A Direct Imaging Method for Half-Space Inverse Elastic Scattering Problems

Zhiming Chen, Shiqi Zhou

LSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China and School of Mathematical Science, University of Chinese Academy of Sciences, Beijing 100049, China.

Abstract. We propose a direct imaging method based on the reverse time migration to reconstruct extended obstacles in the half space with finite aperture elastic scattering data at a fixed frequency. We prove the resolution of the reconstruction method in terms of the aperture and the depth of the obstacle embedded in the half space. The resolution analysis is studied by virtue of the point spread function and implies that the imaginary part of the cross-correlation imaging function always peaks on the upper boundary of the obstacle. Numerical examples are included to illustrate the effectiveness of the method.

1. Introduction

Inverse elastic wave scattering problems have considerable interests in diverse application fields including non-destructive testings, medical imaging, and seismic exploration. The purpose of this paper is to propose and study a direct imaging method to find the shape and location of unknown obstacles embedded in the half-space isotropic and homogeneous elastic medium. We assume the obstacles are far away from the surface of the medium where the sources and receivers are located. The imaging method is based on the idea of reverse time migration (RTM) and does not require the knowledge of physical properties of the obstacles such as penetrable or non-penetrable, and for non-penetrable obstacles, the type of boundary conditions on the boundary of the obstacle.

Let $D \subset \mathbb{R}^2_+ = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 > 0\}$ be a bounded Lipschitz domain with the unit outer normal ν to its boundary Γ_D. We assume the incident wave is emitted by a point source at x_s on the surface $\Gamma_0 = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 = 0\}$, along the polarization direction $q \in \mathbb{R}^2$. Let $\mathbb{N}(x, x_s)$ be the Neumann Green tensor for the half-space elastic scattering problem with free surface condition on Γ_0 (see section 2 below). The measured data is $u_q(x_r, x_s) = u^s_q(x_r, x_s)+\mathbb{N}(x_r, x_s)q, x_r \in \Gamma_0$, where $u^s_q(x, x_s)$ satisfies the following equations

$$\Delta_e u^s_q(x, x_s) + \rho\,\omega^2 u^s_q(x, x_s) = 0 \quad \text{in } \mathbb{R}^2_+\backslash\bar{D}, \tag{1.1}$$

$$u^s_q(x, x_s) = -\mathbb{N}(x, x_s)q \;\text{ on } \Gamma_D, \quad \sigma(u^s_q(x, x_s))e_2 = 0 \;\text{ on } \Gamma_0, \tag{1.2}$$

where $\Delta_e u := (\lambda + \mu)\nabla \mathrm{div} u + \mu\Delta u$ is the linear elastic operator with Lamé constants λ and μ satisfying $\lambda > 0, \mu > 0$, ρ is the density, $\omega > 0$ is the circular frequency, and e_i is the unit vector along the x_i axis, $i = 1, 2$. In the following, we will always assume $\rho = 1$. In the boundary condition (1.2), $\sigma(u) \in \mathbb{C}^{2\times2}$ is the stress tensor, which relates the strain tensor $\varepsilon(u) \in \mathbb{C}^{2\times2}$ through the following constitutive law

$$\sigma(u) = 2\mu\varepsilon(u) + \lambda \mathrm{div} u\, \mathbb{I}, \quad \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T).$$

Here $\mathbb{I} \in \mathbb{R}^{2\times2}$ is the identity matrix.

The equations (1.1)-(1.2) must be complemented by appropriate boundary conditions at infinity to make the problem well-posed. In this paper we shall take the method of limiting absorption principle to define the scattering solution of the problem (1.1)-(1.2). The limiting absorption principle defines the scattering solution of (1.1)-(1.2) as the limit of the solution of the same equations with the complex frequency $\omega(1 + \mathbf{i}\varepsilon)$ when $\varepsilon \to 0^+$. The limiting absorption principle for the half-space elastic scattering problems is proved in [18], [29] for the scatterer with traction free boundary conditions. The results can be easily extended to cover penetrable scatterers or non-penetrable scatterers with Dirichlet or impedance boundary conditions. We also refer to [20], [27] and the references therein for the study of radiation conditions for half-space elastic scattering problems.

The RTM method, whose imaging function is defined as the cross-correction between the incident wave field and the back-propagated wave field using the complex conjugated data, is nowadays widely used in exploration geophysics [15, 7, 6]. In [9, 10, 11, 12], the RTM method for reconstructing extended targets using acoustic, electromagnetic and elastic waves at a fixed frequency in the free space is proposed and studied. The resolution analysis in [9, 10, 11, 12] is achieved without using the small inclusion or geometrical optics assumption previously made in the literature (e.g. [3, 6]).

In the geophysics literature, the RTM method for elastic scattering data usually consists of three steps: 1) back-propagating the received elastic data on the surface to the medium using the full elastic wave equation; 2) decomposing the back-propagated wave field to obtain the p-wave and s-wave components by Helmholtz decomposition; and 3) cross-correlating each decomposed mode component with the corresponding mode component of the incident field to output the imaging profile, see e.g. [28, 31, 17, 14, 26]. In this paper, we study the elastic wave RTM method without using the wave field separation, i.e., the imaging condition is defined as the cross-correlation between the back-propagated wave field with the incident wave field [8]. More precisely, we study the following imaging function (see (4.10) below):

$$\hat{I}_d(z) = \mathrm{Im} \sum_{q=e_1,e_2} \int_{\Gamma_0^d} \int_{\Gamma_0^d} [\mathbb{T}_D(x_s,z)^T q][\mathbb{T}_D(x_r,z)^T \overline{u_q^s(x_r,x_s)}]\, ds(x_r) ds(x_s).$$

Here $\Gamma_0^d = \{x \in \Gamma_0 : x_1 \in (-d,d)\}$, $d > 0$, is the interval where the data are collected and $\mathbb{T}_D(x,z)$ is the traction tensor on Γ_0 associated with the Dirichlet Green tensor (see (2.22) below).

Our resolution analysis, which extends the study in [12] for the half-space acoustic scattering data, indicates that the imaging function always peaks on the illuminating part of the obstacle. The important Rayleigh surface wave is considered in our resolution analysis which shows that the contribution of the surface wave decays exponentially in the imaging function. The elastic wave RTM method based on the wave separation can be studied using the techniques developed in this paper and will be considered in a forthcoming work.

The layout of the paper is as follows. In section 2 we study the Neumann and Dirichlet Green tensors for the half-space elastic scattering problem by using the method of Fourier transform. In section 3 we study the point spread function defined by the RTM method. In section 4 we study the resolution of our RTM method for locating extended targets. In section 5 the extension of our resolution results to other types of obstacles are briefly considered. In section 6 we report extensive numerical results of our RTM method for synthesized scattering data. In section 7 we prove a technical result used in the resolution analysis which is of independent interest.

2. Elastic Green tensors in the half space

In this section we introduce the elastic Green tensors and study their horizontal asymptotic behavior on the surface Γ_0, which will play a crucial role in the resolution analysis for the RTM method to be proposed in this paper. Throughout the paper, we will assume that for $z \in \mathbb{C}\backslash\{0\}$, $z^{1/2}$ is the analytic branch of $\sqrt{z}$ such that $\mathrm{Im}\,(z^{1/2}) \geq 0$. This corresponds to the right half real axis as the branch cut in the complex plane. For $z = z_1 + \mathbf{i}z_2, z_1, z_2 \in \mathbb{R}$, we have

$$z^{1/2} = \mathrm{sgn}(z_2)\sqrt{\frac{|z| + z_1}{2}} + \mathbf{i}\sqrt{\frac{|z| - z_1}{2}}, \quad \forall z \in \mathbb{C}\backslash\bar{\mathbb{R}}_+. \tag{2.1}$$

For z on the upper or lower side of the right half real axis $\mathbb{R}_+ = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0, \mathrm{Im}\, z = 0\}$, we take $z^{1/2}$ accordingly as the limit of $(z + \mathbf{i}\varepsilon)^{1/2}$ or $(z - \mathbf{i}\varepsilon)^{1/2}$ as $\varepsilon \to 0^+$.

We start by introducing the Neumann Green tensor $\mathbb{N}(x, y) \in \mathbb{C}^{2\times 2}, y \in \mathbb{R}^2_+$, which satisfies, for any $q \in \mathbb{R}^2$,

$$\Delta_e[\mathbb{N}(x; y)q] + \omega^2[\mathbb{N}(x, y)q] = -\delta_y(x)q \ \text{ in } \mathbb{R}^2_+, \tag{2.2}$$

$$\sigma(\mathbb{N}(x, y)q)e_2 = 0 \ \text{ on } \Gamma_0, \tag{2.3}$$

where $\delta_y(x)$ is the Dirac source at y. We use the method of Fourier transform to derive a formula of the Neumann Green tensor which is equivalent to that in [20] but is more convenient for our purpose. Let

$$\hat{\mathbb{N}}(\xi, x_2; y_2) = \int_{\mathbb{R}} \mathbb{N}(x_1, x_2; y)e^{-\mathbf{i}(x_1-y_1)\xi}dx_1, \quad \forall \xi \in \mathbb{C}, \tag{2.4}$$

be the spectral Neumann Green tensor. Let $\mathbb{G}(x, y)$ be the fundamental solution tensor of the elastic equation [23] whose Fourier transform is $\hat{\mathbb{G}}(\xi, x_2; y_2) = \hat{\mathbb{G}}_s(\xi, x_2; y_2) + \hat{\mathbb{G}}_p(\xi, x_2; y_2)$ with

$$\hat{\mathbb{G}}_s(\xi, x_2; y_2) = \frac{\mathbf{i}}{2\omega^2}\begin{pmatrix} \mu_s & -\xi\frac{x_2-y_2}{|x_2-y_2|} \\ -\xi\frac{x_2-y_2}{|x_2-y_2|} & \frac{\xi^2}{\mu_s} \end{pmatrix} e^{\mathbf{i}\mu_s|x_2-y_2|}, \tag{2.5}$$

$$\hat{\mathbb{G}}_p(\xi, x_2; y_2) = \frac{\mathbf{i}}{2\omega^2}\begin{pmatrix} \frac{\xi^2}{\mu_p} & \xi\frac{x_2-y_2}{|x_2-y_2|} \\ \xi\frac{x_2-y_2}{|x_2-y_2|} & \mu_p \end{pmatrix} e^{\mathbf{i}\mu_p|x_2-y_2|}. \tag{2.6}$$

Here $\mu_\alpha = (k_\alpha^2 - \xi^2)^{1/2}$ for $\alpha = s, p$, $k_p = \omega/\sqrt{\lambda + 2\mu}, k_s = \omega/\sqrt{\mu}$ are the p and s wave numbers. Using the spectral fundamental solution tensor, one can write the spectral Neumann Green tensor as

$$\hat{\mathbb{N}}(\xi, x_2; y_2) = \hat{\mathbb{G}}(\xi, x_2; y_2) - \hat{\mathbb{G}}(\xi, x_2; -y_2) + \frac{\mathbf{i}}{\omega^2\delta(\xi)}\sum_{\alpha,\beta=p,s}\mathbb{A}_{\alpha\beta}(\xi)e^{\mathbf{i}(\mu_\alpha x_2+\mu_\beta y_2)}, \tag{2.7}$$

where $\varphi(\xi) = k_s^2 - 2\xi^2$, $\delta(\xi) = \varphi(\xi)^2 + 4\xi^2\mu_s\mu_p$, and

$$\mathbb{A}_{ss}(\xi) = \begin{pmatrix} \varphi^2\mu_s & -4\xi^3\mu_s\mu_p \\ -\xi\varphi^2 & 4\xi^4\mu_p \end{pmatrix}, \quad \mathbb{A}_{sp}(\xi) = \begin{pmatrix} 2\xi^2\varphi\mu_s & -2\xi\varphi\mu_s\mu_p \\ -2\xi^3\varphi & 2\xi^2\varphi\mu_p \end{pmatrix},$$

$$\mathbb{A}_{ps}(\xi) = \begin{pmatrix} 2\xi^2\varphi\mu_s & 2\xi^3\varphi \\ 2\xi\varphi\mu_s\mu_p & 2\xi^2\varphi\mu_p \end{pmatrix}, \quad \mathbb{A}_{pp}(\xi) = \begin{pmatrix} 4\xi^4\mu_s & \xi\varphi^2 \\ 4\xi^3\mu_s\mu_p & \varphi^2\mu_p \end{pmatrix}.$$

The desired Neumann Green tensor should be obtained by taking the inverse Fourier transform of the spectral Green tensor $\hat{\mathbb{N}}(\xi, x_2; y_2)$. Unfortunately, one cannot simply take the inverse Fourier transform in the above formula because $\delta(\xi)$ have zeros in the real axis [1, 22].

Lemma 2.1 *The Rayleigh equation $\delta(\xi)=0$ has only two zeros $\pm k_R$, $k_R>k_s$, in the complex plane.*

Proof. For the sake of completeness, we include a proof here. By (2.1), It is clear that $\delta(\xi)$ is analytic outside the branch cuts $C_l=\{\xi=\xi_1+\mathbf{i}\xi_2\in\mathbb{C}:\xi_1\in[-k_s,-k_p],\xi_2=0\}$ and $C_r=\{\xi=\xi_1+\mathbf{i}\xi_2\in\mathbb{C}:\xi_1\in[k_p,k_s],\xi_2=0\}$. On the branch cuts,

$$\delta(\xi)=(k_s^2-2\xi^2)^2+\mathbf{i}\,[4\xi^2(k_s^2-\xi^2)^{1/2}(\xi^2-k_p^2)^{1/2}],\quad \forall\xi\in C_l\cup C_r.$$

Thus, $\delta(\xi)$ has no zeros in $C_l\cup C_r$ and at least two real zeros $\pm k_R$, $k_R>k_s$, since $\delta(\pm k_s)>0$, $\delta(\pm\infty)<0$. The upper and lower sides of C_l,C_r are denoted by $C_l^\pm,C_r^\pm$, respectively.

To conclude the proof, we now show that $\delta(\xi)$ has only two zeros in the complex plane by the principle of argument [2]. Let Γ_R be a circle with sufficiently large radius R. We consider the domain $\mathcal{D}$ surrounded by the contour C consisting of Γ_R, Γ_l from $-k_s$ to $-k_p$ along C_l^+ and then from $-k_p$ to $-k_s$ along C_l^-, and Γ_r from k_p to k_s along C_r^+ and then from k_s to k_p along C_r^-. Since $\delta(\xi)$ has no poles in the complex plane, we know from the principle of argument that the number of zeros in $\mathcal{D}$ is

$$Z=\frac{1}{2\pi\mathbf{i}}\int_C\frac{\delta'(\xi)}{\delta(\xi)}d\xi. \tag{2.8}$$

It is clear that $\delta(\xi)=\delta^\pm(\xi)$ for $\xi\in C_r^\pm$, where

$$\delta^\pm(\xi)=(k_s^2-2\xi^2)^2\mp\mathbf{i}\,[4\xi^2(k_s^2-\xi^2)^{1/2}(\xi^2-k_p^2)^{1/2}\,]:=f_1(\xi)\mp\mathbf{i}f_2(\xi).$$

Then we have

$$\begin{aligned}\int_{\Gamma_r}\frac{\delta'(\xi)}{\delta(\xi)}d\xi=\int_{k_p}^{k_s}\left(\frac{\delta_+'(\xi)}{\delta_+(\xi)}-\frac{\delta_-'(\xi)}{\delta_-(\xi)}\right)d\xi&=2\mathbf{i}\int_{k_p}^{k_s}\frac{f_1'(\xi)f_2(\xi)-f_1(\xi)f_2'(\xi)}{f_1^2(\xi)+f_2^2(\xi)}d\xi\\&=-2\mathbf{i}\arctan\frac{f_2(\xi)}{f_1(\xi)}\bigg|_{k_p}^{k_s}=0.\end{aligned}$$

Similarly, we have $\int_{\Gamma_l}\frac{\delta'(\xi)}{\delta(\xi)}d\xi=0$. Moreover, for $|\xi|$ large, we have $\delta(\xi)=-2(k_p^2+3k_s^2)\xi^2+O(1)$, and consequently $\int_{\Gamma_R}\frac{\delta'(\xi)}{\delta(\xi)}d\xi=4\pi$ for $R\gg1$. This yields $Z=2$ and completes the proof. $\square$

Let $\mathbb{N}_{\omega(1+\mathbf{i}\varepsilon)}(x,y)$ be the Neumann Green tensor with complex circular frequency $\omega(1+\mathbf{i}\varepsilon)$, that is, ω in (2.2) is replaced by $\omega(1+\mathbf{i}\varepsilon)$. Let $\mathbb{N}_{\omega(1+\mathbf{i}\varepsilon)}(\xi,x_2;y_2)$ be the corresponding spectral Neumann Green tensor which are obtained by replacing k_s,k_p in (2.7) by $k_s(1+\mathbf{i}\varepsilon),k_p(1+\mathbf{i}\varepsilon)$, respectively. The Neumann Green tensor $\mathbb{N}(x,y)$ is defined by the limit absorption principle as

$$\mathbb{N}(x,y)=\lim_{\varepsilon\to0^+}\mathbb{N}_{\omega(1+\mathbf{i}\varepsilon)}(x,y)=\lim_{\varepsilon\to0^+}\frac{1}{2\pi}\int_{\mathbb{R}}\hat{\mathbb{N}}_{\omega(1+\mathbf{i}\varepsilon)}(\xi,x_2;y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi. \tag{2.9}$$

The above limit can be computed by the following lemma on the Cauchy principal value (cf. e.g. [24, Chapter 4, Theorem 5]).

Lemma 2.2 *Let $a, b \in \mathbb{R}, a < b$, and $t_0 \in (a, b)$. If γ is Hölder continuous in $[a, b]$, that is, there exists a constant $\alpha \in (0, 1]$ and a constant $C > 0$ such that for any $s, t \in [a, b]$, $|\gamma(s) - \gamma(t)| \le C|s - t|^\alpha$, then*

$$\lim_{z \to t_0, \pm \mathrm{Im}\, z > 0} \int_a^b \frac{\gamma(t)}{t - z} dt = \mathrm{p.v.} \int_a^b \frac{\gamma(t)}{t - t_0} dt \pm \pi \mathbf{i} \gamma(t_0),$$

where p.v. $\int_a^b$ *denotes the Cauchy principal value of the integral.*

Lemma 2.2 and (2.9) yield the following representation formula for the Neumann Green function

$$\mathbb{N}(x, y) = \frac{1}{2\pi} \mathrm{p.v.} \int_{\mathbb{R}} \hat{\mathbb{N}}(\xi, x_2; y_2) e^{\mathbf{i}(x_1 - y_1)\xi} d\xi - \frac{1}{2\omega^2} \left[\sum_{\alpha, \beta = p, s} \frac{\mathbb{A}_{\alpha\beta}(\xi)}{\delta'(\xi)} e^{\mathbf{i}(\mu_\alpha x_2 + \mu_\beta y_2) + \mathbf{i}(x_1 - y_1)\xi} \right]_{-k_R}^{k_R}, \quad \forall x, y \in \mathbb{R}^2_+,$$

where $[f(\xi)]_a^b := f(b) - f(a)$.

For $x_s \in \Gamma_0$, we define $\mathbb{N}(x, x_s), x \in \mathbb{R}^2_+$, as the limit of $\mathbb{N}(x, y)$ when $y \in \mathbb{R}^2_+, y \to x_s$. It is also easy to check that $\mathbb{N}(x, y) = \mathbb{N}(y, x)^T$ for any $x, y \in \mathbb{R}^2_+$.

In the following we are mostly interested in the Neumann Green tensor $\mathbb{N}(x, y)$ when $x \in \Gamma_0, y \in \mathbb{R}^2_+$. In this case, (2.7) simplifies to

$$\begin{aligned} \hat{\mathbb{N}}(\xi, 0; y_2) &= \frac{\mathbf{i}}{\mu\delta(\xi)} \left[\begin{pmatrix} 2\xi^2\mu_s & -2\xi\mu_s\mu_p \\ -\xi\varphi & \mu_p\varphi \end{pmatrix} e^{\mathbf{i}\mu_p y_2} + \begin{pmatrix} \mu_s\varphi & \xi\varphi \\ 2\xi\mu_s\mu_p & 2\xi^2\mu_p \end{pmatrix} e^{\mathbf{i}\mu_s y_2} \right] \\ &:= \frac{1}{\delta(\xi)} (\mathbb{N}_p(\xi) e^{\mathbf{i}\mu_p y_2} + \mathbb{N}_s(\xi) e^{\mathbf{i}\mu_s y_2}), \end{aligned} \tag{2.10}$$

and consequently, for $x \in \Gamma_0, y \in \mathbb{R}^2_+$,

$$\mathbb{N}(x, y) = \frac{1}{2\pi} \mathrm{p.v.} \int_{\mathbb{R}} \hat{\mathbb{N}}(\xi, 0; y_2) e^{\mathbf{i}(x_1 - y_1)\xi} d\xi + \frac{\mathbf{i}}{2} \left[\sum_{\alpha = p, s} \frac{\mathbb{N}_\alpha(\xi)}{\delta'(\xi)} e^{\mathbf{i}\mu_\alpha y_2 + \mathbf{i}(x_1 - y_1)\xi)} \right]_{-k_R}^{k_R} \tag{2.11}$$

The following representation is useful in studying the horizontal asymptotic behavior of the Neumann Green tensor on Γ_0.

Lemma 2.3 *Let $x \in \Gamma_0, y \in \mathbb{R}^2_+$ and $\phi \in (-\pi/2, \pi/2)$ such that $y_2 = |x - y| \cos\phi, x_1 - y_1 = |x - y| \sin\phi$. Assume that $x_1 \ne y_1$, then we have*

$$\mathbb{N}(x, y) = \frac{1}{2\pi} \int_L \mathbb{N}_0(t) \cos(t + \phi) e^{\mathbf{i}\lambda \cos t} dt \pm \mathbf{i} \left[\sum_{\alpha = p, s} \frac{\mathbb{N}_\alpha(\xi)}{\delta'(\xi)} e^{\mathbf{i}\mu_\alpha y_2 + \mathbf{i}(x_1 - y_1)\xi} \right]_{\xi = \pm k_R} \tag{2,12}$$

where $\lambda = k_s|x - y|$, L is the integral path from $-\pi/2 + \mathbf{i}\infty$ to $-\pi/2$, $-\pi/2$ to $\pi/2$, and $\pi/2$ to $\pi/2 - \mathbf{i}\infty$ in the complex plan (see Figure 1), the sign $\pm$ is taken according to $\mathrm{sgn}(x_1 - y_1) = \pm 1$*, and*

$$\mathbb{N}_0(t) = \sum_{\alpha = p, s} k_s \frac{\mathbb{N}_\alpha(k_s \sin(t + \phi))}{\delta(k_s(\sin(t + \phi))}. \tag{2.13}$$

Proof. Without loss of generality, we assume $x_1 > y_1$ and so $\mathrm{sgn}(x_1 - y_1) = 1$. Notice that $\hat{\mathbb{N}}(\xi, 0; y_2) = \sum_{\alpha=p,s} \frac{\mathbb{N}_\alpha(\xi)}{\delta(\xi)} e^{\mathbf{i}(y_2\mu_\alpha + (x_1-y_1)\xi)}$. For $\alpha = p, s$, we use the classical transform $\xi = k_\alpha \sin t$ to obtain

$$\frac{1}{2\pi}\,\mathrm{p.v.}\int_{\mathbb{R}} \hat{\mathbb{N}}(\xi, 0; y_2) e^{\mathbf{i}(x_1-y_1)\xi} d\xi = \frac{1}{2\pi}\,\mathrm{p.v.}\int_L \sum_{\alpha=p,s} k_s \frac{\mathbb{N}_\alpha(k_s \sin t)}{\delta(k_s \sin t)} \cos t\, e^{\mathbf{i}\lambda\cos(t-\phi)} dt.$$

Let $L_{-\phi}$ be the integral path which is the shift of L by $-\phi$, then

$$\frac{1}{2\pi}\,\mathrm{p.v.}\int_{\mathbb{R}} \hat{\mathbb{N}}(\xi, 0; y_2) e^{\mathbf{i}(x_1-y_1)\xi} d\xi = \frac{1}{2\pi}\,\mathrm{p.v.}\int_{L_{-\phi}} \mathbb{N}_0(t) \cos(t+\phi)\, e^{\mathbf{i}\lambda\cos t} dt, \quad (2.14)$$

where $\mathbb{N}_0(t)$ is defined in (2.14).

Let $t_R = \pi/2 - \mathbf{i}s_R \in L$, $s_R > 0$, such that $k_R = k_s \sin t_R$. Thus k_R is the image of t_R under the integral transform $\xi = k_s \sin t$. For any $\varepsilon > 0$, let L^ε be the integral path from $-\pi/2 + \mathbf{i}\infty \to -\pi/2 + \mathbf{i}(s_R + \varepsilon) \cup \partial B_\varepsilon^+(-t_R) \to -\pi/2 + \mathbf{i}(s_R - \varepsilon) \to -\pi/2 \to \pi/2 \to \pi/2 - \mathbf{i}(s_R - \varepsilon) \to \partial B_\varepsilon^+(t_R) \to \pi/2 - \mathbf{i}(s_R + \varepsilon) \to \pi/2 - \mathbf{i}\infty$, where $\partial B_\varepsilon^+(\pm t_R)$ is the right half circle of radius ε centered at $\pm t_R$ (see Figure 1). Let $L_{-\phi}^\varepsilon$ be the shift L^ε by $-\phi$. Then by the definition of Cauchy principle value, we know that

$$\frac{1}{2\pi}\,\mathrm{p.v.}\int_{L_{-\phi}} \mathbb{N}_0(t)\cos(t+\phi)\, e^{\mathbf{i}\lambda\cos t} dt = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{L_{-\phi}^\varepsilon} \mathbb{N}_0(t)\cos(t+\phi)\, e^{\mathbf{i}\lambda\cos t} dt + \frac{\mathbf{i}}{2}\sum_{t'=\pm t_R} \mathrm{Res}(\mathbb{N}_0(t)\cos(t+\phi) e^{\mathbf{i}\lambda\cos t}, t').$$

It is easy to see that the residue

$$\frac{\mathbf{i}}{2}\sum_{t'=\pm t_R} \mathrm{Res}(\mathbb{N}_0(t)\cos(t+\phi) e^{\mathbf{i}\lambda\cos t}, t') = \frac{\mathbf{i}}{2}\sum_{\xi=\pm k_R}\sum_{\alpha=p,s} \frac{\mathbb{N}_\alpha(\xi)}{\delta'(\xi)} e^{\mathbf{i}(y_2\mu_\alpha + (x_1-y_1)\xi)}.$$

On the other hand, by Cauchy integral theorem we have

$$\frac{1}{2\pi}\int_{L_{-\phi}^\varepsilon} \mathbb{N}_0(t)\cos(t+\phi)\, e^{\mathbf{i}\lambda\cos t} dt = \frac{1}{2\pi}\int_L \mathbb{N}_0(t)\cos(t+\phi)\, e^{\mathbf{i}\lambda\cos t} dt.$$

This completes the proof of the lemma by (2.11) and (2.14). □

This lemma is the starting point of our estimate of the decay behavior of $\mathbb{N}(x, y)$ as $x \to \infty$ on Γ_0. We recall first the following Van der Corput lemma for the oscillatory integral [21, P.152].

Lemma 2.4 *Let $\lambda \geq 1$, $f \in C[a, b]$ with absolutely integrable derivative, and $u \in C^k[a, b]$, where $k \geq 1$ and $a < b$.*

1. *If $|u'(t)| \geq 1$ for $t \in (a, b)$ and u' is monotone in (a, b), then*

$$\left|\int_a^b f(t) e^{\mathbf{i}\lambda u(t)} dt\right| \leq 3\lambda^{-1}\left(|f(b)| + \int_a^b |f'(t)| dt\right).$$

2. *For $k \geq 2$, if $|u^{(k)}(t)| \geq 1$ for $t \in (a, b)$, then*

$$\left|\int_a^b f(t) e^{\mathbf{i}\lambda u(t)} dt\right| \leq 12k\lambda^{-1/k}\left(|f(b)| + \int_a^b |f'(t)| dt\right).$$

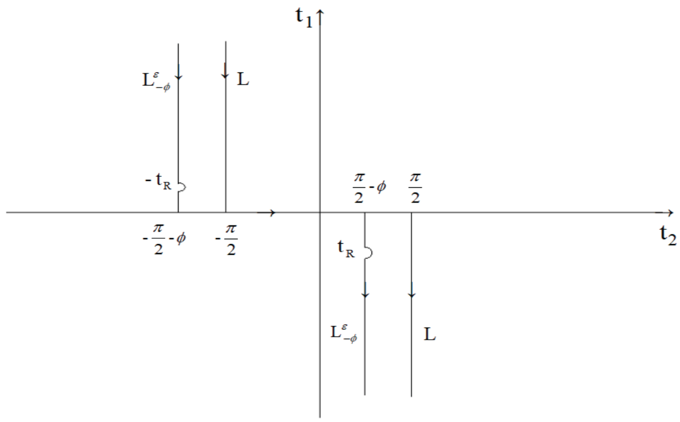

Figure 1. The integral path L and $L^{\varepsilon}_{-\phi}$.

The following lemma, which will be useful in the subsequent analysis, shows that the Van der Corput lemma is still valid when the singular points of the integrand $\phi(t)$ have a gap to the stationary phase points.

Lemma 2.5 *Let $\lambda \geq 1$ and $f \in C[-\pi/2, \pi/2]$ having absolutely integrable derivative. Then for any $(a, b) \subset (-\pi/2, \pi/2)$, we have*

$$\left| \int_a^b f(t) e^{\mathrm{i}\lambda \cos t} dt \right| \leq C \lambda^{-1/2} \left(|f(0)| + \int_a^b |f'(t)| dt \right), \tag{2.15}$$

where the constant C is independent of a, b, λ and the integrand f. Moreover, let $\kappa \in (0, 1)$ and $\phi \in (-\pi/2, \pi/2)$ such that $|\phi| \geq \phi^* > \arcsin \kappa := \phi_\kappa$, we have

$$\left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)(\kappa^2 - \sin^2(t+\phi))^{-1/2} e^{\mathrm{i}\lambda \cos t} dt \right| \leq C \lambda^{-1/2} \left(|f(0)| + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f'(t)| dt \right), \tag{2.16}$$

where C depends only on ϕ^* and κ.

Proof. The estimate (2.15) follows directly from Lemma 2.4 since the interval (a, b) can be divided into several subintervals so that in each subinterval, either $|\cos t| \geq 1/\sqrt{2}$ or $|\sin t| \geq 1/\sqrt{2}$ and $-\sin t$ is monotone.

Let $g(t) = \kappa^2 - \sin^2(t + \phi)$. It is easy to see that $g(t)$ has two zeros t_1, t_2 in $(-\pi/2, \pi/2]$, where $t_1 = \phi_\kappa - \phi$ and $t_2 = -\phi_\kappa - \phi$ or $t_2 = \pi - \phi_\kappa - \phi$ depending on whether $\phi + \phi_\kappa < \pi/2$ or $\phi + \phi_\kappa \geq \pi/2$. Without loss of generality, we assume the later case and thus $t_2 = \pi - \phi_\kappa - \phi$.

Let $\varepsilon_0 = \min(\frac{\phi^* - \phi_\kappa}{2}, \frac{\phi_\kappa}{2}) > 0$. Obviously, $t_1 - \varepsilon_0 \leq -\pi/2, t_1 + \varepsilon_0 \leq -(\phi^* - \phi_\kappa)/2$ and $t_2 - \varepsilon_0 \geq (\phi^* - \phi_\kappa)/2$. We divide $(-\pi/2, \pi/2)$ into five intervals: $I_1 = (-\pi/2, t_1 - \varepsilon_0), I_2 = (t_1 - \varepsilon_0, t_1 + \varepsilon_0), I_3 = (t_1 + \varepsilon_0, t_2 - \varepsilon_0), I_4 = (t_2 - \varepsilon_0, t_2 + \min(t_2 + \varepsilon_0, \pi/2))$ and

$I_5 = (\min(t_2+\varepsilon_0, \pi/2), \pi/2)$. By (2.15) we have

$$\left|\int_{I_1\cup I_3\cup I_5} f(t)(\kappa^2-\sin^2(t+\phi))^{-1/2}e^{\mathbf{i}\lambda\cos t}dt\right| \le C\lambda^{-1/2}\left(|f(0)|+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|f'(t)|dt\right), \quad (2.17)$$

where the constant C depends only on ϕ^* and κ.

Now we estimate the integral in I_2, I_4. We first observe that $|\sin t| \ge \sin((\phi^*-\phi_\kappa)/2)$ in $I_2\cup I_4$. Moreover, $|g'(t)| = |\sin(2(t+\phi))| \ge \min(\sin\phi_\kappa, \sin(\phi^*+\phi_\kappa))$ in $I_2\cup I_4$. Let $\delta\in(0,\varepsilon_0)$ be sufficiently small. Since $g(t_j)=0, j=1,2$, by the mean value theorem, we have

$$|g(t)| \ge \min(\sin\phi_\kappa, \sin(\phi^*+\phi_\kappa))\delta, \quad \forall \delta \le |t-t_j| \le \varepsilon_0, j=1,2.$$

By integration by parts we then obtain

$$\left|\int_{t_1-\varepsilon_0}^{t_1-\delta} f(t)g(t)^{-1/2}e^{\mathbf{i}\lambda\cos t}dt\right| \le C\delta^{-1/2}\lambda^{-1}\left(|f(0)|+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|f'(t)|dt\right).$$

Similarly,

$$\left|\int_{t_1+\delta}^{t_1+\varepsilon_0} f(t)g(t)^{-1/2}e^{\mathbf{i}\lambda\cos t}dt\right| \le C\delta^{-1/2}\lambda^{-1}\left(|f(0)|+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|f'(t)|dt\right).$$

Finally,

$$\begin{aligned}\left|\int_{t_1-\delta}^{t_1+\delta} f(t)g(t)^{-1/2}e^{\mathbf{i}\lambda\cos t}dt\right| &\le C\max_{t\in(-\pi/2,\pi/2)}|f(t)|\int_{-\delta}^{\delta}|\kappa-\sin(\phi_\kappa+t)|^{-1/2}dt \\ &\le C\delta^{1/2}\max_{t\in(-\pi/2,\pi/2)}|f(t)|.\end{aligned}$$

In conclusion, by taking $\delta=\lambda^{-1}$, we obtain

$$\left|\int_{I_2} f(t)(\kappa^2-\sin^2(t+\phi))^{-1/2}e^{\mathbf{i}\lambda\cos t}dt\right| \le C\lambda^{-1/2}\left(|f(0)|+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|f'(t)|dt\right).$$

The integral in I_4 can be estimated similarly. This completes the proof by combining with the estimate in (2.17). □

In the remainder of this paper we denote $\kappa = k_p/k_s \in (0,1)$. The following lemma collects some facts about the Rayleigh function $\delta(\xi)$.

Lemma 2.6 *Let $d_R = (k_R-k_s)/2$, where k_R is the wave number of the Rayleigh surface wave. There exist constants C_1, C_2 depending only on κ such that $|\delta^{(k)}(\xi)| \le C_2 k_s^{4-k}, k=0,1,2$, $|\delta(\xi)| \ge C_1 k_s^4$ for any $|\xi| \le k_R+d_R$, and $|\delta(\xi)| \ge 2k_s^2(|\xi|^2-k_R^2)$ for any $|\xi|\ge k_R$. Moreover, let $\delta(\xi)=\delta_1(\xi)(\xi^2-k_R^2)$ for $\xi\in\mathbb{R}$, then $|\delta_1(\xi)|\ge C_1k_s^2$ for any $k_R-d_R\le|\xi|\le k_R+d_R$.*

Proof. By definition, we know that for $|\xi|\ge k_s$, $\delta(\xi)=k_s^4 f(\xi^2/k_s^2)$, where

$$f(t) = (2t-1)^2 - 4t\sqrt{t-1}\sqrt{t-\kappa^2}, \quad \forall t\ge 1.$$

It is easy to see that $f'(t)\le -2$ for $t\ge 1$, $f(1)=1$ and $f((2-\kappa^2)/(1-\kappa^2))<0$. Thus $k_R^2 \le \frac{2-\kappa^2}{1-\kappa^2}k_s^2$. The estimates for $\delta(\xi)$ follows easily.

Next by the mean value theorem,

$$\min_{k_R-d_R\le|\xi|\le k_R+d_R}|\delta_1(\xi)|\ge\min_{k_R-d_R\le|\xi|\le k_R+d_R}\frac{|\delta'(\xi)|}{|\xi|+k_R}\ge C_1k_s^2,$$

where we have used the fact $f'(t)\le -2$ for $t\ge 1$. This completes the proof. □

Lemma 2.7 *Let $\phi\in(0,\pi/2)$ and H be the hyperbola $\{\xi=\xi_1+\mathbf{i}\xi_2\in\mathbb{C}:(\xi_1/(k_s\cos\phi))^2-(\xi_2/(k_s\sin\phi))^2=1\}$. Let $f(\xi)$ be analytic on H. Then there exists a constant C depending only κ such that*

$$\left|\int_{L\backslash[-\pi/2,\pi/2]}f(k_s\sin(t+\phi))e^{\mathbf{i}\lambda\cos t}dt\right|+\left|\int_{L\backslash[-\pi/2,\pi/2]}f(k_s\sin(t+\phi))\cos te^{\mathbf{i}\lambda\cos t}dt\right|$$
$$\le C\lambda^{-1}\max_{\xi\in H}(|f(\xi)|+k_s|f'(\xi)|).$$

Proof. Notice that for $t=-\pi/2+\mathbf{i}s$, $s>0$, $k_s\sin(t+\phi)=-\cosh(s)\cos\phi+\mathbf{i}\sinh(s)\sin\phi\in H$. Thus

$$\int_{-\pi/2}^{-\pi/2+\mathbf{i}\infty}f(k_s\sin(t+\phi))e^{\mathbf{i}\lambda\cos t}dt$$
$$=\mathbf{i}\int_0^\infty f(-\cosh(s)\cos\phi+\mathbf{i}\sinh(s)\sin\phi)e^{-\lambda\sinh(s)}ds.$$

This implies the estimate on the integral path $-\pi/2+\mathbf{i}\infty\to-\pi/2$ by integration by parts. The estimate on the path $\pi/2\to\pi/2-\mathbf{i}\infty$ is similar. Thus the proof of the first term in the lemma follows. The second term can be proved similarly. Here we omit the details. This completes the proof. □

The following theorem is the first main result in this section.

Theorem 2.1 *Let $x\in\Gamma_0$, $y\in\mathbb{R}^2_+$ satisfy $|x_1-y_1|/|x-y|\ge(1+\kappa)/2$ and $k_sy_2\ge 1$. There exists a constant C depending only on κ such that*

$$|\mathbb{N}(x,y)|+k_s^{-1}|\nabla_y\mathbb{N}(x,y)|\le\frac{C}{\mu}\left(\frac{k_sy_2}{(k_s|x-y|)^{3/2}}+e^{-\sqrt{k_R^2-k_s^2}y_2}\right).$$

Proof. We only prove the estimate for $\mathbb{N}(x,y)$. The estimate of $|\nabla_y\mathbb{N}(x,y)|$ can be proved similarly. The starting point is (2.12) in Lemma 2.3. Without loss of generality, we assume $x_1>y_1$ and thus $\phi\in(0,\pi/2)$ which satisfies $\phi\ge\phi^*=\arcsin(1+\kappa)/2>\phi_\kappa$. It is easy to see from Lemma 2.6 that the second term in (2.12) is bounded by $C\mu^{-1}e^{-\sqrt{k_R^2-k_s^2}y_2}$.

For the first term in (2.12) we first note that

$$\frac{1}{2\pi}\int_L\mathbb{N}_0(t)\cos(t+\phi)e^{\mathbf{i}\lambda\cos t}dt=\frac{1}{2\pi}\int_L\sum_{\alpha=p,s}k_s\frac{\mathbb{N}_\alpha(k_s\sin(t+\phi))}{\delta(k_s\sin(t+\phi))}\cos(t+\phi)e^{\mathbf{i}\lambda\cos t}dt.$$

We shall only estimate the term including $[\mathbb{N}_p(k_s\sin(t+\phi))]_{22}=\mu^{-1}(\varphi\mu_p)(k_s\sin(t+\phi))$. The other terms can be proved similarly. Thus denote by

$$g(t)=k_s\frac{[\mathbb{N}_p(k_s\sin(t+\phi))]_{22}}{\delta(k_s\sin(t+\phi))}:=f(t)(\kappa^2-\sin^2(t+\phi))^{1/2},\quad f(t)=\frac{k_s^2}{\mu}\frac{\varphi(k_s\sin(t+\phi))}{\delta(k_s\sin(t+\phi))}.$$

Note first by integration by parts that

$$\begin{aligned}\int_L g(t)\cos(t+\phi)e^{\mathbf{i}\lambda\cos t}dt =& \cos\phi\int_L g(t)\cos te^{\mathbf{i}\lambda\cos t}dt-\sin\phi\int_L g(t)\sin te^{\mathbf{i}\lambda\cos t}dt\\ =& \cos\phi\int_L g(t)\cos te^{\mathbf{i}\lambda\cos t}dt-\frac{\sin\phi}{\mathbf{i}\lambda}\int_L g'(t)e^{\mathbf{i}\lambda\cos t}dt\\ =& \mathrm{I}_1+\mathrm{I}_2.\end{aligned}$$

By Lemma 2.6 and (2.15) in Lemma 2.5, we know that

$$\left|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(t)\cos te^{\mathbf{i}\lambda\cos t}dt\right|\le C\lambda^{-1/2}\left(|g(0)|+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|(g(t)\cos t)'|dt\right)\le C\mu^{-1}\lambda^{-1/2}.$$

Since $g(t)=k_s([\mathbb{N}_p(k_s\sin(t+\phi))]_{22}\delta^{-1}(k_s\sin(t+\phi))$, $|[\mathbb{N}_p(\xi)]_{22}|\le C|\xi|^3$, $|[\mathbb{N}_p'(\xi)]_{22}|\le C|\xi|^2$, and $\delta(\xi)\ge Ck_s^2|\xi|^2$ on the hyperbola H, we conclude from Lemma 2.7 that

$$\left|\int_{L\backslash[-\frac{\pi}{2},\frac{\pi}{2}]} g(t)\cos te^{\mathbf{i}\lambda\cos t}dt\right|\le C\mu^{-1}\lambda^{-1}.$$

Thus $|\mathrm{I}_1|\le C\mu^{-1}\lambda^{-1/2}\cos\phi$. Similarly, we can obtain $|\mathrm{I}_2|\le C\mu^{-1}\lambda^{-3/2}$. Indeed, the only difference is that since $g'(t)=f'(t)(\kappa^2-\sin^2(t+\phi))^{1/2}-f(t)(\kappa^2-\sin^2(t+\phi))^{-1/2}\sin(t+\phi)\cos(t+\phi)$, one has to use both (2.15) and (2.16) in Lemma 2.5 to obtain

$$\left|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g'(t)e^{\mathbf{i}\lambda\cos t}dt\right|\le C\mu^{-1}\lambda^{-1/2}.$$

Thus $|\mathrm{I}_1+\mathrm{I}_2|\le C\mu^{-1}\lambda^{-1/2}\cos\phi+C\mu^{-1}\lambda^{-3/2}\le C\mu^{-1}(k_sy_2)/(k_s|x-y|)^{3/2}$, where have used the condition $k_sy_2\ge 1$. This completes the proof. □

Now we introduce the Dirichlet Green tensor $\mathbb{D}(x,y), y\in\mathbb{R}^2_+$ which satisfies [4]

$$\Delta_e[\mathbb{D}(x,y)q]+\omega^2[\mathbb{D}(x,y)q]=-\delta_y(x)q\quad\text{in } \mathbb{R}^2_+, \tag{2.18}$$

$$\mathbb{D}(x,y)q=0\quad\text{on } \Gamma_0. \tag{2.19}$$

The spectral Dirichlet Green tensor $\hat{\mathbb{D}}(\xi,x_2;y_2)$ is defined similar to the spectral Neumann Green tensor in (2.4) and it follows that

$$\hat{\mathbb{D}}(\xi,x_2;y_2)=\hat{\mathbb{G}}(\xi,x_2;y_2)-\hat{\mathbb{G}}(\xi,x_2;-y_2)+\frac{\mathbf{i}}{\omega^2\gamma(\xi)}\sum_{\alpha,\beta=s,p}\mathbb{B}_{\alpha\beta}(\xi)e^{\mathbf{i}(x_2\mu_\alpha+y_2\mu_\beta)}, \tag{2.20}$$

where $\gamma(\xi)=\xi^2+\mu_s\mu_p$, $\mathbb{B}_{sp}(\xi)=-\mathbb{B}_{ss}(\xi), \mathbb{B}_{ps}(\xi)=-\mathbb{B}_{pp}(\xi)$, and

$$\mathbb{B}_{ss}(\xi)=\begin{pmatrix}\xi^2\mu_s & -\xi\mu_s\mu_p\\ -\xi^3 & \xi^2\mu_p\end{pmatrix},\qquad \mathbb{B}_{pp}(\xi)=\begin{pmatrix}\xi^2\mu_s & \xi^3\\ \xi\mu_s\mu_p & \xi^2\mu_p\end{pmatrix}.$$

The Dirichlet Green tensor $\mathbb{D}(x,y)$ is obtained as the limit of $\mathbb{D}_{\omega(1+\mathbf{i}\varepsilon)}(x,y)$ when $\varepsilon\to 0^+$, where $\mathbb{D}_{\omega(1+\mathbf{i}\varepsilon)}(x,y)$ is Dirichlet Green tensor with the complex circular frequency $\omega(1+\mathbf{i}\varepsilon)$, that is ω in (2.18) is replaced by $\omega(1+\mathbf{i}\varepsilon)$. The corresponding spectral Dirichle Green tensor $\hat{\mathbb{D}}_{\omega(1+\mathbf{i}\varepsilon)}(x,y)$ are obtained by replacing k_s,k_p in (2.20) by $k_s(1+\mathbf{i}\varepsilon), k_p(1+\mathbf{i}\varepsilon)$, respectively. Thus

$$\mathbb{D}(x,y)=\lim_{\varepsilon\to 0^+}\mathbb{D}_{\omega(1+\mathbf{i}\varepsilon)}(x,y)=\lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int_{\mathbb{R}}\hat{\mathbb{D}}_{\omega(1+\mathbf{i}\varepsilon)}(\xi,x_2;y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi.$$

We have the following representation of Dirichlet Green tensor

$$\mathbb{D}(x,y)=\mathbb{G}(x,y)-\mathbb{G}(x,y')+\frac{\mathbf{i}}{2\pi\omega^2}\int_{\mathbb{R}}\sum_{\alpha,\beta=s,p}\frac{\mathbb{B}_{\alpha\beta}(\xi)}{\gamma(\xi)}e^{\mathbf{i}(\mu_\alpha x_2+\mu_\beta y_2)+\mathbf{i}(x_1-y_1)\xi}d\xi. \quad (2.21)$$

It is easy to check that $\mathbb{D}(x,y)=\mathbb{D}(y,x)^T$ for $x,y\in\mathbb{R}^2_+$.

For $x\in\Gamma_0, y\in\mathbb{R}^2_+$, let $\mathbb{T}_D(x,y)\in\mathbb{C}^{2\times 2}$ denote the traction tensor of $\mathbb{D}(x,y)$ in the direction e_2 with respect to x, that is, $\mathbb{T}_D(x,y)q=\sigma(\mathbb{D}(x,y)q)e_2, \forall q\in\mathbb{R}^2$. By (2.21) we have

$$\mathbb{T}_D(x,y)=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{\mathbb{T}}_D(\xi,0;y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi, \qquad \forall x\in\Gamma_0, \quad (2.22)$$

where

$$\begin{aligned}\hat{\mathbb{T}}_D(\xi,0;y_2)=&\ \frac{1}{\gamma(\xi)}\left[\begin{pmatrix}\xi^2 & -\xi\mu_p\\ -\xi\mu_s & \mu_p\mu_s\end{pmatrix}e^{\mathbf{i}\mu_p y_2}+\begin{pmatrix}\mu_s\mu_p & \xi\mu_p\\ \xi\mu_s & \xi^2\end{pmatrix}e^{\mathbf{i}\mu_s y_2}\right]\\ :=&\ \mathbb{T}_p(\xi)e^{\mathbf{i}\mu_p y_2}+\mathbb{T}_s(\xi)e^{\mathbf{i}\mu_s y_2}. \quad (2.23)\end{aligned}$$

The following theorem on the decay behavior of the traction tensor on Γ_0 improves [4, Lemma 2.2] in the sense that we provide exact dependence on y_2 in the numerator. It can be proved by the same (and simpler) argument as that in the proof of Theorem 2.1. We omit the details.

Theorem 2.2 *Let $x\in\Gamma_0$, $y\in\mathbb{R}^2_+$ satisfy $|x_1-y_1|/|x-y|\geq(1+\kappa)/2$ and $k_s y_2\geq 1$. There exists a constant C depending only on κ such that*

$$|\mathbb{T}_D(x,y)|+k_s^{-1}|\nabla_y\mathbb{T}_D(x,y)|\leq C\frac{k_s^2y_2}{(k_s|x-y|)^{3/2}}.$$

3. The point spread function

In this section we introduce the point spread function for imaging a point source embedded in the half-space elastic medium, which extends the study in [12] for acoustic waves. Let $\mathbb{N}(x,y)$ be the Neumann Green tensor which is the data collected on the surface $\Gamma_0^d=\{(x_1,x_2)^T\in\Gamma_0 : x_1\in(-d,d)\}$ with a point source $y\in\mathbb{R}^2_+$, where $d>0$ is the aperture. The finite aperture point spread function $\mathbb{J}_d(x,y)$, $x,y\in\mathbb{R}^2_+$, is a $\mathbb{C}^{2\times 2}$ matrix, which is the back-propagated field with $\mathbb{N}(x,y)\chi_{(-d,d)}$ as the Dirichlet boundary condition, where $\chi_{(-d,d)}$ is the characteristic function of the interval $(-d,d)$. More precisely, $\mathbb{J}_d(x,y)e_j, j=1,2$, is the scattering solution of the following problem

$$\begin{aligned}&\Delta_e[\mathbb{J}_d(x,y)e_j]+\omega^2[\mathbb{J}_d(x,y)e_j]=0 \ \text{ in } \mathbb{R}^2_+,\\ &\mathbb{J}_d(x,y)e_j=[\overline{\mathbb{N}(x,y)}e_j]\chi_{(-d,d)} \ \text{ on } \Gamma_0.\end{aligned}$$

By the integral representation formula, for any $z,y\in\mathbb{R}^2_+$,

$$[\mathbb{J}_d(z,y)]_{ij}=e_i\cdot[\mathbb{J}_d(z,y)e_j]=\int_{\Gamma_0^d}\mathbb{T}_D(x,z)e_i\cdot\overline{\mathbb{N}(x,y)}e_jds(x), \quad i,j=1,2,$$

or, more concisely,

$$\mathbb{J}_d(z,y)=\int_{\Gamma_0^d}\mathbb{T}_D(x,z)^T\overline{\mathbb{N}(x,y)}ds(x). \quad (3.1)$$

By Theorem 2.1 and Theorem 2.2, we know that the integral converges as $d \to \infty$. Thus we can define the half-space elastic point spread function $\mathbb{J}(x,y) \in \mathbb{C}^{2\times 2}$, $x, y \in \mathbb{R}^2_+$, as

$$\mathbb{J}(z,y) = \int_{\Gamma_0} \mathbb{T}_D(x,z)^T \overline{\mathbb{N}(x,y)} ds(x). \tag{3.2}$$

By the limiting absorption principle, we know that

$$\mathbb{J}(z,y) = \lim_{\varepsilon\to 0^+} \int_{\Gamma_0} \mathbb{T}_D^{\omega(1+\mathbf{i}\varepsilon)}(x,z)^T\, \overline{\mathbb{N}_{\omega(1+\mathbf{i}\varepsilon)}(x,y)} ds(x),$$

where $\mathbb{T}_D^{\omega(1+\mathbf{i}\varepsilon)}(x,z)q = \sigma(\mathbb{D}_{\omega(1+\mathbf{i}\varepsilon)}(x,z)q)e_2, \forall q \in \mathbb{R}^2$. By using Parserval identity, Lemma 2.2, (2.10) and (2.23), we obtain

$$\begin{aligned}\mathbb{J}(z,y) =& \frac{1}{2\pi}\sum_{\alpha,\beta=p,s} \text{p.v.} \int_{\mathbb{R}} \frac{\mathbb{T}_\alpha(\xi)^T \overline{\mathbb{N}_\beta(\xi)}}{\overline{\delta(\xi)}} e^{\mathbf{i}(\mu_\alpha z_2 - \overline{\mu}_\beta y_2) + \mathbf{i}(y_1 - z_1)\xi} d\xi \\ & - \frac{\mathbf{i}}{2}\sum_{\alpha,\beta=p,s} \left[\frac{\mathbb{T}_\alpha(\xi)^T \overline{\mathbb{N}_\beta(\xi)}}{\overline{\delta'(\xi)}} e^{\mathbf{i}(\mu_\alpha z_2 - \overline{\mu}_\beta y_2) + \mathbf{i}(y_1 - z_1)\xi} \right]_{-k_R}^{k_R}.\end{aligned} \tag{3.3}$$

To proceed, we define

$$\begin{aligned}\mathbb{F}(z,y) =& \frac{1}{2\pi}\int_{-k_p}^{k_p} \frac{\mathbb{T}_p(\xi)^T \overline{\mathbb{N}_p(\xi)}}{\overline{\delta(\xi)}} e^{\mathbf{i}\mu_p(z_2 - y_2) + \mathbf{i}(y_1 - z_1)\xi} d\xi \\ & + \frac{1}{2\pi}\int_{-k_s}^{k_s} \frac{\mathbb{T}_s(\xi)^T \overline{\mathbb{N}_s(\xi)}}{\overline{\delta(\xi)}} e^{\mathbf{i}\mu_s(z_2 - y_2) + \mathbf{i}(y_1 - z_1)\xi} d\xi.\end{aligned} \tag{3.4}$$

Let Ω be the imaging domain and $h = \text{dist}(\Omega, \Gamma_0)$ be the distance between Ω and Γ_0. We assume there exist constants $0 < c_1 < 1, c_2 > 0$ such that

$$|x_1| \le c_1 d, \quad |x - y| \le c_2 h, \qquad \forall x, y \in \Omega. \tag{3.5}$$

We remark that this assumption is rather mild in practical applications. The aim of this section is to show that for $z, y \in \Omega$, $\mathbb{F}(z,y)$ is the main contribution in $\mathbb{J}_d(z,y)$. Moreover, $\mathbb{F}(z,y)$ decays as $|z - y| \to \infty$ and the imaginary part of the function $|\text{Im}\,\mathbb{F}_{ii}(z,y)|, i = 1, 2$, peaks when $z = y$.

We start with the following lemma.

Lemma 3.1 *Let $k_s h \ge 1$ and $d \gg h$. For any $z, y \in \Omega$, we have*

$$\begin{aligned}&|\mathbb{J}(z,y) - \mathbb{J}_d(z,y)| + k_s^{-1}|\nabla_y(\mathbb{J}(z,y) - \mathbb{J}_d(z,y))| \\ \le & \frac{C}{\mu}\left[\left(\frac{h}{d}\right)^2 + (k_s h)^{1/2} e^{-k_s h\sqrt{\kappa_R^2 - 1}} \left(\frac{h}{d}\right)^{1/2}\right],\end{aligned}$$

where the constant C depends only on κ.

Proof. By Theorem 2.1 and Theorem 2.2, when $k_s h \ge 1$ and $d \gg h$, we have

$$\begin{aligned}&\left|\int_d^\infty \left[\mathbb{T}_D(x,z)^T \overline{\mathbb{N}(x,y)}\right]_{x_2=0} dx_1\right| \\ \le & \frac{C}{\mu}\int_d^\infty \frac{k_s^{1/2} z_2}{|x-z|^{3/2}} \left(\frac{k_s^{-1/2} y_2}{|x-y|^{3/2}} + e^{-\sqrt{k_R^2 - k_s^2} y_2}\right) dx_1\end{aligned}$$

$$\le \frac{C}{\mu}\int_{(1-c_1)d/h}^{\infty}\left(\frac{1}{(1+t^2)^{3/2}}+\frac{(k_sh)^{1/2}}{(1+t^2)^{3/4}}e^{-\sqrt{k_R^2-k_s^2}h}\right)dt$$
$$\le \frac{C}{\mu}\left[\left(\frac{h}{d}\right)^2+\frac{(k_sh)^{1/2}}{e^{\sqrt{k_R^2-k_s^2}h}}\left(\frac{h}{d}\right)^{1/2}\right].$$

Here we have used the first inequality in (3.5). Similarly, we can prove that the estimate for the integral in $(-\infty,-d)$. This shows the estimate for $\mathbb{J}(z,y)-\mathbb{J}_d(z,y)$. The estimate for $\nabla_y(\mathbb{J}(z,y)-\mathbb{J}_d(z,y))$ can be proved similarly. □

The following lemma shows the second term on the right-hand side of (3.3) is small.

Lemma 3.2 *There exists a constant C depending only on κ such that for any $z,y\in\Omega$,*

$$\left|\sum_{\alpha,\beta=p,s}\left[\frac{\mathbb{T}_\alpha(\xi)^T\overline{\mathbb{N}_\beta(\xi)}}{\overline{\delta'(\xi)}}e^{\mathbf{i}(\mu_\alpha z_2-\overline{\mu}_\beta y_2)+\mathbf{i}(y_1-z_1)\xi}\right]_{-k_R}^{k_R}\right|\le\frac{C}{\mu}e^{-\sqrt{k_R^2-k_s^2}h}.$$

Proof. We know that from (2.23), (2.10) that for $\alpha=p,s$, $|\mathbb{T}_\alpha(\pm k_R)|\le Ck_R^2/k_s^2\le C$, $|\mathbb{N}_\alpha(\pm k_R)|\le Ck_R^3$. The lemma now follows easily by using Lemma 2.6. □

Lemma 3.3 *Let $g\in C^1(\mathbb{R})\cap L^1(\mathbb{R})$. There exists a constant C depending only on κ such that for any $z,y\in\Omega$,*

$$\left|\text{p.v.}\int_{|\xi|>k_s}\frac{g(\xi)}{\delta(\xi)}d\xi\right|\le Ck_s^{-4}\int_{|\xi|>k_s}|g(\xi)|d\xi+Ck_s^{-3}\max_{\xi\in(k_R-d_R,k_R+d_R)}(|g(\xi)|+k_s|g'(\xi)|).$$

where $d_R=(k_R-k_s)/2$.

Proof. Without loss of generality, we prove the lemma for the integral in (k_s,∞). We write $\delta(\xi)=(\xi^2-k_R^2)\delta_1(\xi)$ as in Lemma 2.6, where $\delta_1(\xi)\neq 0$ for $\xi>k_s$. By the definition of the Cauchy principle value

$$\text{p.v.}\int_{k_s}^{\infty}\frac{g(\xi)}{\delta(\xi)}d\xi=\int_{k_s}^{k_R-d_R}\frac{g(\xi)}{\delta(\xi)}d\xi+\int_{k_R+d_R}^{\infty}\frac{g(\xi)}{\delta(\xi)}d\xi$$
$$+\int_{k_R-d_R}^{k_R+d_R}\frac{g(\xi)((\xi+k_R)\delta_1(\xi))^{-1}-g(k_R)(2k_R\delta_1(k_R))^{-1}}{(\xi-k_R)}d\xi. \quad (3.6)$$

By Lemma 2.6 we obtain easily

$$\left|\int_{k_s}^{k_R-d_R}\frac{g(\xi)}{\delta(\xi)}d\xi+\int_{k_R+d_R}^{\infty}\frac{g(\xi)}{\delta(\xi)}d\xi\right|\le Ck_s^{-4}\int_{k_s}^{\infty}|g(\xi)|d\xi.$$

The last term in (3.6) can be proved by using the mean value theorem and the bounds for $\delta(\xi),\delta_1(\xi)$ in Lemma 2.6. This completes the proof. □

Lemma 3.4 *Let $k_sh\ge 1$. There exists a constant C depending only on κ such that for any $z,y\in\Omega$,*

$$\left|\sum_{\alpha,\beta=p,s}\text{p.v.}\int_{|\xi|>k_s}\frac{\mathbb{T}_\alpha(\xi)^T\overline{\mathbb{N}_\beta(\xi)}}{\overline{\delta(\xi)}}e^{\mathbf{i}(\mu_\alpha z_2-\overline{\mu}_\beta y_2)+\mathbf{i}(y_1-z_1)\xi}d\xi\right|\le\frac{C}{\mu}(k_sh)^{-1}.$$

Proof. For $\alpha, \beta = p, s$, we denote $g_{\alpha\beta}(\xi) = \mathbb{T}_\alpha(\xi)^T \overline{\mathbb{N}_\beta(\xi)} e^{\mathbf{i}(\mu_\alpha z_2 - \overline{\mu}_\beta y_2) + \mathbf{i}(y_1 - z_1)\xi}$. By using Lemma 3.3, we obtain easily

$$\begin{aligned}\left|\text{p.v.} \int_{|\xi|>k_s} \frac{g_{\alpha\beta}(\xi)}{\overline{\delta(\xi)}} d\xi\right| &\leq \frac{C}{k_s^6 \mu} \int_{k_s}^{\infty} |\xi|^5 e^{-\sqrt{\xi^2 - k_s^2}(y_2+z_2)} d\xi + \frac{C}{\mu}(k_s h) e^{-\sqrt{(k_R-d_R)^2 - k_s^2}(y_2+z_2)} \\ &\leq \frac{C}{\mu} \int_1^{\infty} t^5 e^{-\sqrt{t^2-1}(y_2+z_2)} dt + \frac{C}{\mu}(k_s h) e^{-\sqrt{(k_R-d_R)^2 - k_s^2}(y_2+z_2)} \\ &\leq \frac{C}{\mu}(k_s h)^{-1} + \frac{C}{\mu}(k_s h) e^{-\sqrt{(k_R-d_R)^2 - k_s^2}(y_2+z_2)},\end{aligned}$$

where we have used that $y_2, z_2 \geq h$ and $d_R = (k_R - k_s)/2 \geq C_1 k_s$ for some constant $C_1 > 0$ depending only on κ. This completes the proof as the second term decays exponentially in $k_s h$. □

Lemma 3.5 *Let $\phi(t) = \sqrt{1-t^2} - \tau\sqrt{\kappa^2 - t^2} + \nu t$, where $\kappa \in (0,1), \tau \geq \tau_0 > 0, \nu \in \mathbb{R}$. There exists a constant C depending only on κ, τ_0 but independent of ν such that for any $\lambda \geq 1$ and $f \in C[0,\kappa]$ with absolutely integrable derivative,*

$$\left|\int_{-\kappa}^{\kappa} f(t) e^{\mathbf{i}\lambda\phi(t)} dt\right| + \left|\int_{-\kappa}^{\kappa} f(t) e^{-\mathbf{i}\lambda\phi(t)} dt\right| \leq C\lambda^{-1/4}\left(|f(0)| + \int_{-\kappa}^{\kappa} |f'(t)| dt\right).$$

Proof. We only prove the estimate for the first integral in the interval $(0,\kappa)$. The other cases can be proved similarly. It is easy to check that for $t \in (0,\kappa), m \geq 2$, the m-th derivative $\phi^{(m)}(t) = \tau\kappa^{-(m-1)}\psi_m(t/\kappa) - \psi_m(t)$, where

$$\psi_2(t) = (1-t^2)^{-3/2}, \quad \psi_3(t) = 3t(1-t^2)^{-5/2}, \quad \psi_4(t) = 3(1+4t^2)(1-t^2)^{-7/2}.$$

Obviously, $\psi_m(t), m \geq 2$, are increasing functions in $(0,\kappa)$.

We first consider the case when $\tau \geq \kappa^2$. This implies $\tau\kappa^{-3} \geq \kappa^{-1}$ and thus

$$\phi^{(4)}(t) \geq (\kappa^{-1} - 1)\psi_4(t) \geq 3(\kappa^{-1} - 1).$$

By using the Van der Corput Lemma 2.4, we have

$$\left|\int_0^{\kappa} f(t) e^{\mathbf{i}\lambda\phi(t)} dt\right| \leq C\lambda^{-1/4}\left(|f(0)| + \int_{-\kappa}^{\kappa} |f'(t)| dt\right). \tag{3.7}$$

Next we consider the case when $\tau < \kappa^2$. Now $\phi''(t)$ has only one zero in $(0,\kappa)$ at $t = t_2$ and either $\phi'''(t) \geq 0$ in $(0,\kappa)$ when $\kappa^3 \leq \tau < \kappa$ or $\phi'''(t)$ has only one zero in $(0,\kappa)$ at t_3 when $\tau < \kappa^3$, where

$$t_2^2 = \kappa^2 - \frac{1-\kappa^2}{(\tau\kappa^2)^{-2/3} - 1}, \quad t_3^2 = \kappa^2 - \frac{1-\kappa^2}{(\tau\kappa^2)^{-2/5} - 1}.$$

When $\kappa^3 \leq \tau < \kappa$, $\phi''(t)$ is increasing in $(0,\kappa)$. Thus for sufficiently small $\delta > 0$,

$$|\phi''(t)| \geq \min(|\phi''(t_2+\delta)|, |\phi''(t_2-\delta)|), \quad \forall t \in (0, t_2-\delta) \cup (t_2+\delta, \kappa). \tag{3.8}$$

On the other hand, when $\tau < \kappa^3$, we have $t_3 < t_2$ and $\phi'''(t) \geq 0$ for $t \geq t_3$ and $\phi'''(t) \leq 0$ for $t \leq t_3$. Therefore, $\phi''(t)$ is increasing in (t_3,κ) and decreasing in $(0,t_3)$. Thus

$$|\phi''(t)| \geq \min(|\phi''(t_2+\delta)|, |\phi''(t_2-\delta)|, |\phi''(0)|), \quad \forall t \in (0, t_2-\delta) \cup (t_2+\delta, \kappa). \tag{3.9}$$

To estimate the lower bound of $|\phi''(t_2 \pm \delta)|$, we observe that since $\tau\kappa^2 < \kappa^4$, $t_2^2 \geq \kappa^2 - (1-\kappa^2)/(\kappa^{-8/3}-1)$, and thus $|\phi'''(t_2)| \geq c_0\tau \geq c_0\tau_0$ for some constant c_0 depending only on κ. Moreover, for any $t \in [t_2-\delta, t_2+\delta]$, $|\phi'''(t) - \phi'''(t_2)| \leq C_1\delta$ for some constant C_1 depending only on κ. Thus, if $\delta \leq c_0\tau_0/(2C_1)$, $|\phi'''(t)| \geq c_0\tau_0/2$ in $[t_2-\delta, t_2+\delta]$. This implies by using the mean value theorem that $|\phi''(t_2 \pm \delta)| \geq (c_0\tau_0/2)\delta$. Notice that $|\phi''(0)| = 1-\tau\kappa^{-1} \geq 1-\kappa$, from (3.8)-(3.9) we conclude that for sufficiently small $\delta > 0$,

$$|\phi''(t)| \geq (c_0\tau_0/2)\delta, \quad \forall t \in (0, t_2-\delta) \cup (t_2+\delta, \kappa). \tag{3.10}$$

Now we split the integral

$$\begin{aligned}\int_0^\kappa f(t)e^{\mathbf{i}\lambda\phi(t)}dt = &\int_0^{t_2-\delta} f(t)e^{\mathbf{i}\lambda\phi(t)}dt + \int_{t_2-\delta}^{t_2+\delta} f(t)e^{\mathbf{i}\lambda\phi(t)}dt + \int_{t_2+\delta}^{\kappa} f(t)e^{\mathbf{i}\lambda\phi(t)}dt \\ :=& \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3.\end{aligned}$$

From (3.10), by Van der Corput Lemma 2.4, we have

$$|\mathrm{II}_1 + \mathrm{II}_3| \leq C(\lambda\delta)^{-1/2}\left(|f(0)| + \int_0^\kappa |f'(t)|dt\right).$$

It is obvious that $|\mathrm{II}_2| \leq 2\delta \max_{t\in(0,\kappa)} |f(t)|$. This yields after taking $\delta = \lambda^{-1/3}$,

$$\left|\int_0^\kappa f(t)e^{\mathbf{i}\lambda\phi(t)}dt\right| \leq C\lambda^{-1/3}\left(|f(0)| + \int_{-\kappa}^\kappa |f'(t)|dt\right).$$

This completes the proof by noticing (3.7). □

The following theorem is the main result of this section.

Theorem 3.1 *Let $k_sh \geq 1$. There exists a constant C depending only on κ such that for any $z, y \in \Omega$,*

$$|\mathbb{J}(z,y) - \mathbb{F}(z,y)| + k_s^{-1}|\nabla_y(\mathbb{J}(z,y) - \mathbb{F}(z,y))| \leq \frac{C}{\mu}(k_sh)^{-1/4}.$$

Proof. By Lemma 3.2, Lemma 3.4 and the definitions of $\mathbb{J}(z,y), \mathbb{F}(z,y)$ in (3.3)-(3.4), we know that we are left to estimate

$$\begin{aligned}&\frac{1}{2\pi}\sum_{\alpha,\beta=p,s}\int_{-k_s}^{k_s} \frac{\mathbb{T}_\alpha(\xi)^T\overline{\mathbb{N}_\beta(\xi)}}{\overline{\delta(\xi)}} e^{\mathbf{i}(\mu_\alpha z_2 - \overline{\mu}_\beta y_2) + \mathbf{i}(y_1-z_1)\xi}d\xi - \mathbb{F}(z,y) \\ =& \frac{1}{2\pi}\sum_{\substack{\alpha,\beta=p,s\\(\alpha,\beta)\neq(s,s)}}\int_{(-k_s,k_s)\backslash[-k_p,k_p]} \frac{\mathbb{T}_\alpha(\xi)^T\overline{\mathbb{N}_\beta(\xi)}}{\overline{\delta(\xi)}} e^{\mathbf{i}(\mu_\alpha z_2 - \overline{\mu}_\beta y_2) + \mathbf{i}(y_1-z_1)\xi}d\xi \\ &+ \frac{1}{2\pi}\int_{-k_p}^{k_p}\left[\frac{\mathbb{T}_p(\xi)\overline{\mathbb{N}_s(\xi)}}{\overline{\delta(\xi)}}e^{\mathbf{i}(\mu_p y_2 - \mu_s z_2)} + \frac{\mathbb{T}_s(\xi)\overline{\mathbb{N}_p(\xi)}}{\overline{\delta(\xi)}}e^{\mathbf{i}(\mu_s y_2 - \mu_p z_2)}\right]e^{\mathbf{i}(y_1-z_1)\xi}d\xi \\ :=& \mathrm{II}_1 + \mathrm{II}_2.\end{aligned}$$

For $k_p < |\xi| < k_s$, we know that $|\delta(\xi)| \geq Ck_s^4$ by Lemma 2.6, and for $\alpha, \beta = p, s$, $|\mathbb{T}_\alpha(\xi)| \leq C, |\mathbb{N}_\beta(\xi)| \leq C\mu^{-1}k_s^2$. This implies

$$|\mathrm{II}_1| \leq \frac{C}{k_s\mu}\int_{k_p}^{k_s} e^{-\sqrt{\xi^2-k_p^2}h}d\xi \leq \frac{C}{\mu}(k_sh)^{-1}.$$

For the term II_2 we use Lemma 3.5. The first term in II_2 can be reduced to the integral in Lemma 3.5 by setting

$$f(t) = k_s \frac{\mathbb{T}_p(k_s t)\overline{\mathbb{N}_s(k_s t)}}{\overline{\delta(k_s t)}}, \quad \lambda = k_s z_2, \tau = \frac{y_2}{z_2}, \nu = \frac{y_1 - z_1}{z_2}.$$

By the assumption (3.5), it is then straightforward by using Lemma 3.5 to see that

$$\left| \int_{-k_p}^{k_p} \frac{\mathbb{T}_p(\xi)\overline{\mathbb{N}_s(\xi)}}{\overline{\delta(\xi)}} e^{\mathbf{i}(\mu_p y_2 - \mu_s z_2)} d\xi \right| \leq \frac{C}{\mu}(k_s h)^{-1/4}.$$

The second integral in II_2 can be estimated similarly. This completes the proof. □

The following theorem shows that $\mathbb{F}(z, y)$ has the similar behavior as the imaginary part of the elastic fundamental solution $\operatorname{Im}\mathbb{G}(z, y)$.

Theorem 3.2 *For any $z, y \in \mathbb{R}^2_+$, $\mathbb{F}(z, y)^T = \mathbb{F}(z, y)$. When $z = y$, $\operatorname{Im}[\mathbb{F}(z, y)]_{12} = \operatorname{Im}[\mathbb{F}(z, y)]_{21} = 0$ and*

$$-\operatorname{Im}[\mathbb{F}(z, y)]_{ii} \geq \frac{1}{4(\lambda + 2\mu)}, \quad i = 1, 2. \tag{3.11}$$

When $z \neq y$,

$$|\mathbb{F}(z, y)| \leq \frac{C}{\mu}\left(\frac{1}{(k_s|z - y|)^{1/2}} + \frac{1}{k_s|z - y|} \right), \tag{3.12}$$

where constant C depends only on κ.

Proof. Substitute (2.23) and (2.10) into (3.4), we obtain

$$\begin{aligned}
\mathbb{F}(z, y) = \; & -\frac{1}{2\pi} \int_{-k_p}^{k_p} \frac{\mathbf{i}k_s^2 \mu_s}{\mu\gamma(\xi)\delta(\xi)} \begin{pmatrix} \xi^2 & -\xi\mu_p \\ -\xi\mu_p & \mu_p^2 \end{pmatrix} e^{\mathbf{i}\mu_p(z_2 - y_2) + \mathbf{i}\xi(y_1 - z_1)} d\xi \\
& -\frac{1}{2\pi} \int_{-k_p}^{k_p} \frac{\mathbf{i}k_s^2 \mu_p}{\mu\gamma(\xi)\delta(\xi)} \begin{pmatrix} \mu_s^2 & \xi\mu_s \\ \xi\mu_s & \xi^2 \end{pmatrix} e^{\mathbf{i}\mu_s(z_2 - y_2) + \mathbf{i}\xi(y_1 - z_1)} d\xi \\
& -\frac{1}{2\pi} \int_{(-k_s, k_s)\setminus[-k_p, k_p]} \frac{\mathbf{i}(k_s^2 - 4\xi^2)\mu_p}{\mu\gamma(\xi)\overline{\delta(\xi)}} \begin{pmatrix} \mu_s^2 & \xi\mu_s \\ \xi\mu_s & \xi^2 \end{pmatrix} e^{\mathbf{i}\mu_s(z_2 - y_2) + \mathbf{i}\xi(y_1 - z_1)} d\xi \\
:= \; & \mathrm{III}_1 + \mathrm{III}_2 + \mathrm{III}_3.
\end{aligned} \tag{3.13}$$

It is easy to show that $\operatorname{Im}[\mathbb{F}(z, y)]_{12} = \operatorname{Im}[\mathbb{F}(z, y)]_{21} = 0$ when $z = y$.

Now we show the inequality (3.11) for the case of $i = j = 1$. The other case is similar. Notice that for $\xi \in (-k_p, k_p)$, $\delta(\xi) \leq k_s^4$ and $\mu_p \leq \mu_s$. Then, if $z = y$,

$$-\operatorname{Im}(\mathrm{III}_1 + \mathrm{III}_2) = \frac{1}{2\pi\mu} \int_{-k_p}^{k_p} \frac{k_s^2 \mu_s}{\delta(\xi)} d\xi \geq \frac{1}{2\pi\mu} \int_{-k_p}^{k_p} \frac{\mu_p}{k_s^2} d\xi = \frac{1}{4(\lambda + 2\mu)}.$$

If $\xi \in (-k_s, k_s)\setminus[-k_p, k_p]$, $\mu_p = \mathbf{i}\sqrt{\xi^2 - k_p^2}$, we have

$$-\mathrm{III}_3 = \frac{1}{2\pi\mu} \int_{(-k_s, k_s)\setminus(-k_p, k_p)} \frac{\mu_s^2 \sqrt{\xi^2 - k_p^2}(k_s^2 - 4\xi^2)}{(\xi^2 + \mathbf{i}\mu_s\sqrt{\xi^2 - k_p^2})(\varphi^2 - \mathbf{i}4\xi^2\mu_s\sqrt{\xi^2 - k_p^2})} d\xi.$$

A simple computation shows that $\mathrm{Im}\,[(\xi^2+\mathbf{i}\mu_s\sqrt{\xi^2-k_p^2})(\varphi^2-\mathbf{i}4\xi^2\mu_s\sqrt{\xi^2-k_p^2})]=k_s^2\mu_s\sqrt{\xi^2-k_p^2}(k_s^2-4\xi^2)$. It is then clear that $-\mathrm{Im}\,(\mathrm{III}_3)\geq 0$. This shows $-\mathrm{Im}\,[\mathbb{F}(z,y)]_{11}\geq 1/[4(\lambda+2\mu)]$ when $z=y$.

For $z\neq y$, we denote $y-z=|y-z|(\cos\phi,\sin\phi)^T$ for some $0\leq\phi\leq\pi$. Then it is easy to see that

$$\mathrm{III}_1=\frac{1}{\mu}\int_0^\pi A(\theta,\kappa)e^{\mathbf{i}k_s|z-y|\cos(\theta-\phi)}d\theta,$$

for some function $A(\theta,\kappa)$. If $k_s|z-y|\geq 1$, since $(0,\pi)$ can be divided into several intervals so that in each interval, either $|\cos(\theta-\phi)|\geq 1/\sqrt{2}$ or $|\sin(\theta-\phi)|\geq 1/\sqrt{2}$ and $-\sin(\theta-\phi)$ is monotone, we can show easily by Van der Corput Lemma 2.4 that

$$|\mathrm{III}_1|\leq\frac{C}{\mu}\left(\frac{1}{(k_s|z-y|)^{1/2}}+\frac{1}{k_s|z-y|}\right).$$

If $k_s|z-y|<1$, then $|\mathrm{III}_1|\leq C\mu^{-1}\leq C\mu^{-1}(k_s|z-y|)^{-1}$. The estimate for $\mathrm{III}_2+\mathrm{III}_3$ can be proved similarly. This completes the proof. □

4. The reverse time migration algorithm

We start by introducing some notation. For any Lipschitz domain $\mathcal{D}\subset\mathbb{R}^2$ with boundary $\Gamma_\mathcal{D}$, let $\|u\|_{H^1(\mathcal{D})}=(\|\nabla\phi\|^2_{L^2(\mathcal{D})}+d_\mathcal{D}^{-2}\|\phi\|^2_{L^2(\mathcal{D})})^{1/2}$ be the weighted $H^1(\mathcal{D})$ norm and $\|v\|_{H^{1/2}(\Gamma_\mathcal{D})}=(d_\mathcal{D}^{-1}\|v\|^2_{L^2(\Gamma_\mathcal{D})}+|v|^2_{\frac12,\Gamma_\mathcal{D}})^{1/2}$ be the weighted $H^{1/2}(\Gamma_\mathcal{D})$ norm, where $d_\mathcal{D}$ is the diameter of $\mathcal{D}$ and

$$|v|_{\frac12,\Gamma_\mathcal{D}}=\left(\int_{\Gamma_\mathcal{D}}\int_{\Gamma_\mathcal{D}}\frac{|v(x)-v(y)|^2}{|x-y|^2}ds(x)ds(y)\right)^{1/2}.$$

By the scaling argument and trace theorem we know that there exists a constant $C>0$ independent of $d_\mathcal{D}$ such that for any $\phi\in C^1(\bar{\mathcal{D}})^2$ [12, corollary 3.1],

$$\|\phi\|_{H^{1/2}(\Gamma_\mathcal{D})}+\|\sigma(\phi)\nu\|_{H^{-1/2}(\Gamma_\mathcal{D})}\leq C\max_{x\in\bar{\mathcal{D}}}(|\phi(x)|+d_\mathcal{D}|\nabla\phi(x)|). \tag{4.1}$$

In this paper, for any Sobolev space X, we still denote X the vector valued space X^2 or tensor valued space $X^{2\times2}$. The norms of $X,X^2,X^{2\times2}$ are all denoted by $\|\cdot\|_X$.

Lemma 4.1 *Let $k_sh\geq 1, d\gg h$, there exists a constant C depending only on κ but independent of k_s,h,d,d_D such that for any $z\in\Omega$, $j=1,2$,*

$$\|\mathbb{F}(z,\cdot)e_j\|_{H^{1/2}(\Gamma_D)}+\|\sigma(\mathbb{F}(z,\cdot)e_j)\nu\|_{H^{-1/2}(\Gamma_D)}\leq\frac{C}{\mu}(1+k_sd_D),$$

$$\|\mathbb{R}_d(z,\cdot)e_j\|_{H^{1/2}(\Gamma_D)}+\|\sigma(\mathbb{R}_d(z,\cdot)e_j)\nu\|_{H^{-1/2}(\Gamma_D)}\leq\frac{C}{\mu}(1+k_sd_D)\left[\left(\frac{h}{d}\right)^2+(k_sh)^{-1/4}\right],$$

where $\mathbb{R}_d(z,\cdot)=\mathbb{J}_d(z,\cdot)-\mathbb{F}(z,\cdot)$.

Proof. The first estimate follows easily from (4.1) and the definition of $\mathbb{F}(z,\cdot)$ in (3.4). The second estimate follows from (4.1), Lemma 3.1 and Theorem 3.1. This completes the proof. □

Now we briefly recall the classical argument of limiting absorption principle (see e.g. [25, 30, 18]) to define the scattering solution for the exterior elastic scattering problem in the half space:

$$\Delta_e u + \omega^2 u = 0 \quad \text{in } \mathbb{R}^2_+ \backslash \bar{D}, \tag{4.2}$$

$$u = g \quad \text{on } \Gamma_D, \qquad \sigma(u)e_2 = 0 \quad \text{on } \Gamma_0, \tag{4.3}$$

where $g \in H^{1/2}(\Gamma_D)$. Let $\varepsilon > 0$ and u_ε be the solution of the problem

$$\Delta_e u_\varepsilon + [\omega(1+\mathbf{i}\varepsilon)]^2 u_\varepsilon = 0 \quad \text{in } \mathbb{R}^2_+ \backslash \bar{D}, \tag{4.4}$$

$$u_\varepsilon = g \quad \text{on } \Gamma_D, \qquad \sigma(u_\varepsilon)e_2 = 0 \quad \text{on } \Gamma_0. \tag{4.5}$$

By the Lax-Milgram lemma, the problem (4.4)-(4.5) has a unique solution $u_\varepsilon \in H^1(\mathbb{R}^2_+ \backslash \bar{D})$. Let $\mathcal{D}(\Delta_e) = \{v \in H^1(\mathbb{R}^2_+ \backslash \bar{D}) : \Delta_e v \in L^2(\mathbb{R}^2_+ \backslash \bar{D}), v = 0 \text{ on } \Gamma_D, \sigma(v)e_2 = 0 \text{ on } \Gamma_0\}$ as the domain of the operator $-\Delta_e$, it is shown in [18] that if ω^2 is not the eigenvalue for $-\Delta_e$ in the domain $\mathcal{D}(\Delta_e)$, u_ε converges to some function u satisfying (4.2)-(4.3) in $H^{1,-s}(\mathbb{R}^2_+ \backslash \bar{D})$, $s > 1/2$, where the weighted Sobolev space $H^{1,s}(\mathbb{R}^2_+ \backslash \bar{D})$, $s \in \mathbb{R}$, is defined as the set of functions in $L^{2,s}(\mathbb{R}^2_+ \backslash \bar{D}) = \{v \in L^2_{\text{loc}}(\mathbb{R}^2_+ \backslash \bar{D}) : (1+|x|^2)^{s/2} v \in L^2(\mathbb{R}^2_+ \backslash \bar{D})\}$ whose first derivatives are also in $L^{2,s}(\mathbb{R}^2_+ \backslash \bar{D})$. The norm $\|v\|_{H^{1,s}(\mathbb{R}^2_+ \backslash \bar{D})} = (\|v\|^2_{L^{2,s}(\mathbb{R}^2_+ \backslash \bar{D})} + \|\nabla v\|^2_{L^{2,s}(\mathbb{R}^2_+ \backslash \bar{D})})^{1/2}$, where $\|v\|_{L^{2,s}(\mathcal{D})} = (\int_{\mathcal{D}} (1+|x|^2)^s |v|^2 dx)^{1/2}$. The absence of the positive eigenvalue for the operator $-\Delta_e$ is proved in [29] in the domain $\mathcal{D}'(\Delta_e) = \{v \in H^1(\mathbb{R}^2_+ \backslash \bar{D}), \Delta_e v \in L^2(\mathbb{R}^2_+ \backslash \bar{D}), \sigma(v)\nu = 0 \text{ on } \Gamma_D, \sigma(v)e_2 = 0 \text{ on } \Gamma_0\}$. One can easily extend the argument in [29] to show the absence of the positive eigenvalue for $-\Delta_e$ also in the domain $\mathcal{D}(\Delta_e)$ and thus obtain the following theorem for the forward scattering problem.

Theorem 4.1 *Let $g \in H^{1/2}(\Gamma_D)$. The half-space elastic scattering problem (4.2)-(4.3) admits a unique solution $u \in H^1_{\text{loc}}(\mathbb{R}^2_+ \backslash \bar{D})$. Moreover, for any bounded open set $\mathcal{O} \subset \mathbb{R}^2_+ \backslash \bar{D}$ there exists a constant $C > 0$ such that $\|u\|_{H^1(\mathcal{O})} \le C\|g\|_{H^{1/2}(\Gamma_D)}$.*

For the sake of convenience, we introduce the following notation: for any $u, v \in H^1(\mathbb{R}^2 \backslash \bar{D})$ such that $\Delta_e u, \Delta_e v \in L^2(\mathbb{R}^2 \backslash \bar{D})$,

$$\mathcal{G}(u,v) = \int_{\Gamma_D} [u(x) \cdot \sigma(v(x))\nu - \sigma(u(x))\nu \cdot v(x)] ds(x). \tag{4.6}$$

Using this notation, the integral representation formula for the solution of the half-space elastic scattering problem reads:

$$u(y) \cdot q = \mathcal{G}(u(\cdot), \mathbb{N}(\cdot, y)q), \quad \forall y \in \mathbb{R}^2_+ \backslash \bar{D}, \quad \forall q \in \mathbb{R}^2. \tag{4.7}$$

Now we introduce the RTM algorithm for the half-space inverse elastic scattering problem. Assume that there are $N_s \ge 1$ sources and $N_r \ge 1$ receivers uniformly distributed on Γ^d_0.

For any $q \in \mathbb{R}^2$, let u^i_q be the incident field which satisfies

$$\Delta_e u^i_q(x, x_s) + \omega^2 u^i_q(x, x_s) = 0 \quad \text{in } \mathbb{R}^2_+, \quad u^i_q(x, x_s) = q\delta_{x_s}(x) \quad \text{on } \Gamma_0.$$

By the integral representation formula, $u_q^i(x,x_s) = \mathbb{T}_D(x_s,x)^T q$. The following algorithm extends the algorithm in [12, 33] for acoustic waves.

Algorithm 4.1 (RTM ALGORITHM FOR HALF-SPACE ELASTIC SCATTERING DATA) *Given the data $u_q^s(x_r,x_s)$ which is the measurement of the scattered field at x_r when the source is emitted at x_s along the polarized direction $q = e_1, e_2$, $s = 1,\dots,N_s$, $r = 1,\dots,N_r$.*

1° *Back-propagation: Compute $v_q(x,x_s)$ as the scattering solution of the following half-space elastic scattering problem:*

$$\Delta_e v_q(x,x_s) + \omega^2 v_q(x,x_s) = 0 \quad \text{in } \mathbb{R}^2_+,$$
$$v_q(x,x_s) = \frac{|\Gamma_0^d|}{N_r}\sum_{r=1}^{N_r} \overline{u_q^s(x_r,x_s)}\delta_{x_r}(x) \quad \text{on } \Gamma_0.$$

2° *Cross-correlation: For each $z \in \Omega$, compute the imaging function*

$$I_d(z) = \mathrm{Im} \sum_{q=e_1,e_2} \left\{ \frac{|\Gamma_0^d|}{N_s} \sum_{s=1}^{N_s} u_q^i(z,x_s)\cdot v_q(z,x_s) \right\}. \tag{4.8}$$

By the integral representation formula, we know that

$$v_q(x,x_s)\cdot e_j = \frac{|\Gamma_0^d|}{N_r}\sum_{r=1}^{N_r} \mathbb{T}_D(x_r,x)e_j \cdot \overline{u_q^s(x_r,x_s)},$$

which yields

$$I_d(z) = \mathrm{Im} \sum_{q=e_1,e_2} \left\{ \frac{|\Gamma_0^d|^2}{N_s N_r} \sum_{s=1}^{N_s}\sum_{r=1}^{N_r} [\mathbb{T}_D(x_s,z)^T q]\cdot[\mathbb{T}_D(x_r,z)^T \overline{u_q^s(x_r,x_s)}] \right\}. \tag{4.9}$$

This is the formula will be used in our numerical examples in section 6. By letting $N_s, N_r \to \infty$, we know that (4.8) can be viewed as an approximation of the following continuous integral:

$$\hat{I}_d(z) = \mathrm{Im} \sum_{q=e_1,e_2} \int_{\Gamma_0^d}\int_{\Gamma_0^d} [\mathbb{T}_D(x_s,z)^T q]\cdot[\mathbb{T}_D(x_r,z)^T \overline{u_q^s(x_r,x_s)}]\, ds(x_r)ds(x_s). \tag{4.10}$$

The following theorem which extends [12, Theorem 4.1] for acoustic waves will be proved in the Appendix of this paper. It shows that the difference between the half-space scattering solution and the full space scattering solution is small when the scatterer is far away from the boundary Γ_0.

Theorem 4.2 *Let $g \in H^{1/2}(\Gamma_D)$ and u_1, u_2 be the scattering solution of following problems:*

$$\Delta_e u_1 + \omega^2 u_1 = 0 \quad \text{in } \mathbb{R}^2_+\backslash\bar{D}, \quad u_1 = g \quad \text{on } \Gamma_D, \quad \sigma(u_1)e_2 = 0 \quad \text{on } \Gamma_0, \tag{4.11}$$
$$\Delta_e u_2 + \omega^2 u_2 = 0 \quad \text{in } \mathbb{R}^2\backslash\bar{D}, \quad u_2 = g \quad \text{on } \Gamma_D. \tag{4.12}$$

Then there exits a constant C depending only on κ but independent of k_s, h, d_D such that

$$\|\sigma(u_1-u_2)\nu\|_{H^{-1/2}(\Gamma_D)} \le \frac{C}{\mu}(1+\|T_1\|)(1+\|T_2\|)(1+k_s d_D)^2(k_s h)^{-1/2}\|g\|_{H^{1/2}(\Gamma_D)}.$$

Here $T_1, T_2 : H^{1/2}(\Gamma_D) \to H^{-1/2}(\Gamma_D)$ are the Dirichlet to Neumann mapping associated with the elastic scattering problem (4.11) and (4.12), respectively. $\|T_1\|, \|T_2\|$ denote their operator norms.

We remark that the well-posedness of the full space elastic scattering problem (4.12) under the so-called Sommerfeld-Kupradze radiation condition is well known (cf. e.g. [23]). It is equivalent to the solution defined by the limiting absorption principle [25, 13].

The following theorem, which relates the imaging function $\hat{I}_d(z)$ to the point spread function in section 3, shows the resolution of our RTM imaging algorithm for the half-space inverse elastic scattering problems.

Theorem 4.3 *For any $z \in \Omega$, let $\mathbb{U}(z,x) \in \mathbb{C}^{2\times 2}$ such that $\mathbb{U}(z,x)e_j$, $j = 1, 2$, is the scattering solution of the problem:*

$$\Delta_e[\mathbb{U}(z,x)e_j] + \omega^2[\mathbb{U}(z,x)e_j] = 0 \quad in\ \mathbb{R}^2\backslash\bar{D}, \qquad \mathbb{U}(z,x)e_j = -\overline{\mathbb{F}(z,x)}e_j \quad on\ \Gamma_D.$$

Then, we have

$$\hat{I}_d(z) = \mathrm{Im}\sum_{j=1}^{2}\int_{\Gamma_D}[\sigma(\mathbb{U}(z,x)e_j + \overline{\mathbb{F}(z,x)}e_j)\nu]\cdot[\overline{\mathbb{F}(z,x)}e_j]ds(x) + R_d(z), \tag{4.13}$$

where $|R_d(z)| \le C\mu^{-2}(1 + \|T_1\|)(1 + \|T_2\|)(1 + k_s d_D)^3\left[\left(\frac{h}{d}\right)^2 + (k_s h)^{-1/4}\right]$ for some constant C depending only on κ but independent of k_s, k_p, h, d, d_D.

Proof. From (4.10) we know that

$$\hat{I}_d(z) = \mathrm{Im}\sum_{q=e_1,e_2}\int_{\Gamma_0^d}[\mathbb{T}_D(x_s,z)^T q]\cdot\hat{v}_q(z,x_s)ds(x_s), \tag{4.14}$$

where for $j = 1, 2$,

$$\hat{v}_q(z,x_s)\cdot e_j = \int_{\Gamma_0^d}\mathbb{T}_D(x_r,z)e_j\cdot\overline{u_q^s(x_r,x_s)}ds(x_r).$$

By (4.7) we know that $u_q^s(x_r,x_s)\cdot e_i = \mathcal{G}(u_q^s(\cdot,x_s),\mathbb{N}(\cdot,x_r)e_i), i = 1, 2$, and thus

$$\hat{v}_q(z,x_s)\cdot e_j = \mathcal{G}(\overline{u_q^s(\cdot,x_s)},\left[\int_{\Gamma_0^d}\sum_{i=1}^{2}[\mathbb{T}_D(x_r,z)]_{ij}\overline{\mathbb{N}(\cdot,x_r)}e_i ds(x_r)\right]).$$

By using the reciprocity relation $\mathbb{N}(x,x_r) = \mathbb{N}(x_r,x)^T$ and the definition of $\mathbb{J}_d(\cdot,\cdot)$ in (3.1), we obtain

$$\int_{\Gamma_0^d}\sum_{i=1}^{2}[\mathbb{T}_D(x_r,z)]_{ij}\overline{\mathbb{N}(\cdot,x_r)}e_i ds(x_r) = \mathbb{J}_d(z,x)^T e_j. \tag{4.15}$$

This implies $\hat{v}_q(z,x_s)e_j = \mathcal{G}(\overline{u_q^s(\cdot,x_s)},\mathbb{J}_d(z,\cdot)^T e_j)$. Substitute it into (4.14) we have

$$\hat{I}_d(z) = \mathrm{Im}\sum_{j=1}^{2}\mathcal{G}(\mathbb{W}(z,\cdot)e_j,\mathbb{J}_d(z,\cdot)^T e_j), \tag{4.16}$$

where $\mathbb{W}(z,x)\in\mathbb{C}^{2\times 2}$ is the tensor defined by

$$\mathbb{W}(z,x)e_j=\int_{\Gamma_0^d}\sum_{k=1}^{2}[\mathbb{T}_D(x_s,z)]_{kj}\overline{u^s_{e_k}(x,x_s)}ds(x_s),\quad j=1,2.$$

Notice that $\overline{\mathbb{W}(z,x)}e_j$ can be viewed as the weighted superposition of $u^s_{e_k}(x,x_s)$ and thus it satisfies

$$\Delta_e[\overline{\mathbb{W}(z,x)}e_j]+\omega^2[\overline{\mathbb{W}(z,x)}e_j]=0\ \text{ in }\mathbb{R}^2_+\backslash\bar{D},\qquad \sigma(\overline{\mathbb{W}(z,x)}e_j)e_2=0\ \text{ on }\Gamma_0.\tag{4.17}$$

On the boundary of the obstacle Γ_D, by using (4.15) we have

$$\overline{\mathbb{W}(z,x)}e_j=-\int_{\Gamma_0^d}\sum_{k=1}^{2}[\overline{\mathbb{T}_D(x_s,z)}]_{kj}\mathbb{N}(x,x_s)e_k ds(x_s)=-\overline{\mathbb{J}_d(z,x)}^Te_j.\tag{4.18}$$

Now define the tensor $\mathbb{W}_d(z,x)\in\mathbb{C}^{2\times 2}$ such that $\mathbb{W}_d(z,x)e_j$, $j=1,2$, is the scattering solution of the problem

$$\Delta_e[\mathbb{W}_d(z,x)e_j]+\omega^2[\mathbb{W}_d(z,x)e_j]=0\quad \text{ in }\mathbb{R}^2_+\backslash\bar{D},\tag{4.19}$$

$$\mathbb{W}_d(z,x)e_j=-\overline{\mathbb{F}(z,x)}e_j\ \text{ on }\Gamma_D,\qquad \sigma(\mathbb{W}_d(z,x)e_j)e_2=0\ \text{ on }\Gamma_0.\tag{4.20}$$

By (4.16) we deduce

$$\begin{aligned}\hat{I}_d(z)&=\mathrm{Im}\sum_{j=1}^{2}\mathcal{G}(\mathbb{W}(z,\cdot)e_j,J_d(z,\cdot)^Te_j-\mathbb{F}(z,\cdot)e_j)\\&\quad+\mathrm{Im}\sum_{j=1}^{2}\mathcal{G}(\mathbb{W}(z,\cdot)e_j-\overline{\mathbb{W}_d(z,\cdot)}e_j,\mathbb{F}(z,\cdot)e_j)\\&\quad+\mathrm{Im}\sum_{j=1}^{2}\mathcal{G}(\overline{\mathbb{W}_d(z,\cdot)}e_j-\overline{\mathbb{U}(z,\cdot)}e_j,\mathbb{F}(z,\cdot)e_j)\\&\quad+\mathrm{Im}\sum_{j=1}^{2}\mathcal{G}(\overline{\mathbb{U}(z,\cdot)}e_j,\mathbb{F}(z,\cdot)e_j):=\mathrm{VI}_1+\cdots+\mathrm{VI}_4.\end{aligned}\tag{4.21}$$

Recall that $\mathbb{F}(z,y)^T=\mathbb{F}(z,y)$ by Theorem 3.2. By Lemma 4.1,

$$\|\mathbb{J}_d(z,\cdot)e_j\|_{H^{1/2}(\Gamma_D)}+\|\sigma(\mathbb{J}_d(z,\cdot)e_j)\nu\|_{H^{-1/2}(\Gamma_D)}\le\frac{C}{\mu}(1+k_sd_D).$$

This implies, by (4.17)-(4.18) and Lemma 4.1, that

$$\begin{aligned}|\mathrm{VI}_1|&\le\sum_{j=1}^{2}\Big(\|\mathbb{W}(z,\cdot)e_j\|_{H^{1/2}(\Gamma_D)}\|\sigma(\mathbb{J}_d(z,\cdot)^Te_j-\mathbb{F}(z,\cdot)e_j)e_2\|_{H^{-1/2}(\Gamma_D)}\\&\qquad+\|\sigma(\mathbb{W}(z,\cdot)e_j)e_2\|_{H^{-1/2}(\Gamma_D)}\|\mathbb{J}_d(z,\cdot)^Te_j-\mathbb{F}(z,\cdot)e_j\|_{H^{1/2}(\Gamma_D)}\Big)\\&\le\frac{C}{\mu^2}(1+\|T_1\|)(1+k_sd_D)^2\left[\left(\frac{h}{d}\right)^2+(k_sh)^{-1/4}\right].\end{aligned}$$

From (4.17)-(4.18) and (4.19)-(4.20), we obtain by using Lemma 4.1 that

$$|\mathrm{VI}_2|\le\frac{C}{\mu^2}(1+\|T_1\|)(1+k_sd_D)^2\left[\left(\frac{h}{d}\right)^2+(k_sh)^{-1/4}\right].$$

To estimate the third term, we use Theoremn 4.2 and Lemma 4.1 to obtain

$$|\mathrm{VI}_3| \leq \frac{C}{\mu^2}(1+\|T_1\|)(1+\|T_2\|)(1+k_sd_D)^3(k_sh)^{-1/2}.$$

Finally, since $\mathbb{U}(z,x)e_j = -\overline{\mathbb{F}(z,x)}e_j$ on Γ_D,

$$\begin{aligned}
\mathrm{IV}_4 &= \mathrm{Im}\sum_{j=1}^{2}\int_{\Gamma_D}(\overline{\mathbb{U}(z,x)}e_j\cdot\sigma(\mathbb{F}(z,x)e_j)\nu - \sigma(\overline{\mathbb{U}(z,x)}e_j)\nu\cdot\mathbb{F}(z,x)e_j)ds(x)\\
&= -\,\mathrm{Im}\sum_{j=1}^{2}\int_{\Gamma_D}\sigma(\overline{\mathbb{U}(z,x)}e_j+\mathbb{F}(z,x)e_j)\nu\cdot\mathbb{F}(z,x)e_jds(x)\\
&= \mathrm{Im}\sum_{j=1}^{2}\int_{\Gamma_D}\sigma(\mathbb{U}(z,x)e_j+\overline{\mathbb{F}(z,x)}e_j)\nu\cdot\overline{\mathbb{F}(z,x)}e_jds(x).
\end{aligned}$$

The theorem follows now from (4.21). □

We remark that Theorem 4.3 indicates for extended obstacles, that is, when $k_sd_D \approx 1$, the error $R_d(z)$ in the imaging function is small when the obstacle is far away from the surface $k_sh \gg 1$ and the aperture is large $d \gg h$. Thus the first term on the right-hand side of (4.13) dominates in the imaging function $\hat{I}_d(z)$.

By (3.13) we know that for any fixed $z \in \Omega$ and some functions $A_j(\xi), B_j(\xi)$, $j = 1, 2$,

$$\begin{aligned}
\mathbb{F}(z,x)e_j &= \int_{-k_p}^{k_p} A_j(\xi)\begin{pmatrix}-\xi\\ \mu_p\end{pmatrix}e^{\mathbf{i}(z-x)\cdot(-\xi,\mu_p)^T}d\xi + \int_{-k_s}^{k_s} B_j(\xi)\begin{pmatrix}\mu_s\\ \xi\end{pmatrix}e^{\mathbf{i}(z-x)\cdot(-\xi,\mu_s)^T}d\xi\\
&= \int_0^{\pi}\left[\tilde{A}_j(\theta)\tau(\theta)e^{\mathbf{i}k_p(z-x)\cdot\tau(\theta)} + \tilde{B}_j(\theta)\tau(\theta)^{\perp}e^{\mathbf{i}k_s(z-x)\cdot\tau(\theta)}\right]d\theta,
\end{aligned}$$

where $\tilde{A}_j(\theta) = k_pA_j(k_p\cos\theta)\sin\theta, \tilde{B}_j(\theta) = k_sB_j(k_s\cos\theta)\sin\theta,\ \tau(\theta) = (-\cos\theta,\sin\theta)^T$ and $\tau(\theta)^{\perp} = (\sin\theta,\cos\theta)^T$. Thus $\overline{\mathbb{F}(z,x)}e_j$ is the weighted superposition of planar p and s waves and thus satisfies the elastic wave equation. Therefore, $\mathbb{U}(z,x)e_j$ can be viewed as the scattering solution of the elastic equation with the incident wave $\overline{\mathbb{F}(z,x)}e_j$. By Theorem 3.2 we know that $\overline{\mathbb{F}(z,x)}$ decays as $|x-z|$ becomes large. Thus the imaging function $\hat{I}_d(z)$ becomes small when z moves away from the boundary Γ_D if $k_sh \gg 1$ and $d \gg h$.

To understand the behavior of the imaging function when z is close to the boundary of the scatterer, we introduce the concept of the scattering coefficient for incident plane waves.

Definition 4.1 *For any unit vector $\tau \in \mathbb{R}^2$, let $u_p^i = \tau e^{\mathbf{i}k_px\cdot\tau}$, $u_s^i = \tau^{\perp}e^{\mathbf{i}k_sx\cdot\tau}$ be the incident planar p and s wave. Let $u_\alpha^s(x) = u_\alpha^s(x;\tau), \alpha = p, s$, be the corresponding scattering solution of the elastic wave equation:*

$$\Delta_e u_\alpha^s + \omega^2 u_\alpha^s = 0 \quad in\ \mathbb{R}^2\backslash\bar{D}, \qquad u_\alpha^s = -u_\alpha^i \quad on\ \Gamma_D. \tag{4.22}$$

The scattering coefficient $R_\alpha(x;\tau)$, $x \in \Gamma_D$, is defined by the relation

$$\sigma(u_\alpha^s(x) + u_\alpha^i(x))\nu(x) = \mathbf{i}k_\alpha R_\alpha(x;\tau)e^{\mathbf{i}k_\alpha x\cdot\tau} \quad on\ \Gamma_D.$$

Here for $\tau = (\tau_1, \tau_2)^T \in \mathbb{R}^2$, $\tau^\perp = (\tau_2, -\tau_1)^T$.

With this definition we deduce from Theorem 4.3 that for any $z \in \Gamma_D$,

$$\hat{I}_d(z) \approx \mathrm{Im} \sum_{j=1}^{2} \int_{\Gamma_D} \left[\int_0^\pi \overline{\tilde{A}_j(\theta)} \mathbf{i} k_p R_p(x; \tau(\theta)) e^{\mathbf{i} k_p (x-z)\cdot\tau(\theta)} d\theta \right] \cdot \overline{\mathbb{F}(z,x)} e_j ds(x)$$
$$+ \mathrm{Im} \sum_{j=1}^{2} \int_{\Gamma_D} \left[\int_0^\pi \overline{\tilde{B}_j(\theta)} \mathbf{i} k_s R_s(x; \tau(\theta)) e^{\mathbf{i} k_s (x-z)\cdot\tau(\theta)} d\theta \right] \cdot \overline{\mathbb{F}(z,x)} e_j ds(x).$$

In the case of Kirchhoff high-frequency approximation for convex scatterers, the scattering coefficient is approximately zero in the shadow region of the obstacle:

$$R_\alpha(x;\tau) \approx 0 \ \text{ if } x \in \Gamma_D^+(\tau) = \{x \in \Gamma_D, \nu(x)\cdot\tau > 0\}, \quad \alpha = p, s. \tag{4.23}$$

In the illuminating region where $\nu(x)\cdot\tau < 0$, the Kirchhoff approximation approximates the total field by considering the boundary at $x \in \Gamma_D$ locally as a line with the normal $\nu(x)$. By using the analytical representation for the incident plane wave over the line [1, P. 172], the total field of (4.22) with the incident p-wave is approximated by

$$u_p^{\text{total}} = u_p^i + u_p^s = \tau e^{\mathbf{i} k_p x\cdot\tau} + A_1 d_1 e^{\mathbf{i} k_p x\cdot d_1} + A_2 d_2^\perp e^{\mathbf{i} k_s x\cdot d_2},$$

where

$$d_1 = \tau - 2(\tau\cdot\nu)\nu, d_2 = \kappa\tau - [\kappa(\tau\cdot\nu) + \mathrm{sgn}(\tau\cdot\nu)\sqrt{1-\kappa^2(\tau\cdot\nu^\perp)^2}]\nu,$$
$$A_1 = -(\tau\cdot d_2)/(d_1\cdot d_2), A_2 = 2(\tau\cdot\nu)(\tau\cdot\nu^\perp)/(d_1\cdot d_2).$$

Here $\nu = \nu(x)$. Similarly, the total field of (4.22) with the incident s-wave is approximated by

$$u_s^{\text{total}} = u_s^i + u_s^s = \tau^\perp e^{\mathbf{i} k_s x\cdot\tau} + A_3 d_3 e^{\mathbf{i} k_p x\cdot d_3} + A_4 d_4^\perp e^{\mathbf{i} k_s x\cdot d_4},$$

where

$$d_3 = \kappa^{-1}\tau - [\kappa^{-1}(\tau\cdot\nu) + \mathrm{sgn}(\tau\cdot\nu)\sqrt{1-\kappa^{-2}(\tau\cdot\nu^\perp)^2}]\nu, d_4 = \tau - 2(\tau\cdot\nu)\nu,$$
$$A_3 = -2(\tau\cdot\nu)(\tau\cdot\nu^\perp)/(d_3\cdot d_4), A_4 = -(\tau\cdot d_3)/(d_3\cdot d_4).$$

This implies by direct calculation that under Kirchhoff approximation,

$$R_p(x;\tau) \approx \lambda\nu + 2\mu(\tau\cdot\nu)\tau + A_1[\lambda\nu + 2\mu(d_1\cdot\nu)d_1]$$
$$+ A_2\mu[(d_2\cdot\nu)d_2^\perp + (d_2^\perp\cdot\nu)d_2], \tag{4.24}$$
$$R_s(x;\tau) \approx \mu[(\tau\cdot\nu)\tau^\perp + (\tau^\perp\cdot\nu)\tau] + A_3[\lambda\nu + 2(d_3\cdot\nu)d_3]$$
$$+ A_4\mu[(d_4\cdot\nu)d_4^\perp + (d_4^\perp\cdot\nu)d_4]. \tag{4.25}$$

Let $x(s)$, $0 < s < L$, be the arc length parametrization of the boundary Γ_D and $x_\pm(\theta)$ be the points on Γ_D such that $\nu(x_\pm(\theta)) = \pm\tau(\theta)$. By using the method of the stationary phase and the Kirchhoff approximation in (4.23) we can obtain as in [12] that

$$\hat{I}_d(z) \approx \mathrm{Im} \sum_{j=1}^{2} \sqrt{2\pi k_p} \int_0^\pi \overline{\tilde{A}_j(\theta)} e^{\mathbf{i} k_p (x_-(\theta)-z)\cdot\tau(\theta) + \mathbf{i}\frac{\pi}{4}} \frac{R_p(x_-(\theta);\tau(\theta)) \cdot \overline{\mathbb{F}(z, x_-(\theta))} e_j}{\sqrt{\kappa(x_-(\theta))}} d\theta$$
$$\mathrm{Im} \sum_{j=1}^{2} \sqrt{2\pi k_s} \int_0^\pi \overline{\tilde{B}_j(\theta)} e^{\mathbf{i} k_s (x_-(\theta)-z)\cdot\tau(\theta) + \mathbf{i}\frac{\pi}{4}} \frac{R_s(x_-(\theta);\tau(\theta)) \cdot \overline{\mathbb{F}(z, x_-(\theta))} e_j}{\sqrt{\kappa(x_-(\theta))}} d\theta.$$

Here $\kappa(x)$ is the curvature of Γ_D. Now for z in the part of Γ_D which is back to Γ_0, i.e., $\nu(z)\cdot\tau(\theta)>0$ for any $\theta\in(0,\pi)$, we know that z and $x_-(\theta)$ are far away and thus $\hat{I}_d(z)\approx 0$. This indicates that one cannot image the back part of the obstacle with only the data collected on Γ_0. This is confirmed in our numerical examples in section 6.

On the other hand, if z is in the illuminating part of Γ_D where $\nu(z)\cdot\tau(\theta)<0$ for some $\theta\in(0,\pi)$, by $\nu(x_-(\theta))=-\tau(\theta)$ we know from (4.24)-(4.25) that

$$R_p(x_-(\theta);\tau(\theta))\approx -2(\lambda+2\mu)\tau(\theta),\quad R_s(x_-(\theta);\tau(\theta))\approx -2\mu\tau(\theta)^\perp.$$

This implies that $\hat{I}_d(z)$ is the weighted sum of $[\kappa(x_-(\theta))]^{-1/2}$ for $x_-(\theta)$ close to z on Γ_D due to the property of $\mathbb{F}(z,x_-(\theta))$ in Theorem 3.2.

5. Extensions

In this section we consider the reconstruction of non-penetrable obstacles with the impedance boundary condition and penetrable obstacles in the half space by the RTM algorithm 4.1. For non-penetrable obstacles with the impedance boundary condition on the obstacle, the measured data $u_q(x_r,x_s)=u_q^s(x_r,x_s)+\mathbb{N}(x_r,x_s)q$, $q=e_1,e_2$, where $u_q^s(x,x_s)$ is the scattering solution of the following problem:

$$\begin{aligned}&\Delta_e u_q^s(x,x_s)+\omega^2 u_q^s(x,x_s)=0\ \text{ in } \mathbb{R}^2_+\backslash\bar{D},\\&\sigma(u_q^s(x,x_s))\nu+\mathbf{i}\eta(x)u_q^s(x,x_s)=-[\sigma(\mathbb{N}(x,x_s)q)\nu+\mathbf{i}\eta(x)\mathbb{N}(x,x_s)q]\ \text{ on } \Gamma_D,\\&\sigma(u_q^s(x,x_s))e_2=0\ \text{ on } \Gamma_0,\end{aligned}$$

where $\eta\in L^\infty(\Gamma_D)$ and $\eta\geq 0$ on Γ_D. By modifying the argument in the proof of Theorem 4.3, we can show the following theorem whose proof is omitted.

Theorem 5.1 *For any $z\in\Omega$, let $\mathbb{U}(z,x)\in\mathbb{C}^{2\times 2}$ such that $\mathbb{U}(z,x)e_j$, $j=1,2$, is the scattering solution of the problem:*

$$\begin{aligned}&\Delta_e[\mathbb{U}(z,x)e_j]+\omega^2[\mathbb{U}(z,x)e_j]=0\ \text{ in } \mathbb{R}^2\backslash\bar{D},\\&\sigma(\mathbb{U}(z,x)e_j)\nu+\mathbf{i}\eta(x)[\mathbb{U}(z,x)e_j]=-[\sigma(\overline{\mathbb{F}(z,x)}e_j)\nu+\mathbf{i}\eta(x)\overline{\mathbb{F}(z,x)}e_j]\ \text{ on } \Gamma_D.\end{aligned}$$

Then the imaging function (4.10) for the half-space elastic scattering data $u_q^s(x_r,x_s)$ of the non-penetrable obstacle with the impedance boundary condition satisfies

$$\begin{aligned}\hat{I}_d(z)=&-\operatorname{Im}\sum_{j=1}^2\int_{\Gamma_D}[\mathbb{U}(z,x)e_j+\overline{\mathbb{F}(z,x)}e_j]\cdot[\sigma(\overline{\mathbb{F}(z,x)}e_j)\nu+\mathbf{i}\eta(x)\overline{\mathbb{F}(z,x)}e_j]ds(x)\\&+R_d(z),\quad\forall z\in\Omega,\end{aligned}$$

where $|R_d(z)|\leq C\mu^{-2}(1+k_sd_D)^3\left[\left(\frac{h}{d}\right)^2+(k_sh)^{-1/4}\right]$ for some constant C depending only on κ but independent of k_s,k_p,h,d,d_D.

For the penetrable obstacle, the measured data is $u_q(x_r.x_s)=u_q^s(x_r,x_s)+\mathbb{N}(x_r,x_s)q$, $q=e_1,e_2$, where $u_q^s(x,x_s)$ is the scattering solution of the following problem:

$$\begin{aligned}&\Delta_e u_q^s(x,x_s)+\omega^2 n(x)u_q^s(x,x_s)=-\omega^2(n(x)-1)\mathbb{N}(x,x_s)q\ \text{ in } \mathbb{R}^2_+,\\&\sigma(u_q^s(x,x_s))e_2=0\ \text{ on } \Gamma_0,\end{aligned}$$

where $n(x) \in L^\infty(\mathbb{R}^2_+)$ is a positive function which is equal to 1 outside D. By modifying the argument in Theorem 4.3, the following theorem can be proved.

Theorem 5.2 *For any $z \in \Omega$, let $\mathbb{U}(z,x) \in \mathbb{C}^{2\times 2}$ such that $\mathbb{U}(z,x)e_j$, $j = 1, 2$, is the scattering solution of the problem:*

$$\Delta_e[\mathbb{U}(z,x)e_j] + \omega^2 n(x)[\mathbb{U}(z,x)e_j] = -\omega^2(n(x)-1)\overline{\mathbb{F}(z,x)}e_j \quad \text{in } \mathbb{R}^2.$$

Then the imaging function (4.10) for the half-space elastic scattering data $u_q^s(x_r, x_s)$ of the penetrable obstacle satisfies

$$\hat{I}_d(z) = \mathrm{Im}\sum_{j=1}^{2}\int_D \omega^2(n(x)-1)[(\mathbb{U}(z,x)e_j + \overline{\mathbb{F}(z,x)}e_j)\cdot\overline{\mathbb{F}(z,x)}e_j]dx + R_d(z),$$

where $|R_d(z)| \leq C\mu^{-2}(1+k_s d_D)^3\left[\left(\frac{h}{d}\right)^2 + (k_s h)^{-1/4}\right]$ for some constant C depending only on κ but independent of k_s, k_p, h, d, d_D.

6. Numerical examples

In this section we present several numerical examples to show the effectiveness of our RTM algorithm. To synthesize the scattering data we compute the solution $u_q^s(x, x_s)$ of the scattering problems by representing the ansatz solution as the single layer potential with the Neumann Green tensor $\mathbb{N}(x, y)$ as the kernel and discretizing the integral equation by standard Nyström methods [16]. The boundary integral equations on Γ_D are solved on a uniform mesh over the boundary with ten points per probe wavelength. The sources and receivers are both equally placed on the surface Γ_0^d. In all our numerical examples we choose $h = 10, d = 50$ and Lamé constant $\lambda = 1/2$, $\mu = 1/4$. The boundaries of the obstacles used in our numerical experiments are parameterized as follows,

Circle: $x_1 = \rho\cos(\theta)$, $x_2 = \rho\sin(\theta)$;

Kite: $x_1 = \cos(\theta) + 0.65\cos(2\theta) - 0.65$, $x_2 = 1.5\sin(\theta)$;

p-leaf: $r(\theta) = 1 + 0.2\cos(p\theta)$;

peanut: $x_1 = \cos\theta + 0 : 2\cos 3\theta; x_2 = \sin\theta + 0 : 2\sin 3\theta$;

square: $x_1 = \cos 3\theta + \cos\theta; x_2 = \sin 3\theta + \sin\theta$.

where $\theta \in [0, 2\pi]$. The numerical imaging function is (4.9) in section 4.

In the following by Dirichlet, Neumann or impedance obstacle we mean the non-penetrable obstacle that satisfies Dirichlet, Neumann or impedance boundary condition on the boundary of the obstacle.

Example 1. We consider imaging of a Dirichlet, a Neumann, an impedance, and a penetrable obstacle. The imaging domain $\Omega = (-2, 2) \times (8, 12)$ with the sampling grid 201×201. We set $N_s = N_r = 401$. The angular frequency $\omega = 2\pi$.

The imaging results are shown in Figure 2. It demonstrates clearly that our RTM algorithm can effectively image the upper boundary illuminated by the sources and

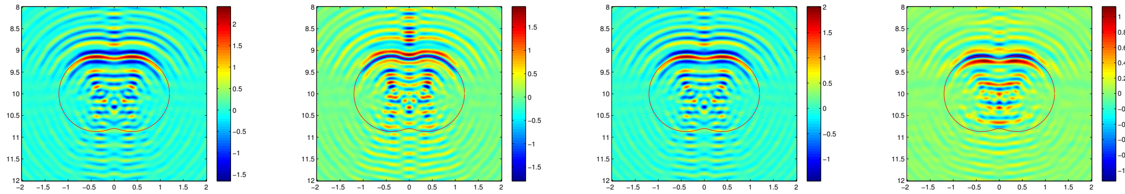

Figure 2. Example 1: From left to right: imaging results of a Dirichlet, a Neumann, an impedance with $\eta(x) = 1$, and a penetrable obstacle with diffractive index $n(x) = 0.25$.

receivers distributed along the boundary Γ_0 for non-penetrable obstacles. The imaging values decrease on the shadow part of the obstacles and at the points away from the boundary of the obstacle.

Example 2. We consider the imaging of Dirichlet obstacles with different shapes including a circle, a peanut, a p-leaf and a rounded square. The imaging domain $\Omega = (-2, 2) \times (8, 12)$ with the sampling grid 201×201. We set $N_s = N_r = 401$. The angular frequency $\omega = 3\pi, 4\pi$ for the single frequency and $\omega = \pi \times [2 : 0.5 : 8]$ for the test of multiple frequencies.

Example 3 We consider the imaging of two Neumann obstacles. The first model consists of two circles along horizontal direction and the second one is a circle and a peanut along the vertical direction. The angular frequency $\omega = 3\pi$ for the test of the single frequency and $\omega = \pi \times [2 : 0.5 : 8]$ for the test of multiple frequencies. Figure 4 shows the imaging result of the first model. The imaging domain $\Omega = (-4, 4) \times (8, 12)$ with mesh size 401×201. We set $N_s = N_r = 301$. Figure 5 shows the imaging result of the second model. The imaging domain $\Omega = (-4, 4) \times (8, 12)$ with mesh size 401×401. We set $N_s = N_r = 301$. The multi-frequency RTM imaging results in Figure 4 and Figure 5 are obtained by adding the inmaging results from different frequencies. We observe from these two figures that imaging results can be greatly improved by stacking the multiple single frequency imaging results.

Example 4 We consider the stability of our half-space RTM imaging algorithm with respect to the complex additive Gaussian random noise. We introduce the additive Gaussian noise as $u_{\text{noise}} = u_s + \nu_{\text{noise}}$, where u_s is the synthesized data and ν_{noise} is the Gaussian noise with mean zero and standard deviation σ times the maximum of the data $|u_s|$, i.e. $\nu_{\text{noise}} = \frac{\sigma \max |u_s|}{\sqrt{2}}(\varepsilon_1 + \mathbf{i}\varepsilon_2)$ and $\varepsilon_i \sim \mathcal{N}(0, 1)$.

Figure 6 shows the imaging results using single frequency data added with additive Gaussian noise. The imaging quality can be greatly improved by using multi-frequency data $\omega = \pi \times [2 : 0.5 : 8]$.

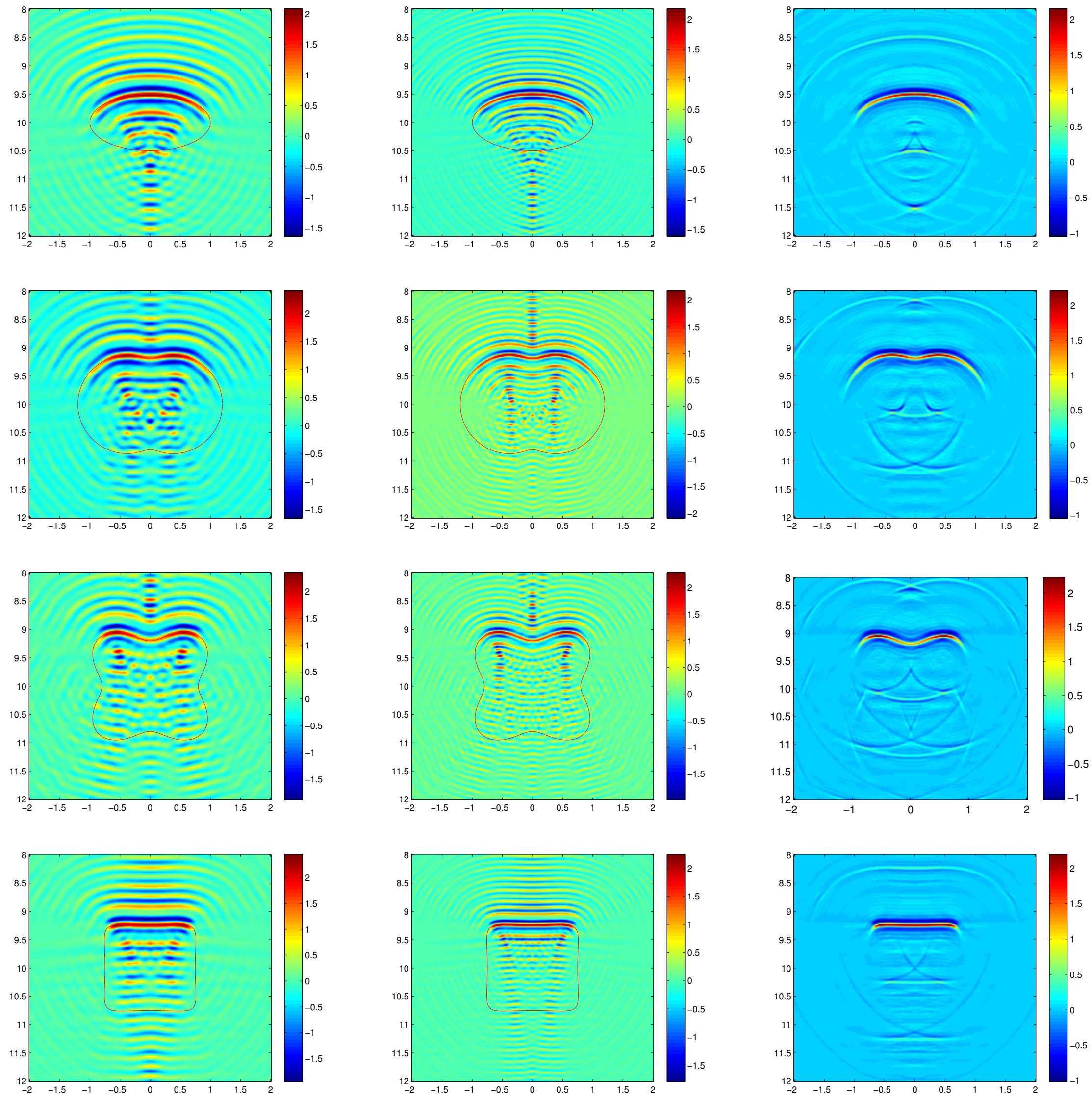

Figure 3. Example 2: Imaging results of Dirichlet obstacles with different shapes from top to below. The left column is imaged with single frequency data $\omega = 3\pi$, The middle column is imaged with single frequency data $\omega = 5\pi$ and the right column is imaged with multiple frequency data.

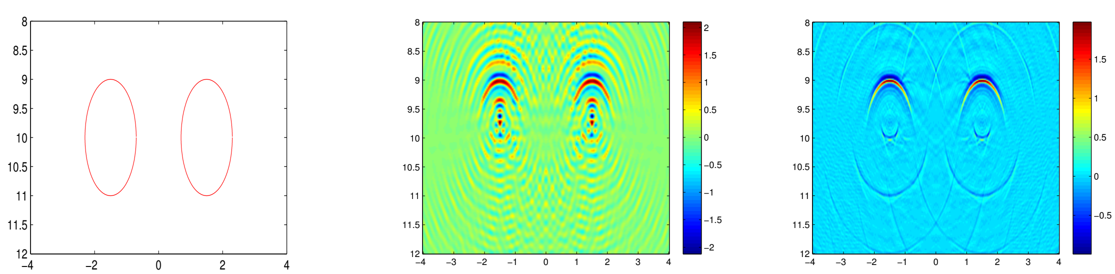

Figure 4. Example 3: From left to right, true obstacle model with two circles, the imaging result with single frequency data $\omega = 3\pi$, the imaging result with multiple frequency data.

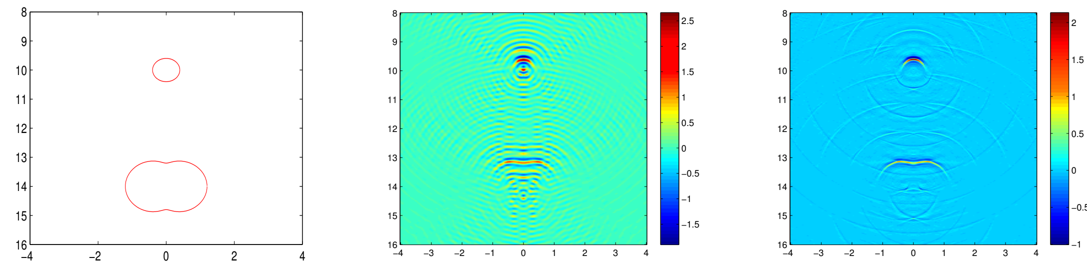

Figure 5. Example 3: From left to right, true obstacle model with one circle and one peanut, the imaging result with single frequency data where $\omega = 3\pi$, the imaging result with multiple frequency data.

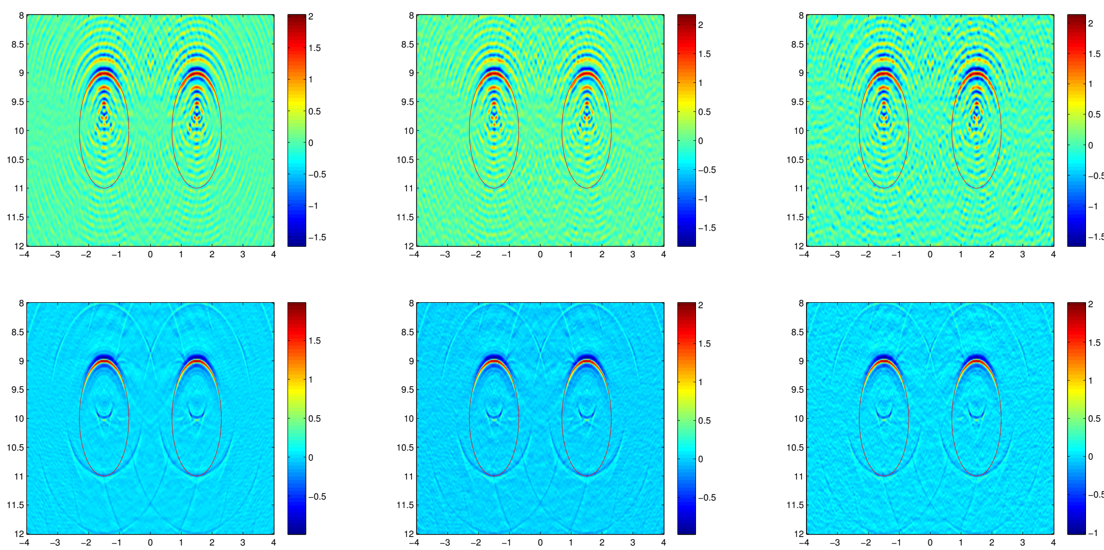

Figure 6. Example 4: Imaging results of a Dirichlet obstacle with noise levels $\mu = 0.2; 0.3; 0.4$ (from left to right). The top row is imaged with single frequency data $\omega = 4\pi$, and the bottom row is imaged with multi-frequency data.

7. Appendix: Proof of Theorem 4.2

In this section we prove Theorem 4.2. Let $w(x)$ be the scattering solution of the problem:

$$\Delta_e w + \omega^2 w = 0 \ \text{ in } \mathbb{R}^2_+, \quad \sigma(w)e_2 = -\sigma(u_2)e_2 \ \text{ on } \Gamma_0. \tag{7.1}$$

Then $u_1 - u_2 - w$ is the scattering solution of the problem (4.11) with the boundary condition $u_1 - u_2 - w = -w$ on Γ_D. Thus by Theorem 4.1 and (4.1), we have

$$\begin{aligned}\|\sigma(u_1 - u_2)\nu\|_{H^{-1/2}(\Gamma_D)} &\leq \|T_1(u_1 - u_2 - w)\|_{H^{-1/2}(\Gamma_D)} + \|\sigma(w)\nu\|_{H^{-1/2}(\Gamma_D)} \\ &\leq C(1 + \|T_1\|) \max_{x\in\bar{D}}(|w(x)| + d_D|\nabla w(x)|),\end{aligned} \tag{7.2}$$

where we recall that $T_1 : H^{1/2}(\Gamma_D) \to H^{-1/2}(\Gamma_D)$ is the Dirichlet to Neumann mapping associated to the half-space elastic scattering problem (4.11) and $\|T_1\|$ denotes its operator norm.

By the integral representation formula, the scattering solution of the problem (7.1) satisfies

$$w(y)\cdot e_j=\int_{\Gamma_0}\sigma(u_2(x))e_2\cdot\mathbb{N}(x,y)e_j ds(x),\quad \forall y\in\mathbb{R}^2,\ j=1,2. \tag{7.3}$$

On the other hand, by the integral representation formula, we have

$$u_2(x)\cdot e_j=\mathcal{G}(u_2(\cdot),\mathbb{G}(\cdot,x)e_j),\quad \forall x\in\Gamma_0,\ j=1,2,$$

where $\mathcal{G}(\cdot,\cdot)$ is defined in (4.6) and $\mathbb{G}(\cdot,\cdot)$ is the fundamental solution tensor of the elastic wave equation introduced in section 2. For any $x\in\Gamma_0, z\in\mathbb{R}^2$, denote by $\mathbb{T}(z,x)\in\mathbb{C}^{2\times 2}$ the traction tensor, $\mathbb{T}(z,x)q=\sigma(\mathbb{G}(z,x)q)e_2,\forall q\in\mathbb{R}^2$. The (i,j)-th element of $\mathbb{T}(z,x)$ is

$$[\mathbb{T}(z,x)]_{ij}=[\sigma(\mathbb{G}(z,x)e_j)e_2]e_i,\quad i,j=1,2.$$

Simple calculation shows that

$$\sigma(u_2(x))e_2\cdot e_i=\mathcal{G}(u_2(\cdot),\mathbb{T}(\cdot,x)^Te_i),\quad \forall x\in\Gamma_0, i=1,2,$$

which yields from (7.3) that

$$\begin{aligned} w(y)\cdot e_j&=\mathcal{G}(u_2(\cdot),\left[\int_{\Gamma_0}\sum_{i=1}^2[\mathbb{T}(\cdot,x)^Te_i]\cdot[e_i^T\mathbb{N}(x,y)e_j]ds(x)\right])\\ &=\mathcal{G}(u_2(\cdot),\mathbb{V}(\cdot,y)e_j), \end{aligned} \tag{7.4}$$

where

$$\mathbb{V}(z,y)=\int_{\Gamma_0}\mathbb{T}(z,x)^T\mathbb{N}(x,y)ds(x),\quad \forall y\in\mathbb{R}^2, z\in\Gamma_D.$$

Notice that $\|\sigma(u_2)\nu\|_{H^{-1/2}(\Gamma_D)}\le\|T_2\|\|g\|_{H^{1/2}(\Gamma_D)}$, where $T_2: H^{1/2}(\Gamma_D)\to H^{-1/2}(\Gamma_D)$ is the Dirichlet to Neumann mapping associated to the scattering problem (4.12) and $\|T_2\|$ denotes its operator norm. We obtain from (7.4) and (4.1) that

$$|w(y)|+d_D|\nabla w(y)|\le C(1+\|T_2\|)\|g\|_{H^{1/2}(\Gamma_D)}\max_{z\in\Gamma_D}\sum_{i,j=0}^1 d_D^{i+j}|\nabla_z^i\nabla_y^j\mathbb{V}(z,y)|. \tag{7.5}$$

To estimate the term involving $\mathbb{V}(z,y)$, we use Parserval identity and Lemma 2.2 to obtain

$$\begin{aligned} \mathbb{V}(z,y)=&\frac{1}{2\pi}\,\text{p.v.}\int_{\mathbb{R}}\hat{\mathbb{T}}(z_2;\xi,0)^T\hat{\mathbb{N}}(\xi,0;y_2)e^{-\mathbf{i}\xi(y_1+z_1)}d\xi\\ &-\frac{\mathbf{i}}{2}\left[\hat{\mathbb{T}}(z_2;\xi,0)^T\hat{\mathbb{N}}(\xi,0;y_2)e^{-\mathbf{i}\xi(y_1+z_1)}\right]_{-k_R}^{k_R}. \end{aligned}$$

It is easy to see from (2.5)-(2.6) that

$$\begin{aligned} \hat{\mathbb{T}}(z_2;\xi,0)=&\ \frac{\mu}{2\omega^2}\begin{pmatrix}\varphi & \frac{\xi\varphi}{\mu_s}\\ 2\xi\mu_s & 2\xi^2\end{pmatrix}e^{\mathbf{i}\mu_s z_2}+\frac{\mu}{2\omega^2}\begin{pmatrix}2\xi^2 & -2\xi\mu_p\\ -\frac{\xi\varphi}{\mu_p} & \varphi\end{pmatrix}e^{\mathbf{i}\mu_p z_2}\\ :=&\ \tilde{\mathbb{T}}_s(\xi)e^{\mathbf{i}\mu_p z_2}+\tilde{\mathbb{T}}_p(\xi)e^{\mathbf{i}\mu_s z_2}. \end{aligned}$$

Now by using (2.10) we have

$$\mathbb{V}(z,y)=\frac{1}{2\pi}\sum_{\alpha,\beta=p,s}\text{p.v.}\int_{\mathbb{R}}\frac{\tilde{\mathbb{T}}_\alpha(\xi)^T\mathbb{N}_\beta(\xi)}{\delta(\xi)}e^{\mathbf{i}(\mu_\alpha z_2+\mu_\beta y_2)-\mathbf{i}(y_1+z_1)\xi}d\xi$$
$$-\frac{\mathbf{i}}{2}\sum_{\alpha,\beta=p,s}\left[\frac{\tilde{\mathbb{T}}_\alpha(\xi)^T\mathbb{N}_\beta(\xi)}{\delta'(\xi)}e^{\mathbf{i}(\mu_\alpha z_2+\mu_\beta y_2)-\mathbf{i}(y_1+z_1)\xi}\right]_{-k_R}^{k_R}:=\mathrm{V}_1+\mathrm{V}_2.$$

To estimate V_1 we split the integral into two domains $(-k_s,k_s)$ and $\mathbb{R}\backslash[-k_s,k_s]$ and use the Van der Corput lemma 2.4 to estimate the integral in the first interval and the argument in Lemma 3.4 to estimate the integral in $\mathbb{R}\backslash[-k_s,k_s]$. This yields $|\mathrm{V}_1|\le C\mu^{-1}(k_sh)^{-1/2}$. By the same argument as in Lemma 3.2 we can show $|\mathrm{V}_2|\le C\mu^{-1}e^{-\sqrt{k_R-k_s}h}$. This shows

$$\max_{z\in\Gamma_D}|\mathbb{V}(z,y)|\le\frac{C}{\mu}(k_sh)^{-1/2},\quad\forall y\in\bar{D}.$$

A similar argument shows that

$$\max_{z\in\Gamma_D}k_s^{i+j}|\nabla_z^i\nabla_y^j\mathbb{V}(z,y)|\le\frac{C}{\mu}(k_sh)^{-1/2},\quad\forall y\in\bar{D},\ i,j=0,1.$$

Substitute the above two estimates into (7.5) we obtain

$$\max_{y\in\bar{D}}(|w(y)|+d_D|\nabla w(y)|)\le\frac{C}{\mu}(1+\|T_2\|)(1+k_sd_D)^2(k_sh)^{-1/2}\|g\|_{H^{1/2}(\Gamma_D)}.$$

This completes the proof of the theorem from (7.2). □

Acknowledgement. The first author is grateful for the partial support by the China NSF under the grant 118311061. The authors are indebted to Dr. Guanghui Huang from Michigan State University for helpful discussions.

References

[1] Achenbach J 1980 *Wave Propagation in Elastic Solids* (North-Holland)

[2] Ahlfors L V 1979 *Complex Analysis: An introduction to the theory of analytic functions of one complex variable* (McGraw-Hill)

[3] Ammari H, Garnier J, Jing W, Kang H, Lim M, Solna K and Wang H 2013 *Mathematical and Statistical Methods for Multistatic Imaging* (Springer)

[4] Arens T 1999 A New Integral Equation Formulation for the Scattering of Plane Elastic Waves by Diffraction Gratings *Journal of Integral Equations and Applications* **11** 232-245.

[5] Baysal E, Kosloff D D and Sherwood J W C. 1983 Reverse time migration *Geophysics* **48** 1514–1524.

[6] Bleistein N, Cohen J and Stockwell J 2001 *Mathematics of Multidimensional Seismic Imaging, Migration, and Inversion* (New York: Springer)

[7] Berkhout A 1984 *Seismic Migration: Imaging of Acoustic Energy by Wave Field Extrapolation* (New York: Elsevier)

[8] Chang W-F and McMechan G A 1987 Elastic reverse-time migration *Geophysical Prospecting* **37** 243 – 256

[9] Chen J, Chen Z and Huang G 2013 Reverse time migration for extended obstacles: acoustic waves *Inverse Problems* **29** 085005 (17pp)

[10] Chen J, Chen Z and Huang G 2013 Reverse time migration for extended obstacles: electromagnetic waves *Inverse Problems* **29** 085006 (17pp)

[11] Chen Z and Huang G 2014 Reverse time migration for extended obstacles: Elastic waves (in Chinese)*Science China Mathematics* **45** 1103–1114

[12] Chen Z and Huang G 2015 Reverse time migration for reconstructing extended obstacles in the half space *Inverse Problems* **31** 055007 (19pp)

[13] Chen Z, Xiang X and Zhang X 2016 Convergence of the PML method for elastic wave scattering problems *Math. Comp.* **85** 2687-2714

[14] Chung W, Pyun S, Bae H S, Shin C and Marfurt K J 2012 Implementation of elastic reverse-time migration using wavefield separation in the frequency domain *Geophysical Journal International* **189** 1611–1625

[15] Claerbout J F 1985 *Imaging the Earth's Interior* (Oxford: Blackwell Scientific Publication)

[16] Colton D and Kress R 1998 *Inverse Acoustic and Electromagnetic Scattering Problems* (Heidelberg: Springer)

[17] Denli H and Huang L 2008 Elastic-wave reverse-time migration with a wavefield-separation imaging condition *78th Annual International Meeting, SEG, Expanded Abstracts* 2346-2350

[18] Dermenjian Y and Cuillot J C 1998 Scattering of elastic waves in a perturbed isotropic half space with a free boundary. The limiting absorption principle *Mathematical Methods in the Applied Sciences* **10** 87-124

[19] Duran M, Godoy E and Nedelec J-C 2010 Theoretical aspects and numerical computation of the time-harmonic Green's function for an isotropic elastic half-plane with an impedance boundary condition *SAIM: Math. Model. Numer. Anal.* **44** 671-692

[20] Duran M, Muga I and Nedelec J-C 2011 The outgoing time-harmonic elastic wave in a half-plane with free boundary *SIAM Journal on Applied Mathematics* **71** 255-277

[21] Grafakos L 2004 *Classical and Modern Fourier Analysis* (London: Pearson)

[22] Harris J 2001 *Linear elastic waves* (Cambridge University Press)

[23] Kupradze V D 1963 *Progress in solid mechanics: Dynamical problems in elasticity* (North-Holland Publishing Company)

[24] Kuroda ST 1978 *An Introduction to Scattering Theory* (Matematik Institut, Aarhus Universitet, Aarhus)

[25] Leis R 1986 *Initial Boundary Value Problems in Mathematical Physics* (Stuttgart: B.G. Teubner)

[26] Li Y E, Du Y, Yong J, Cheng A and Fang X 2017 Elastic reverse time migration using acoustic propagators *Geophysics* **83** S399-S408

[27] Madyarov A I and Guzina B B 2006 A Radiation Condition for Layered Elastic Media *Journal of Elasticity* **82** 73-98

[28] Sun R and McMechan G A 2001 Scalar reverse-time migration of prestack elastic seismic data *Geophysics* **66** 1519-1527

[29] Sini M 2004 Absence of positive eigenvalues for the linearized elasticity system *Integral Equations and Operator Theory* **49** 255–277

[30] Wilcox C H 2006 *Scattering theory for the d'Alembert equation in exterior domains* (Springer)

[31] Yan J and Sava P 2008 Isotropic angle-domain elastic reverse-time migration *Geophysics* **73** S229-S239.

[32] Zhang Y and Sun J 2009 Particle issues in reverse time migration: true amplitude gathers, noise removal and harmonic source encoding *First Break* **26** 29-35

[33] Zhang Y, Xu S, Bleistein N and Zhang G 2007 True-amplitude, angle-domain, common-image gathers from one-way wave-equation migration *Geophysics* **72** S49-S58