

A Balanced Oversampling Finite Element Method for Elliptic Problems with Observational Boundary Data ^{*1)}

Zhiming Chen

LSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China and School of Mathematical Science, University of Chinese Academy of Sciences, Beijing 100049, China. zmchen@lsec.cc.ac.cn

Rui Tuo

Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China. tuorui@amss.ac.cn

Wenlong Zhang

School of Mathematical Science, University of Chinese Academy of Sciences, Beijing 100190, China. zhangwl@lsec.cc.ac.cn.

Abstract

In this paper we propose a finite element method for solving elliptic equations with observational Dirichlet boundary data which may subject to random noises. The method is based on the weak formulation of Lagrangian multiplier and requires balanced oversampling of the measurements of the boundary data to control the random noises. We show the convergence of the random finite element error in expectation and, when the noise is sub-Gaussian, in the Orlicz ψ_2 -norm which implies the probability that the finite element error estimates are violated decays exponentially. Numerical examples are included.

Key words: Observational boundary data, elliptic equation, sub-Gaussian random variable

1. Introduction

In many scientific and engineering applications involving partial differential equations (PDEs), the input data such as sources or boundary conditions are usually given through the measurements which may subject to random noises. For example, in the application of seismic imaging/inversion problems, the measurements are the wave field collected on the surface of the ground which are inevitably polluted by random noises. The collected data will serve as the boundary value for the governing wave equation or Helmholtz equation to find the subsurface structure by some imaging or inversion algorithms (see, e.g., [21], [6]). In this paper, as the first effort to design numerical methods which control both the discretization and random noise error simultaneously for solving PDEs with random input data, we consider the Laplace equation with Dirichlet observational boundary data.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary Γ . In this paper we consider the problem to find $u \in H^1(\Omega)$ such that

$$-\Delta u = f \text{ in } \Omega, \quad u = g_0 \text{ on } \Gamma. \quad (1.1)$$

Here $f \in L^2(\Omega)$ is given but the boundary condition $g_0 \in H^2(\Gamma)$ is generally unknown. Here we assume the boundary condition $g_0 \in H^2(\Gamma)$ since our measurements are pointwise on the boundary, this is also what one usually encounters in practice. We assume we know the measurements

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$g_i = g_0(x_i) + e_i$, $i = 1, 2, \dots, n$, where $\mathbb{T} = \{x_i : 1 \leq i \leq n\}$ is the set of the measurement locations on the boundary Γ and e_i , $i = 1, 2, \dots, n$, are independent identically distributed random variables over some probability space $(\mathfrak{X}, \mathcal{F}, \mathbb{P})$ satisfying $\mathbb{E}[e_i] = 0$ and $\mathbb{E}[e_i^2] = \sigma > 0$. In this paper \mathbb{P} denotes the probability measure and $\mathbb{E}[X]$ denotes the expectation of the random variable X . We remark that for simplicity we only consider the problem of observational Dirichlet boundary data in this paper and the problem with observational sources f or other type of boundary conditions can be studied by the same method. We also remark that the extension of the results in this paper to higher dimension PDEs is an interesting problem worth of future investigations.

The classical problem to find a smooth function from the knowledge of its observation at scattered locations subject to random noises is well studied in the literature [24]. One popular model to tackle this classical problem is to use the thin plate spline model [11, 22] which can be efficiently solved by using finite element methods [1, 7, 17]. The scattered data in our problem (1.1) are defined on the boundary of the domain and a straightforward application of the method developed in [1, 7, 11, 17, 22] would lead to solving a fourth order elliptic equation on the boundary which would be much more expansive than the method proposed in this paper.

If the boundary condition g_0 is smooth and it does not have noise, the standard finite element method to solve (1.1) would be to enforce the Dirichlet boundary condition nodewise on the boundary. If the boundary data contain noise, however, the aforementioned standard finite element method does not work and our crucial finding is that one must enforce the Dirichlet condition in some weak sense. Our method is based on the following weak formulation of Lagrangian multiplier for (1.1) in [2]: Find $(u, \lambda) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$(\nabla u, \nabla v) + \langle \lambda, v \rangle = (f, v), \quad \forall v \in H^1(\Omega), \quad (1.2)$$

$$\langle \mu, u \rangle = \langle \mu, g_0 \rangle, \quad \forall \mu \in H^{-1/2}(\Gamma), \quad (1.3)$$

where (\cdot, \cdot) is the duality pairing between $H^1(\Omega)$ and $H^1(\Omega)'$ which is an extension of the inner product of $L^2(\Omega)$ and $\langle \cdot, \cdot \rangle$ is the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ which is an extension of the inner product of $L^2(\Gamma)$. We remark that while the method of Lagrangian multiplier is one of the standard ways in enforcing Dirichlet boundary condition on smooth domains, it is essential here for solving the problem with Dirichlet observational boundary data even when the domain Ω is a polygon. One may apply the techniques developed in this paper to other weak formulations such as mixed finite element method or those in [18] to deal with observational Dirichlet boundary data.

Let Ω_h be a polygonal domain which approximates the domain Ω . Let $V_h \subset H^1(\Omega_h)$ and $Q_h \subset L^2(\Gamma)$ be the finite element spaces for approximating the field variable and the Lagrangian multiplier. Our finite element method is defined as follows: Find $(u_h, \lambda_h) \in V_h \times Q_h$ such that

$$\begin{aligned} (\nabla u_h, \nabla v_h)_{\Omega_h} + \langle \lambda_h, v_h \rangle_n &= (I_h f, v_h)_{\Omega_h}, \quad \forall v_h \in V_h, \\ \langle \mu_h, u_h \rangle_n &= \langle \mu_h, g \rangle_n, \quad \forall \mu_h \in Q_h, \end{aligned}$$

where $(\cdot, \cdot)_{\Omega_h}$ is the inner product of $L^2(\Omega_h)$, $\langle \cdot, \cdot \rangle_n$ is some quadrature rule for approximating $\langle \cdot, \cdot \rangle$, and I_h is some finite element interpolation operator (we refer to section 2 for the precise definitions). Our analysis in section 3 shows that

$$\mathbb{E} [\|u - u_h \circ \Phi_h^{-1}\|_{L^2(\Omega)}] \leq Ch^2 |\ln h| \Delta(u, f, g_0) + C |\ln h| (\sigma n^{-1/2}), \quad (1.4)$$

where $\Delta(u, f, g_0) = \|u\|_{H^2(\Omega)} + \|f\|_{H^2(\Omega)} + \|g_0\|_{H^2(\Gamma)}$ and $\Phi_h : \Omega_h \rightarrow \Omega$ is the Lenoir homeomorphism defined in section 3. This error estimate suggests that in order to achieve the optimal

convergence, one should take the number of sampling points satisfying $\sigma n^{-1/2} \leq Ch^2$ to compute the solution over a finite element mesh of the mesh size h . For problems having Neumann or Robin boundary conditions, the same method of the analysis in this paper yields this relation should be changed to $\sigma n^{-1/2} \leq Ch$. This suggests the importance of appropriate balance between the number of measurements and the finite element mesh sizes for solving PDEs with random observational data.

If the random variables $e_i, 1 \leq i \leq n$, are also sub-Gaussian, we prove by resorting to the theory of empirical processes that for any $z > 0$,

$$\mathbb{P} \left(\|u - u_h \circ \Phi_h^{-1}\|_{L^2(\Omega)} \geq \left[h^2 |\ln h| \Delta(u, f, g_0) + |\ln h| (\sigma n^{-1/2}) \right] z \right) \leq 2e^{-Cz^2}.$$

This implies that the probability of the random error $\|u - u_h\|_{L^2(\Omega)}$ violating the error estimate in (1.4) decays exponentially.

To conclude this section we remark that a different perspective of solving PDEs with uncertain input data due to incomplete knowledge or inherent variability in the system has drawn considerable interests in recent years (see e.g. [3, 10, 13, 19] and the references therein). The goal of those studies is to learn about the uncertainties in system outputs of interest, given information about the uncertainties in the system inputs which are modeled as random field. This goal usually leads to the mathematical problem of breaking the curse of dimensionality for solving PDEs having large number of parameters.

The layout of the paper is as follows. In section 2 we introduce our finite element formulation and derive an error estimate based on the Babuška-Brezzi theory. In section 3 we study the random finite element error in terms of the expectation. In section 4 we show the stochastic convergence of our method when the random noise is sub-Gaussian. In section 5 we report some numerical examples to confirm our theoretical analysis.

2. The finite element method

We start by introducing the finite element meshes. Let \mathcal{M}_h be a mesh over Ω consisting of curved triangles. We assume each element $K \in \mathcal{M}_h$ has at most one curved edge and the edge of the element K is curved only when its two vertices all lie on the boundary Γ . For any $K \in \mathcal{M}_h$, we denote \tilde{K} the straight triangle which has the same vertices as K . We set $\Omega_h = \cup_{K \in \mathcal{M}_h} \tilde{K}$ and assume the mesh $\tilde{\mathcal{M}}_h = \{\tilde{K} : K \in \mathcal{M}_h\}$ over Ω_h is shape regular and quasi-uniform:

$$h_{\tilde{K}} \leq C\rho_{\tilde{K}}, \quad \forall K \in \mathcal{M}_h, \quad h_{\tilde{K}} \leq Ch_{\tilde{K}'}, \quad \forall K, K' \in \mathcal{M}_h, \quad (2.1)$$

where $h_{\tilde{K}}$ and $\rho_{\tilde{K}}$ are the diameter of \tilde{K} and the diameter of the biggest circle contained in \tilde{K} . The finite element space for the field variable is then defined as

$$V_h = \{v_h \in C(\bar{\Omega}_h) : v_h|_{\tilde{K}} \in P_1(\tilde{K}), \forall \tilde{K} \in \tilde{\mathcal{M}}_h\},$$

where $P_1(\tilde{K})$ is the set of the linear polynomials on \tilde{K} . As usual, we denote $h = \max_{\tilde{K} \in \tilde{\mathcal{M}}_h} h_{\tilde{K}}$.

Let $\mathcal{E}_h = \{K \cap \Gamma : K \in \mathcal{M}_h\}$ be the mesh of Γ which is induced from \mathcal{M}_h . We assume that each element $E \in \mathcal{E}_h$ is the image of the reference element $\hat{E} = [0, 1]$ under a smooth mapping F_E . Since the boundary Γ is smooth, the argument in [8, Theorem 4.3.3] implies that if the diameter of the element h_E is sufficiently small,

$$\|\hat{D}F_E\|_{L^\infty(\hat{E})} \leq Ch_E, \quad \|D_T F_E^{-1}\|_{L^\infty(E)} \leq Ch_E^{-1}, \quad \forall E \in \mathcal{E}_h, \quad (2.2)$$

where \hat{D} is the derivative in \hat{E} and D_T is the tangential derivative on Γ . It is then obvious that there are constants C_1, C_2 independent of the mesh \mathcal{M}_h such that $C_1 h \leq h_E \leq C_2 h$, $\forall E \in \mathcal{E}_h$. We use the following finite element space for the Lagrangian multiplier [18]:

$$Q_h = \{\mu_h \in C(\Gamma) : \mu_h|_E = \hat{\mu}_h \circ F_E^{-1} \text{ for some } \hat{\mu}_h \in P_1(\hat{E}), \forall E \in \mathcal{E}_h\}, \quad (2.3)$$

where $P_1(\hat{E})$ is the set of linear polynomials over \hat{E} .

We assume that the measurement locations \mathbb{T} are quasi-uniformly distributed over Γ in the sense that [22] there exists a constant $B > 0$ such that $\frac{s_{\max}}{s_{\min}} \leq B$, where

$$s_{\max} = \sup_{x \in \Gamma} \inf_{1 \leq i \leq n} s(x, x_i), \quad s_{\min} = \inf_{1 \leq i \neq j \leq n} s(x_i, x_j),$$

where $s(x, y)$ is the arc length between $x, y \in \Gamma$. It is easy to see that there exist constants B_1, B_2 such that $B_1 n^{-1} \leq s_{\max} \leq B s_{\min} \leq B_2 n^{-1}$.

We introduce the empirical inner product between the data and any function $v \in C(\Gamma)$ as $\langle g, v \rangle_n = \sum_{i=1}^n \alpha_i g_i v(x_i)$. We also write $\langle u, v \rangle_n = \sum_{i=1}^n \alpha_i u(x_i) v(x_i)$ for any $u, v \in C(\Gamma)$ and the empirical norm $\|u\|_n = (\sum_{i=1}^n \alpha_i u^2(x_i))^{1/2}$ for any $u \in C(\Gamma)$. We remark that the empirical norm is in fact a semi-norm on $C(\Gamma)$. The weights α_i , $i = 1, 2, \dots, n$, are chosen such that $\langle u, v \rangle_n$ is a good quadrature formula for the inner product $\langle u, v \rangle$ that we describe now.

Let $\mathbb{T}_E = \mathbb{T} \cap E$ be the measurement points in $E \in \mathcal{E}_h$. Since the measurement locations are quasi-uniformly distributed, $n_E = \#\mathbb{T}_E \sim n h_E$. We further assume $t_{j,E} = F_E^{-1}(x_j)$, $j = 1, 2, \dots, n_E$, are ordered as $0 = t_{0,E} \leq t_{1,E} < t_{2,E} < \dots < t_{n_E,E} \leq t_{n_E+1,E} = 1$. We remark that the vertices of the element E need not be at the measurement locations. Denote $\Delta t_{j,E} = t_{j,E} - t_{j-1,E}$, $j = 1, 2, \dots, n_E + 1$. We define the following quadrature formula

$$Q_{\mathbb{T}_E}(w) = \sum_{j=1}^{n_E} \omega_{j,E} w(t_{j,E}), \quad \forall w \in C(\hat{E}), \quad (2.4)$$

where $\omega_{1,E} = \Delta t_{1,E} + \frac{1}{2} \Delta t_{2,E}$, $\omega_{j,E} = \frac{1}{2} (\Delta t_{j,E} + \Delta t_{j+1,E})$, $j = 2, \dots, n_E - 1$, $\omega_{n_E,E} = \frac{1}{2} \Delta t_{n_E,E} + \Delta t_{n_E+1,E}$.

Lemma 2.1. *There exists a constant C independent of \mathbb{T}_E such that*

$$\begin{aligned} \left| \int_0^1 w(t) dt - Q_{\mathbb{T}_E}(w) \right| &\leq C \int_0^1 |w''(t)| dt + \frac{1}{2} \Delta t_{1,E} \int_{t_{0,E}}^{t_{1,E}} |w'(t)| dt \\ &+ \frac{1}{2} \Delta t_{n_E+1,E} \int_{t_{n_E,E}}^{t_{n_E+1,E}} |w'(t)| dt, \quad \forall w \in W^{2,1}(\hat{E}). \end{aligned}$$

Proof. We introduce the standard piecewise trapezoid quadrature rule

$$\tilde{Q}_{\mathbb{T}_E}(w) = \sum_{j=1}^{n_E+1} \Delta t_{j,E} \frac{w(t_{j-1,E}) + w(t_{j,E})}{2}, \quad (2.5)$$

which is exact for linear functions. By the Bramble-Hilbert lemma we know that there exists a constant C such that

$$\left| \int_0^1 w(t) dt - \tilde{Q}_{\mathbb{T}_E}(w) \right| \leq C \int_0^1 |w''(t)| dt, \quad \forall w \in W^{2,1}(\hat{E}).$$

Now the lemma follows since

$$\begin{aligned} & Q_{\mathbb{T}_E}(w) - \tilde{Q}_{\mathbb{T}_E}(w) \\ &= \frac{1}{2}\Delta t_{1,E}(w(t_{1,E}) - w(t_{0,E})) + \frac{1}{2}\Delta t_{n_E+1,E}(w(t_{n_E,E}) - w(t_{n_E+1,E})). \end{aligned} \quad (2.6)$$

This completes the proof.

Now for any $v \in C(\Gamma)$ we can define the following quadrature rule which defines the weights α_j , $j = 1, 2, \dots, n$, in the empirical inner product,

$$\begin{aligned} \int_{\Gamma} v ds &= \sum_{E \in \mathcal{E}_h} \int_0^1 v(F_E(t)) |F'_E(t)| dt \\ &\approx \sum_{E \in \mathcal{E}_h} \sum_{j=1}^{n_E} \omega_{j,E} |F'_E(t_{j,E})| v(x_j) \\ &= \sum_{j=1}^n \alpha_j v(x_j), \quad \alpha_j = \sum_{E \in \mathcal{E}_h, x_j \in \mathbb{T}_E} \omega_{j,E} |F'_E(t_{j,E})|. \end{aligned} \quad (2.7)$$

Since $\Delta t_{1,E} \leq C\Delta t_{2,E}$, $\Delta t_{n_E+1,E} \leq C\Delta t_{n_E,E}$, and $\Delta t_{j,E}/\Delta t_{k,E} \leq C$ for any $j, k = 2, \dots, n_E$, because the points in \mathbb{T} are quasi-uniformly distributed, we have $\omega_{j,E} \sim 1/n_E \sim 1/(nh_E)$. This implies by (2.2) there exist constants B_3, B_4 such that

$$B_3 n^{-1} \leq \alpha_j \leq B_4 n^{-1}, \quad j = 1, 2, \dots, n. \quad (2.8)$$

Let $y_j, j = 1, 2, \dots, J$, be the nodes of the mesh \mathcal{M}_h on Γ . For any $v_h \in V_h$, we define $\Pi_h v_h \in Q_h$ such that $\Pi_h v_h(y_j) = v_h(y_j), j = 1, 2, \dots, J$. For any $E \in \mathcal{E}_h$, let \tilde{E} be the segment connecting two vertices of E and denote $F_{\tilde{E}} : \hat{E} \rightarrow \tilde{E}$ the affine mapping from the reference element \hat{E} to \tilde{E} . Then $(\Pi_h v_h)(F_E(t)) = v_h(F_{\tilde{E}}(t)), \forall t \in \tilde{E}$.

Now we are in the position to define the finite element solution for the problem (1.2)-(1.3). Given $f \in H^2(\Omega)$ and the observation g_i at x_i of the boundary value $g_0(x_i), i = 1, 2, \dots, n$, find $(u_h, \lambda_h) \in V_h \times Q_h$ such that

$$(\nabla u_h, \nabla v_h)_{\Omega_h} + \langle \lambda_h, \Pi_h v_h \rangle_n = (I_h f, v_h)_{\Omega_h}, \quad \forall v_h \in V_h, \quad (2.9)$$

$$\langle \mu_h, \Pi_h u_h \rangle_n = \langle \mu_h, g \rangle_n, \quad \forall \mu_h \in Q_h, \quad (2.10)$$

where $(\cdot, \cdot)_{\Omega_h}$ is the inner product of $L^2(\Omega_h)$ and $I_h : C(\bar{\Omega}) \rightarrow V_h$ is the standard Lagrange interpolation operator. The interpolation operator I_h can be replaced by the Clément interpolant [9] if the source f is less regular. We remark that the computation in (2.9)-(2.10) does not involve any geometric representation of the boundary Γ due to the introduction of the quadrature.

Following [16, 18] we introduce the following mesh-dependent Sobolev norms

$$\|v\|_{1/2,h}^2 = \sum_{E \in \mathcal{E}_h} h_E^{-1} \|v\|_{L^2(E)}^2, \quad \|v\|_{-1/2,h}^2 = \sum_{E \in \mathcal{E}_h} h_E \|v\|_{L^2(E)}^2, \quad \forall v \in L^2(\Gamma).$$

We use the following norms for functions $v_h \in V_h, \mu_h \in Q_h$

$$\|v_h\|_{V_h} = \left(\|\nabla v_h\|_{L^2(\Omega_h)}^2 + \|\Pi_h v_h\|_{1/2,h}^2 \right)^{1/2}, \quad \|\mu_h\|_{Q_h} = \|\mu_h\|_{-1/2,h}.$$

We consider now the well-posedness of the discrete problem (2.9)-(2.10) in the framework of Babuška-Brezzi theory. We start from the following simple lemma.

Lemma 2.2. *There exists a constant C such that*

$$|\langle 1, v \rangle - \langle 1, v \rangle_n| \leq C \sum_{E \in \mathcal{E}_h} \int_0^1 (h_E |\hat{v}_E''| + h_E^2 |\hat{v}_E'| + h_E^3 |\hat{v}_E|) dt, \quad \forall v \in W^{2,1}(\Gamma),$$

where $\hat{v}_E(t) = v|_E(F_E(t))$ for any $E \in \mathcal{E}_h$.

Proof. We first note that since Γ is smooth, we have $|F_E''(t)| \leq Ch_E^2$, $|F_E'''(t)| \leq Ch_E^3$ for any $E \in \mathcal{E}_h$. Since

$$|\langle 1, v \rangle - \langle 1, v \rangle_n| \leq \sum_{E \in \mathcal{E}_h} \left| \int_E v ds - Q_{\mathbb{T}_E}(\hat{v}_E(t)|F_E'(t)) \right|,$$

the lemma follows easily from Lemma 2.1 by taking $w = \hat{v}_E(t)|F_E'(t)$ in each element $E \in \mathcal{E}_h$. We omit the details.

Lemma 2.3. *Let $K_h = \{v_h \in V_h : \langle \Pi_h v_h, \mu_h \rangle_n = 0, \quad \forall \mu_h \in Q_h\}$. There exists a constant $\alpha > 0$ independent of h, n such that*

$$(\nabla v_h, \nabla v_h)_{\Omega_h} \geq \alpha \|v_h\|_{V_h}^2, \quad \forall v_h \in K_h.$$

Proof. For simplicity we write $\tilde{v}_h = \Pi_h v_h \in Q_h$ for any $v_h \in V_h$. Then $\langle \tilde{v}_h, \tilde{v}_h \rangle_n = 0$ for any $v_h \in K_h$. By Lemma 2.2 for $v = \tilde{v}_h^2$ we obtain after some simple computations

$$|\langle \tilde{v}_h, \tilde{v}_h \rangle - \langle \tilde{v}_h, \tilde{v}_h \rangle_n| \leq Ch(\|\nabla v_h\|_{L^2(\Omega_h)}^2 + Ch^{1/2}\|\nabla v_h\|_{L^2(\Omega_h)}\|\tilde{v}_h\|_{L^2(\Gamma)}).$$

Thus

$$\begin{aligned} \|\tilde{v}_h\|_{1/2,h}^2 &\leq Ch^{-1}\|\tilde{v}_h\|_{L^2(\Gamma)}^2 = Ch^{-1}|\langle \tilde{v}_h, \tilde{v}_h \rangle - \langle \tilde{v}_h, \tilde{v}_h \rangle_n| \\ &\leq C\|\nabla v_h\|_{L^2(\Omega_h)}^2 + Ch\|\nabla v_h\|_{L^2(\Omega_h)}\|\tilde{v}_h\|_{1/2,h}. \end{aligned}$$

This shows $\|\nabla v_h\|_{L^2(\Omega_h)}^2 \geq C\|\tilde{v}_h\|_{1/2,h}^2$ and completes the proof.

Lemma 2.4. *There exists constants $C_1, C_2 > 0, h_0 > 0$ independent of h, n such that for $h \leq h_0$,*

$$C_1\|\mu_h\|_{L^2(\Gamma)} \leq \|\mu_h\|_n \leq C_2\|\mu_h\|_{L^2(\Gamma)}, \quad \forall \mu_h \in Q_h.$$

Proof. Since $\hat{\mu}_h(t) = \mu_h(F_E(t))$ is linear in \hat{E} for any $E \in \mathcal{E}_h$, we use Lemma 2.2 for $v = \mu_h^2$ to obtain

$$|\langle \mu_h, \mu_h \rangle - \langle \mu_h, \mu_h \rangle_n| \leq C \sum_{E \in \mathcal{E}_h} \int_{\hat{E}} h_E |\hat{\mu}_h|^2 dt \leq C\|\mu_h\|_{L^2(\Gamma)}^2.$$

This shows the right inequality. Next by definition we have

$$\langle \mu_h, \mu_h \rangle_n = \sum_{E \in \mathcal{E}_h} Q_{\mathbb{T}_E}(\hat{\mu}_h^2|F_E'). \quad (2.11)$$

From (2.5) and (2.2) we know that for any $E \in \mathcal{E}_h$,

$$\begin{aligned} \tilde{Q}_{\mathbb{T}_E}(\hat{\mu}_h^2|F'_E|) &= \frac{1}{2} \sum_{j=0}^{n_E} \int_{t_j}^{t_{j+1}} (\hat{\mu}_h(t_j)^2|F'_E(t_j)| + \hat{\mu}_h^2(t_{j+1})|F'_E(t_{j+1})|) dt \\ &\geq Ch_E \sum_{j=0}^{n_E} \int_{t_j}^{t_{j+1}} (\hat{\mu}_h(t_j)^2 + \hat{\mu}_h(t_{j+1})^2) dt \\ &\geq Ch_E \sum_{j=0}^{n_E} \int_{t_j}^{t_{j+1}} |\hat{\mu}_h(t)|^2 dt, \end{aligned}$$

where in the last inequality we have used the fact that $\hat{\mu}_h$ is linear in \hat{E} and the Jensen inequality for convex functions. Thus $|\tilde{Q}_{\mathbb{T}_E}(\hat{\mu}_h^2|F'_E|)| \geq C\|\mu_h\|_{L^2(E)}^2, \forall E \in \mathcal{E}_h$. On the other hand, by (2.6) we have

$$|Q_{\mathbb{T}_E}(\hat{\mu}_h^2|F'_E|) - \tilde{Q}_{\mathbb{T}_E}(\hat{\mu}_h^2|F'_E|)| \leq Ch_E \|\mu_h\|_{L^2(E)}^2.$$

Therefore, by (2.11), $\|\mu_h\|_n \geq C\|\mu_h\|_{L^2(\Gamma)}$ for sufficiently small h . This completes the proof.

We have the following inf-sup condition for the empirical inner product.

Lemma 2.5. *There exists a constant $h_0, \beta > 0$ independent of h, n such that for $h \leq h_0$,*

$$\sup_{v_h \in V_h \setminus \{0\}} \frac{\langle \Pi_h v_h, \mu_h \rangle_n}{\|v_h\|_{V_h}} \geq \beta \|\mu_h\|_{Q_h}, \quad \forall \mu_h \in Q_h.$$

Proof. The proof follows an idea in [15] where the inf-sup condition for the bilinear form $\langle v_h, \mu_h \rangle$ is proved. Let $y_j, j = 1, 2, \dots, J$, be the nodes of the mesh \mathcal{M}_h on Γ and denote $\psi_j, j = 1, 2, \dots, J$, the corresponding nodal basis function of V_h .

For any $\mu_h \in Q_h$, we define $v_h(x) = \sum_{j=1}^J \mu_h(y_j) \psi_j(x) \in V_h$. It is easy to check that

$$\|v_h\|_{V_h}^2 \leq C \sum_{j=1}^J |\mu_h(y_j)|^2 \leq Ch^{-1} \|\mu_h\|_{L^2(\Gamma)}^2. \quad (2.12)$$

From the definition of $\Pi_h v_h \in Q_h$ we know that $\Pi_h v_h = \mu_h$ on Γ . Thus by Lemma 2.4,

$$\langle \Pi_h v_h, \mu_h \rangle_n = \|\mu_h\|_n^2 \geq C\|\mu_h\|_{L^2(\Gamma)}^2.$$

This completes the proof by using (2.12).

By Lemma 2.4 we know that for any $v_h \in V_h, \mu_h \in Q_h$

$$|\langle \Pi_h v_h, \mu_h \rangle_n| \leq C\|\Pi_h v_h\|_{L^2(\Gamma)} \|\mu_h\|_{L^2(\Gamma)} \leq C\|v_h\|_{V_h} \|\mu_h\|_{Q_h}.$$

Now by the standard Babuška-Brezzi theory (cf., e.g., [4, Proposition 5.5.4]) we obtain the following theorem.

Theorem 2.1. *There exists a constant $h_0 > 0$ independent of h, n such that for any $h \leq h_0$, the discrete problem (2.9)-(2.10) has a unique solution $(u_h, \lambda_h) \in V_h \times Q_h$. Moreover, for any $(u_I, \lambda_I) \in V_h \times Q_h$, we have*

$$\|u_h - u_I\|_{V_h} + \|\lambda_h - \lambda_I\|_{Q_h} \leq C \sum_{i=1}^3 M_{ih},$$

where the errors M_{1h}, M_{2h}, M_{3h} are defined by

$$M_{1h} = \sup_{v_h \in V_h \setminus \{0\}} \frac{|(\nabla u_I, \nabla v_h)_{\Omega_h} + \langle \lambda_I, \Pi_h v_h \rangle_n - (I_h f, v_h)_{\Omega_h}|}{\|v_h\|_{V_h}},$$

$$M_{2h} = \sup_{\mu_h \in Q_h \setminus \{0\}} \frac{|\langle \mu_h, \Pi_h u_I - g_0 \rangle_n|}{\|\mu_h\|_{Q_h}}, \quad M_{3h} = \sup_{\mu_h \in Q_h \setminus \{0\}} \frac{|\langle \mu_h, e \rangle_n|}{\|\mu_h\|_{Q_h}}.$$

3. Convergence of the finite element method

We will use the Lenoir homeomorphism $\Phi_h : \Omega_h \rightarrow \Omega$ [14]. The mapping Φ_h is defined elementwise: for any $\tilde{K} \in \tilde{\mathcal{M}}_h$, $\Phi_h|_{\tilde{K}} = \Psi_K$ is a C^2 -diffeomorphism from \tilde{K} to K . If no edge of K belongs to $\partial\Omega_h$, $\Psi_K = I$, the identity. If one edge \tilde{E} of \tilde{K} lies on $\partial\Omega_h$ which corresponds to the curved edge E of $K \in \mathcal{M}_h$, Ψ_K maps \tilde{E} onto E and $\Psi_K = I$, the identity, alongs the other two edges of \tilde{K} . We need the following properties of Ψ_K from [14] in the following lemma.

Lemma 3.1. *The following assertions are valid for any $\tilde{K} \in \tilde{\mathcal{M}}_h$ and $K \in \mathcal{M}_h$.*

1° *The mapping $\Psi_K : \tilde{K} \rightarrow K$ satisfies the following estimates*

$$\|D^s(\Psi_K - I)\|_{L^\infty(\tilde{K})} \leq Ch^{2-s}, \quad \forall s \leq 2, \quad \sup_{x \in \tilde{K}} |J(\Psi_K)(x) - 1| \leq Ch,$$

where $J(\Psi_K)$ denotes the modulus of the Jacobi determinant of Ψ_K .

2° *The mapping $\Psi_K^{-1} : K \rightarrow \tilde{K}$ satisfies*

$$\|D^s(\Psi_K^{-1} - I)\|_{L^\infty(K)} \leq Ch^{2-s}, \quad \forall s \leq 2, \quad \sup_{x \in K} |J(\Psi_K^{-1})(x) - 1| \leq Ch.$$

Let $r_h : L^1(\Omega_h) \rightarrow V_h$ be the Clément interpolant [8] which enjoys the following properties

$$\|v - r_h v\|_{H^j(\tilde{K})} \leq Ch^{m-j} |v|_{H^m(\Delta_{\tilde{K}})}, \quad \forall \tilde{K} \in \tilde{\mathcal{M}}_h, 0 \leq j \leq m, m = 1, 2, \quad (3.1)$$

$$\|v - r_h v\|_{H^j(e)} \leq Ch^{m-j-1/2} |v|_{H^m(\Delta_e)}, \quad \forall e \in \tilde{\mathcal{E}}_h, 0 \leq j < m, m = 1, 2, \quad (3.2)$$

where $\tilde{\mathcal{E}}_h$ is the set of all sides of the mesh $\tilde{\mathcal{M}}_h$, and for any set $A \subset \Omega_h$, Δ_A is the union of the elements surrounding A . We remark that (3.1) is proved in [8] and (3.2) is the consequence of (3.1) and the following scaled trace inequality

$$\|v\|_{L^2(e)} \leq Ch^{-1/2} \|v\|_{L^2(\Delta_e)} + Ch^{1/2} \|\nabla v\|_{L^2(\Delta_e)}, \quad \forall v \in H^1(\Omega_h).$$

We will assume in this section the solution $u \in H^2(\Omega)$ and thus $\lambda \in H^{1/2}(\Gamma)$. By the trace theorem, there exists a function $\tilde{\lambda} \in H^1(\Omega)$ such that $\tilde{\lambda} = \lambda$ on Γ and $\|\tilde{\lambda}\|_{H^1(\Omega)} \leq C\|\lambda\|_{H^{1/2}(\Gamma)}$. Now we define the following interpolation operator $R_h : L^2(\Omega) \rightarrow L^2(\Omega)$

$$R_h v = [r_h(v \circ \Phi_h)] \circ \Phi_h^{-1}, \quad \forall v \in L^2(\Omega).$$

We notice that similar interpolation functions are used in [14] where the Clément interpolation operator is replaced by the Lagrangian interpolation operator. The following theorem can be easily proved by using Lemma 3.1 and (3.1)-(3.2).

Lemma 3.2. *For any $v \in H^2(\Omega)$, we have $\|v - R_h v\|_{H^j(\Omega)} \leq Ch^{m-j} \|v\|_{H^m(\Omega)}$, $\|v - R_h v\|_{H^j(\Gamma)} \leq Ch^{m-j-1/2} \|v\|_{H^m(\Omega)}$, $0 \leq j \leq m-1$, $m = 1, 2$.*

For any $v_h \in V_h$, we denote $\check{v}_h = v_h \circ \Phi_h^{-1}$ which is a function defined in Ω . Let $\Omega^* = \cup_{K \in \mathcal{M}_h^*} K$, where \mathcal{M}_h^* is the set of all elements having one curved edge. Obviously, $|\Omega^*| \leq Ch$. By definition $\Phi_h = \Psi_K$ is identity for $K \in \mathcal{M}_h \setminus \mathcal{M}_h^*$, it is easy to check by using Lemma 3.1 that (cf. [14, Lemma 8]) for any $v_h, w_h \in V_h$,

$$|(\nabla v_h, \nabla w_h)_{\Omega_h} - (\nabla \check{v}_h, \nabla \check{w}_h)| \leq Ch \|\check{v}_h\|_{H^1(\Omega^*)} \|\check{w}_h\|_{H^1(\Omega^*)}. \quad (3.3)$$

Now by the Poincaré inequality, it is easy to see that $\|v\|_{L^2(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)} + C \|v\|_{1/2,h}, \forall v \in H^1(\Omega)$. Thus by (3.3)

$$\|\check{v}_h\|_{H^1(\Omega)} \leq \|\nabla \check{v}_h\|_{L^2(\Omega)} + C \|\check{v}_h\|_{1/2,h} \leq C \|v_h\|_{V_h} + Ch^{1/2} \|\check{v}_h\|_{H^1(\Omega)},$$

which implies, for sufficiently small h ,

$$\|\check{v}_h\|_{H^1(\Omega)} \leq C \|v_h\|_{V_h}, \quad \forall v_h \in V_h. \quad (3.4)$$

Lemma 3.3. *Let $(u, \lambda) \in H^2(\Omega) \times H^{1/2}(\Gamma)$ be the solution of (1.2)-(1.3). We have*

$$\|u - u_h \circ \Phi_h^{-1}\|_{H^1(\Omega)} + \|\lambda - \lambda_h\|_{-1/2,h} \leq Ch \|u\|_{H^2(\Omega)} + \sum_{i=1}^3 M_{ih},$$

where $M_{ih}, i = 1, 2, 3$, are defined in Theorem 2.1 with $u_I = r_h(u \circ \Phi_h) \in V_h$ and $\lambda_I = R_h \tilde{\lambda} \in Q_h$.

Proof. We first observe that by Lemma 3.2

$$\|\lambda - \lambda_I\|_{L^2(\Gamma)} \leq Ch^{1/2} \|\tilde{\lambda}\|_{H^1(\Omega)} \leq Ch^{1/2} \|\lambda\|_{H^{1/2}(\Gamma)} \leq Ch^{1/2} \|u\|_{H^2(\Omega)}.$$

Notice that $\check{u}_h = R_h u$, we obtain by Lemma 3.2, (3.4), and Theorem 2.1 that

$$\begin{aligned} & \|u - u_h \circ \Phi_h^{-1}\|_{H^1(\Omega)} + \|\lambda - \lambda_h\|_{-1/2,h} \\ & \leq \|u - R_h u\|_{H^1(\Omega)} + \|\lambda - R_h \tilde{\lambda}\|_{-1/2,h} + C(\|u_h - u_I\|_{V_h} + \|\lambda_h - \lambda_I\|_{Q_h}) \\ & \leq Ch \|u\|_{H^2(\Omega)} + C \sum_{i=1}^3 M_{ih}. \end{aligned}$$

This completes the proof.

Lemma 3.4. *We have $M_{1h} \leq Ch |\ln h|^{1/2} (\|u\|_{H^2(\Omega)} + \|f\|_{H^2(\Omega)})$.*

Proof. We first note that by (1.2) we have

$$(\nabla u, \nabla \check{v}_h) + \langle \lambda, \check{v}_h \rangle = (f, \check{v}_h), \quad \forall v_h \in V_h.$$

Now since $\Pi_h v_h = \check{v}_h$ on Γ , for any $v_h \in V_h$, we have

$$\begin{aligned} & |(\nabla u_I, \nabla v_h)_{\Omega_h} + \langle \lambda_I, \Pi_h v_h \rangle_n - (I_h f, v_h)_{\Omega_h}| \\ & \leq |(f, \check{v}_h) - (I_h f, v_h)_{\Omega_h}| + |(\nabla u_I, \nabla v_h)_{\Omega_h} - (\nabla u, \nabla \check{v}_h)| + |\langle \lambda, \check{v}_h \rangle - \langle \lambda_I, \check{v}_h \rangle_n|. \end{aligned}$$

Since $\Phi_h = \Psi_K$ is identity for $K \in \mathcal{M}_h \setminus \mathcal{M}_h^*$, we have

$$(f, \check{v}_h) - (I_h f, v_h)_{\Omega_h} = \sum_{K \in \mathcal{M}_h^*} \int_{\tilde{K}} ((f \circ \Psi_K) v_h J(\Psi_K) - I_h(f \circ \Psi_K) v_h) dx,$$

which implies by using Lemma 3.1 that

$$\begin{aligned} & |(f, \check{v}_h) - (I_h f, v_h)_{\Omega_h}| \\ & \leq Ch \|f\|_{L^2(\Omega^*)} \|\check{v}_h\|_{L^2(\Omega^*)} + Ch^2 \|f\|_{H^2(\Omega)} \|v_h\|_{L^2(\Omega)}. \end{aligned}$$

Obviously, $\|f\|_{L^2(\Omega^*)} \leq Ch^{1/2} \|f\|_{L^\infty(\Omega)} \leq Ch^{1/2} \|f\|_{H^2(\Omega)}$. Moreover, by the well-known embedding theorem [20]

$$\|v\|_{L^p(\Omega)} \leq Cp^{1/2} \|v\|_{H^1(\Omega)}, \forall v \in H^1(\Omega), \forall p > 2,$$

we have

$$\|v\|_{L^2(\Omega^*)} \leq C |\Omega^*|^{\frac{1}{2} - \frac{1}{p}} p^{1/2} \|v\|_{H^1(\Omega)} \leq Ch^{\frac{1}{2} - \frac{1}{p}} p^{1/2} \|v\|_{H^1(\Omega)}.$$

By taking $p = \ln(h^{-1})$ we obtain then

$$\|v\|_{L^2(\Omega^*)} \leq Ch^{1/2} |\ln h|^{1/2} \|v\|_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega). \quad (3.5)$$

This implies

$$|(f, \check{v}_h) - (I_h f, v_h)_{\Omega_h}| \leq Ch^2 |\ln h|^{1/2} \|f\|_{H^2(\Omega)} \|v_h\|_{V_h}. \quad (3.6)$$

By Lemma 3.2, (3.3) and (3.4) we have

$$\begin{aligned} & |(\nabla u_I, \nabla v_h)_{\Omega_h} - (\nabla u, \nabla \check{v}_h)| \\ & \leq |(\nabla u_I, \nabla v_h)_{\Omega_h} - (\nabla \check{u}_I, \nabla \check{v}_h)| + |(\nabla(u - \check{u}_I), \nabla \check{v}_h)| \\ & \leq Ch \|u\|_{H^2(\Omega)} \|v_h\|_{V_h}. \end{aligned} \quad (3.7)$$

By using Lemma 2.2 one can prove

$$|\langle \check{v}_h, \check{w}_h \rangle_n - \langle \check{v}_h, \check{w}_h \rangle| \leq Ch \|v_h\|_{H^1(\Omega_h)} \|w_h\|_{H^1(\Omega_h)}, \quad \forall v_h, w_h \in V_h. \quad (3.8)$$

Thus

$$\begin{aligned} |\langle \lambda_I, \check{v}_h \rangle_n - \langle \lambda_I, \check{v}_h \rangle| & \leq Ch \|r_h(\tilde{\lambda} \circ \Phi_h)\|_{H^1(\Omega_h)} \|v_h\|_{H^1(\Omega_h)} \\ & \leq Ch \|u\|_{H^2(\Omega)} \|v_h\|_{V_h}, \end{aligned}$$

which implies by using Lemma 3.2 that

$$|\langle \lambda, \check{v}_h \rangle - \langle \lambda_I, \check{v}_h \rangle_n| \leq Ch \|u\|_{H^2(\Omega)} \|v_h\|_{V_h}. \quad (3.9)$$

The estimate for M_{1h} now follows from (3.6), (3.7) and (3.9).

Lemma 3.5. *We have $M_{2h} \leq Ch \|u\|_{H^2(\Omega)}$.*

Proof. We first observe that the argument in the proof of Lemma 2.1 implies that

$$\left| \int_0^1 w(t) dt - Q_{\mathbb{T}_E}(w) \right| \leq C \|w'\|_{L^2(\hat{E})}, \quad \forall w \in H^1(\hat{E}).$$

For any $v \in H^1(\Gamma)$, by taking $w(t) = \hat{v}_E(t) |F'_E(t)|$ in each element $E \in \mathcal{E}_h$, where $\hat{v}_E(t) = v|_E(F_E(t))$, we know that

$$|\langle 1, v \rangle - \langle 1, v \rangle_n| \leq C \sum_{E \in \mathcal{E}_h} (h_E \|\hat{v}'_E\|_{L^2(\hat{E})} + h_E^2 \|\hat{v}_E\|_{L^2(\hat{E})}).$$

We use the above inequality for $v = \mu_h \varphi$, where $\varphi = u - \tilde{u}_I$ in Γ , to obtain

$$|\langle \mu_h, \varphi \rangle - \langle \mu_h, \varphi \rangle_n| \leq C \sum_{E \in \mathcal{E}_h} \|\hat{\mu}_h\|_{L^2(\hat{E})} (h_E \|\hat{\varphi}_E\|_{L^2(\hat{E})} + h_E^2 \|\hat{\varphi}'_E\|_{L^2(\hat{E})}),$$

where we have used the fact $\|\hat{\mu}_h\|_{W^{1,\infty}(\hat{E})} \leq C \|\hat{\mu}_h\|_{L^2(\hat{E})}$ since $\hat{\mu}_h \in P_1(\hat{E})$. This implies by using Lemma 3.2 again

$$\begin{aligned} & |\langle \mu_h, u - \tilde{u}_I \rangle - \langle \mu_h, u - \tilde{u}_I \rangle_n| \\ & \leq C \|\mu_h\|_{L^2(\Gamma)} (\|u - R_h u\|_{L^2(\Gamma)} + h \|u - R_h u\|_{H^1(\Gamma)}) \\ & \leq Ch^{3/2} \|u\|_{H^2(\Omega)} \|\mu_h\|_{L^2(\Gamma)}. \end{aligned}$$

This completes the proof.

The following theorem shows the convergence of the finite element solution in the sense of expectation.

Theorem 3.1. *We have*

$$\begin{aligned} & \mathbb{E} \left[\|u - u_h \circ \Phi_h^{-1}\|_{H^1(\Omega)} + h^{1/2} \|\lambda - \lambda_h\|_{L^2(\Gamma)} \right] \\ & \leq Ch \ln h^{1/2} (\|u\|_{H^2(\Omega)} + \|f\|_{H^2(\Omega)}) + Ch^{-1} (\sigma n^{-1/2}). \end{aligned}$$

Proof. By Lemmas 3.3-3.5 we are left to estimate $\mathbb{E}[M_{3h}]$. We first observe that

$$\mathbb{E} \left[\sup_{\mu_h \in Q_h \setminus \{0\}} \frac{|\langle \mu_h, e \rangle_n|^2}{\|\mu_h\|_{Q_h}^2} \right] \leq Ch^{-1} \mathbb{E} \left[\sup_{\mu_h \in Q_h \setminus \{0\}} \frac{|\langle \mu_h, e \rangle_n|^2}{\|\mu_h\|_{L^2(\Gamma)}^2} \right]. \quad (3.10)$$

Let N_h be the dimension of Q_h and $\{\psi_j\}_{j=1}^{N_h}$ be the orthonormal basis of Q_h in the $L^2(\Gamma)$ inner product. Then for any $\mu_h = \sum_{j=1}^{N_h} (\mu_h, \psi_j) \psi_j$, by Cauchy-Schwarz inequality and (2.8)

$$|\langle \mu_h, e \rangle_n|^2 \leq \frac{C}{n^2} \|\mu_h\|_{L^2(\Gamma)}^2 \sum_{j=1}^{N_h} \left(\sum_{i=1}^n e_i \psi_j(x_i) \right)^2.$$

Since $e_i, i = 1, 2, \dots, n$, are independent and identically distributed random variables, we have

$$\mathbb{E} \left[\sup_{\mu_h \in Q_h \setminus \{0\}} \frac{|\langle \mu_h, e \rangle_n|^2}{\|\mu_h\|_{L^2(\Gamma)}^2} \right] \leq C \frac{\sigma^2}{n^2} \sum_{j=1}^{N_h} \sum_{i=1}^n \psi_j(x_i)^2.$$

Since the number of measurement points in E , $\#\mathbb{T}_E \leq Cnh_E$ and $N_h \leq Ch^{-1}$, we obtain by using the inverse estimate that

$$\sum_{j=1}^{N_h} \sum_{i=1}^n \psi_j(x_i)^2 \leq Cnh \sum_{j=1}^{N_h} \sum_{E \in \mathcal{E}_h} \|\psi_j\|_{L^\infty(E)}^2 \leq CN_h n \leq Cnh^{-1}.$$

Therefore

$$\mathbb{E} \left[\sup_{\mu_h \in Q_h \setminus \{0\}} \frac{|\langle \mu_h, e \rangle_n|^2}{\|\mu_h\|_{L^2(\Gamma)}^2} \right] \leq Ch^{-1} (\sigma^2 n^{-1}). \quad (3.11)$$

This, together with (3.10), yields

$$\mathbb{E} \left[\sup_{\mu_h \in Q_h \setminus \{0\}} \frac{|\langle \mu_h, e \rangle_n|^2}{\|\mu_h\|_{Q_h}^2} \right] \leq Ch^{-2}(\sigma^2 n^{-1}).$$

This completes the proof.

The following lemma will be useful in deriving the improved estimate for $\|u - u_h \circ \Phi_h^{-1}\|_{L^2(\Omega)}$.

Lemma 3.6. *We have*

$$\mathbb{E} \left[\sup_{\mu_h \in Q_h \setminus \{0\}} \frac{|\langle e, \mu_h \rangle_n|}{\|\mu_h\|_{H^{1/2}(\Gamma)}} \right] \leq C |\ln h| (\sigma n^{-1/2}).$$

Proof. Let $h_0 = h \leq 1$ and $h_i = h^{(p+1-i)/(p+1)}$ for $1 \leq i \leq p$, where $p \geq 1$ is an integer to be determined later. Obviously $h_i \leq h_{i+1}$, $0 \leq i \leq p$. Let \mathcal{E}_{h_i} be a quasi-uniform mesh over the boundary Γ and Q_{h_i} the finite element space defined in (2.3) over the mesh Q_{h_i} . Let $\{y_{h_i}^k\}_{k=1}^{N_{h_i}}$ be the nodes of the mesh \mathcal{E}_{h_i} , $i = 0, \dots, p$. We introduce the following Clément-type interpolation operator $\pi_{h_i} : L^1(\Gamma) \rightarrow Q_{h_i}$ such that for any $v \in L^1(\Gamma)$,

$$(\pi_{h_i} v)(y_{h_i}^k) = \frac{1}{|S(y_{h_i}^k)|} \int_{S(y_{h_i}^k)} v(x) ds(x), \quad 1 \leq k \leq N_{h_i},$$

where $S(y_{h_i}^k)$ is the union of the two elements sharing the common node $y_{h_i}^k$. It is easy to show by scaling argument that

$$\|v - \pi_{h_i} v\|_{L^2(\Gamma)} \leq h_i^m \|v\|_{H^m(\Gamma)}, \quad \forall v \in H^1(\Gamma), m = 0, 1.$$

Thus by the theory of real interpolation of Sobolev spaces, e.g., [5, Proposition 12.1.5],

$$\|v - \pi_{h_i} v\|_{L^2(\Gamma)} \leq Ch_i^{1/2} \|v\|_{H^{1/2}(\Gamma)}, \quad \forall v \in H^{1/2}(\Gamma). \quad (3.12)$$

Now we introduce the telescope sum

$$\mu_h = \sum_{i=0}^{p-1} (\mu_{h_i} - \mu_{h_{i+1}}) + \mu_{h_p}, \quad \forall \mu_h \in Q_h = Q_{h_0}, \quad (3.13)$$

where $\mu_{h_i} = \pi_{h_i} \mu_h \in Q_{h_i}$, $0 \leq i \leq p+1$. By (3.12)

$$\|\mu_{h_i} - \mu_{h_{i+1}}\|_{L^2(\Gamma)} \leq Ch_{i+1}^{1/2} \|\mu_h\|_{H^{1/2}(\Gamma)}. \quad (3.14)$$

Then the same argument in proving (3.11) implies

$$\begin{aligned} \mathbb{E} \left[\sup_{\mu_h \in Q_h \setminus \{0\}} \frac{|\langle e, \mu_{h_i} - \mu_{h_{i+1}} \rangle_n|}{\|\mu_h\|_{H^{1/2}(\Gamma)}} \right] &\leq Ch_{i+1}^{1/2} h_i^{-1/2} (\sigma n^{-1/2}), \\ \mathbb{E} \left[\sup_{\mu_h \in Q_h \setminus \{0\}} \frac{|\langle e, \mu_{h_p} \rangle_n|}{\|\mu_h\|_{H^{1/2}(\Gamma)}} \right] &\leq Ch_p^{-1/2} (\sigma n^{-1/2}). \end{aligned}$$

By (3.13) we then obtain

$$\mathbb{E} \left[\sup_{\mu_h \in Q_h \setminus \{0\}} \frac{|\langle e, \mu_h \rangle_n|}{\|\mu_h\|_{H^{1/2}(\Gamma)}} \right] \leq C(p+1) h^{-\frac{1}{2(p+1)}} (\sigma n^{-1/2}).$$

This completes the proof by taking the integer p such that $p < |\ln h| \leq p+1$.

Theorem 3.2. *We have*

$$\begin{aligned} & \mathbb{E} [\|u - u_h \circ \Phi_h^{-1}\|_{L^2(\Omega)}] \\ & \leq Ch^2 |\ln h| (\|u\|_{H^2(\Omega)} + \|f\|_{H^2(\Omega)} + \|g_0\|_{H^2(\Gamma)}) + C |\ln h| (\sigma n^{-1/2}). \end{aligned}$$

Proof. Let $(w, p) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ be the solution of the following problem

$$(\nabla w, \nabla v) + \langle p, v \rangle = (u - \check{u}_h, v), \quad \forall v \in H^1(\Omega), \quad (3.15)$$

$$\langle \mu, w \rangle = 0, \quad \forall \mu \in H^{-1/2}(\Gamma). \quad (3.16)$$

By the regularity theory of elliptic equations, $(w, p) \in H^2(\Omega) \times H^1(\Omega)$ and satisfies

$$\|w\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \leq C \|u - \check{u}_h\|_{L^2(\Omega)}. \quad (3.17)$$

Let $w_I = I_h(w \circ \Phi_h) \in V_h$ be the Lagrange interpolation of $w \in H^2(\Omega)$ and $p_I = r_h(p \circ \Phi_h) \in V_h$ be the Clément interpolation of $p \in H^1(\Omega)$. By (3.16) we know that $w = 0$ on Γ and consequently, $w_I = 0$ on Γ_h , $\check{w}_I = w_I \circ \Phi_h^{-1} = 0$ on Γ . Now by using (1.2)-(1.3), (2.9)-(2.10) we obtain

$$\begin{aligned} & \|u - \check{u}_h\|_{L^2(\Omega)}^2 \\ & = (\nabla(w - \check{w}_I), \nabla(u - \check{u}_h)) + \langle p - \check{p}_I, u - \check{u}_h \rangle \\ & + [(f, \check{w}_I) - (I_h f, \check{w}_I)_{\Omega_h}] + [(\nabla w_I, \nabla u_h)_{\Omega_h} - (\nabla \check{w}_I, \nabla \check{u}_h)] \\ & + [\langle \check{p}_I, u - \check{u}_h \rangle - \langle \check{p}_I, u - \check{u}_h \rangle_n] - \langle \check{p}_I, e \rangle_n \\ & := \text{I} + \dots + \text{VI}. \end{aligned} \quad (3.18)$$

By Lemma 3.1 and (3.17) we have

$$|\text{I}| + |\text{II}| \leq Ch \|u - \check{u}_h\|_{H^1(\Omega)} \|u - \check{u}_h\|_{L^2(\Omega)}. \quad (3.19)$$

By (3.6) and (3.17)

$$\begin{aligned} |\text{III}| & \leq Ch^2 |\ln h|^{1/2} \|f\|_{H^2(\Omega)} \|w_I\|_{V_h} \\ & \leq Ch^2 |\ln h|^{1/2} \|f\|_{H^2(\Omega)} \|u - \check{u}_h\|_{L^2(\Omega)}. \end{aligned} \quad (3.20)$$

Since $\Phi_h|_K = I$ for $K \in \mathcal{M}_h \setminus \mathcal{M}_h^*$, by (3.3), Lemma 3.1 and (3.17) we have

$$|\text{IV}| \leq Ch \|\check{w}_I\|_{H^1(\Omega^*)} \|\check{u}_h\|_{H^1(\Omega^*)}$$

Now by using (3.5), Lemma 3.1, and (3.17), we have

$$\begin{aligned} \|\check{u}_h\|_{H^1(\Omega^*)} & \leq \|u - \check{u}_h\|_{H^1(\Omega)} + Ch^{1/2} |\ln h|^{1/2} \|u\|_{H^2(\Omega)}, \\ \|\check{w}_I\|_{H^1(\Omega^*)} & \leq Ch \|u - \check{u}_h\|_{L^2(\Omega)} + Ch^{1/2} |\ln h|^{1/2} \|u - \check{u}_h\|_{L^2(\Omega)}. \end{aligned}$$

This implies

$$|\text{IV}| \leq C [h \|u - \check{u}_h\|_{H^1(\Omega)} + h^2 |\ln h| \|u\|_{H^2(\Omega)}] \|u - \check{u}\|_{L^2(\Omega)}. \quad (3.21)$$

To estimate the term V we first use the triangle inequality

$$|\text{V}| \leq |\langle \check{p}_I, u - \check{u}_I \rangle - \langle \check{p}_I, u - \check{u}_I \rangle_n| + |\langle \check{p}_I, u_I - \check{u}_h \rangle - \langle \check{p}_I, \check{u}_I - \check{u}_h \rangle_n|$$

By using Lemma 2.2 for $v = \check{p}_I(u - \check{u}_I)$ one obtains easily

$$\begin{aligned} |\langle \check{p}_I, u - \check{u}_I \rangle - \langle \check{p}_I, u - \check{u}_I \rangle_n| &\leq Ch^2 \|g_0\|_{H^2(\Gamma)} \|\check{p}_I\|_{L^2(\Gamma)} \\ &\leq Ch^2 \|g_0\|_{H^2(\Gamma)} \|u - \check{u}_h\|_{L^2(\Omega)}, \end{aligned}$$

where we have used the estimate $\|\check{p}_I\|_{L^2(\Gamma)} \leq C\|p\|_{H^1(\Omega)} \leq C\|u - \check{u}_h\|_{L^2(\Omega)}$. By (3.8) and (3.17) we have

$$\begin{aligned} |\langle \check{p}_I, u_I - \check{u}_h \rangle - \langle \check{p}_I, \check{u}_I - \check{u}_h \rangle_n| &\leq Ch \|p_I\|_{H^1(\Omega_h)} \|u_I - u_h\|_{H^1(\Omega_h)} \\ &\leq Ch \|u_I - u_h\|_{H^1(\Omega_h)} \|u - \check{u}_h\|_{L^2(\Omega)}. \end{aligned}$$

Thus

$$|V| \leq Ch^2 (h^{-1} \|u - \check{u}_h\|_{H^1(\Omega)} + \|u\|_{H^2(\Omega)} + \|g_0\|_{H^2(\Gamma)}) \|u - \check{u}_h\|_{L^2(\Omega)}. \quad (3.22)$$

By inserting (3.19)-(3.22) into (3.18) we obtain finally

$$\begin{aligned} \|u - \check{u}_h\|_{L^2(\Omega)} &\leq Ch^2 |\ln h| [\|u\|_{H^2(\Omega)} + \|f\|_{H^2(\Omega)} + \|g_0\|_{H^2(\Gamma)}] \\ &\quad + Ch \|u - \check{u}_h\|_{H^1(\Omega)} + \sup_{\mu_h \in Q_h \setminus \{0\}} \frac{|\langle e, \mu_h \rangle_n|}{\|\mu_h\|_{H^{1/2}(\Gamma)}}. \end{aligned} \quad (3.23)$$

The lemma now follows from Theorem 3.1 and Lemma 3.6.

4. Sub-Gaussian random errors

In this section, we will study the convergence of our finite element method when the random errors added to the boundary data are sub-Gaussian. We will use the theory of empirical processes [22, 23].

Definition 4.1. *A random variable X is called sub-Gaussian with parameter σ if*

$$\mathbb{E}[e^{\lambda(X - \mathbb{E}[X])}] \leq e^{\sigma^2 \lambda^2 / 2}, \quad \forall \lambda \in \mathbb{R}.$$

The following definition on the Orlicz ψ_2 -norm will be used in our analysis.

Definition 4.2. *Let $\psi_2 = e^{x^2} - 1$ and X be a random variable. The ψ_2 norm of X is defined as*

$$\|X\|_{\psi_2} = \inf \left\{ C > 0 : \mathbb{E} \left[\psi_2 \left(\frac{|X|}{C} \right) \right] \leq 1 \right\}.$$

It is known that [23, Lemma 2.2.1] if $\|X\|_{\psi_2} \leq K$, then

$$\mathbb{P}(|X| > z) \leq 2 \exp \left(-\frac{z^2}{K^2} \right), \quad \forall z > 0. \quad (4.1)$$

Inversely, if

$$\mathbb{P}(|X| > z) \leq C \exp \left(-\frac{z^2}{K^2} \right), \quad \forall z > 0, \quad (4.2)$$

then $\|X\|_{\psi_2} \leq \sqrt{1 + CK}$.

Definition 4.3. Let (T, d) be a semi-metric space, a stochastic process $\{X_t : t \in T\}$ is called a sub-Gaussian process respect to the semi-metric d , if

$$\mathbb{P}(|X_t - X_s| > z) \leq 2 \exp\left(-\frac{1}{2} \frac{z^2}{d^2(t, s)}\right), \quad \forall s, t \in T, z > 0.$$

For a semi-metric space (T, d) , the covering number $N(\varepsilon, T, d)$ is the minimum number of ε -balls that covers T . A set is called ε -separated if the distance of any two points in the set is strictly greater than ε . The packing number $D(\varepsilon, T, d)$ is the maximum number of ε -separated points in T . It is easy to check that [23, P.98]

$$N(\varepsilon, T, d) \leq D(\varepsilon, T, d) \leq N\left(\frac{\varepsilon}{2}, T, d\right). \quad (4.3)$$

The following maximum inequality can be found in [23, Section 2.2.1].

Lemma 4.1. If $\{X_t : t \in T\}$ is a separable sub-Gaussian process respect to the semi-metric d , then

$$\left\| \sup_{s, t \in T} |X_t - X_s| \right\|_{\psi_2} \leq K \int_0^{\text{diam } T} \sqrt{D(\varepsilon, T, d)} d\varepsilon.$$

Here $K > 0$ is some constant.

The following lemma provides the estimate of the covering number for finite dimensional subsets [12, Corollary 2.6].

Lemma 4.2. Let G be a finite dimensional subspace of $L^2(D)$ of dimension $N > 0$ and $G_R = \{f \in G : \|f\|_{L^2(D)} \leq R\}$. Then

$$N(\varepsilon, G_R, \|\cdot\|_{L^2(D)}) \leq (1 + 4R/\varepsilon)^N, \quad \forall \varepsilon > 0.$$

Theorem 4.1. We have

$$\begin{aligned} & \left\| \|u - u_h \circ \Phi_h^{-1}\|_{H^1(\Omega)} \right\|_{\psi_2} + h^{1/2} \|\lambda - \lambda_h\|_{L^2(\Gamma)} \left\| \right\|_{\psi_2} \\ & \leq Ch |\ln h|^{1/2} (\|u\|_{H^2(\Omega)} + \|f\|_{H^2(\Omega)}) + Ch^{-1} (\sigma h^{-1/2}). \end{aligned}$$

Proof. By Lemmas 3.3-3.5 we are left to estimate $\|M_{3h}\|_{\psi_2}$. Let $F_h = \{\mu_h \in Q_h : \|\mu_h\|_{L^2(\Gamma)} \leq 1\}$, then

$$\|M_{3h}\|_{\psi_2} \leq h^{-1/2} \left\| \sup_{\mu_h \in F_h} \langle \mu_h, e \rangle_n \right\|_{\psi_2}. \quad (4.4)$$

For any $\mu_h \in F_h$, denote by $E_n(\mu_h) = \langle \mu_h, e \rangle_n$. Then $E_n(\mu_h) - E_n(\mu'_h) = \sum_{i=1}^n c_i e_i$, where $c_i = \alpha_i (\mu_h - \mu'_h)(x_i)$, $i = 1, 2, \dots, n$. For any $\lambda > 0$, since $\alpha_i \leq B_4 n^{-1}$ by (2.8),

$$\mathbb{E} \left[e^{\lambda \sum_{i=1}^n c_i e_i} \right] \leq e^{\frac{1}{2} \lambda^2 \sigma^2 \sum_{i=1}^n c_i^2} \leq e^{\frac{1}{2} B_4 \lambda^2 \sigma^2 n^{-1} \|\mu_h - \mu'_h\|_n^2} = e^{\frac{1}{2} \sigma_1^2 \lambda^2},$$

where $\sigma_1 = B_4 \sigma n^{-1/2} \|\mu_h - \mu'_h\|_n$. Thus $E_n(\mu_h) - E_n(\mu'_h)$ is a sub-Gaussian process with the parameter σ_1 . This implies by (4.1) that

$$\mathbb{P}(|E_n(\mu_h - \mu'_h)| > z) \leq 2e^{-z^2/2\sigma_1^2}, \quad \forall z > 0.$$

Thus $E_n(\mu_h)$ is a sub-Gaussian random process with respect to the semi-distance $d(\mu_h, \mu'_h) = \|\mu_h - \mu'_h\|_n^*$, where $\|\mu_h\|_n^* = B_4 \sigma n^{-1/2} \|\mu_h\|_n$.

By Lemma 2.4 we know that the diameter of F_h in terms of the semi-distance d is bounded by $2C_2 B_4 (\sigma n^{-1})$. By maximal inequality in Lemma 4.1 and (4.3) we have

$$\begin{aligned} \left\| \sup_{\mu_h \in F_h} |\langle \mu_h, e \rangle_n| \right\|_{\psi_2} &\leq K \int_0^{2C_2 B_4 \sigma n^{-1/2}} \sqrt{\log N\left(\frac{\varepsilon}{2}, F_h, \|\cdot\|_n^*\right)} d\varepsilon \\ &= K \int_0^{2C_2 B_4 \sigma n^{-1/2}} \sqrt{\log N\left(\frac{\varepsilon}{2B_4 \sigma n^{-1/2}}, F_h, \|\cdot\|_n\right)} d\varepsilon. \end{aligned}$$

By Lemma 2.4 and Lemma 4.2 we know that for any $\delta > 0$,

$$N(\delta, F_h, \|\cdot\|_n) \leq N(C_1^{-1} \delta, F_h, \|\cdot\|_{L^2(\Gamma)}) \leq (1 + 4C_1/\delta)^{N_h},$$

where N_h is the dimension of Q_h which is bounded by Ch^{-1} . Therefore,

$$\begin{aligned} \left\| \sup_{\mu_h \in F_h} |\langle \mu_h, e \rangle_n| \right\|_{\psi_2} &\leq Ch^{-1/2} \int_0^{2C_2 B_4 \sigma n^{-1/2}} \sqrt{\log\left(1 + \frac{C\sigma n^{-1/2}}{\varepsilon}\right)} d\varepsilon \\ &\leq Ch^{-1/2} (\sigma n^{-1/2}). \end{aligned} \quad (4.5)$$

This shows $\|M_{3h}\|_{\psi_2} \leq Ch^{-1} (\sigma n^{-1/2})$ by (4.4).

By (4.2), Theorem 4.1 implies that the probability of the H^1 -finite element error violating the convergence order $O(h |\ln h|^{1/2} (\|u\|_{H^2(\Omega)} + \|f\|_{H^2(\Omega)} + h^{-1} (\sigma n^{-1/2})))$ decays exponentially.

Theorem 4.2. *We have*

$$\begin{aligned} &\| \|u - u_h \circ \Phi_h^{-1}\|_{L^2(\Omega)} \|_{\psi_2} \\ &\leq Ch^2 |\ln h| (\|u\|_{H^2(\Omega)} + \|f\|_{H^2(\Omega)} + \|g_0\|_{H^2(\Gamma)}) + C |\ln h| (\sigma n^{-1/2}). \end{aligned}$$

Proof. Let $G_h = \{\mu_h \in Q_h : \|\mu_h\|_{H^{1/2}(\Gamma)} \leq 1\}$. By (3.23) we are left to show

$$\left\| \sup_{\mu_h \in G_h} |\langle \mu_h, e \rangle_n| \right\|_{\psi_2} \leq C |\ln h| (\sigma n^{-1/2}). \quad (4.6)$$

We again use the telescope sum in (3.13) and obtain

$$\begin{aligned} &\left\| \sup_{\mu_h \in G_h} |\langle \mu_h, e \rangle_n| \right\|_{\psi_2} \\ &\leq \sum_{i=0}^{p-1} \left\| \sup_{\mu_h \in G_h} |\langle \mu_{h_i} - \mu_{h_{i+1}}, e \rangle_n| \right\|_{\psi_2} + \left\| \sup_{\mu_h \in G_h} |\langle \mu_{h_p}, e \rangle_n| \right\|_{\psi_2}. \end{aligned} \quad (4.7)$$

By the same argument in proving (4.5) and using (3.14) we have

$$\begin{aligned} \left\| \sup_{\mu_h \in G_h} |\langle \mu_{h_i} - \mu_{h_{i+1}}, e \rangle_n| \right\|_{\psi_2} &\leq Ch_{i+1}^{1/2} (h_i^{-1/2} + h_{i+1}^{-1/2}) (\sigma n^{-1/2}), \\ \left\| \sup_{\mu_h \in G_{h_p}} |\langle \mu_h, e \rangle_n| \right\|_{\psi_2} &\leq Ch_p^{-1/2} (\sigma n^{-1/2}). \end{aligned}$$

Inserting the estimates to (4.7) shows (4.6) by taking p such that $|\ln h| < p \leq |\ln h| + 1$.

By (4.2), Theorem 4.2 implies that the probability of the L^2 -finite element error violating the convergence order $O(h^2 |\ln h| (\|u\|_{H^2(\Omega)} + \|f\|_{H^2(\Omega)} + \|g_0\|_{H^2(\Gamma)}) + |\ln h| (\sigma n^{-1/2}))$ decays exponentially.

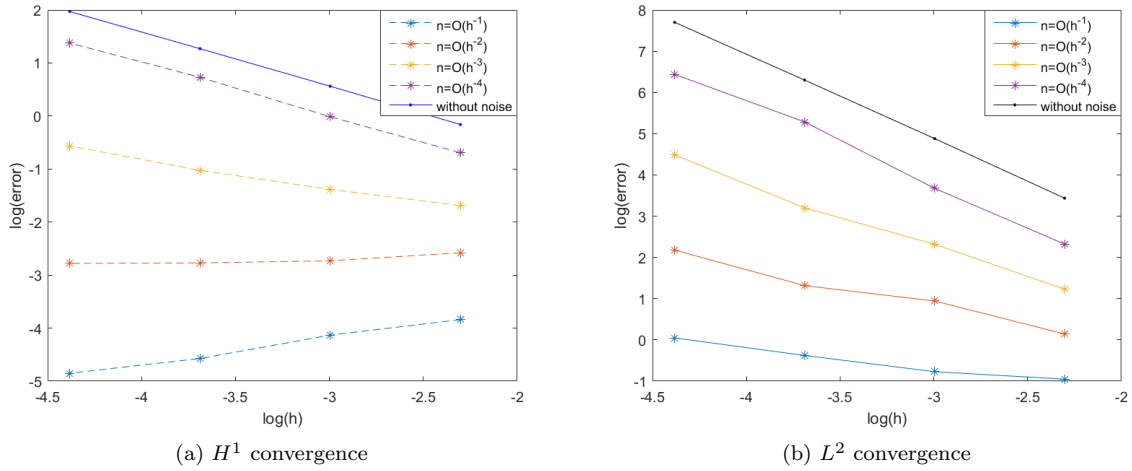


Fig. 5.1. The log-log plot of the convergence rate on the unit square.

5. Numerical examples

In this section, we show several numerical experiments to verify the theoretical analysis in this paper. We start by remarking that the results in this paper extend trivially to the case when Ω is a convex polygonal domain and in fact in this case the construction of the quadrature formula in section 2 and the analyses in sections 3 and 4 are simpler. As stated in the section 1, the weak formulation of Lagrangian multiplier is also crucial for the case of polygonal domains. The analyses in sections 3 and 4 suggest that the optimal convergence rate can be achieved by taking $n = O(h^{-4})$. For the examples below, we take the exact solution $u = \sin(5x + 1) \sin(5y + 1)$.

Example 5.1. We take $\Omega = (0, 1) \times (0, 1)$. We construct the finite element mesh by first dividing the domain into $h^{-1} \times h^{-1}$ uniform rectangles and then connecting the lower left and upper right angle. We set $\{x_i\}_{i=1}^n$ being quasi-uniformly distributed on Γ , and e_i , $i = 1, 2, \dots, n$, being independent normal random variables with variance $\sigma = 2$. We take different $n = h^{-i}$, $i = 1, 2, 3, 4$. Figure 5.1 shows the convergence rate of the error in the H^1 and L^2 norm for each choice of n . Table 5.1 show the convergence rate α in the H^1 norm and the convergence rate β in the L^2 norm.

We observe the numerical results confirm our theoretical analysis. The optimal convergence rate is achieved when choosing $n = h^{-4}$ while the other choices do not achieve optimal convergence. For example, when $n = h^{-2}$, the L^2 error is approximately $O(h)$ and no convergence for the H^1 error.

Example 5.2. We take Ω being a unit circle. The mesh is depicted in Figure 5.2. We set $\{x_i\}_{i=1}^n$ being quasi-uniformly distributed on Γ , and let $e_i = \eta_i + \alpha_i$, $i = 1, 2, \dots, n$, where η_i and α_i are independent normal random variables with variance $\sigma_1 = 1$ and $\sigma_2 = 10e_i$, $i = 1, 2, \dots, n$. We take different $n = h^{-i}$, $i = 1, 2, 3, 4$. Figure 5.3 shows the convergence rate of the error in the H^1 and L^2 norm for each choice of n . Table 5.2 show the convergence rate α in

n	h	H^1 error	α	L^2 error	β
$n = h^{-1}$	0.1000	8.8686		0.3978	
	0.0125	24.3951	0.4866	0.1394	-0.5043
$n = h^{-2}$	0.1000	2.8101		0.1348	
	0.0125	2.7125	-0.0170	0.0167	-1.0037
$n = h^{-3}$	0.1000	0.9637		0.0537	
	0.0125	0.3094	-0.5464	0.0017	-1.6649
$n = h^{-4}$	0.1000	0.6325		0.0380	
	0.0125	0.0838	-0.9721	6.3816e-4	-1.9656

Table 5.1: The convergence rate α in the H^1 norm and β in the L^2 norm on the unit square.

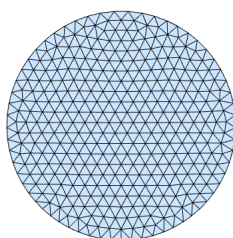


Fig. 5.2. The uniform mesh for the unit circle with mesh size $h = 0.1$.

the H^1 norm and the convergence rate β in the L^2 norm. Here again we observe the numerical results confirm our theoretical analysis.

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n	h	H^1 error	α	L^2 error	β
$n = h^{-1}$	0.1000	46.5037		2.5872	
	0.0125	127.832	0.4863	0.9527	-0.4804
$n = h^{-2}$	0.1000	13.1775		0.8668	
	0.0125	16.1040	0.0964	0.1133	-0.9787
$n = h^{-3}$	0.1000	5.4157		0.2924	
	0.0125	1.7581	-0.5410	0.0113	-1.5665
$n = h^{-4}$	0.1000	2.0009		0.0980	
	0.0125	0.2527	-0.9950	0.0016	-1.9790

Table 5.2: The convergence rate α in the H^1 norm and β in the L^2 norm on the unit circle.

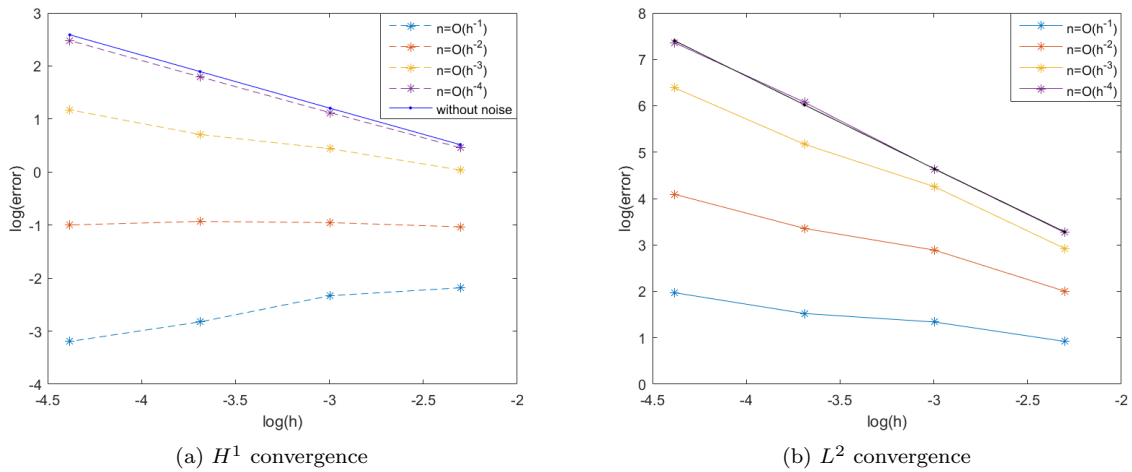


Fig. 5.3. The log-log plot of the convergence rate on the unit circle.

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