldsymbol{U}$ constructs a Hilbert space under the inner product and norm

$$(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{U}} = \int_{\Omega_1} (\mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \bar{\boldsymbol{v}} + \boldsymbol{u} \cdot \bar{\boldsymbol{v}}) + \int_{\Gamma_1} \gamma_T \boldsymbol{u} \cdot \gamma_T \bar{\boldsymbol{v}}, \qquad \|\boldsymbol{u}\|_{\boldsymbol{U}} := (\boldsymbol{u}, \boldsymbol{u})_{\boldsymbol{U}}^{1/2}.$$

By [30, Theorem 4.7], we have the direct-sum decomposition $\boldsymbol{U}_0 = \hat{\boldsymbol{U}}_0 + \nabla S_0$ where

$$\hat{\boldsymbol{U}}_0 = \left\{ \boldsymbol{v} \in \boldsymbol{U}_0 : \operatorname{div}(k^2 \boldsymbol{v}) = 0 \right\}, \quad S_0 = \left\{ v \in H^1_{\Gamma_D}(\Omega_1) : v|_{\Gamma_1} = \text{Const.} \right\}. \tag{2.25}$$

Moreover, $\hat{\boldsymbol{U}}_0$ is embedded compactly into $\boldsymbol{L}^2(\Omega_1)$.

Let $b_+ \colon \boldsymbol{U} \times \boldsymbol{U} \to \mathbb{C}$ be the sesquilinear form defined by

$$b_+(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega_1} \left(\mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \bar{\boldsymbol{v}} + k^2 \boldsymbol{u} \cdot \bar{\boldsymbol{v}} \right) - \mathbf{i} \int_{\Gamma_1} \gamma_T \boldsymbol{u} \cdot \gamma_T \bar{\boldsymbol{v}}. \tag{2.26}$$

Clearly b_+ is continuous and coercive on $\boldsymbol{U}$. There is a unique $\boldsymbol{E}_g \in \boldsymbol{U}$ satisfying

$$b_+(\boldsymbol{E}_g, \boldsymbol{v}) = 0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{U}_0, \qquad \gamma_t \boldsymbol{E}_g = \boldsymbol{g} \quad \text{on } \Gamma_D. \tag{2.27}$$

Furthermore, there is a constant $C > 0$ depending only on k and Ω_1 such that

$$\|\boldsymbol{E}_g\|_{\boldsymbol{H}(\mathbf{curl}, \Omega_1)} \le C \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)}. \tag{2.28}$$

A weak formulation for (1.1) reads: Find $\boldsymbol{E}_1 := \boldsymbol{E} - \boldsymbol{E}_g \in \boldsymbol{U}_0$ such that

$$b(\boldsymbol{E}_1, \boldsymbol{v}) + b_1(\boldsymbol{E}_1, \boldsymbol{v}) = -b(\boldsymbol{E}_g, \boldsymbol{v}) - b_1(\boldsymbol{E}_g, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \boldsymbol{U}_0, \tag{2.29}$$

where the sesquilinear forms are defined by

$$b(\boldsymbol{u}, \boldsymbol{v}) = b_+(\boldsymbol{u}, \boldsymbol{v}) - \int_{\Omega_1} 2k^2 \boldsymbol{u} \cdot \bar{\boldsymbol{v}}, \quad b_1(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_1} [\gamma_t\, \mathbf{curl}\, \mathcal{P}(\boldsymbol{u}) + \mathbf{i} \gamma_T \mathcal{P}(\boldsymbol{u})] \cdot \gamma_T \bar{\boldsymbol{v}}.$$

Clearly b, b_1 are continuous on $\boldsymbol{U}$. It suffices to show that (2.29) has a solution.

Clearly (1.1a), (1.1c) yield $\mathbf{curl}\, \boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl}, \Omega_1)$ and $\operatorname{div}(k^2 \boldsymbol{E}) = 0$ in Ω_1. Taking $\boldsymbol{v} = \nabla \varphi$, $\varphi \in S_0$ in (2.27), we also have $\operatorname{div}(k^2 \boldsymbol{E}_g) = 0$ in Ω_1. This implies $\boldsymbol{E}_1 \in \hat{\boldsymbol{U}}_0$. Using (2.27), $\boldsymbol{E}_1$ can be solved in the subspace: Find $\boldsymbol{E}_1 \in \hat{\boldsymbol{U}}_0$ such that

$$b(\boldsymbol{E}_1, \boldsymbol{v}) + b_1(\boldsymbol{E}_1, \boldsymbol{v}) = 2(k^2 \boldsymbol{E}_g, \boldsymbol{v})_{\Omega_1} - b_1(\boldsymbol{E}_g, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \hat{\boldsymbol{U}}_0. \tag{2.30}$$

Let $K_1, K_2 \colon \boldsymbol{L}^2(\Omega_1) \to \hat{\boldsymbol{U}}_0$ be the linear operators defined by

$$b_+(K_1(\boldsymbol{u}), \boldsymbol{v}) = 2(k^2 \boldsymbol{u}, \boldsymbol{v})_{\Omega_1}, \quad b_+(K_2(\boldsymbol{u}), \boldsymbol{v}) = b_1(\boldsymbol{u}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \hat{\boldsymbol{U}}_0. \tag{2.31}$$

Since b_+ is coercive, K_1, K_2 are well-defined and $\|K_1(\boldsymbol{u})\|_{\boldsymbol{U}} \le C \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega_1)}$. By the compact embedding of $\hat{\boldsymbol{U}}_0$ into $\boldsymbol{L}^2(\Omega_1)$, K_1 is a compact operator.

By (2.7)–(2.8), $\mathbb{G}$ and its partial derivatives are uniformly bounded for $\boldsymbol{x} \in \mathbb{R}^3_\pm \backslash B_1$ and $\boldsymbol{y} \in \bar{B}_0$. There is a constant $C > 0$ depending only on k, B_0 such that

$$\|\nabla_{\boldsymbol{x}} \mathbb{G}(k; \boldsymbol{x}, \cdot)\|_{W^{2,\infty}(B_0)} + \|\mathbb{G}(k; \boldsymbol{x}, \cdot)\|_{W^{2,\infty}(B_0)} \le C \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^3_\pm \backslash B_1.$$

By the definition of $\mathcal{P}$ and the Cauchy-Schwarz inequality, we have

$$\|K_2(\boldsymbol{u})\|_{\boldsymbol{U}} \le C \left\|\boldsymbol{n} \times \mathbf{curl}\, \mathcal{P}(\boldsymbol{u}) + \mathbf{i}\gamma_T \mathcal{P}(\boldsymbol{u})\right\|_{\boldsymbol{L}^2(\Gamma_1)} \le C \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega_1)}.$$

Therefore, K_2 is also a compact mapping from $\hat{\boldsymbol{U}}_0$ to $\boldsymbol{L}^2(\Omega_1)$.

Now we can write (2.30) into an operator equation

$$\boldsymbol{E}_1 - K_1(\boldsymbol{E}_1) + K_2(\boldsymbol{E}_1) = K_1(\boldsymbol{E}_g) - K_2(\boldsymbol{E}_g)\,. \tag{2.32}$$

This is a Fredholm equation on $\boldsymbol{L}^2(\Omega_1)$. Since the scattering solution $\boldsymbol{E}$ is unique, the solution $\boldsymbol{E}_1$ of (2.30) is also unique. By the Fredholm alternative, we conclude that (2.32) attains a unique solution $\boldsymbol{E}_1 \in \boldsymbol{L}^2(\Omega_1)$. From (2.32) we know that $\boldsymbol{E}_1 \in \hat{\boldsymbol{U}}_0$. Therefore, the weak problem (2.30) or (2.29) has a unique solution.

For the stability of the solution, from (2.32) and (2.28) we know that

$$\|\boldsymbol{E}_1\|_{\boldsymbol{L}^2(\Omega_1)} = \left\|(I - K_1 + K_2)^{-1}(K_1 - K_2)\boldsymbol{E}_g\right\|_{\boldsymbol{L}^2(\Omega_1)} \le C \|\boldsymbol{E}_g\|_{\boldsymbol{L}^2(\Omega_1)}.$$

By (2.28), this shows $\|\boldsymbol{E}\|_{\boldsymbol{L}^2(\Omega_1)} \le C \|\boldsymbol{E}_g\|_{\boldsymbol{L}^2(\Omega_1)} \le C \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}$. Finally, taking $\boldsymbol{v} = \boldsymbol{E}_1$ in (2.29), we find that

$$\|\mathbf{curl}\, \boldsymbol{E}_1\|^2_{\boldsymbol{L}^2(\Omega_1)} \le k_-^2 \left|(\boldsymbol{E}, \boldsymbol{E}_1)_{\Omega_1}\right| + |b_1(\boldsymbol{E}, \boldsymbol{E}_1)| + \left|(\mathbf{curl}\, \boldsymbol{E}_g, \mathbf{curl}\, \boldsymbol{E}_1)_{\Omega_1}\right|.$$

We conclude that $\|\mathbf{curl}\, \boldsymbol{E}_1\|_{\boldsymbol{L}^2(\Omega_1)} \le C \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}$. By (2.28), this yields

$$\|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)} \le C \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}.$$

Finally, for any bounded domain Ω, we need only choose R_1 large enough such that $\Omega \subset \Omega_1$. Then (2.24) follows clearly from the above estimate. ☐

3. The Cagniard-de Hoop transform.

In this section, we shall derive a new integral form of $\mathbb{P}$ by the Cagniard de-Hoop transform [20, Page 215]. It plays the key role in proving the exponential decay of the solution in PML. The Green's function $\mathbb{G}$ is defined indeed by the method of limiting absorption principle, that is,

$$\mathbb{G}(k; \boldsymbol{x}, \boldsymbol{y}) = \lim_{\varepsilon \to 0} \mathbb{G}(k(\varepsilon); \boldsymbol{x}, \boldsymbol{y}) \qquad \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3,\ \boldsymbol{x} \ne \boldsymbol{y}. \tag{3.1}$$

where $k(\varepsilon) := k + \mathbf{i}\varepsilon/k$ is the complex wave number. Clearly the column vectors of $\mathbb{G}(k(\varepsilon); \cdot, \cdot)$ satisfy the time-harmonic Maxwell's equations in a lossy material

$$\begin{aligned}
&\mathbf{curl}\,\mathbf{curl}\, \mathbf{g}_j(k(\varepsilon); \boldsymbol{x}, \cdot) - k^2(\varepsilon)\mathbf{g}_j(k(\varepsilon); \boldsymbol{x}, \cdot) = \delta_{\boldsymbol{x}} \boldsymbol{e}_j \quad \text{in } \mathbb{R}^3_\pm,\\
&[\boldsymbol{n} \times \mathbf{curl}\, \mathbf{g}_j(k(\varepsilon); \boldsymbol{x}, \cdot)] = [\boldsymbol{n} \times \mathbf{g}_j(k(\varepsilon); \boldsymbol{x}, \cdot)] = 0 \qquad \text{on } \Sigma,\\
&|\mathbf{g}_j(k(\varepsilon); \boldsymbol{x}, \boldsymbol{y})| \text{ is bounded as } |\boldsymbol{y}| \to \infty.
\end{aligned}$$

In this and the next sections, we shall study $\mathbb{G}(k(\varepsilon); \cdot, \cdot)$ for a fixed parameter $0 < \varepsilon \ll \min(1, k_+)$. The theories will be extended to $\mathbb{G}(k; \cdot, \cdot)$ by passage $\varepsilon \to 0$.

Write $k_\pm(\varepsilon) := k_\pm + \mathbf{i}\varepsilon/k_\pm$ and $\mathbb{H}(k(\varepsilon); \cdot, \cdot) = \mathbb{S}(k(\varepsilon); \cdot, \cdot) - \mathbb{P}(k(\varepsilon); \cdot, \cdot)$. Note that

$$\mathbb{G}(k(\varepsilon); \boldsymbol{x}, \cdot) = \mathbb{H}(k(\varepsilon); \boldsymbol{x}, \cdot) + k_\pm^{-2}(\varepsilon) \nabla\, \mathrm{div}\, \mathbb{H}(k(\varepsilon); \boldsymbol{x}, \cdot) \quad \text{in } \mathbb{R}^3_\pm,$$

where $\mathbb{S}(k(\varepsilon);\cdot,\cdot)$ and $\mathbb{P}(k(\varepsilon);\cdot,\cdot)$ are given in (2.8)–(2.15) by replacing $k_\pm$ with $k_\pm(\varepsilon)$. We shall derive the Cagniard-de Hoop representation of the perturbation tensor $\mathbb{P}$. Without loss of generality, we only consider P_{33} for $\boldsymbol{x}\in\mathbb{R}^3_+$ and $\boldsymbol{y}\in\mathbb{R}^3_-$. The results can be extended straightforwardly to other cases of $\boldsymbol{x},\boldsymbol{y}$ and to other entries of $\mathbb{P}$.

LEMMA 3.1. *For any $\boldsymbol{x}_3\in\mathbb{R}^3_+$ and $\boldsymbol{y}\in\mathbb{R}^3_-$, write $x_1-y_1=r\cos\phi$, $x_2-y_2=r\sin\phi$ with $r=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$ and $\phi\in[0,2\pi]$. Then*

$$\frac{\partial^{l+m+n}\mathrm{P}_{33}(k(\varepsilon);\boldsymbol{x},\boldsymbol{y})}{\partial x_1^l\partial x_2^m\partial x_3^n}=\mathbf{i}^{l+m+n}\frac{1+\mathbf{i}\varepsilon}{2\pi^2}\int_0^\infty\int_{-\infty}^\infty\frac{\lambda_1^l\lambda_2^m\mu_+^n e^{\mathbf{i}(r\xi+x_3\mu_+-y_3\mu_-)}}{\mu_++k_-^{-2}(\varepsilon)k_+^2(\varepsilon)\mu_-}\mathrm{d}\xi\mathrm{d}q,$$

for any integers $l,m,n\geq 0$, where $\mu_\pm=\mu_\pm(\xi,(\varepsilon-\mathbf{i})q)$ and

$$\lambda_1=\xi\cos\phi+(\varepsilon-\mathbf{i})q\sin\phi,\qquad\lambda_2=\xi\sin\phi-(\varepsilon-\mathbf{i})q\cos\phi.$$

Proof. We consider the rotational transform $\xi=\lambda_1\cos\phi+\lambda_2\sin\phi$, $\eta=\lambda_1\sin\phi-\lambda_2\cos\phi$. The definition of $\mu_\pm$ indicates

$$\mu_\pm(\lambda_1,\lambda_2)=\left[k_\pm^2(\varepsilon)-\lambda_1^2-\lambda_2^2\right]^{1/2}=\left[k_\pm^2(\varepsilon)-\xi^2-\eta^2\right]^{1/2}=\mu_\pm(\xi,\eta).$$

Write $N=l+m+n$. Then from (2.15) we have

$$\frac{\partial^N\mathrm{P}_{33}(k(\varepsilon);\boldsymbol{x},\boldsymbol{y})}{\partial x_1^l\partial x_2^m\partial x_3^n}=\frac{\mathbf{i}^{N+1}}{2\pi^2}\int_0^\infty F(\eta)\mathrm{d}\eta,\quad F(\eta)=\int_{-\infty}^\infty\frac{\lambda_1^l\lambda_2^m\mu_+^n e^{\mathbf{i}(r\xi+x_3\mu_+-y_3\mu_-)}}{\mu_++k_-^{-2}(\varepsilon)k_+^2(\varepsilon)\mu_-}\mathrm{d}\xi.$$

For any fixed $\eta\in\mathbb{C}$, when $|\xi|\gg k_-+|\eta|$, we have

$$\operatorname{Im}\mu_\pm(\xi,\eta)=\sqrt{\frac{|\mu_\pm(\xi,\eta)^2|-\operatorname{Re}\mu_\pm(\xi,\eta)^2}{2}}\geq\sqrt{\xi^2-k_\pm^2-|\eta|^2}\geq\frac{1}{2}|\xi|\,.$$

Thus the dξ-integration in $F(\eta)$ converges for any $|x_3|+|y_3|>0$. Since $\operatorname{Re}\mu_\pm>0$ and $(\operatorname{Im}\mu_+)(\operatorname{Im}\mu_-)>0$, we have $\mu_++k_-^{-2}(\varepsilon)k_+^2(\varepsilon)\mu_-\neq 0$. Therefore, F defines an analytic function of η.

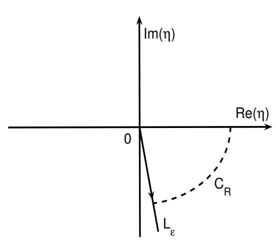

FIG. 3.1. *The deformation of integral path from the real axis to L_ε.*

Now we define a half line in the fourth quadrant of the complex η-plane

$$L_\varepsilon=\{(\varepsilon-\mathbf{i})q:q\geq 0\}=\left\{te^{-\mathbf{i}\theta_\varepsilon}:t\geq 0\right\},\qquad\theta_\varepsilon=\arcsin\frac{1}{\sqrt{1+\varepsilon^2}}.$$

Let $C_R=\{Re^{\mathbf{i}\theta}:0<\theta<\theta_\varepsilon\}$ be the arc of radius R which is bounded by L_ε and the real axis (See Fig. 3.1). We orient L_ε to the downward direction. Suppose that

$$\lim_{R\to\infty}\int_{C_R}F(\eta)\mathrm{d}\eta=0.\tag{3.2}$$

Then the result follows from Cauchy's theorem and the fact that

$$\int_0^\infty F(\eta)\mathrm{d}\eta = \mathbf{i}\int_{L_\varepsilon} F(\eta)\mathrm{d}\eta = (1+\mathbf{i}\varepsilon)\int_0^\infty \int_{-\infty}^\infty \frac{\lambda_1^l\lambda_2^m\mu_+^n e^{\mathbf{i}(r\xi+x_3\mu_+-y_3\mu_-)}}{\mu_+ + k_-^{-2}(\varepsilon)k_+^2(\varepsilon)\mu_-}\mathrm{d}\xi\mathrm{d}q.$$

It is left to show (3.2). Assume $R \gg 2(1+k_-)$ and recall

$$\operatorname{Im}\mu_\pm^2 = \operatorname{Im}\left[k_\pm^2(\varepsilon) - \xi^2 - R^2 e^{-2\mathbf{i}\theta}\right] = 2\varepsilon + R^2\sin 2\theta > 0 \qquad \forall\, \theta \in (0,\theta_\varepsilon).$$

We have $\operatorname{Re}\mu_\pm \geq 0$, $\operatorname{Im}\mu_\pm \geq 0$. There is a constant C independent of ξ,η such that

$$\left|F(Re^{\mathbf{i}\theta})\right| \leq CR^N \int_0^\infty (1+\xi)^N e^{-x_3\operatorname{Im}\mu_+}\mathrm{d}\xi = CR^N[F_1(R,\theta)+F_2(R,\theta)], \tag{3.3}$$

where $F_1(R,\theta) = \int_0^{2R}(1+\xi)^N e^{-x_3\operatorname{Im}\mu_+}\mathrm{d}\xi$ and $F_2(R,\theta) = \int_{2R}^\infty (1+\xi)^N e^{-x_3\operatorname{Im}\mu_+}\mathrm{d}\xi$.

We consider $F_1(R,\theta)$ first. For any $0 \leq \theta \leq \theta_\varepsilon/2$, we have

$$\operatorname{Im}\mu_\pm \geq \frac{1}{2}\sqrt{\left|\mu_\pm^2\right| - \operatorname{Re}\mu_\pm^2} \geq \frac{1}{2}\sqrt{\left|\mu_\pm^2\right| + \xi^2 + R^2\cos 2\theta - k_\pm^2} \geq \frac{1}{4}R.$$

For any $\theta_\varepsilon/2 < \theta < \theta_\varepsilon$, from (2.5) we know that

$$\operatorname{Im}\mu_+ = \frac{\operatorname{Im}\mu_+^2}{2\operatorname{Re}\mu_+} \geq \frac{R^2\sin 2\theta}{2(|\xi|+k_-+R)} \geq \frac{R}{8}\min(\sin 2\theta_\varepsilon, \sin\theta_\varepsilon).$$

This shows

$$\lim_{R\to\infty} R^{N+1}\int_0^{\theta_\varepsilon} F_1(R,\theta) = 0. \tag{3.4}$$

As for $F_2(R,\theta)$, since $\xi \geq 2R$, we deduce that

$$(\operatorname{Im}\mu_\pm)^2 \geq \frac{1}{2}\left(\left|\mu_\pm^2\right| + \xi^2 + R^2\cos 2\theta - k_\pm^2\right) \geq \xi^2 + R^2\cos 2\theta - k_\pm^2 \geq \frac{1}{4}\xi^2.$$

This indicates that

$$F_2(R,\theta) \leq \int_{2R}^{+\infty}(1+\xi)^N e^{-\frac{1}{2}x_3\xi}\mathrm{d}\xi \leq Cx_3^{-N-1}e^{-x_3R}.$$

We conclude that

$$\lim_{R\to\infty} R^{N+1}\int_0^{\theta_\varepsilon} F_2(R,\theta) = 0. \tag{3.5}$$

Finally, the proof is finished by combining (3.4)–(3.5) with (3.3). □

The integral form in Lemma 3.1 is still unfavorable to the PML analysis. We are going to derive the Cagniard-de Hoop representation of $\mathbb{P}$. This will be fulfilled by deforming the $\mathrm{d}\xi$-integration from the real axis to a hyperbolic integral path.

For convenience in notation, we write

$$\kappa_\pm(\varepsilon,q) := \left[k_\pm^2(\varepsilon) - (\varepsilon-\mathbf{i})^2q^2\right]^{1/2} = \left[k_\pm^2(\varepsilon) + (1+\mathbf{i}\varepsilon)^2q^2\right]^{1/2} \qquad \forall\, q \geq 0.$$

We shall always abbreviate the notations to $\kappa_\pm := \kappa_\pm(\varepsilon, q)$ without specifying their dependency on ε, q in this and the next sections. From (2.5), we know that $\operatorname{Re}\kappa_\pm$, $\operatorname{Im}\kappa_\pm$ are positive and satisfy

$$\operatorname{Im}\kappa_- \le \operatorname{Im}\kappa_+ \le \frac{\varepsilon q^2}{\sqrt{k_+^2 + q^2 - \varepsilon^2(k_+^{-2} + q^2)}}, \tag{3.6}$$

$$\operatorname{Re}\kappa_- \ge \operatorname{Re}\kappa_+ \ge \sqrt{k_+^2 + q^2 - \varepsilon^2(k_+^{-2} + q^2)}. \tag{3.7}$$

For convenience, we also use

$$\mu_1(\xi) = \left(\kappa_+^2 - \xi^2\right)^{1/2} = \mu_+(\xi, (\varepsilon - \mathbf{i})q), \quad \mu_2(\xi) = \left(\kappa_-^2 - \xi^2\right)^{1/2} = \mu_-(\xi, (\varepsilon - \mathbf{i})q),$$

without specifying their dependency on ε and q. By (2.5), the branch cuts for the square roots $\mu_{1,2} = (\kappa_\pm + \xi)^{1/2}(\kappa_\pm - \xi)^{1/2}$ are given by four half-lines

$$C_\pm^R = \{\xi : \xi = \kappa_\pm + t,\ t \ge 0\}, \qquad C_\pm^L = \{\xi : \xi = -\kappa_\pm - t,\ t \ge 0\}.$$

Then for any fixed $q \ge 0$, μ_1, μ_2 are analytic functions of ξ in $\mathbb{C}\backslash(C_+^L \cup C_+^R)$ and $\mathbb{C}\backslash(C_-^L \cup C_-^R)$ respectively. For convenience, we write $C_{\varepsilon,q} = C_+^L \cup C_+^R \cup C_-^L \cup C_-^R$.

Given a function h in the complex ξ-plane, define

$$I(h; r, z) = \int_{-\infty}^{+\infty} h(\xi) e^{\mathbf{i}(r\xi + z\mu_1)} \mathrm{d}\xi \qquad \forall\, r > 0,\ z > 0.$$

We shall rewrite the integral by the Cagniard-de Hoop transform. Then the theory will be applied to the Perturbation tensor.

Lemma 3.2. *For any $q > 0$ and $0 < \varepsilon \ll 1$, let h be an analytic function in $\mathbb{C}\backslash C_{\varepsilon,q}$ and satisfy $|h(\xi)| \le C(1 + |\xi|)^m$ for some integer m and some constant $C > 0$. Then for any r, z satisfying $z \ge 2\varepsilon r > 0$,*

$$I(h; r, z) = -\mathbf{i}\int_1^\infty \left[h(\xi_+(t))\Lambda_+(t) + h(\xi_-(t))\Lambda_-(t)\right] \frac{e^{\mathbf{i}\kappa_+\rho t}}{\sqrt{t^2 - 1}} \mathrm{d}t, \tag{3.8}$$

where $\rho = \sqrt{r^2 + z^2}$ and

$$\xi_\pm(t) = \frac{\kappa_+}{\rho}\left(rt \pm \mathbf{i}z\sqrt{t^2 - 1}\right), \qquad \Lambda_\pm(t) = \frac{\kappa_+}{\rho}\left(zt \mp \mathbf{i}r\sqrt{t^2 - 1}\right). \tag{3.9}$$

Proof. First we define a hyperbolic integral path $\Gamma = \Gamma_+ \cup \Gamma_-$ where

$$\Gamma_\pm = \{\xi_\pm(t) :\ t \ge 1\}\ .$$

Notice that $\Lambda_\pm^2(t) = \kappa_+^2 - \xi_\pm^2(t)$ for any $\xi_\pm(t) \in \Gamma$. From (3.6)–(3.7), we have

$$\rho \operatorname{Re}\Lambda_\pm(t) \ge zt \operatorname{Re}\kappa_+ - r\sqrt{t^2 - 1}\operatorname{Im}\kappa_+ \ge \frac{\varepsilon rt}{\operatorname{Re}\kappa_+}\left[2(1 - \varepsilon^2)q^2 - q^2\right] \ge 0.$$

By the convention in (2.5), we have $\Lambda_\pm(t) = \mu_1(\xi_\pm(t))$.

For any $R > 0$, let O_R^+, O_R^- be the parts of the circle $\{\xi : |\xi| = R\}$ that are bounded by the real axis and $\Gamma_\pm$ respectively (see Fig. 3.2). Suppose that

$$\lim_{R\to\infty} F_\pm(R) = 0, \qquad F_\pm(R) := \int_{O_R^\pm} h(\xi) e^{\mathbf{i}(r\xi + z\mu_1)} \mathrm{d}\xi. \tag{3.10}$$

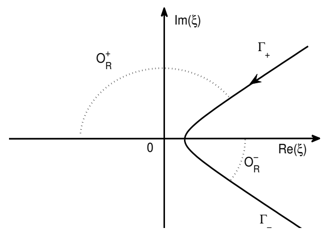

FIG. 3.2. *The Cagniard-de Hoop transform from the real axis to $\Gamma_+ \cup \Gamma_-$.*

By Cauchy's theorem, (3.8) follows from the fact that

$$I(h; r, z) = \int_\Gamma h(\xi) e^{\mathbf{i}(r\xi + z\mu_1)} \mathrm{d}\xi.$$

It is left to show (3.10). We only prove the limit for $F_+(R)$. The proof for $F_-(R)$ is similar and omitted here. Let $\xi_R = Re^{\mathbf{i}\theta_R}$ be the intersection point of O_R^+ and Γ_+. Then $F_+(R)$ can be written into

$$F_+(R) = \mathbf{i} \int_{\theta_R}^{\pi} h(Re^{\mathbf{i}\theta}) e^{-rR\sin\theta} e^{\mathbf{i}(rR\cos\theta + \mu_1 z)} Re^{\mathbf{i}\theta} \mathrm{d}\theta,$$

where $\mu_1 = \mu_1(Re^{\mathbf{i}\theta})$. Since $\left|h(Re^{\mathbf{i}\theta})\right| \le C(1+R)^m$, we have

$$|F_+(R)| \le CR^{m+1} \int_{\theta_R}^{\pi} e^{-(rR\sin\theta + z\,\mathrm{Im}\,\mu_1)} \mathrm{d}\theta\ . \tag{3.11}$$

Without loss of generality, we assume that $R \gg 2(k_- + q)$ and define

$$\theta_0 := \frac{\pi}{2} - \frac{1}{2}\arcsin\frac{2\varepsilon(1+q^2)}{R^2}. \tag{3.12}$$

Clearly $\pi/4 < \theta_0 < \pi/2$. Since $\mathrm{Im}\,\mu_1^2 = 2\varepsilon(1+q^2) - R^2\sin 2\theta$, by (2.5), we have

$$\mathrm{Im}\,\mu_1 > 0 \quad \forall\,\theta \in (\theta_0, \pi); \qquad \mathrm{Im}\,\mu_1 < 0 \quad \forall\,\theta \in (\theta_R, \theta_0). \tag{3.13}$$

From (2.5), we also have $\mathrm{Im}\,\mu_1 \ge R/8$ for any $5\pi/6 \le \theta \le \pi$. Then

$$\int_{\theta_0}^{\pi} e^{-(rR\sin\theta + z\,\mathrm{Im}\,\mu_1)} \mathrm{d}\theta \le \int_{\theta_0}^{\frac{5\pi}{6}} e^{-rR\sin\theta} \mathrm{d}\theta + \int_{\frac{5\pi}{6}}^{\pi} e^{-z\,\mathrm{Im}\,\mu_1} \mathrm{d}\theta \le \pi e^{-\frac{rR}{4}} + \pi e^{-\frac{zR}{8}}\ .$$

This shows that

$$\lim_{R\to\infty} R^{m+1} \int_{\theta_0}^{\pi} e^{-(rR\sin\theta + z\,\mathrm{Im}\,\mu_1)} \mathrm{d}\theta = 0. \tag{3.14}$$

For sufficiently large R and fixed $q > 0$, careful calculations show that

$$\frac{\partial}{\partial\theta}\,\mathrm{Im}\left(\kappa_+^2 - R^2 e^{2\mathbf{i}\theta}\right)^{1/2} \ge 0 \qquad \forall\,\theta \in (\theta_R, \theta_0).$$

Thus μ_1 is increasing with respect to $\theta \in (\theta_R, \theta_0)$. Let $r = \rho\cos\phi$, $z = \rho\sin\phi$ with $\phi \in (0, \pi/2)$. Since $\xi_R = \xi_+(t_R)$ for some $t_R \ge 1$, we have

$$\xi_R = \kappa_+\left(\cos\phi\, t_R + \mathbf{i}\sin\phi\sqrt{t_R^2 - 1}\right), \quad \mu_1(\xi_R) = \kappa_+\left(\sin\phi\, t_R - \mathbf{i}\cos\phi\sqrt{t_R^2 - 1}\right).$$

Note that $R = |\kappa_+| \sqrt{t_R^2 \cos^2\phi + (t_R^2 - 1)\sin^2\phi} \le |\kappa_+|\, t_R$. We get

$$r R \sin\theta + z \operatorname{Im}\mu_1 \ge r \operatorname{Im}\xi_R + z \operatorname{Im}\mu_1(\xi_R) = \rho t_R \operatorname{Im}\kappa_+ \ge \frac{\operatorname{Im}\kappa_+}{|\kappa_+|}\rho R\,.$$

This yields

$$\lim_{R\to\infty} R^{m+1} \int_{\theta_R}^{\theta_0} e^{-(rR\sin\theta + z\operatorname{Im}\mu_1)} \mathrm{d}\theta = 0. \tag{3.15}$$

Finally, we obtain (3.10) by substituting (3.14) and (3.15) into (3.11). □

Now we apply Lemma 3.2 to the perturbation tensor and its derivatives. For convenience, we write $r = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$, $z = |x_3| + |y_3|$, and $\rho = \sqrt{r^2+z^2}$. Let the Cagniard-de Hoop transform be defined in (3.9). Since $\operatorname{Re}\Lambda_\pm \ge 0$, it is easy to see that

$$\mu_1(\xi_\pm) = \Lambda_\pm, \qquad \mu_2(\xi_\pm) = \left[k_-^2(\varepsilon) - k_+^2(\varepsilon) + \Lambda_\pm^2\right]^{1/2}.$$

For a function $f(\xi)$, we define

$$J_{\mathrm{cdh}}(f;\boldsymbol{x},\boldsymbol{y}) = \frac{\varepsilon - \mathbf{i}}{2\pi^2} \int_0^\infty \int_1^\infty \left[\Lambda_+(t) f(\xi_+(t)) + \Lambda_-(t) f(\xi_-(t))\right] \frac{e^{\mathbf{i}\kappa_+\rho t}}{\sqrt{t^2-1}} \mathrm{d}t\mathrm{d}q.$$

Let h_1, h_2, h_3 be defined by (2.12) with μ_+, μ_- replaced by μ_1, μ_2 respectively. Then the diagonal entries of the perturbation tensor $\mathbb{P}$ are given by, for $j = 1, 2$,

$$\mathrm{P}_{jj}(k(\varepsilon);\boldsymbol{x},\boldsymbol{y}) = \begin{cases} J_{\mathrm{cdh}}(h_1;\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}_+^3, \\ J_{\mathrm{cdh}}(h_1 e^{\mathbf{i}(\mu_1-\mu_2)y_3};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x} \in \mathbb{R}_+^3,\ \boldsymbol{y} \in \mathbb{R}_-^3, \\ J_{\mathrm{cdh}}(h_1 e^{\mathbf{i}(\mu_1-\mu_2)x_3};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x} \in \mathbb{R}_-^3,\ \boldsymbol{y} \in \mathbb{R}_+^3, \\ J_{\mathrm{cdh}}(h_1 e^{\mathbf{i}(\mu_1-\mu_2)(x_3+y_3)};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}_-^3, \end{cases} \tag{3.16}$$

$$\mathrm{P}_{33}(k(\varepsilon);\boldsymbol{x},\boldsymbol{y}) = \begin{cases} k_-^2(\varepsilon) J_{\mathrm{cdh}}(h_2;\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}_+^3, \\ k_-^2(\varepsilon) J_{\mathrm{cdh}}(h_2 e^{\mathbf{i}(\mu_1-\mu_2)y_3};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x} \in \mathbb{R}_+^3,\ \boldsymbol{y} \in \mathbb{R}_-^3, \\ k_+^2(\varepsilon) J_{\mathrm{cdh}}(h_2 e^{\mathbf{i}(\mu_1-\mu_2)x_3};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x} \in \mathbb{R}_-^3,\ \boldsymbol{y} \in \mathbb{R}_+^3, \\ k_+^2(\varepsilon) J_{\mathrm{cdh}}(h_2 e^{\mathbf{i}(\mu_1-\mu_2)(x_3+y_3)};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}_-^3. \end{cases} \tag{3.17}$$

Write $x_1 - y_1 = r\cos\phi$, $x_2 - y_2 = r\sin\phi$. Then $\mathrm{P}_{23} = \tan\phi\, \mathrm{P}_{13}$ where

$$\frac{\mathrm{P}_{13}(k(\varepsilon);\boldsymbol{x},\boldsymbol{y})}{\cos\phi} = \begin{cases} J_{\mathrm{cdh}}(\xi h_3;\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}_+^3, \\ J_{\mathrm{cdh}}(\xi h_3 e^{\mathbf{i}(\mu_1-\mu_2)y_3};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x} \in \mathbb{R}_+^3,\ \boldsymbol{y} \in \mathbb{R}_-^3, \\ J_{\mathrm{cdh}}(\xi h_3 e^{\mathbf{i}(\mu_1-\mu_2)x_3};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x} \in \mathbb{R}_-^3,\ \boldsymbol{y} \in \mathbb{R}_+^3, \\ J_{\mathrm{cdh}}(\xi h_3 e^{\mathbf{i}(\mu_1-\mu_2)(x_3+y_3)};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}_-^3. \end{cases} \tag{3.18}$$

Similarly, we can obtain the Cagniard-de Hoop representations for derivatives of $\mathbb{P}$. We only give the derivatives of $\mathrm{P}_{33}(\boldsymbol{x},\boldsymbol{y})$ for $\boldsymbol{x} \in \mathbb{R}_+^3$ and $\boldsymbol{y} \in \mathbb{R}_-^3$. The other cases are similar. By Lemma 3.2, we have, for any $z \ge 2\varepsilon r$,

$$\frac{\partial^{l+m+n}\mathrm{P}_{33}(k(\varepsilon);\boldsymbol{x},\boldsymbol{y})}{\partial x_1^l \partial x_2^m \partial x_3^n} = \mathbf{i}^{l+m+n} k_-^2(\varepsilon) J_{\mathrm{cdh}}(\lambda_1^l \lambda_2^m \mu_1^n h_2 e^{\mathbf{i}(\mu_1-\mu_2)y_3};\boldsymbol{x},\boldsymbol{y}), \tag{3.19}$$

$$\frac{\partial^{l+m+n}\mathrm{P}_{33}(k(\varepsilon);\boldsymbol{x},\boldsymbol{y})}{\partial y_1^l \partial y_2^m \partial y_3^n} = (-\mathbf{i})^{l+m+n} k_-^2(\varepsilon) J_{\mathrm{cdh}}(\lambda_1^l \lambda_2^m \mu_2^n h_2 e^{\mathbf{i}(\mu_1-\mu_2)y_3};\boldsymbol{x},\boldsymbol{y}), \tag{3.20}$$

where $\lambda_1 = \xi\cos\phi + (\varepsilon - \mathbf{i})q\sin\phi$ and $\lambda_2 = \xi\sin\phi - (\varepsilon - \mathbf{i})q\cos\phi$.

4. Perfectly matched layer. Now we introduce the wave-absorbing material, or, the perfectly matched layer. To make the main theme more focused on the layered medium, we only consider the spherical PML in this paper.

4.1. Complex coordinate stretching. Let $B_0 := B(R_0)$ be the ball of radius $R_0 \geq 1$ which contains D and where the scattering field is interested. For any $\boldsymbol{x} \in \mathbb{R}^3$, let $\rho = |\boldsymbol{x}|$ and $\hat{\boldsymbol{x}} = \boldsymbol{x}/\rho$. The complex stretching is defined by $\tilde{\boldsymbol{x}} = \tilde{\rho}\hat{\boldsymbol{x}}$ where

$$\tilde{\rho} = \rho\alpha(\rho), \qquad \alpha(\rho) = 1 + \frac{\mathbf{i}}{\rho}\int_0^\rho \sigma(t)\mathrm{d}t, \tag{4.1}$$

The PML medium property σ is defined piecewise by

$$\sigma(R_0 + sR_0/4) = \sigma_0\hat{\sigma}(s), \qquad \hat{\sigma}(s) = \begin{cases} 0 & \text{if } s \leq 0, \\ e^{\frac{2s-1}{s}} & \text{if } 0 < s \leq 0.5, \\ 2 - e^{\frac{2s-1}{s-1}} & \text{if } 0.5 < s < 1, \\ 2 & \text{if } s \geq 1. \end{cases} \tag{4.2}$$

Clearly σ is C^1-smooth and satisfies $\sigma(t) = 2\sigma_0$ for all $t \geq 1.25R_0$. Here $\sigma_0 \geq 1$ is the medium property parameter. It is well-known that larger value of σ_0 means faster decay of the scattering solution in the PML. For theoretical analysis, we assume $\sigma_0 \geq 4$ in the rest of the paper. Our theory allows more general definitions of the medium property σ (see [6, 12]).

Write the complex stretching by $\boldsymbol{F}(\boldsymbol{x}) := \tilde{\boldsymbol{x}}$. Clearly $\boldsymbol{F}$ is C^2-smooth. In the rest of the paper, both $\boldsymbol{F}(\boldsymbol{x})$ and $\tilde{\boldsymbol{x}}$ will denote the same complex vector. It is easy to see that the Jacobi matrix is given by

$$\mathbb{B} := D\boldsymbol{F} = \alpha(\rho)\mathbb{I} + \rho\alpha'(\rho)\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top, \tag{4.3}$$

Clearly $\mathbb{B}$ is symmetric and C^1-smooth. The determinant is given by

$$J = \det(\mathbb{B}) = \alpha^2(\alpha + \rho\alpha').$$

The analytic continuations of 2D and 3D distance functions are given by

$$r(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = \left[(\tilde{x}_1 - \tilde{y}_2)^2 + (\tilde{x}_2 - \tilde{y}_2)^2\right]^{1/2}, \qquad d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = \left[(\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{y}}) \cdot (\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{y}})\right]^{1/2}.$$

Direct calculations show $\operatorname{Im} r(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \geq 0$ and

$$|\boldsymbol{x} - \boldsymbol{y}| \leq |d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})| \leq \sqrt{1 + 16\sigma_0^2}\; |\boldsymbol{x} - \boldsymbol{y}|, \tag{4.4}$$

$$|r(\boldsymbol{x}, \boldsymbol{y})| \leq |r(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})| \leq \sqrt{1 + 16\sigma_0^2}\; |r(\boldsymbol{x}, \boldsymbol{y})|. \tag{4.5}$$

Moreover, if $\max(|\boldsymbol{x}|, |\boldsymbol{y}|) \geq 2R_0$, we also have

$$\operatorname{Im} d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \geq \frac{1}{2}\sigma_0 |\boldsymbol{x} - \boldsymbol{y}|. \tag{4.6}$$

LEMMA 4.1. *Let $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ be the complex stretching of $\boldsymbol{x} \in \mathbb{R}^3_+, \boldsymbol{y} \in \mathbb{R}^3_-$ respectively. If $\max(|\boldsymbol{x}|, |\boldsymbol{y}|) \geq 2R_0$ and $\operatorname{Im}(\tilde{x}_3 - \tilde{y}_3) \geq x_3 - y_3$, then $|d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})| \geq |r(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})|$.*

Proof. For convenience, we write $\tilde{r} = r(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = r_1 + \mathbf{i}r_2$, $\tilde{z} = \tilde{x}_3 - \tilde{y}_3 = z_1 + \mathbf{i}z_2$ with $r_1, r_2 \geq 0$ and $z_2 \geq z_1 \geq 0$. If $r_2 \geq r_1$, we find that

$$|d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})|^4 = |\tilde{r}|^4 + |\tilde{z}|^4 + 8r_1r_2z_1z_2 + 2(z_2^2 - z_1^2)(r_2^2 - r_1^2) \geq |\tilde{r}|^4 + |\tilde{z}|^4.$$

If $r_2 < r_1$, we have $\operatorname{Re}\tilde{r}^2 \geq 0$ and $|\tilde{r}|^2 \leq 2|\boldsymbol{x} - \boldsymbol{y}|^2$. The proof is completed by (4.6). ∎

4.2. Dyadic Green's function for real wave-number. For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3_\pm$, let $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ be their complex stretching respectively. Similar to (2.7), the analytic continuation of the dyadic Green's function is defined by

$$\mathbb{G}(k(\varepsilon); \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = \mathbb{H}(k(\varepsilon); \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) + \frac{1}{k^2(\varepsilon)} \nabla_{\tilde{\boldsymbol{y}}} \operatorname{div}_{\tilde{\boldsymbol{y}}} \mathbb{H}(k(\varepsilon); \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}),$$

where $\mathbb{H}(k(\varepsilon); \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = \mathbb{S}(k(\varepsilon); \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) - \mathbb{P}(k(\varepsilon); \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ is the analytic continuation of the Hertz tensor, $\mathbb{S}(k(\varepsilon); \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ is defined by replacing $k, \boldsymbol{x}, \boldsymbol{y}$ with $k(\varepsilon), \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ in (2.8), and $\mathbb{P}(k(\varepsilon); \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ is defined by replacing $\boldsymbol{x}, \boldsymbol{y}$ with $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ in (3.16)–(3.18).

We are going to show that $\mathbb{G}(k(\varepsilon); \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ respects the limiting absorption principle and converges to $\mathbb{G}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ as $\varepsilon \to 0$, where $\mathbb{G}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ is the dyadic Green's function for real wave-number and defined by

$$\mathbb{G}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = \mathbb{H}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) + \frac{1}{k^2} \nabla_{\tilde{\boldsymbol{y}}} \operatorname{div}_{\tilde{\boldsymbol{y}}} \mathbb{H}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}).$$

Similarly, $\mathbb{H}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = \mathbb{S}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) - \mathbb{P}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ where $\mathbb{S}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ is given by replacing $\boldsymbol{x}, \boldsymbol{y}$ with $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ in (2.8), and $\mathbb{P}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ is given by replacing $k(\varepsilon), \boldsymbol{x}, \boldsymbol{y}$ with $k, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ in (3.16)–(3.18) respectively.

LEMMA 4.2. *Suppose $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3_\pm$ satisfying $\max(|\boldsymbol{x}|, |\boldsymbol{y}|) \geq 2R_0$. Let $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ be the complex stretching of $\boldsymbol{x}, \boldsymbol{y}$ and write $\boldsymbol{\zeta} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$. Then*

$$\frac{\partial^{m+n}}{\partial \zeta_i^m \partial \zeta_j^n} \mathbb{H}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = \lim_{\varepsilon \to 0^+} \frac{\partial^{m+n}}{\partial \zeta_i^m \partial \zeta_j^n} \mathbb{H}(k(\varepsilon); \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}), \quad 1 \leq i, j \leq 6, \quad m, n \geq 0.$$

Proof. From (2.8), it is easy to see that, for any fixed $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{y}}$,

$$\frac{\partial^{m+n}}{\partial \zeta_i^m \partial \zeta_j^n} \mathbb{S}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) := \lim_{\varepsilon \to 0} \frac{\partial^{m+n}}{\partial \zeta_i^m \partial \zeta_j^n} \mathbb{S}(k(\varepsilon); \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}).$$

It is left to prove the limits for $\mathbb{P}(k(\varepsilon); \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$. Without loss of generality, we only study $\mathrm{P}_{33}(k(\varepsilon); \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ for $\boldsymbol{x} \in \mathbb{R}^3_+$ and $\boldsymbol{y} \in \mathbb{R}^3_-$. The results can be extended straightforwardly to other entries of $\mathbb{P}$ and to other cases of $\boldsymbol{x}, \boldsymbol{y}$. Moreover, in view of (3.19)–(3.20), we can only study the derivatives of $\mathrm{P}_{33}(k(\varepsilon); \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ with respect to $\tilde{\boldsymbol{x}}$. The limits for other derivatives are similar and omitted here.

For convenience, we write $\tilde{r} = r(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$, $\tilde{z} = \tilde{x}_3 - \tilde{y}_3$, and $\tilde{d} = d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$. The PML extension of the Cagniard-de Hoop transform in (3.9) is defined by

$$\xi_\pm(t) = \frac{\kappa_+}{\tilde{d}} \left(\tilde{r}t \pm \mathbf{i}\tilde{z}\sqrt{t^2 - 1} \right), \quad \Lambda_\pm(t) = \frac{\kappa_+}{\tilde{d}} \left(\tilde{z}t \mp \mathbf{i}\tilde{r}\sqrt{t^2 - 1} \right) \qquad \forall\, t \geq 1.$$

Remember that $\kappa_+ = \left[k_+^2(\varepsilon) + (1 + \mathbf{i}\varepsilon)^2 q^2\right]^{1/2}$. Let C denote the generic constant which is independent of $\tilde{\boldsymbol{x}}$, $\tilde{\boldsymbol{y}}$, ε, q, and t. By Lemma 4.1, we have

$$|\xi_\pm(t)| + |\Lambda_\pm(t)| \leq C \frac{|\tilde{r}| + |\tilde{z}|}{|\tilde{d}|} (k_+ + q)t \leq C(k_+ + q)t. \tag{4.7}$$

Define $\mu_1(\xi_\pm) = \left(k_+^2(\varepsilon) - \xi_\pm^2\right)^{1/2}$ and $\mu_2(\xi_\pm) = \left(k_-^2(\varepsilon) - \xi_\pm^2\right)^{1/2}$. From (2.5), we have

$$\operatorname{Re}\mu_1 \geq 0, \qquad \operatorname{Re}\mu_2 \geq 0, \qquad \operatorname{sign}(\operatorname{Im}\mu_1) = \operatorname{sign}(\operatorname{Im}\mu_2).$$

This shows $|\mu_1 - \mu_2| \le |\mu_1 + \mu_2|$. Since $\mu_1^2 - \mu_2^2 = k_+^2(\varepsilon) - k_-^2(\varepsilon)$, we have

$$|\mu_1(\xi_\pm) - \mu_2(\xi_\pm)| \le \left|k_+^2(\varepsilon) - k_-^2(\varepsilon)\right|^{1/2} \le 4k_-.$$

Moreover, for $\varepsilon < k_+^2/16$, we have

$$\begin{aligned}\left|k_-^2(\varepsilon)\mu_1 + k_+^2(\varepsilon)\mu_2\right| &\ge \left|(k_-^2 - \varepsilon^2 k_-^{-2})\mu_1 + (k_+^2 - \varepsilon^2 k_+^{-2})\mu_2\right| - 2\varepsilon\left|\mu_1 + \mu_2\right| \\ &\ge (k_+^2 - \varepsilon^2 k_+^{-2} - 2\varepsilon)\left|\mu_1 + \mu_2\right| \ge (k_+^2 - 3\varepsilon)\left|\mu_1^2 - \mu_2^2\right|^{1/2} \\ &\ge \frac{1}{2}k_+^2(k_- - k_+).\end{aligned}$$

It follows that

$$\left|k_-^2(\varepsilon)\mu_1(\xi_\pm) + k_+^2(\varepsilon)\mu_2(\xi_\pm)\right|^{-1} \le 2\left|k_+^2(k_- - k_+)\right|^{-1}. \tag{4.8}$$

Let $z = x_3 - y_3$, $r = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$, and $d = |\boldsymbol{x} - \boldsymbol{y}|$. Replacing $\boldsymbol{x}, \boldsymbol{y}$ with $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ in (3.19), we get, for $z \ge \varepsilon r$ and $1 \le i, j \le 3$,

$$\left|\frac{\partial^{m+n}\mathrm{P}_{33}(k(\varepsilon);\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})}{\partial\tilde{x}_i^m\partial\tilde{x}_j^n}\right| \le Ce^{4k_-|y_3|}\int_0^\infty\int_1^\infty [(k_+ + q)t]^{m+n+1}\frac{\left|e^{\mathbf{i}\kappa_+\tilde{d}t}\right|}{\sqrt{t^2-1}}\mathrm{d}t\mathrm{d}q. \tag{4.9}$$

From (4.4) and (4.6), we have $\operatorname{Im}\tilde{d} \ge \frac{1}{2}\sigma_0 d \ge 2d$. This indicates $\operatorname{Re}\tilde{d}^2 \le 0$ and $\operatorname{Im}\tilde{d} \ge \operatorname{Re}\tilde{d}$. Then using (3.6)–(3.7), we have $\left|e^{\mathbf{i}\kappa_+\tilde{d}t}\right| \le e^{-\frac{1}{8}\sigma_0 d(k_+ + q)t}$. Inserting the estimates into (4.9), we get

$$\left|\frac{\partial^{m+n}\mathrm{P}_{33}(k(\varepsilon);\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})}{\partial\tilde{x}_i^m\partial\tilde{x}_j^n}\right| \le Ce^{4k_-|y_3|}\int_0^\infty\int_1^\infty \frac{[(k_+ + q)t]^{m+n+1}}{\sqrt{t^2-1}}e^{-\frac{1}{8}\sigma_0 d(k_+ + q)t}\mathrm{d}t\mathrm{d}q.$$

The integral on the righthand side is convergent and independent of ε. By the dominated convergence theorem, we get, for any $z > 0$,

$$\frac{\partial^{m+n}\mathrm{P}_{33}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})}{\partial\tilde{x}_i^m\partial\tilde{x}_j^n} = \lim_{\varepsilon\to 0^+}\frac{\partial^{m+n}\mathrm{P}_{33}(k(\varepsilon);\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})}{\partial\tilde{x}_i^m\partial\tilde{x}_j^n},$$

that is, $\dfrac{\partial^{m+n}\mathrm{P}_{33}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})}{\partial\tilde{x}_i^m\partial\tilde{x}_j^n}$ is given by setting $\varepsilon = 0$ in $\dfrac{\partial^{m+n}\mathrm{P}_{33}(k(\varepsilon);\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})}{\partial\tilde{x}_i^m\partial\tilde{x}_j^n}$ for any integers $m, n \ge 0$ and $1 \le i, j \le 3$. ☐

4.3. Exponential decay of the solution in PML. Now we study the exponential decay of the scattering solution in the wave-absorbing material. First we extend the Cagniard-de Hoop transform from real coordinates to complex coordinates and prove some useful lemmas. It suffices to study the dyadic Green's function for real wave-number. This shows that $\kappa_\pm = \kappa_\pm(0,q) = \sqrt{k_\pm^2 + q^2} > 0$.

LEMMA 4.3. *For any $\boldsymbol{x} \in \mathbb{R}_+^3$ and $\boldsymbol{y} \in \mathbb{R}_-^3$, let $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ be the complex stretching of $\boldsymbol{x}, \boldsymbol{y}$ and write $\tilde{r} = r(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$, $\tilde{z} = \tilde{x}_3 - \tilde{y}_3$, $\tilde{d} = d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$. Assume* $\max(|\boldsymbol{x}|, |\boldsymbol{y}|) \ge 2R_0$ *and* $\operatorname{Im}\tilde{z} \ge \operatorname{Re}\tilde{z}$. *Then*

$$\left(\kappa^2 - \xi_\pm^2(t)\right)^{1/2} = \kappa\tilde{d}^{-1}\left(\tilde{z}t \mp \mathbf{i}\tilde{r}\sqrt{t^2-1}\right) \qquad \forall\, t \ge 1,$$

where $\kappa > 0$ *and* $\xi_\pm(t) = \kappa\tilde{d}^{-1}\big(\tilde{r}t \pm \mathbf{i}\tilde{z}\sqrt{t^2-1}\big)$.

Proof. Write $\Lambda_\pm = \kappa\tilde{d}^{-1}\left(\tilde{z}t \mp \mathbf{i}\tilde{r}t_1\right)$ with $t_1 = \sqrt{t^2-1}$. It is clear $\Lambda_\pm^2 + \xi_\pm^2 = \kappa^2$. By the convention in (2.5), it suffices to show $\operatorname{Re}\Lambda_\pm \geq 0$. We only prove $\operatorname{Re}\Lambda_+ \geq 0$. The proof for $\operatorname{Re}\Lambda_- \geq 0$ is similar. Without loss of generality, we assume $|\boldsymbol{x}| \geq |\boldsymbol{y}|$.

Let $\tilde{r} = r_1 + \mathbf{i}r_2$, $\tilde{z} = z_1 + \mathbf{i}z_2$, $\tilde{d} = d_1 + \mathbf{i}d_2$ with $r_i, d_i \geq 0$ and $z_2 \geq z_1 \geq 0$. Then

$$\kappa^{-1}\big|\tilde{d}\big|^2 \operatorname{Re}\Lambda_+ = d_1(z_1 t + r_2 t_1) + d_2(z_2 t - r_1 t_1) \geq t_1(M-N),$$

where $M = d_1 z_1 + d_1 r_2 + d_2 z_2$ and $N = d_2 r_1$. Using Lemma 4.1 and $z_2 \geq z_1$, we know that $\operatorname{Re}\tilde{z}^2 \leq 0$ and $\big|\tilde{d}\big| \geq |\tilde{r}|$. From (4.4) and (4.6), we have $\operatorname{Re}\tilde{d}^2 \leq 0$. Then the convention in (2.5) shows that

$$\begin{aligned}\frac{1}{2}\left(M^2 - N^2\right) &\geq \frac{1}{2}\left(d_1^2 z_1^2 + d_2^2 z_2^2 + d_1^2 r_2^2 - d_2^2 r_1^2\right)\\ &= \big|\tilde{d}^2\big|\,\big|\tilde{z}^2\big| + \operatorname{Re}\tilde{d}^2\operatorname{Re}\tilde{z}^2 + \operatorname{Re}\tilde{d}^2\,\big|\tilde{r}^2\big| - \big|\tilde{d}^2\big|\operatorname{Re}\tilde{r}^2\\ &\geq \big|\tilde{d}^2\big|\big(\big|\tilde{z}^2\big| + \operatorname{Re}\tilde{z}^2\big) + \operatorname{Re}\tilde{d}^2\big(\big|\tilde{r}^2\big| - \big|\tilde{d}^2\big|\big) \geq \big|\tilde{d}^2\big|\big(\big|\tilde{z}^2\big| + \operatorname{Re}\tilde{z}^2\big).\end{aligned}$$

Therefore, $M \geq N$, namely, $\operatorname{Re}\Lambda_+ \geq 0$. □

Lemma 4.4. *Let* $\xi_\pm$ *and the assumptions be same to those in Lemma 4.3. For either* $\xi = \xi_+$ *or* $\xi = \xi_-$*, define* $\mu_j = \left(\kappa_j^2 - \xi^2\right)^{1/2}$, $j = 1, 2$ *with* $\kappa_2 \geq \kappa_1 = \kappa$*. Then*

$$\operatorname{Im}\left[(\mu_2 - \mu_1)(a + \mathbf{i}b)\right] \leq 0 \qquad \forall\, b \geq a \geq 0.$$

Proof. We only prove the lemma for $\xi = \kappa_1\tilde{d}^{-1}\left(\tilde{r}t + \mathbf{i}\tilde{z}t_1\right)$ where $t_1 = \sqrt{t^2-1}$. The proof for $\xi = \kappa_1\tilde{d}^{-1}\left(\tilde{r}t - \mathbf{i}\tilde{z}t_1\right)$ is similar and omitted here.

Write $\mu_j = \alpha_j + \mathbf{i}\beta_j$ with $\alpha_j, \beta_j \in \mathbb{R}$, $j = 1, 2$. Since $\mu_2^2 - \mu_1^2 = \kappa_2^2 - \kappa_1^2$, we have

$$\alpha_2^2 - \beta_2^2 = \kappa_2^2 - \kappa_1^2 + \alpha_1^2 - \beta_1^2, \qquad \alpha_1\beta_1 = \alpha_2\beta_2.$$

We recall (2.5) and deduce that

$$\sqrt{2}\alpha_2 = \left[\sqrt{\left(\kappa_2^2 - \kappa_1^2 + \alpha_1^2 - \beta_1^2\right)^2 + 4\alpha_1^2\beta_1^2} + \left(\kappa_2^2 - \kappa_1^2 + \alpha_1^2 - \beta_1^2\right)\right]^{1/2}, \tag{4.10}$$

$$\sqrt{2}\,|\beta_2| = \left[\sqrt{\left(\kappa_2^2 - \kappa_1^2 + \alpha_1^2 - \beta_1^2\right)^2 + 4\alpha_1^2\beta_1^2} - \left(\kappa_2^2 - \kappa_1^2 + \alpha_1^2 - \beta_1^2\right)\right]^{1/2}. \tag{4.11}$$

Since $\alpha_1 \geq 0$ by the convention in (2.5), direct calculations show that

$$\alpha_2 \geq \alpha_1 \geq 0, \qquad |\beta_2| \leq |\beta_1|, \qquad \operatorname{sign}(\beta_1) = \operatorname{sign}(\beta_2). \tag{4.12}$$

Since

$$\operatorname{Im}\left[(\mu_1 - \mu_2)(a + \mathbf{i}b)\right] = a(\beta_1 - \beta_2) + b(\alpha_1 - \alpha_2), \tag{4.13}$$

the lemma now follows obviously for $\beta_1 \leq 0$.

Now we assume $\beta_1 > 0$. By Lemma 4.3, we have $\mu_1 = \kappa_1\tilde{d}^{-1}\left(\tilde{z}t - \mathbf{i}\tilde{r}t_1\right)$. Similar to the proof of Lemma 4.3, we write $\tilde{d} = d_1 + \mathbf{i}d_2$, $\tilde{r} = r_1 + \mathbf{i}r_2$, $\tilde{z} = z_1 + \mathbf{i}z_2$ with $r_1, r_2 \geq 0$, $d_2 \geq d_1 \geq 0$, and $z_2 \geq z_1 \geq 0$. Then

$$\begin{aligned}\kappa_1^{-1}|\tilde{d}|^2\alpha_1 &= t(\rho_1 z_1 + \rho_2 z_2) + t_1(\rho_1 r_2 - \rho_2 r_1),\\ \kappa_1^{-1}|\tilde{d}|^2\beta_1 &= t(\rho_1 z_2 - \rho_2 z_1) - t_1(\rho_1 r_1 + \rho_2 r_2).\end{aligned}$$

We deduce that

$$\kappa_1^{-1}|\tilde{d}|^2(\alpha_1-\beta_1) \geq (\rho_2-\rho_1)(z_1+z_2+r_2-r_1) \geq 0.$$

Since $\beta_1>0$ and $\alpha_1\beta_1=\alpha_2\beta_2$, by (4.13), we have

$$\operatorname{Im}\left[(\mu_1-\mu_2)(a+\mathbf{i}b)\right] = \frac{1}{\beta_1}(\beta_2-\beta_1)(b\alpha_2-a\beta_1) \leq \frac{a}{\beta_1}(\beta_2-\beta_1)(\alpha_2-\beta_1) \leq 0,$$

where we have used $0\leq a\leq b$ and (4.12). The proof is complete. ☐

LEMMA 4.5. *Let $\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}$ be the complex stretching of $\boldsymbol{x},\boldsymbol{y}$ and write $\boldsymbol{\zeta}=(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{y}_1,\tilde{y}_2,\tilde{y}_3)$. There is a constant C depending only on k such that*

$$\left|\frac{\partial^{m+n}}{\partial\zeta_i^m\partial\zeta_j^n}\mathbb{S}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})\right| \leq C\times\begin{cases}1+|\boldsymbol{x}-\boldsymbol{y}|^{-m-n-1}, & \text{if } \max(|\boldsymbol{x}|,|\boldsymbol{y}|)<2R_0,\\ e^{-\frac{1}{2}k_+\sigma_0|\boldsymbol{x}-\boldsymbol{y}|}, & \text{otherwise,}\end{cases}$$

for any $1\leq i,j\leq 6$ and $m,n\geq 0$.

Proof. Since $\Phi(k_\pm;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})=[4\pi d(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})]^{-1}e^{ik_\pm d(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})}$, the lemma comes directly from (2.8), (4.4), and (4.6). ☐

LEMMA 4.6. *For any $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_\pm$ satisfying $\max(|\boldsymbol{x}|,|\boldsymbol{y}|)\geq 2R_0$, let $\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}$ be the complex stretching of $\boldsymbol{x},\boldsymbol{y}$ and write $\boldsymbol{\zeta}=(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{y}_1,\tilde{y}_2,\tilde{y}_3)$. Then there is a constant C depending only on k such that*

$$\left|\frac{\partial^{m+n}}{\partial\zeta_i^m\partial\zeta_j^n}\mathbb{P}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})\right| \leq Ce^{-\frac{1}{2}k_+\sigma_0|\boldsymbol{x}-\boldsymbol{y}|}, \qquad 1\leq i,j\leq 6, \quad m,n\geq 0.$$

Proof. Without loss of generality, we assume $\boldsymbol{x}\in\mathbb{R}^3_+$, $\boldsymbol{y}\in\mathbb{R}^3_-$ and only study the derivatives of $\mathrm{P}_{33}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ with respect to $\tilde{\boldsymbol{x}}$. The results can be extended straightforwardly to other entries of $\mathbb{P}$, to other derivatives, and to other cases of $\boldsymbol{x},\boldsymbol{y}$.

Write $\tilde{r}=r(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$, $\tilde{z}=\tilde{x}_3-\tilde{y}_3$, and $\tilde{d}=d(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ for convenience. Define the PML extension of the Cagniard-de Hoop transform by

$$\xi_\pm(t)=\kappa_+\tilde{d}^{-1}\left(\tilde{r}t\pm\mathbf{i}\tilde{z}\sqrt{t^2-1}\right), \quad \Lambda_\pm(t)=\kappa_+\tilde{d}^{-1}\left(\tilde{z}t\mp\mathbf{i}\tilde{r}\sqrt{t^2-1}\right), \quad \forall\, t\geq 1,$$

where $\kappa_\pm=\sqrt{k_\pm^2+q^2}$. Write $\mu_1(\xi)=\left(\kappa_+^2-\xi^2\right)^{1/2}$ and $\mu_2(\xi)=\left(\kappa_-^2-\xi^2\right)^{1/2}$. Then from Lemma 4.3 we know that

$$\mu_1(\xi_\pm)=\Lambda_\pm, \qquad \mu_2(\xi_\pm)=\left(k_-^2-k_+^2+\Lambda_\pm^2\right)^{1/2}.$$

Replacing $\boldsymbol{x},\boldsymbol{y}$ with $\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}$ and setting $\varepsilon=0$ in (3.19), we find that

$$\frac{\partial^{l+m+n}\mathrm{P}_{33}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})}{\partial\tilde{x}_1^l\partial\tilde{x}_2^m\partial\tilde{x}_3^n} = \mathbf{i}^{l+m+n-1}\frac{k_-^2}{2\pi^2}\left[F_+(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})+F_-(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})\right], \tag{4.14}$$

where

$$F_\pm(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})=\int_0^\infty\int_1^\infty \lambda_1^l(\xi_\pm)\lambda_2^m(\xi_\pm)\Lambda_\pm^{n+1}\frac{e^{\mathbf{i}[\mu_1(\xi_\pm)-\mu_2(\xi_\pm)]y_3}}{k_-^2\mu_1(\xi_\pm)+k_+^2\mu_2(\xi_\pm)}\frac{e^{\mathbf{i}\kappa_+\tilde{d}t}}{\sqrt{t^2-1}}\mathrm{d}t\mathrm{d}q.$$

Here $\lambda_1(\xi)=\xi\cos\phi-\mathbf{i}q\sin\phi$, $\lambda_2(\xi)=\xi\sin\phi+\mathbf{i}q\cos\phi$ and the polar angle satisfies

$$x_1-y_1=r\cos\phi, \qquad x_2-y_2=r\sin\phi, \qquad r=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}.$$

It suffices to estimate $F_\pm(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$. Similar to (4.7), there is a generic constant C independent of $\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}$ such that

$$|\lambda_1(\xi_\pm)|+|\lambda_2(\xi_\pm)|+|\xi_\pm|+|\Lambda_\pm|\le C(k_++q)t,$$

By Lemma 4.4 and (4.8), we have $\left|\frac{e^{\mathbf{i}[\mu_1(\xi_\pm)-\mu_2(\xi_\pm)]y_3}}{k_-^2\mu_1(\xi_\pm)+k_+^2\mu_2(\xi_\pm)}\right|\le C$. Using (4.6), we have $|F_\pm(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})|\le Ce^{-\frac{1}{2}k_+\sigma_0|\boldsymbol{x}-\boldsymbol{y}|}$. The proof is completed by using (4.14). ▯

LEMMA 4.7. *Let $\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}$ be given in Lemma 4.6 and $\boldsymbol{\zeta}=(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{y}_1,\tilde{y}_2,\tilde{y}_3)$. Then there is a constant C depending only on k such that*

$$\left|\frac{\partial^{m+n}}{\partial\zeta_i^m\partial\zeta_j^n}\mathbb{G}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})\right|\le Ce^{-\frac{1}{2}k_+\sigma_0|\boldsymbol{x}-\boldsymbol{y}|},\qquad 1\le i,j\le 6,\quad m,n\ge 0.$$

Proof. Since $\mathbb{G}=\mathbb{H}+k_\pm^{-2}\nabla_{\tilde{\boldsymbol{y}}}\operatorname{div}_{\tilde{\boldsymbol{y}}}\mathbb{H}$ and $\mathbb{H}=\mathbb{S}-\mathbb{P}$, the theorem is a direct consequence of Lemma 4.5 and Lemma 4.6. ▯

Now we prove the main result of this section, that is, the exponential decay of the scattering solution in PML. From [30, Section 12.4.3], the solution $\boldsymbol{E}$ of (1.1) admits the integral representation

$$\boldsymbol{E}=\boldsymbol{\Psi}_{\rm SL}(\boldsymbol{\mu})+\boldsymbol{\Psi}_{\rm DL}(\boldsymbol{g})\qquad\text{in }\ \mathbb{R}^3\backslash\bar{D},$$

where $\boldsymbol{g}=\gamma_t\boldsymbol{E}$ and $\boldsymbol{\mu}=\gamma_t(\mathbf{curl}\,\boldsymbol{E})$ are the Dirichlet trace and the Neumann trace of the solution on Γ_D. The Maxwell single and double layer potentials are defined by

$$\boldsymbol{\Psi}_{\rm SL}(\boldsymbol{\mu})=\int_{\Gamma_D}\mathbb{G}^\top(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\mu}(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}},\quad \boldsymbol{\Psi}_{\rm DL}(\boldsymbol{g})=\int_{\Gamma_D}(\mathbf{curl}_{\boldsymbol{y}}\,\mathbb{G})^\top(\boldsymbol{x},\boldsymbol{y})\,\boldsymbol{g}(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}.$$

The analytic continuation of the scattering solution is defined by

$$\boldsymbol{E}(\tilde{\boldsymbol{x}})=\boldsymbol{\Psi}_{\rm SL}(\boldsymbol{\mu})(\tilde{\boldsymbol{x}})+\boldsymbol{\Psi}_{\rm DL}(\boldsymbol{g})(\tilde{\boldsymbol{x}})\tag{4.15}$$

THEOREM 4.8. *There is a constant $C>0$ depending only on k and R_0 such that, for any $\boldsymbol{x}\in\mathbb{R}^3_\pm$ satisfying $|\boldsymbol{x}|\ge 2R_0$,*

$$|\boldsymbol{E}(\tilde{\boldsymbol{x}})|+|\mathbf{curl}_{\tilde{\boldsymbol{x}}}\,\boldsymbol{E}(\tilde{\boldsymbol{x}})|\le Ce^{-\frac{1}{2}k_+\sigma_0|\boldsymbol{x}|}\left\|\boldsymbol{g}\right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}.$$

Proof. Since γ_t, γ_T are bounded operators, using (1.1a) and Theorem 2.2, we have

$$\begin{aligned}|\boldsymbol{\Psi}_{\rm SL}(\boldsymbol{\mu})(\tilde{\boldsymbol{x}})|&\le\|\boldsymbol{\mu}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}\|\gamma_T\mathbb{G}(\tilde{\boldsymbol{x}},\cdot)\|_{\boldsymbol{H}^{-1/2}(\mathrm{Curl},\Gamma_D)}\\&\le\|\mathbf{curl}\,\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_0)}\|\mathbb{G}(\tilde{\boldsymbol{x}},\cdot)\|_{\boldsymbol{H}(\mathbf{curl},\Omega_0)}\\&\le C\|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_0)}\|\mathbb{G}(\tilde{\boldsymbol{x}},\cdot)\|_{\boldsymbol{H}(\mathbf{curl},\Omega_0)}\\&\le Ce^{-\frac{1}{2}k_+\sigma_0|\boldsymbol{x}|}\|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)},\end{aligned}$$

where we have used $\tilde{\boldsymbol{y}}=\boldsymbol{y}$ in $\Omega_0=B_0\backslash\bar{D}$. Similarly, we have

$$\begin{aligned}|\boldsymbol{\Psi}_{\rm DL}(\boldsymbol{g})(\tilde{\boldsymbol{x}})|&\le\|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}\|\gamma_T(\mathbf{curl}\,\mathbb{G})(\tilde{\boldsymbol{x}},\cdot)\|_{\boldsymbol{H}^{-1/2}(\mathrm{Curl},\Gamma_D)}\\&\le\|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}\|\mathbf{curl}\,\mathbb{G}(\tilde{\boldsymbol{x}},\cdot)\|_{\boldsymbol{H}(\mathbf{curl},\Omega_0)}\\&\le Ce^{-\frac{1}{2}k_+\sigma_0|\boldsymbol{x}|}\|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}.\end{aligned}$$

This yields $|\boldsymbol{E}(\tilde{\boldsymbol{x}})|\le Ce^{-\frac{1}{2}k_+\sigma_0|\boldsymbol{x}|}\|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}$. The estimate for $\mathbf{curl}_{\tilde{\boldsymbol{x}}}\,\boldsymbol{E}(\tilde{\boldsymbol{x}})$ is similar and omitted here. ▯

5. The Maxwell exterior problem. In this section, we shall prove that the scattering problem (1.1) is equivalent to an exterior problem of Maxwell's equations under the complex stretching. Then we shall propose a weak formulation of the exterior problem on a bounded domain. It is the solution of the exterior problem that will be approximated by the PML method.

5.1. The exterior problem. First we define the stretched dyadic Green's function by $\tilde{\mathbb{G}}(k;\tilde{\boldsymbol{x}},\boldsymbol{y}) = \mathbb{B}^\top(\boldsymbol{y})\mathbb{G}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$. Using the argument in the proof of [28, Theorem 2.8], we have

$$\mathbf{curl}[\mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}}(k;\tilde{\boldsymbol{x}},\cdot)] - k_\pm^2\mathbb{A}^{-1}\tilde{\mathbb{G}}(k;\tilde{\boldsymbol{x}},\cdot) = \mathbb{B}^{-1}\delta_{\boldsymbol{x}} \qquad \text{in } \mathbb{R}^3_\pm, \tag{5.1}$$

where $\mathbb{A} := J^{-1}\mathbb{B}^\top\mathbb{B}$. Since $\boldsymbol{F}$ is C^2-smooth, the tangential continuities in (2.6b) yield

$$\left[\mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}}(k;\tilde{\boldsymbol{x}},\cdot)\times\boldsymbol{n}\right] = \left[\tilde{\mathbb{G}}(k;\tilde{\boldsymbol{x}},\cdot)\times\boldsymbol{n}\right] = 0 \qquad \text{on } \Sigma. \tag{5.2}$$

Define the pull-back of $\boldsymbol{E}$ by $\tilde{\boldsymbol{E}} := \mathbb{B}^\top\boldsymbol{E}\circ\boldsymbol{F}$. Similarly we have

$$\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\tilde{\boldsymbol{E}}) - k^2\mathbb{A}^{-1}\tilde{\boldsymbol{E}} = 0 \quad \text{in } \mathbb{R}^3_\pm\setminus\bar{D}.$$

THEOREM 5.1. *Let $\tilde{\boldsymbol{E}}$ be the pull-back of the scattering solution $\boldsymbol{E}$. Then $\tilde{\boldsymbol{E}}$ is the unique solution of the exterior problem: $\tilde{\boldsymbol{E}}\in\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\backslash\bar{D})$,*

$$\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\tilde{\boldsymbol{E}}) - k^2\mathbb{A}^{-1}\tilde{\boldsymbol{E}} = 0 \quad \textit{in } \mathbb{R}^3_\pm\setminus\bar{D}, \tag{5.3a}$$

$$\left[\mathbb{A}\,\mathbf{curl}\,\tilde{\boldsymbol{E}}\times\boldsymbol{n}\right] = \left[\tilde{\boldsymbol{E}}\times\boldsymbol{n}\right] = 0 \quad \textit{on } \Sigma, \tag{5.3b}$$

$$\gamma_t\tilde{\boldsymbol{E}} = \boldsymbol{g} \quad \textit{on } \Gamma_D. \tag{5.3c}$$

Proof. From Theorem 2.2, the scattering problem (1.1) has a unique solution $\boldsymbol{E}$. We already proved that the pull-back $\tilde{\boldsymbol{E}}$ satisfies (5.3a). Since $\tilde{\boldsymbol{E}} = \boldsymbol{E}$ in $B_0\backslash\bar{D}$, (5.3c) follows directly from (1.1c). Furthermore, since $\boldsymbol{F}$ is C^2-smooth, (5.3b) follows directly from (1.1c). From Theorem 4.8, we also know that $\tilde{\boldsymbol{E}}\in\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\backslash\bar{D})$. Therefore, $\tilde{\boldsymbol{E}}$ is one solution of (5.3). It is left to show the uniqueness of the solution.

Now we suppose $\tilde{\boldsymbol{E}}_1,\tilde{\boldsymbol{E}}_2$ are two solutions of (5.3). Then $\tilde{\boldsymbol{w}} := \tilde{\boldsymbol{E}}_1 - \tilde{\boldsymbol{E}}_2 \in \boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\backslash\bar{D})$ satisfies the exterior problem

$$\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\tilde{\boldsymbol{w}}) - k^2\mathbb{A}^{-1}\tilde{\boldsymbol{w}} = 0 \quad \text{in } \mathbb{R}^3_\pm\setminus\bar{D}, \tag{5.4a}$$

$$\left[\mathbb{A}\,\mathbf{curl}\,\tilde{\boldsymbol{w}}\times\boldsymbol{n}\right] = \left[\tilde{\boldsymbol{w}}\times\boldsymbol{n}\right] = 0 \quad \text{on } \Sigma, \tag{5.4b}$$

$$\gamma_t\tilde{\boldsymbol{w}} = 0 \quad \text{on } \Gamma_D. \tag{5.4c}$$

For any $\boldsymbol{x}\in\mathbb{R}^3_\pm\backslash\bar{D}$, let $B(\rho)$ be a sufficiently large ball which contains $\boldsymbol{x}$. Write $\Omega_\rho = B(\rho)\backslash\bar{D}$ and $\Gamma_\rho = \partial B(\rho)$. By (5.1) and (5.4a), an integration by part yields

$$\begin{aligned}\mathbb{B}^{-1}(\boldsymbol{x})\tilde{\boldsymbol{w}}(\boldsymbol{x}) &= \int_{\Omega_\rho\cap\mathbb{R}^3_\pm}\left[\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}}(k;\tilde{\boldsymbol{x}},\cdot)) - k_\pm^2\mathbb{A}^{-1}\tilde{\mathbb{G}}(k;\tilde{\boldsymbol{x}},\cdot)\right]^\top\tilde{\boldsymbol{w}}\\ &= \int_{\Gamma_D}\left[\tilde{\mathbb{G}}^\top(k;\tilde{\boldsymbol{x}},\cdot)\gamma_t(\mathbb{A}\,\mathbf{curl}\,\tilde{\boldsymbol{w}}) + (\mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}})^\top(k;\tilde{\boldsymbol{x}},\cdot)\gamma_t\tilde{\boldsymbol{w}}\right] - I(\rho),\end{aligned}$$

where the second term is defined by

$$I(\rho) = \int_{\partial B(\rho)}\left[\tilde{\mathbb{G}}^\top(k;\boldsymbol{x},\cdot)\gamma_t(\mathbb{A}\,\mathbf{curl}\,\tilde{\boldsymbol{w}}) + (\mathbb{A}\,\mathbf{curl}_{\boldsymbol{y}}\,\tilde{\mathbb{G}})^\top(k;\boldsymbol{x},\cdot)\gamma_t\tilde{\boldsymbol{w}}\right].$$

By Theorem 4.7 and the Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned}|I(\rho)| &\le Ce^{-\frac{1}{2}k_+\sigma_0\rho}\left[\|\mathbb{A}\,\mathbf{curl}\,\tilde{\boldsymbol{w}}\|_{\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\backslash\bar{D})}+\|\tilde{\boldsymbol{w}}\|_{\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\backslash\bar{D})}\right]\\ &\le Ce^{-\frac{1}{2}k_+\sigma_0\rho}\,\|\tilde{\boldsymbol{w}}\|_{\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\backslash\bar{D})}, \qquad \text{as } \rho\to\infty,\end{aligned}$$

where C depends only on σ_0, R_0, and k. This shows $\lim_{\rho\to\infty} I(\rho)=0$. Since $\mathbb{A}=\mathbb{B}=\mathbb{I}$ on Γ_D, we find that

$$\begin{aligned}\mathbb{B}^{-1}(\boldsymbol{x})\tilde{\boldsymbol{w}}(\boldsymbol{x}) &= \int_{\Gamma_D}\left[\tilde{\mathbb{G}}^\top(k;\tilde{\boldsymbol{x}},\cdot)\gamma_t(\mathbb{A}\,\mathbf{curl}\,\tilde{\boldsymbol{w}})+(\mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}})^\top(k;\tilde{\boldsymbol{x}},\cdot)\gamma_t\tilde{\boldsymbol{w}}\right]\\ &= \int_{\Gamma_D}\left[\mathbb{G}^\top(k;\tilde{\boldsymbol{x}},\cdot)\gamma_t(\mathbf{curl}\,\tilde{\boldsymbol{w}})+(\mathbf{curl}\,\mathbb{G})^\top(k;\tilde{\boldsymbol{x}},\cdot)\gamma_t\tilde{\boldsymbol{w}}\right]. \qquad (5.5)\end{aligned}$$

We replace $\tilde{\boldsymbol{x}}$ with $\boldsymbol{x}$ on the righthand side of (5.5) and define

$$\boldsymbol{w}(\boldsymbol{x})=\int_{\Gamma_D}\left[\mathbb{G}^\top(k;\boldsymbol{x},\cdot)\gamma_t(\mathbf{curl}\,\tilde{\boldsymbol{w}})+(\mathbf{curl}\,\mathbb{G})^\top(k;\boldsymbol{x},\cdot)\gamma_t\tilde{\boldsymbol{w}}\right] \qquad \forall\,\boldsymbol{x}\in\mathbb{R}^3\backslash\bar{D}.$$

Note that $\tilde{\boldsymbol{w}}(\boldsymbol{x})=\boldsymbol{w}(\boldsymbol{x})$ for all $\boldsymbol{x}\in B_0\backslash\bar{D}$. From (5.4c) and [23, Proposition A.13], the integral representation shows that $\boldsymbol{w}$ satisfies the scattering problem

$$\begin{aligned}&\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{w}-k_\pm^2\boldsymbol{w}=0 \quad \text{in } \mathbb{R}^3_\pm\setminus\bar{D},\\ &[\mathbf{curl}\,\boldsymbol{w}\times\boldsymbol{n}]=[\boldsymbol{w}\times\boldsymbol{n}]=0 \quad \text{on } \Sigma,\\ &\gamma_t\boldsymbol{w}=0 \quad \text{on } \Gamma_D,\\ &\lim_{\rho\to\infty}\int_{\partial B(\rho)}|\mathbf{curl}\,\boldsymbol{w}\times\boldsymbol{n}-\mathbf{i}k\boldsymbol{w}|^2=0.\end{aligned}$$

By Theorem 2.2, we conclude $\boldsymbol{w}=0$. Therefore, $\tilde{\boldsymbol{w}}=0$ and $\tilde{\boldsymbol{E}}_1=\tilde{\boldsymbol{E}}_2$. □

5.2. The Dirichlet-to-Neumann map. Let $B_1=B(R_1)$ and $\Gamma_1=\partial B_1$ where $R_1=1.25R_0$. We define the Dirichlet-to-Neumann map $\mathcal{G}$: $\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)\to\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)$ as follows: for any $\boldsymbol{\lambda}\in\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)$,

$$\mathcal{G}(\boldsymbol{\lambda}):=\gamma_t(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}) \quad \text{on } \Gamma_1, \qquad (5.6)$$

where $\boldsymbol{u}\in\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\backslash\bar{B}_1)$ is the solution of the exterior problem

$$\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u})-k^2\mathbb{A}^{-1}\boldsymbol{u}=0 \quad \text{in } \mathbb{R}^3_\pm\setminus\bar{B}_1, \qquad (5.7a)$$

$$[\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}]=[\boldsymbol{u}\times\boldsymbol{n}]=0 \quad \text{on } \Sigma, \qquad (5.7b)$$

$$\gamma_t\boldsymbol{u}=\boldsymbol{\lambda} \quad \text{on } \Gamma_1. \qquad (5.7c)$$

LEMMA 5.2. *The exterior problem* (5.7) *has a unique solution which satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\backslash\bar{B}_1)}\le C\sigma_0\,\|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)},$$

where the constant $C>0$ depends only on k and R_0. Moreover, it admits the integral representation, for any $\boldsymbol{x}\in\mathbb{R}^3\backslash\bar{B}_1$,

$$\boldsymbol{u}(\boldsymbol{x})=\mathbb{B}(\boldsymbol{x})\int_{\Gamma_1}\left[\tilde{\mathbb{G}}^\top(k;\tilde{\boldsymbol{x}},\cdot)\gamma_t(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u})+(\mathbb{A}\,\mathbf{curl}_{\boldsymbol{y}}\,\tilde{\mathbb{G}})^\top(k;\tilde{\boldsymbol{x}},\cdot)\gamma_t\boldsymbol{u}\right].$$

Proof. Multiplying both sides of (5.7a) with any $\boldsymbol{v} \in \boldsymbol{C}_0^\infty(\mathbb{R}^3\backslash\bar{B}_1)$ and using the formula of integral by path, we find that

$$b(\boldsymbol{u},\boldsymbol{v}) := \int_{\mathbb{R}^3\backslash\bar{B}_1} \left(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\bar{\boldsymbol{v}} - k^2\mathbb{A}^{-1}\boldsymbol{u}\cdot\bar{\boldsymbol{v}}\right) = 0.$$

Since $\boldsymbol{C}_0^\infty(\mathbb{R}^3\backslash\bar{B}_1)$ is dense in $\boldsymbol{H}_{\Gamma_1}(\mathbf{curl},\mathbb{R}^3\backslash\bar{B}_1)$, we get an equivalent weak formulation of (5.7): Find $\boldsymbol{u}\in\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\backslash\bar{B}_1)$ such that $\gamma_t\boldsymbol{u}=\boldsymbol{\lambda}$ on Γ_1 and

$$b(\boldsymbol{u},\boldsymbol{v}) = 0 \qquad \forall\,\boldsymbol{v}\in\boldsymbol{H}_{\Gamma_1}(\mathbf{curl},\mathbb{R}^3\backslash\bar{B}_1). \tag{5.8}$$

It is easy to calculate the eigenvalues of $\mathbb{B}$ and get

$$\lambda_1=\lambda_2=\alpha, \qquad \lambda_3=\alpha+\rho\alpha'. \tag{5.9}$$

Write $\boldsymbol{x}=\rho(\cos\theta\cos\phi,\sin\theta\cos\phi,\sin\phi)^\top$ with $\theta\in[0,2\pi]$ and $\phi\in[-\pi/2,\pi/2]$. The associated eigenvectors are real and read as follows

$$\begin{aligned}
\boldsymbol{\xi}_1 &= (-\sin\theta,\cos\theta,0)^\top,\\
\boldsymbol{\xi}_2 &= (\cos\theta\sin\phi,\sin\theta\sin\phi,-\cos\phi)^\top,\\
\boldsymbol{\xi}_3 &= (\cos\theta\cos\phi,\sin\theta\cos\phi,\sin\phi)^\top.
\end{aligned} \tag{5.10}$$

Clearly $\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3$ are also eigenvectors of $\mathbb{A}$ and belong to the eigenvalues

$$\nu_1=\nu_2=\frac{1}{\alpha+\rho\alpha'}, \qquad \nu_3=\frac{\alpha+\rho\alpha'}{\alpha^2}. \tag{5.11}$$

By careful calculations, there exist two constants $C_1>C_0>0$ depending only on R_0 such that, for any $\rho\ge 2R_0$,

$$C_0\le -\operatorname{Im}(\sigma_0\nu_i)\le C_1, \qquad C_0\le \operatorname{Im}\frac{1}{\sigma_0\nu_i}\le C_1, \qquad i=1,2,3. \tag{5.12}$$

For any $\boldsymbol{\xi}=\sum_{i=1}^3 t_i\boldsymbol{\xi}_i$ with $t_1,t_2,t_3\in\mathbb{C}$, we find that

$$\mathbb{A}\boldsymbol{\xi}\cdot\bar{\boldsymbol{\xi}} = \sum_{i=1}^3 t_i\nu_i\boldsymbol{\xi}_i\cdot\sum_{j=1}^3\bar{t}_j\boldsymbol{\xi}_j = \sum_{i=1}^3\nu_i\left|t_i\right|^2, \qquad \mathbb{A}^{-1}\boldsymbol{\xi}\cdot\bar{\boldsymbol{\xi}} = \sum_{i=1}^3\nu_i^{-1}\left|t_i\right|^2. \tag{5.13}$$

Therefore, the sesquilinear form b is coercive on $\boldsymbol{H}_{\Gamma_1}(\mathbf{curl},\mathbb{R}^3\backslash\bar{B}_1)$, namely,

$$|b(\boldsymbol{v},\boldsymbol{v})|\ge -\operatorname{Im} b(\boldsymbol{v},\boldsymbol{v})\ge \frac{\min(1,k_1)}{C_1\sigma_0}\left\|\boldsymbol{v}\right\|^2_{\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\backslash\bar{B}_1)}.$$

Then problem (5.8) attains a unique solution which satisfies

$$\left\|\boldsymbol{u}\right\|_{\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\backslash\bar{B}_1)}\le C\sigma_0\left\|\boldsymbol{\lambda}\right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)}.$$

Finally, the proof for the integral representation of $\boldsymbol{u}$ is parallel to that for (5.5). We omit the details. The proof is complete. ∎

LEMMA 5.3. *There exists a constant $C>0$ depending only on k,R_0 such that*

$$\left\|\mathcal{G}(\boldsymbol{\lambda})\right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)}\le C\sigma_0^2\left\|\boldsymbol{\lambda}\right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)} \qquad \forall\,\boldsymbol{\lambda}\in\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1).$$

Proof. Let $\boldsymbol{u}$ be the solution of (5.7). By Lemma 5.2, we have

$$\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\setminus\bar{B}_1)} \le C\sigma_0 \|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)}.$$

Setting $O = B(3R_0)\setminus\bar{B}_1$ and using (5.7a), we deduce that

$$\begin{aligned}\|\mathcal{G}(\boldsymbol{\lambda})\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)} &= \|\gamma_t(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u})\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)} \le C\|\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},O)} \\ &\le C\left\{\|\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^2(O)} + \left\|k^2\mathbb{A}^{-1}\boldsymbol{u}\right\|_{\boldsymbol{L}^2(O)}\right\} \\ &\le C\sigma_0\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\setminus\bar{B}_1)} \le C\sigma_0^2\|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)}.\end{aligned}$$

The proof is complete. □

6. The weak formulation inf-sup condition for the sesquilinear form. We propose an equivalent weak formulation of (5.3): Find $\tilde{\boldsymbol{E}} \in \boldsymbol{H}(\mathbf{curl},\Omega_1)$ such that $\gamma_t\tilde{\boldsymbol{E}} = \boldsymbol{g}$ on Γ_D and

$$a(\tilde{\boldsymbol{E}},\boldsymbol{v}) = 0 \qquad \forall\,\boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1), \tag{6.1}$$

where the sesquilinear form a: $\boldsymbol{H}(\mathbf{curl},\Omega_1)\times\boldsymbol{H}(\mathbf{curl},\Omega_1)\to\mathbb{C}$ is defined by

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_{\Omega_1}\left(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\bar{\boldsymbol{v}} - k^2\mathbb{A}^{-1}\boldsymbol{u}\cdot\bar{\boldsymbol{v}}\right) + \langle\mathcal{G}(\gamma_t\boldsymbol{u}),\gamma_T\boldsymbol{v}\rangle_{\Gamma_1}.$$

By Theorem 5.1, (6.1) attains a unique solution for any $\boldsymbol{g} \in \boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)$. The purpose of this section is to prove the inf-sup condition for the sesquilinear form a. The inf-sup condition plays the key role in both the well-posedness of the PML problem and the exponential convergence of the approximate solution. First we introduce the stretched gradient, curl, divergence, and Laplace operators as follows [13]

$$\begin{aligned}&\tilde{\nabla}v := \mathbb{B}^{-\top}\nabla v, && \tilde{\nabla}\times\boldsymbol{u} := J^{-1}\mathbb{B}\,\mathbf{curl}(\mathbb{B}^{\top}\boldsymbol{u}), \\ &\tilde{\nabla}\cdot\boldsymbol{u} := J^{-1}\operatorname{div}(J\mathbb{B}^{-1}\boldsymbol{u}), && \tilde{\Delta}v := J^{-1}\operatorname{div}(J\mathbb{B}^{-1}\mathbb{B}^{-\top}\nabla v).\end{aligned} \tag{6.2}$$

LEMMA 6.1. *Let $\boldsymbol{f} \in \boldsymbol{L}^2(\mathbb{R}^3)$ and define*

$$\boldsymbol{u}(\boldsymbol{x}) = \int_{\mathbb{R}^3}\mathbb{H}(k;\tilde{\boldsymbol{y}},\tilde{\boldsymbol{x}})\boldsymbol{f}(\boldsymbol{y})J(\boldsymbol{y})\mathrm{d}\boldsymbol{y}. \tag{6.3}$$

Then $\boldsymbol{u}$ satisfies the stretched Helmholtz equation

$$\tilde{\Delta}\boldsymbol{u} + k_{\pm}^2\boldsymbol{u} = -\boldsymbol{f} \qquad \text{in } \mathbb{R}^3_{\pm}. \tag{6.4}$$

Furthermore, there exists a constant $C > 0$ depending only on k and R_0 such that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\mathbb{R}^3)} \le C\sigma_0^4\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\mathbb{R}^3)}. \tag{6.5}$$

Proof. By (2.8)–(2.9) and similar arguments as in the proof of [28, Theorem 2.8], we have the stretched Helmholtz equation

$$\tilde{\Delta}_{\boldsymbol{x}}\mathbb{H}(k;\tilde{\boldsymbol{y}},\tilde{\boldsymbol{x}}) + k_{\pm}^2\mathbb{H}(k;\tilde{\boldsymbol{y}},\tilde{\boldsymbol{x}}) = -J^{-1}(\boldsymbol{x})\delta_{\boldsymbol{y}}(\boldsymbol{x})\mathbb{I} \qquad \text{in } \mathbb{R}^3_{\pm}. \tag{6.6}$$

Combining (6.3) and (6.6) yields (6.4). It is left to prove (6.5).

From [30, Section 12.4], we know that $\mathbb{H}$ is continuous across the interface Σ. Then (6.3) shows that $\boldsymbol{u}$ is also continuous across Σ. From (6.2), we find that

$$\nabla \boldsymbol{u}(\boldsymbol{x}) = \mathbb{B}^\top(\boldsymbol{x}) \int_{\mathbb{R}^3} \nabla_{\tilde{\boldsymbol{x}}} \mathbb{H}(k; \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{x}}) \boldsymbol{f}(\boldsymbol{y}) J(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}.$$

From Lemma 4.5 and Lemma 4.6, the stretched Hertz tensor satisfies

$$|\mathbb{H}(k; \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{x}})| + |\nabla_{\tilde{\boldsymbol{x}}} \mathbb{H}(k; \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{x}})| \le C e^{-\frac{1}{2} k_+ \sigma_0 |\boldsymbol{x}-\boldsymbol{y}|} \qquad \text{for } |\boldsymbol{x}-\boldsymbol{y}| \ge 2R_0.$$

From Lemma 2.1 and Lemma 4.5, we know that

$$|\mathbb{H}(k; \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{x}})| + |\nabla_{\tilde{\boldsymbol{x}}} \mathbb{H}(k; \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{x}})| \le C \left(1 + |\boldsymbol{x}-\boldsymbol{y}|^{-2}\right) \qquad \text{for } |\boldsymbol{x}-\boldsymbol{y}| < 2R_0,$$

We conclude that

$$|\mathbb{H}(k; \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{x}})| + |\nabla_{\tilde{\boldsymbol{x}}} \mathbb{H}(k; \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{x}})| \le C \left(1 + |\boldsymbol{x}-\boldsymbol{y}|^{-2}\right) e^{-\frac{1}{2} k_+ \sigma_0 |\boldsymbol{x}-\boldsymbol{y}|} \qquad \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3_\pm.$$

Write $w(\boldsymbol{x}, \boldsymbol{y}) = (1 + |\boldsymbol{x}-\boldsymbol{y}|^{-2}) e^{-\frac{1}{4} k_+ \sigma_0 |\boldsymbol{x}-\boldsymbol{y}|}$ for convenience. Then

$$\begin{aligned} \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\mathbb{R}^3)} &\le C\sigma_0^8 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}^3} w(\boldsymbol{x}, \boldsymbol{y}) \left|\boldsymbol{f}(\boldsymbol{y})\right|^2 \mathrm{d}\boldsymbol{y}\right] \left[\int_{\mathbb{R}^3} w(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d}\boldsymbol{y}\right] \mathrm{d}\boldsymbol{x} \\ &\le C\sigma_0^8 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} w(\boldsymbol{x}, \boldsymbol{y}) \left|\boldsymbol{f}(\boldsymbol{y})\right|^2 \mathrm{d}\boldsymbol{y} \mathrm{d}\boldsymbol{x} \le C\sigma_0^8 \left\|\boldsymbol{f}\right\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}. \end{aligned}$$

The proof is complete. ◻

LEMMA 6.2. *For any $\boldsymbol{f} \in \boldsymbol{L}^2(\mathbb{R}^3)$ satisfying $\tilde{\nabla} \cdot \boldsymbol{f} = 0$, there exists a function $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)$ satisfying the modified Maxwell's equation*

$$\tilde{\nabla} \times \tilde{\nabla} \times \boldsymbol{u} - k_\pm^2 \boldsymbol{u} = \boldsymbol{f} \quad \text{in } \mathbb{R}^3_\pm, \tag{6.7a}$$

$$\left[(\tilde{\nabla} \times \boldsymbol{u}) \times \boldsymbol{n}\right] = \left[\boldsymbol{u} \times \boldsymbol{n}\right] = 0 \quad \text{on } \Sigma. \tag{6.7b}$$

Furthermore, there exists a constant $C > 0$ depending only on k and R_0 such that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)} \le C\sigma_0^6 \left\|\boldsymbol{f}\right\|_{\boldsymbol{L}^2(\mathbb{R}^3)}.$$

Proof. Define $\boldsymbol{w} = \int_{\mathbb{R}^3} \mathbb{H}(k; \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{x}}) \boldsymbol{f}(\boldsymbol{y}) J(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}$. Since $\mathbb{H}(k; \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{x}})$ is continuous across Σ with respect to $\boldsymbol{x}$, $\boldsymbol{w}$ is also continuous across Σ. From Lemma 6.1, we have

$$\tilde{\Delta} \boldsymbol{w} + k_\pm^2 \boldsymbol{w} = -\boldsymbol{f} \quad \text{in } \mathbb{R}^3_\pm, \qquad \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\mathbb{R}^3)} \le C\sigma_0^4 \left\|\boldsymbol{f}\right\|_{\boldsymbol{L}^2(\mathbb{R}^3)}. \tag{6.8}$$

Consider the weak formulation: Find $\psi \in H^1(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} J \tilde{\nabla} \psi \cdot \tilde{\nabla} \bar{\varphi} = -\int_{\mathbb{R}^3} J \boldsymbol{w} \cdot \tilde{\nabla} \bar{\varphi} \qquad \forall\, \varphi \in H^1(\mathbb{R}^3). \tag{6.9}$$

By (5.11) and (5.13), we have

$$\mathrm{Re}(J \tilde{\nabla} \varphi \cdot \tilde{\nabla} \bar{\varphi}) = \mathrm{Re}\left(\mathbb{A}^{-1} \nabla \varphi \cdot \nabla \bar{\varphi}\right) \ge \left|\nabla \varphi\right|^2 / 4.$$

By the Lax-Milgram lemma, (6.9) has a unique solution which satisfies

$$\tilde{\nabla} \cdot (\tilde{\nabla}\psi + \boldsymbol{w}) = 0, \qquad \|\nabla\psi\|_{\boldsymbol{L}^2(\mathbb{R}^3)} \le C\sigma_0^2 \|\boldsymbol{w}\|_{L^2(\mathbb{R}^3)}. \tag{6.10}$$

Define $\phi := k^{-2}\tilde{\nabla} \cdot \boldsymbol{w} \in \boldsymbol{L}^2(\mathbb{R}^3)$. Then we deduce that

$$\tilde{\Delta}\phi = k_\pm^{-2}\tilde{\nabla} \cdot (\tilde{\Delta}\boldsymbol{w}) = -k_\pm^{-2}\tilde{\nabla} \cdot (\boldsymbol{f} + k_\pm^2 \boldsymbol{w}) = -\tilde{\nabla} \cdot \boldsymbol{w} = -k_\pm^2 \phi \quad \text{in } \mathbb{R}^3_\pm. \tag{6.11}$$

Combining (6.10) and (6.11) shows that $\tilde{\Delta}\psi = \tilde{\Delta}\phi$ in $\mathbb{R}^3_\pm$.

From [30, Section 12.4], $k^{-2}(\boldsymbol{x})\nabla_{\boldsymbol{x}} \cdot \mathbb{H}(k;\boldsymbol{y},\boldsymbol{x})$ is continuous across Σ with respect to $\boldsymbol{x}$. Since the complex stretching is C^2-smooth and k is constant in horizontal directions, $k^{-2}(\boldsymbol{x})\tilde{\nabla}_{\boldsymbol{x}} \cdot \mathbb{H}(k;\tilde{\boldsymbol{y}},\tilde{\boldsymbol{x}})$ is also continuous across Σ. Then

$$\phi(\boldsymbol{x}) = \int_{\mathbb{R}^3} \left[k^{-2}(\boldsymbol{x})\tilde{\nabla}_{\boldsymbol{x}} \cdot \mathbb{H}(k;\tilde{\boldsymbol{y}},\tilde{\boldsymbol{x}})\right] \boldsymbol{f}(\boldsymbol{y})J(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$$

is continuous across Σ. Since $\psi, \phi \in \boldsymbol{L}^2(\mathbb{R}^3)$, $\tilde{\nabla}(\psi - \phi) \in H^1(\mathbb{R}^3)'$ and satisfies in a distributional sense

$$\int_{\mathbb{R}^3} J\tilde{\nabla}(\psi-\phi) \cdot \tilde{\nabla}\bar{v} = \int_{\mathbb{R}^3} J\tilde{\Delta}(\psi-\phi)\,\bar{v} = 0 \qquad \forall\, v \in C_0^\infty(\mathbb{R}^3).$$

This shows $\tilde{\nabla}\phi = \tilde{\nabla}\psi \in \boldsymbol{L}^2(\mathbb{R}^3)$. By $\psi, \phi \in L^2(\mathbb{R}^3)$, we know that $\psi = \phi$ and satisfies (6.11). Define $\boldsymbol{u} = \boldsymbol{w} + \tilde{\nabla}\psi$. Then (6.10) shows $\tilde{\nabla} \cdot \boldsymbol{u} = 0$ in $\mathbb{R}^3$. By (6.11) and the well-known identity $-\tilde{\Delta} = \tilde{\nabla} \times \tilde{\nabla} \times -\tilde{\nabla}\tilde{\nabla}\cdot$, we deduce that

$$\tilde{\nabla} \times \tilde{\nabla} \times \boldsymbol{u} = -\tilde{\Delta}\boldsymbol{w} - \tilde{\nabla}\tilde{\Delta}\psi = \boldsymbol{f} + k_\pm^2\boldsymbol{w} + k_\pm^2\tilde{\nabla}\psi = \boldsymbol{f} + k_\pm^2\boldsymbol{u} \qquad \text{in } \mathbb{R}^3_\pm.$$

This is (6.7a). The stability estimate for $\boldsymbol{u}$ follows from (6.8) and (6.10).

For the continuities in (6.7b), we recall (5.2) and get

$$\left[\tilde{\nabla} \times \mathbb{H}(k;\tilde{\boldsymbol{y}},\boldsymbol{F}(\cdot)) \times \boldsymbol{n}\right] = \left[\tilde{\nabla} \times \mathbb{G}(k;\tilde{\boldsymbol{y}},\boldsymbol{F}(\cdot)) \times \boldsymbol{n}\right] = \left[\mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}}(k;\tilde{\boldsymbol{y}},\cdot) \times \boldsymbol{n}\right] = 0,$$

where $\tilde{\mathbb{G}}(k;\tilde{\boldsymbol{y}},\cdot) = \mathbb{B}^\top\mathbb{G}(k;\tilde{\boldsymbol{y}},\boldsymbol{F}(\cdot))$. This shows that $\left[(\tilde{\nabla}\times\boldsymbol{u})\times\boldsymbol{n}\right] = \left[(\tilde{\nabla}\times\boldsymbol{w})\times\boldsymbol{n}\right] = 0$ on Σ. Moreover, since both $\boldsymbol{w}$ and ψ are continuous across Σ, we conclude that

$$[\boldsymbol{u} \times \boldsymbol{n}] = [\boldsymbol{w} \times \boldsymbol{n}] + [\nabla\psi \times \mathbb{B}^{-1}\boldsymbol{n}] = (\alpha + \rho\alpha')^{-1}[\nabla\psi \times \boldsymbol{n}] = 0 \quad \text{on } \Sigma.$$

The proof is complete. ◻

LEMMA 6.3. *For any $\boldsymbol{f} \in \boldsymbol{L}^2(\mathbb{R}^3)$, there exists a function $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)$ satisfying the modified Maxwell's equation*

$$\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}) - k_\pm^2\mathbb{A}^{-1}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \mathbb{R}^3_\pm, \tag{6.12a}$$

$$\left[\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n}\right] = \left[\boldsymbol{u} \times \boldsymbol{n}\right] = 0 \quad \text{on } \Sigma. \tag{6.12b}$$

Furthermore, there exists a constant $C > 0$ depending only on k and R_0 such that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3)} \le C\sigma_0^6 \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\mathbb{R}^3)}. \tag{6.13}$$

Proof. First we consider the weak formulation: Find $\psi \in H^1(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} k^2 J\tilde{\nabla}\psi \cdot \tilde{\nabla}\bar{v} = -\int_{\mathbb{R}^3} \mathbb{B}\boldsymbol{f} \cdot \tilde{\nabla}\bar{v} \qquad \forall\, v \in H^1(\mathbb{R}^3).$$

Similar to (6.9), the problem has a unique solution which satisfies

$$\tilde{\nabla} \cdot \boldsymbol{f}_1 = 0, \qquad \left\| \tilde{\nabla}\psi \right\|_{\boldsymbol{L}^2(\mathbb{R}^3)} \le C \left\| \boldsymbol{f} \right\|_{L^2(\mathbb{R}^3)}, \tag{6.14}$$

where $\boldsymbol{f}_1 = J^{-1}\mathbb{B}\boldsymbol{f} + k^2\tilde{\nabla}\psi$. By Lemma 6.2, there is a $\boldsymbol{u}_1 \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)$ such that

$$\begin{aligned} \tilde{\nabla} \times \tilde{\nabla} \times \boldsymbol{u}_1 - k_\pm^2 \boldsymbol{u}_1 = \boldsymbol{f}_1 \quad &\text{in } \mathbb{R}^3_\pm, \\ \big[(\tilde{\nabla} \times \boldsymbol{u}_1) \times \boldsymbol{n}\big] = \big[\boldsymbol{u}_1 \times \boldsymbol{n}\big] = 0 \quad &\text{on } \Sigma. \end{aligned}$$

There exists a constant $C > 0$ depending only on k and R_0 such that

$$\|\boldsymbol{u}_1\|_{\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3)} \le C\sigma_0^6 \|\boldsymbol{f}_1\|_{\boldsymbol{L}^2(\mathbb{R}^3)} \le C\sigma_0^6 \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\mathbb{R}^3)}. \tag{6.16}$$

Define $\boldsymbol{u} = \mathbb{B}^\top(\boldsymbol{u}_1 + \tilde{\nabla}\psi)$. Then (6.12a) is deduced as follows

$$\begin{aligned} \mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}) &= J\mathbb{B}^{-1}\tilde{\nabla} \times \tilde{\nabla} \times \boldsymbol{u} = J\mathbb{B}^{-1}\tilde{\nabla} \times \tilde{\nabla} \times \boldsymbol{u}_1 \\ &= J\mathbb{B}^{-1}(\boldsymbol{f}_1 + k_\pm^2\boldsymbol{u}_1) = \boldsymbol{f} + k_\pm^2\mathbb{A}^{-1}\boldsymbol{u} \qquad \text{in } \mathbb{R}^3_\pm. \end{aligned}$$

Furthermore, (6.13) comes from (6.14) and (6.16). Finally, we have

$$\begin{aligned} &[\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n}] = [(\tilde{\nabla} \times \boldsymbol{u}_1) \times (\mathbb{B}^{-1}\boldsymbol{n})] = (\alpha + \rho\alpha')^{-1}[(\tilde{\nabla} \times \boldsymbol{u}_1) \times \boldsymbol{n}] = 0, \\ &[\boldsymbol{u} \times \boldsymbol{n}] = [\mathbb{B}^\top \boldsymbol{u}_1 \times \boldsymbol{n}] + [\nabla\psi \times \boldsymbol{n}] = (\alpha + \rho\alpha')[\boldsymbol{u}_1 \times \boldsymbol{n}] = 0. \end{aligned}$$

This completes the proof. ◻

Lemma 6.4. *Let $\boldsymbol{f} \in \boldsymbol{L}^2(\mathbb{R}^3\backslash\bar{D})$. There is a $\boldsymbol{u} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \mathbb{R}^3\backslash\bar{D})$ satisfying*

$$\begin{aligned} \mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}) - k^2\mathbb{A}^{-1}\boldsymbol{u} = \boldsymbol{f} \quad &\textit{in } \mathbb{R}^3_\pm\backslash\bar{D}, \\ [\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n}] = [\boldsymbol{u} \times \boldsymbol{n}] = 0 \quad &\textit{on } \Sigma. \end{aligned}$$

Proof. First we extend $\boldsymbol{f}$ by zero to the interior of D and denote the extension still by $\boldsymbol{f}$. By Lemma 6.3, there exists a $\boldsymbol{u}_0 \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)$ satisfying

$$\begin{aligned} \mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}_0) - k^2\mathbb{A}^{-1}\boldsymbol{u}_0 = \boldsymbol{f} \quad &\text{in } \mathbb{R}^3_\pm, \\ [\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}_0 \times \boldsymbol{n}] = [\boldsymbol{u}_0 \times \boldsymbol{n}] = 0 \quad &\text{on } \Sigma. \end{aligned}$$

By Theorem 5.1, there exists a unique $\boldsymbol{u}_1 \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3\backslash\bar{D})$ such that

$$\begin{aligned} \mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}_1) - k^2\mathbb{A}^{-1}\boldsymbol{u}_1 = 0 \quad &\text{in } \mathbb{R}^3_\pm\backslash\bar{D}, \\ [\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}_1 \times \boldsymbol{n}] = [\boldsymbol{u}_1 \times \boldsymbol{n}] = 0 \quad &\text{on } \Sigma, \\ \gamma_t\boldsymbol{u}_1 = \gamma_t\boldsymbol{u}_0 \quad &\text{on } \Gamma_D. \end{aligned}$$

Clearly $\boldsymbol{u} = \boldsymbol{u}_0 - \boldsymbol{u}_1 \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3\backslash\bar{D})$ and satisfies the lemma. ◻

Now we present the main result of this section.

Theorem 6.5. *There is a constant $C_{\inf} > 0$ depending only on k, R_0, σ_0, and D such that*

$$\sup_{\boldsymbol{v}\in\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1)} \frac{|a(\boldsymbol{w},\boldsymbol{v})|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)}} \ge C_{\inf} \|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)} \quad \forall\, \boldsymbol{w} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega_1). \tag{6.17}$$

Proof. From Lemma 5.3 we know that a is continous on $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1)$. It defines a linear and continuous operator $\mathcal{A}$: $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1) \to [\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1)]'$ such that

$$\langle \mathcal{A}\boldsymbol{w}, \boldsymbol{v}\rangle = a(\boldsymbol{w},\boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1).$$

By the uniqueness of the solution to (6.1), $\mathcal{A}$ is an one-to-one mapping. It suffices to show that $\mathcal{A}$ is also surjective, that is, for any $l \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1)'$, the weak problem

$$a(\boldsymbol{u},\boldsymbol{v}) = \langle l, \boldsymbol{v}\rangle \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1), \tag{6.18}$$

attains a solution $\boldsymbol{u} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1)$.

Since $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\mathbb{R}^3\backslash\bar{D})$ is embedded into $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1)$, l can also be extended to a functional on $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\mathbb{R}^3\backslash\bar{D})$ as follows

$$\langle l_1, \boldsymbol{v}\rangle = \langle l, \boldsymbol{v}|_{\Omega_1}\rangle \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\mathbb{R}^3\backslash\bar{D}).$$

Then (6.18) can be written as

$$a(\boldsymbol{u},\boldsymbol{v}) = \langle l_1, \boldsymbol{v}\rangle \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\mathbb{R}^3\backslash\bar{D}). \tag{6.19}$$

Remember that $\mathcal{G}(\gamma_t\boldsymbol{u}) = \boldsymbol{n}\times\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}$ where $\boldsymbol{\xi} \in \boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\backslash\bar{B}_1)$ solves

$$\begin{aligned} \mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}) - k^2\mathbb{A}^{-1}\boldsymbol{\xi} &= 0 \quad \text{in } \mathbb{R}^3_\pm \setminus \bar{B}_1,\\ \big[\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n}\big] = \big[\boldsymbol{\xi}\times\boldsymbol{n}\big] &= 0 \quad \text{on } \Sigma,\\ \gamma_t\boldsymbol{\xi} &= \gamma_t\boldsymbol{u} \quad \text{on } \Gamma_1. \end{aligned}$$

Using integration by part, we have, for any $\boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\mathbb{R}^3\backslash\bar{D})$,

$$\begin{aligned} \langle \mathcal{G}(\gamma_t\boldsymbol{u}), \gamma_T\boldsymbol{v}\rangle_{\Gamma_1} &= \int_{\Gamma_1} (\boldsymbol{n}\times\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi})\cdot\gamma_T\boldsymbol{v} = \int_{\Gamma_1} (\boldsymbol{n}\times\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi})\cdot\boldsymbol{v}\\ &= \int_{\mathbb{R}^3\backslash\bar{B}_1} [\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi})\cdot\bar{\boldsymbol{v}} - \mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\bar{\boldsymbol{v}}]\\ &= \int_{\mathbb{R}^3\backslash\bar{B}_1} \left[k^2\mathbb{A}^{-1}\boldsymbol{\xi}\cdot\bar{\boldsymbol{v}} - \mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\bar{\boldsymbol{v}}\right]. \end{aligned} \tag{6.20}$$

Let $\boldsymbol{u}_1 \in \boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\backslash\bar{D})$ be defined such that $\boldsymbol{u}_1 = \boldsymbol{u}$ in Ω_1 and $\boldsymbol{u}_1 = \boldsymbol{\xi}$ in $\mathbb{R}^3\backslash\bar{B}_1$. Substituting (6.20) into (6.19) shows that $\boldsymbol{u}_1$ satisfies

$$\int_{\mathbb{R}^3\backslash\bar{D}} \left(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}_1\cdot\mathbf{curl}\,\bar{\boldsymbol{v}} - k^2\mathbb{A}^{-1}\boldsymbol{u}_1\cdot\bar{\boldsymbol{v}}\right) = \langle l_1,\boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\mathbb{R}^3\backslash\bar{D}). \tag{6.21}$$

Therefore, to prove the existence of $\boldsymbol{u}$, it suffices to prove that (6.21) has a solution.

From (5.11) and (5.13), there exist two constants $C_1 > C_0 > 0$ such that

$$C_0\,\|\boldsymbol{\eta}\|^2 \le \mathrm{Re}\,(\mathbb{A}(\boldsymbol{x})\boldsymbol{\eta}\cdot\bar{\boldsymbol{\eta}}) \le |\mathbb{A}(\boldsymbol{x})\boldsymbol{\eta}\cdot\bar{\boldsymbol{\eta}}| \le C_1\,\|\boldsymbol{\eta}\|^2 \qquad \forall\, \boldsymbol{x},\boldsymbol{\eta} \in \mathbb{R}^3.$$

There exists a unique $\boldsymbol{w} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\mathbb{R}^3\backslash\bar{D})$ which solves

$$\int_{\mathbb{R}^3\backslash\bar{D}} \left(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{w}\cdot\mathbf{curl}\,\bar{\boldsymbol{v}} + k^2\mathbb{A}\boldsymbol{w}\cdot\bar{\boldsymbol{v}}\right) = \langle l_1,\boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\mathbb{R}^3\backslash\bar{D}).$$

Since $\boldsymbol{f} = k^2\left(\mathbb{A}+\mathbb{A}^{-1}\right)\boldsymbol{w} \in \boldsymbol{L}^2(\mathbb{R}^3\backslash\bar{D})$, by Lemma 6.4, there exists a $\boldsymbol{u}_2 \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\mathbb{R}^3\backslash\bar{D})$ such that

$$\int_{\mathbb{R}^3\backslash\bar{D}} \left(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}_2\cdot\mathbf{curl}\,\bar{\boldsymbol{v}} - k^2\mathbb{A}^{-1}\boldsymbol{u}_2\cdot\bar{\boldsymbol{v}}\right) = \int_{\mathbb{R}^3\backslash\bar{D}} \boldsymbol{f}\cdot\bar{\boldsymbol{v}} \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\mathbb{R}^3\backslash\bar{D}).$$

Clearly $\boldsymbol{u}_1 := \boldsymbol{u}_2 + \boldsymbol{w}$ solves (6.21). We conclude the existence of $\boldsymbol{u}$. ∎

7. The PML problem. The purpose of this section is to study the PML approximation to the scattering problem (1.1) or to the exterior problem (5.3). We let $R_2 \geq 2R_0$ and define $B_2 = B(R_2)$, $\Omega_2 = B_2 \backslash \bar{D}$, $\Gamma_2 = \partial B_2$. For convenience, we denote by $\Omega_{\mathrm{PML}} := B_2 \backslash \bar{B}_1$ the wave-absorbing layer with constant medium property. Its thickness is denoted by

$$d = \mathrm{distance}(\Gamma_1, \Gamma_2) = R_2 - R_1.$$

We consider the PML problem with homogeneous boundary condition on the truncation boundary:

$$\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\hat{\boldsymbol{E}}) - k^2\mathbb{A}^{-1}\hat{\boldsymbol{E}} = 0 \quad \text{in } \Omega_2, \tag{7.1a}$$

$$\left[\mathbb{A}\,\mathbf{curl}\,\hat{\boldsymbol{E}} \times \boldsymbol{n}\right] = \left[\hat{\boldsymbol{E}} \times \boldsymbol{n}\right] = 0 \quad \text{on } \Sigma, \tag{7.1b}$$

$$\gamma_t \hat{\boldsymbol{E}} = \boldsymbol{g} \quad \text{on } \Gamma_D, \qquad \gamma_t \hat{\boldsymbol{E}} = 0 \quad \text{on } \Gamma_2. \tag{7.1c}$$

7.1. The PML Dirichlet-to-Neumann map. We first introduce the PML Dirichlet-to-Neumann map $\hat{\mathcal{G}}$: $\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1) \to \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)$ as follows: for any $\boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)$, let $\hat{\mathcal{G}}(\boldsymbol{\lambda}) = \boldsymbol{n} \times \mathbb{A}\,\mathbf{curl}\,\hat{\boldsymbol{u}}$ on Γ_1 where $\hat{\boldsymbol{u}}$ solves the Dirichlet boundary value problem

$$\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\hat{\boldsymbol{u}}) - k_\pm^2\mathbb{A}^{-1}\hat{\boldsymbol{u}} = 0 \quad \text{in } \Omega_{\mathrm{PML}} \cap \mathbb{R}^3_\pm, \tag{7.2a}$$

$$\left[\mathbb{A}\,\mathbf{curl}\,\hat{\boldsymbol{u}} \times \boldsymbol{n}\right] = \left[\hat{\boldsymbol{u}} \times \boldsymbol{n}\right] = 0 \quad \text{on } \Sigma, \tag{7.2b}$$

$$\gamma_t \hat{\boldsymbol{u}} = \boldsymbol{\lambda} \quad \text{on } \Gamma_1, \qquad \gamma_t \hat{\boldsymbol{u}} = 0 \quad \text{on } \Gamma_2. \tag{7.2c}$$

LEMMA 7.1. *There exists a constant $C > 0$ depending only on k, R_0 such that*

$$\left\| \hat{\mathcal{G}}(\boldsymbol{\lambda}) \right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)} \leq C\sigma_0^2 \left\| \boldsymbol{\lambda} \right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)}.$$

Proof. The proof is very similar to that of Lemma 5.3. A weak formulation of (7.2) reads: Find $\hat{\boldsymbol{u}} \in \boldsymbol{H}(\mathbf{curl}, \Omega_{\mathrm{PML}})$ such that $\gamma_t \hat{\boldsymbol{u}} = \boldsymbol{\lambda}$ on Γ_1, $\gamma_t \hat{\boldsymbol{u}} = 0$ on Γ_2, and

$$c(\hat{\boldsymbol{u}}, \boldsymbol{v}) = 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_{\mathrm{PML}}), \tag{7.3}$$

where c: $\boldsymbol{H}(\mathbf{curl}, \Omega_{\mathrm{PML}}) \times \boldsymbol{H}(\mathbf{curl}, \Omega_{\mathrm{PML}}) \to \mathbb{C}$ is defined by

$$c(\boldsymbol{w}, \boldsymbol{v}) = \int_{\Omega_{\mathrm{PML}}} \left(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{w} \cdot \mathbf{curl}\,\bar{\boldsymbol{v}} - k^2\mathbb{A}^{-1}\boldsymbol{w} \cdot \bar{\boldsymbol{v}} \right). \tag{7.4}$$

Note that $|\boldsymbol{x}| \geq 2R_0$ for any $\boldsymbol{x} \in \Omega_{\mathrm{PML}}$. From (5.12) and (5.13), there exists a constant $C > 0$ depending only on k, R_0 such that

$$|c(\boldsymbol{v}, \boldsymbol{v})| \geq C\sigma_0^{-1} \left\| \boldsymbol{v} \right\|^2_{\boldsymbol{H}(\mathbf{curl},\Omega_{\mathrm{PML}})} \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega_{\mathrm{PML}}). \tag{7.5}$$

This shows that (7.2) has a unique solution and

$$\left\| \hat{\boldsymbol{u}} \right\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\mathrm{PML}})} \leq C\sigma_0 \left\| \boldsymbol{\lambda} \right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)}.$$

Using (7.2a), we deduce that

$$\begin{aligned} \left\| \hat{\mathcal{G}}(\boldsymbol{\lambda}) \right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)} &= \left\| \gamma_t(\mathbb{A}\,\mathbf{curl}\,\hat{\boldsymbol{u}}) \right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)} \leq C \left\| \mathbb{A}\,\mathbf{curl}\,\hat{\boldsymbol{u}} \right\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\mathrm{PML}})} \\ &\leq C \left\{ \left\| \mathbb{A}\,\mathbf{curl}\,\hat{\boldsymbol{u}} \right\|_{\boldsymbol{L}^2(\Omega_{\mathrm{PML}})} + \left\| k^2\mathbb{A}^{-1}\hat{\boldsymbol{u}} \right\|_{\boldsymbol{L}^2(\Omega_{\mathrm{PML}})} \right\} \\ &\leq C\sigma_0^2 \left\| \boldsymbol{\lambda} \right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)}. \end{aligned}$$

The proof is complete. $\square$

7.2. Exponential convergence of the PML solution. A weak formulation of (7.1) reads: Find $\hat{\boldsymbol{E}} \in \boldsymbol{H}(\mathbf{curl}, \Omega_1)$ such that $\gamma_t \hat{\boldsymbol{E}} = \boldsymbol{g}$ on Γ_D and

$$\hat{a}(\hat{\boldsymbol{E}}, \boldsymbol{v}) = 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega_1), \tag{7.6}$$

where the sesquilinear form $\hat{a}$: $\boldsymbol{H}(\mathbf{curl}, \Omega_1) \times \boldsymbol{H}(\mathbf{curl}, \Omega_1) \to \mathbb{C}$ is defined by

$$\hat{a}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega_1} \big(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\bar{\boldsymbol{v}} - k^2 \mathbb{A}^{-1} \boldsymbol{u} \cdot \bar{\boldsymbol{v}}\big) + \langle \hat{\mathcal{G}}(\gamma_t \boldsymbol{u}), \gamma_T \boldsymbol{v} \rangle_{\Gamma_1}.$$

LEMMA 7.2. *Assume $R_2 \geq 2R_0$ and $\sigma_0 \geq 4$. There is a constant $C > 0$ depending only on k, R_0 such that, for any $\boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)$,*

$$\left\| \mathcal{G}(\boldsymbol{\lambda}) - \hat{\mathcal{G}}(\boldsymbol{\lambda}) \right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)} \leq C \sigma_0^8 d e^{-\frac{1}{2} k_+ \sigma_0 d} \left\| \boldsymbol{\lambda} \right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)}.$$

Proof. Given $\boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)$, let $\boldsymbol{u}$ be the solution of problem (5.7). By Lemma 5.2, $\boldsymbol{u}$ admits an integral representation, for any $\boldsymbol{x} \in \mathbb{R}^3 \backslash \bar{B}_1$,

$$\boldsymbol{u}(\boldsymbol{x}) = \mathbb{B}(\boldsymbol{x}) \int_{\Gamma_1} \left[\gamma_T \tilde{\mathbb{G}}^\top(k; \tilde{\boldsymbol{x}}, \cdot) \mathcal{G}(\boldsymbol{\lambda}) + \gamma_T \big(\mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}}\big)^\top (k; \tilde{\boldsymbol{x}}, \cdot) \boldsymbol{\lambda} \right].$$

By Lemma 5.3 and the boundedness of γ_T, we find that

$$\begin{aligned} |\boldsymbol{u}(\boldsymbol{x})| &\leq C\sigma_0 \left[\left\| \tilde{\mathbb{G}}(k; \tilde{\boldsymbol{x}}, \cdot) \right\|_{\boldsymbol{H}(\mathbf{curl}, B_1)} + \left\| \mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}}(k; \tilde{\boldsymbol{x}}, \cdot) \right\|_{\boldsymbol{H}(\mathbf{curl}, B_1)} \right] \left\| \boldsymbol{\lambda} \right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)} \\ &\leq C \sigma_0^4 e^{-\frac{1}{2} k_+ \sigma_0 d} \left\| \boldsymbol{\lambda} \right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)} \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^3 \backslash B_2. \end{aligned}$$

Moreover, it is easy to see that

$$\mathbf{curl}\,\boldsymbol{u}(\boldsymbol{x}) = J(\boldsymbol{x}) \mathbb{B}^{-1}(\boldsymbol{x}) \tilde{\nabla} \times \int_{\Gamma_1} \left[\tilde{\mathbb{G}}^\top(k; \tilde{\boldsymbol{x}}, \cdot)\, \mathcal{G}(\boldsymbol{\lambda}) + \big(\mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}}\big)^\top (k; \tilde{\boldsymbol{x}}, \cdot)\, \boldsymbol{\lambda} \right].$$

Similarly, from Theorem 4.7 we have

$$|\mathbf{curl}\,\boldsymbol{u}(\boldsymbol{x})| \leq C \sigma_0^5 e^{-\frac{1}{2} k_+ \sigma_0 d} \left\| \boldsymbol{\lambda} \right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)} \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^3 \backslash B_2.$$

Now from (5.6) and (7.2), we know that $\mathcal{G}(\boldsymbol{\lambda}) - \hat{\mathcal{G}}(\boldsymbol{\lambda}) = \boldsymbol{n} \times \mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}$ where $\boldsymbol{\xi}$ solves the Dirichlet boundary value problem in the layer

$$\begin{aligned} &\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}) - k_\pm^2 \mathbb{A}^{-1} \boldsymbol{\xi} = 0 \quad \text{in } \Omega_{\mathrm{PML}} \cap \mathbb{R}^3_\pm, \\ &\big[\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi} \times \boldsymbol{n}\big] = \big[\boldsymbol{\xi} \times \boldsymbol{n}\big] = 0 \quad \text{on } \Sigma, \\ &\gamma_t \boldsymbol{\xi} = 0 \quad \text{on } \Gamma_1, \quad \gamma_t \boldsymbol{\xi} = \gamma_t \boldsymbol{u} \quad \text{on } \Gamma_2. \end{aligned} \tag{7.7}$$

A weak formulation reads: Find $\boldsymbol{\xi} \in \boldsymbol{H}(\mathbf{curl}, \Omega_{\mathrm{PML}})$ such that $\gamma_t \boldsymbol{\xi} = 0$ on Γ_1, $\gamma_t \boldsymbol{\xi} = \gamma_t \boldsymbol{u}$ on Γ_2, and

$$c(\boldsymbol{\xi}, \boldsymbol{v}) = 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_{\mathrm{PML}}),$$

where c is the sesquilinear form defined in (7.4). By the coercivity in (7.5), the weak problem has a unique solution.

Multiplying both sides of (7.7) and using integration by part, we have

$$c(\boldsymbol{\xi},\boldsymbol{\xi}) = -\int_{\Gamma_2} \gamma_t(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi})\cdot\gamma_T\bar{\boldsymbol{\xi}} = -\int_{\Gamma_2}\gamma_t(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi})\cdot\gamma_T\bar{\boldsymbol{u}}.$$

Define $O_- = B(R_2)\backslash\overline{B(R_2-R_0/2)}$ and $O_+ = B(R_2+R_0/2)\backslash\overline{B(R_2)}$. Clearly O_+ and O_- share the boundary Γ_2. Using (7.5), we deduce that

$$\begin{aligned}\|\boldsymbol{\xi}\|^2_{\boldsymbol{H}(\mathbf{curl},\Omega_{\rm PML})} &\le C\sigma_0\,|c(\boldsymbol{\xi},\boldsymbol{\xi})| \le C\sigma_0\,\|\gamma_t(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi})\|_{\boldsymbol{H}^{-\frac12}(\mathrm{Div},\Gamma_2)}\,\|\gamma_T\boldsymbol{u}\|_{\boldsymbol{H}^{-\frac12}(\mathrm{Curl},\Gamma_2)}\\ &\le C\sigma_0\,\|\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}\|_{\boldsymbol{H}(\mathbf{curl},O_-)}\,\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},O_+)}\,.\end{aligned}$$

Furthermore, (7.7) indicates that

$$\|\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}\|_{\boldsymbol{H}(\mathbf{curl},O_-)} \le \|\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}\|_{\boldsymbol{L}^2(O_-)} + \left\|k^2\mathbb{A}^{-1}\boldsymbol{\xi}\right\|_{\boldsymbol{L}^2(O_-)} \le C\sigma_0\,\|\boldsymbol{\xi}\|_{\boldsymbol{H}(\mathbf{curl},O_-)}\,.$$

Combining the above two estimates gives

$$\|\boldsymbol{\xi}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\rm PML})} \le C\sigma_0^2\,\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},O_+)} \le C\sigma_0^7 d e^{-\frac12 k_+\sigma_0 d}\,\|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)}\,.$$

Together with (7.7), this yields

$$\begin{aligned}\left\|\mathcal{G}(\boldsymbol{\lambda})-\hat{\mathcal{G}}(\boldsymbol{\lambda})\right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)} &\le \|\mathbb{A}\,\mathbf{curl}\,\xi\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\rm PML})} \le C\sigma_0\,\|\boldsymbol{\xi}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\rm PML})}\\ &\le Cd\sigma_0^8 e^{-\frac12 k_+\sigma_0 d}\,\|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)}\,.\end{aligned}$$

The proof is complete. □

THEOREM 7.3. *Assume* $\sigma_0 \ge 4$ *and* R_2 *is large enough. The PML problem* (7.1) *attains a unique solution* $\hat{\boldsymbol{E}} \in \boldsymbol{H}(\mathbf{curl},\Omega_1)$. *There exists a constant* $C > 0$ *depending only on* k, R_0, D *such that*

$$\left\|\hat{\boldsymbol{E}}\right\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)} \le CC_{\inf}^{-1}\,\|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}\,,$$

where $C_{\inf}$ *is the inf-sup constant for the sesquilinear form* a *in* (6.17).

Proof. We first prove the inf-sup condition for $\hat{a}$. For any $\boldsymbol{v},\boldsymbol{w}\in\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1)$, Lemma 7.2 shows that

$$\begin{aligned}|\hat{a}(\boldsymbol{w},\boldsymbol{v})| &\ge |a(\boldsymbol{w},\boldsymbol{v})| - \left\|\mathcal{G}(\gamma_t\boldsymbol{w})-\hat{\mathcal{G}}(\gamma_t\boldsymbol{w})\right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)}\,\|\gamma_T\boldsymbol{v}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Curl},\Gamma_1)}\\ &\ge |a(\boldsymbol{w},\boldsymbol{v})| - C_1\sigma_0^8 d e^{-\frac12 k_+\sigma_0 d}\,\|\gamma_t\boldsymbol{w}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)}\,\|\gamma_T\boldsymbol{v}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Curl},\Gamma_1)}\\ &\ge |a(\boldsymbol{w},\boldsymbol{v})| - C_1\sigma_0^8 d e^{-\frac12 k_+\sigma_0 d}\,\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)}\,\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)}\,,\end{aligned}$$

where $C_1 > 0$ depends only on k and R_0. We fix σ_0 and let d be so large that $C_1\sigma_0^8 d e^{-\frac12 k_+\sigma_0 d} \le C_{\inf}/2$. Then Theorem 6.5 indicates that

$$\begin{aligned}\sup_{\boldsymbol{v}\in\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1)}\frac{|\hat{a}(\boldsymbol{w},\boldsymbol{v})|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)}} &\ge \sup_{\boldsymbol{v}\in\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1)}\frac{|a(\boldsymbol{w},\boldsymbol{v})|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)}} - \frac{C_{\inf}}{2}\,\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)}\\ &\ge \frac{C_{\inf}}{2}\,\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)}\qquad\forall\,\boldsymbol{w}\in\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1).\qquad(7.8)\end{aligned}$$

Therefore, $\hat{a}$ satisfies the inf-sup condition. Then problem (7.6) or problem (7.1) has a unique solution. It is left to show the stability estimate.

Since γ_t: $\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D) \to \boldsymbol{H}(\mathbf{curl},\Omega_1)$ is surjective, there exists an $\boldsymbol{E}_g \in \boldsymbol{H}(\mathbf{curl},\Omega_1)$ such that $\gamma_t \boldsymbol{E}_g = \boldsymbol{g}$ on Γ_D, $\mathrm{supp}(\boldsymbol{E}_g) \subset \Omega_0$, and

$$\|\boldsymbol{E}_g\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)} \le C \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}, \tag{7.9}$$

where the constant C depends only on Γ_D and R_0. Using (7.8) and the weak formulation (7.6), we deduce that

$$\frac{C_{\mathrm{inf}}}{2} \left\|\hat{\boldsymbol{E}} - \boldsymbol{E}_g\right\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)} \le \sup_{\boldsymbol{v}\in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1)} \frac{\left|\hat{a}(\hat{\boldsymbol{E}} - \boldsymbol{E}_g, \boldsymbol{v})\right|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)}} \le C \|\boldsymbol{E}_g\|_{\boldsymbol{H}(\mathbf{curl},\Omega_0)}.$$

Together with (7.9), this proves the stability of $\hat{\boldsymbol{E}}$. ☐

Finally, we arrive at the main theorem of this paper.

THEOREM 7.4. *Assume $\sigma_0 \ge 4$ and R_2 is large enough. Let $\boldsymbol{E}$, $\hat{\boldsymbol{E}}$ be the solutions of problem* (1.1) *and problem* (7.1) *respectively. There exists a constant C independent of σ_0 and d such that*

$$\left\|\boldsymbol{E} - \hat{\boldsymbol{E}}\right\|_{\boldsymbol{H}(\mathbf{curl},\Omega_0)} \le C e^{-\frac{1}{3}k_+\sigma_0 d} \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}.$$

Proof. Write $\tilde{\boldsymbol{E}} = \mathbb{B}\boldsymbol{E} \circ \boldsymbol{F}$. Then by Theorem 5.1, $\tilde{\boldsymbol{E}}$ is the unique solution of problem (6.1). From (6.1) and (7.6), we have

$$a(\tilde{\boldsymbol{E}} - \hat{\boldsymbol{E}}, \boldsymbol{v}) = \langle \hat{\mathcal{G}}(\gamma_t \hat{\boldsymbol{E}}) - \mathcal{G}(\gamma_t \hat{\boldsymbol{E}}), \gamma_T \boldsymbol{v}\rangle_{\Gamma_1} \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1).$$

Remember that the inf-sup constant C_{inf} is independent of d. We can choose R_2 so large that $C_{\mathrm{inf}}^{-2}\sigma_0^8 d e^{-\frac{1}{6}k_+\sigma_0 d} \le 1$. Then by (7.8) and Lemma 7.2, we have

$$\begin{aligned}
\left\|\tilde{\boldsymbol{E}} - \hat{\boldsymbol{E}}\right\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)} &\le 2C_{\mathrm{inf}}^{-1} \sup_{\boldsymbol{v}\in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1)} \frac{\left|\hat{a}(\tilde{\boldsymbol{E}} - \hat{\boldsymbol{E}}, \boldsymbol{v})\right|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)}} \\
&\le C C_{\mathrm{inf}}^{-1} \left\|\hat{\mathcal{G}}(\gamma_t \hat{\boldsymbol{E}}) - \mathcal{G}(\gamma_t \hat{\boldsymbol{E}})\right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)} \\
&\le C C_{\mathrm{inf}}^{-1} \sigma_0^8 d e^{-\frac{1}{2}k_+\sigma_0 d} \left\|\hat{\boldsymbol{E}}\right\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)} \\
&\le C e^{-\frac{1}{3}k_+\sigma_0 d} \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}.
\end{aligned}$$

This completes the proof. ☐

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