Communications on Applied Mathematics and Computation
https://doi.org/10.1007/s42967-024-00386-x

ORIGINAL PAPER

On the Regularity of Time-Harmonic Maxwell Equations with Impedance Boundary Conditions

Zhiming Chen[1,2]

Received: 28 September 2023 / Revised: 9 February 2024 / Accepted: 10 February 2024

Abstract
In this paper, we prove the H^2 regularity of the solution to the time-harmonic Maxwell equations with impedance boundary conditions on domains with a C^2 boundary under minimum regularity assumptions on the source and boundary functions.

Keywords Maxwell equations · Impedance boundary condition · H^2 regularity

Mathematics Subject Classification 35B65 · 35D30 · 35Q60

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a C^2 boundary Σ. We consider in this paper the following time-harmonic Maxwell equations with impedance boundary conditions:

$$\nabla \times (\mu^{-1}\nabla \times \boldsymbol{E}) - k^2\varepsilon \boldsymbol{E} = \boldsymbol{J}, \quad \mathrm{div}\,(\varepsilon \boldsymbol{E}) = 0 \quad \text{in } \Omega, \tag{1}$$

$$(\mu^{-1}\nabla \times \boldsymbol{E}) \times \boldsymbol{n} - \mathbf{i}k\lambda_{\mathrm{imp}}\boldsymbol{E}_T = \boldsymbol{g} \times \boldsymbol{n} \quad \text{on } \Sigma, \tag{2}$$

where the relative permeability μ and permittivity ε are assumed to be positive constants, $k = \omega\sqrt{\varepsilon_0\mu_0}$ is the wave number of the vacuum with $\omega > 0$ the angular frequency, and μ_0, ε_0 the permeability and permittivity of the vacuum, respectivley. With this notation, $\boldsymbol{J} = \mathbf{i}k\mu_0\boldsymbol{J}_a$ with $\boldsymbol{J}_a$ being the applied current density. We assume $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$ with div $\boldsymbol{J} = 0$ in Ω, the impedance $\lambda_{\mathrm{imp}} \in C^{0,1}(\Sigma)$ satisfies $\lambda_{\mathrm{imp}} \geqslant \lambda_0$ for some constant $\lambda_0 > 0$, and $\boldsymbol{g} \in \boldsymbol{L}^2(\Sigma)$.

Dedicated to Professor Zhong-Ci Shi.

This work is supported in part by the National Key Technologies R&D Program of China under the Grant 2019YFA0709602 and the National Natural Science Foundation of China under the Grants 11831016, 12288201, and 12201621.

✉ Zhiming Chen
zmchen@lsec.cc.ac.cn

1 LSEC, Institute of Computational Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

2 School of Mathematical Science, University of Chinese Academy of Sciences, Beijing 100049, China

Published online: 03 June 2024

Here $\boldsymbol{n}$ is the unit outer normal to Ω on Σ and $\boldsymbol{E}_T = \boldsymbol{n} \times \boldsymbol{E} \times \boldsymbol{n}$ is the tangential components of $\boldsymbol{E}$ on Σ. Throughout the paper, for any Banach space X, we denote $\boldsymbol{X} = X^3$ and $\|\cdot\|_X$ both the norms of X and $\boldsymbol{X}$.

The existence and uniqueness of the weak solution to the problem (1), (2) are studied in [14]. Wave number explicit stability estimates are considered in [10, 15] under some additional conditions on the domains. In [6, §4.5.d], the H^2 regularity of the solution to (1), (2) is studied when $\boldsymbol{g} = 0$, in which (1), (2) is reformulated as

$$\nabla \times \boldsymbol{E} - \mathbf{i}k\mu \boldsymbol{H} = 0, \quad \nabla \times \boldsymbol{H} + \mathbf{i}k\varepsilon \boldsymbol{E} = \frac{1}{\mathbf{i}k}\boldsymbol{J} \quad \text{in } \Omega, \tag{3}$$

$$\boldsymbol{H} \times \boldsymbol{n} - \lambda_{\text{imp}} \boldsymbol{E}_T = 0 \quad \text{on } \Sigma. \tag{4}$$

Then (4) is treated as essential boundary conditions for the coupled regularized Maxwell equations for $(\boldsymbol{E}, \boldsymbol{H})$. The H^2 regularity is proved for both $(\boldsymbol{E}, \boldsymbol{H})$ under the assumption that $\boldsymbol{J} \in \boldsymbol{H}(\text{curl}; \Omega)$, $\operatorname{div} \boldsymbol{J} \in H^1(\Omega)$, and $\boldsymbol{J} \times \boldsymbol{n} \in \boldsymbol{H}^{1/2}(\Sigma)$. Notice that the assumption that $\boldsymbol{J} \in \boldsymbol{H}(\text{curl}; \Omega)$, $\operatorname{div} \boldsymbol{J} \in H^1(\Omega)$, and $\boldsymbol{J} \times \boldsymbol{n} \in \boldsymbol{H}^{1/2}(\Sigma)$ implies $\boldsymbol{J} \in \boldsymbol{H}^1(\Omega)$ by a well-known embedding inequality (see the proof of Lemma 3 below).

In this paper, we prove the H^2 regularity of the solution to (1), (2) based on the $\boldsymbol{H}(\text{curl})$-coercive Maxwell equations under the minimum assumption that $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$, $\boldsymbol{g} \times \boldsymbol{n} \in \boldsymbol{H}^{1/2}(\Sigma)$, and $\boldsymbol{J} \cdot \boldsymbol{n} - \operatorname{div}_\Sigma(\boldsymbol{g} \times \boldsymbol{n}) \in H^{1/2}(\Sigma)$ (see Remark 4 below). This regularity result is expected to be useful when studying finite element methods for (1), (2) on smooth domains based on the Schatz argument in dealing with the indefiniteness of time-harmonic Maxwell equations with impedance boundary conditions. We refer to [5] for the recent work on unfitted finite element methods for time-harmonic Maxwell interface problems with Dirichlet boundary conditions, in which the H^2 regularity and the associated regularity estimates play an important role. Finally, we remark that we have not considered the wave number explicit estimates in this paper. Interested readers may find results in this respect in [10, 12, 15].

The layout of the paper is as follows. In Sect. 2, we introduce some notation and the main result of the H^2 regularity of the solution to the problem (1), (2). In Sect. 3, we prove the main result.

2 The Time-Harmonic Maxwell Equations

We first recall some notation. For any Lipschitz domain D in $\mathbb{R}^3$ with the boundary ∂D whose unit normal is denoted by $\boldsymbol{n}$, the space

$$\boldsymbol{H}(\text{curl}; D) = \{\boldsymbol{v} \in \boldsymbol{L}^2(D) \colon \nabla \times \boldsymbol{v} \in \boldsymbol{L}^2(D)\}$$

is a Hilbert space under the graph norm. For any $\boldsymbol{v} \in \boldsymbol{H}(\text{curl}; D)$, its tangential trace is defined as $\gamma_\tau(\boldsymbol{v}) = \boldsymbol{v} \times \boldsymbol{n}$ on ∂D. It is shown in Buffa et al. [4] that $\gamma_\tau \colon \boldsymbol{H}(\text{curl}; D) \to \boldsymbol{H}^{-1/2}(\operatorname{div}_{\partial D}; \partial D)$ is bounded and surjective, where

$$\boldsymbol{H}^{-1/2}(\operatorname{div}_{\partial D}; \partial D) = \{\boldsymbol{\lambda} \in \boldsymbol{V}_\pi(\partial D)' \colon \operatorname{div}_{\partial D}(\boldsymbol{\lambda}) \in \boldsymbol{H}^{-1/2}(\partial D)\}.$$

Here $\boldsymbol{V}_\pi(\partial D)'$ is the dual space of $\boldsymbol{V}_\pi(\partial D) = \{\boldsymbol{v}_T = \boldsymbol{n} \times \boldsymbol{v} \times \boldsymbol{n} \colon \boldsymbol{v} \in \boldsymbol{H}^{1/2}(\partial D)\}$ and $\operatorname{div}_{\partial D} \colon \boldsymbol{V}_\pi(\partial D)' \to H^{-3/2}(\partial D)$ is the dual operator of the surface gradient $\nabla_T \colon H^{3/2}(\partial D) \to \boldsymbol{V}_\pi(\partial D)$. It is known that $\operatorname{div}_{\partial D}(\boldsymbol{v} \times \boldsymbol{n}) = (\nabla \times \boldsymbol{v}) \cdot \boldsymbol{n} \in \boldsymbol{H}^{-1/2}(\partial D)$ for any $\boldsymbol{v} \in \boldsymbol{H}(\text{curl}; D)$.

In the following, we always denote $(\cdot,\cdot)_D$ the inner product in $\boldsymbol{L}^2(D)$ or the duality pairing between $\boldsymbol{H}^1(D)'$ and $\boldsymbol{H}^1(D)$, $\langle\cdot,\cdot\rangle_{\partial D}$ the inner product in $\boldsymbol{L}^2(\partial D)$ or the duality pairing between $\boldsymbol{H}^{-1/2}(\partial D)$ and $\boldsymbol{H}^{1/2}(\partial D)$.

We call a symmetric matrix $\mathbb{A} \in \mathbb{R}^{3\times 3}$ with elements in $L^\infty(D)$ strongly positive definite if it satisfies

$$\exists\, a_-, a_+ > 0,\ \ \forall \boldsymbol{y} \in \mathbb{R}^3,\ \ a_-|\boldsymbol{y}|^2 \leqslant \mathbb{A}\boldsymbol{y}\cdot\boldsymbol{y} \leqslant a_+|\boldsymbol{y}|^2 \ \text{ a.e. in } D. \tag{5}$$

Similarly, a symmetric matrix $\mathbb{L} \in \mathbb{R}^{3\times 3}$ with elements in $L^\infty(\partial D)$ is called strongly positive definite if it satisfies

$$\exists\, \lambda_-, \lambda_+ > 0,\ \ \forall \boldsymbol{y} \in \mathbb{R}^3,\ \ \lambda_-|\boldsymbol{y}|^2 \leqslant \mathbb{L}\boldsymbol{y}\cdot\boldsymbol{y} \leqslant \lambda_+|\boldsymbol{y}|^2 \ \text{ a.e. in } \partial D. \tag{6}$$

Lemma 1 *Let D be a bounded Lipschitz domain in $\mathbb{R}^3$. Assume that $\mathbb{A}, \mathbb{B} \in \mathbb{R}^{3\times 3}$ are symmetric and strongly positive definite matrices with elements in $L^\infty(D)$ and $\mathbb{L}_{\mathrm{imp}} \in \mathbb{R}^{3\times 3}$ is a symmetric and strongly positive definite matrix with elements in $L^\infty(\partial D)$.*

Assume that $\boldsymbol{u} \in \boldsymbol{H}(\mathrm{curl}; D)$ satisfies

$$\nabla\times(\mathbb{A}\nabla\times\boldsymbol{u}) + \mathbb{B}\boldsymbol{u} = \boldsymbol{F} \quad \text{in } D, \tag{7}$$

$$(\mathbb{A}\nabla\times\boldsymbol{u})\times\boldsymbol{n} - \mathbf{i}k(\mathbb{L}_{\mathrm{imp}}(\boldsymbol{n}\times\boldsymbol{u}))\times\boldsymbol{n} = \boldsymbol{g}_1\times\boldsymbol{n} \ \text{ on } \partial D. \tag{8}$$

Then for any $\boldsymbol{F} \in \boldsymbol{L}^2(D)$ such that $\mathrm{div}\,\boldsymbol{F} \in H^1(D)'$, $\boldsymbol{g}_1\times\boldsymbol{n} \in \boldsymbol{H}^{-1/2}(\partial D)$, *and* $\boldsymbol{F}\cdot\boldsymbol{n} - \mathrm{div}_{\Sigma}(\boldsymbol{g}_1\times\boldsymbol{n}) \in H^{-1/2}(\partial D)$, *we have*

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathrm{curl};D)} \leqslant\ & C(\|\boldsymbol{F}\|_{H^1(D)'} + \|\mathrm{div}\,\boldsymbol{F}\|_{H^1(D)'} + \|\boldsymbol{g}_1\times\boldsymbol{n}\|_{H^{-1/2}(\partial D)})\\ & + C\|\boldsymbol{F}\cdot\boldsymbol{n} - \mathrm{div}_{\partial D}(\boldsymbol{g}_1\times\boldsymbol{n})\|_{H^{-1/2}(\partial D)},\end{aligned}$$

where the constant C depends only on the domain D, the coefficients $\mathbb{A}, \mathbb{B}, \mathbb{L}_{\mathrm{imp}}$, and the wave number k.

Proof By testing (7) by $\boldsymbol{u} \in \boldsymbol{H}(\mathrm{curl}; D)$, by the symmetric and strongly positive definite conditions on $\mathbb{A}, \mathbb{B}, \mathbb{L}_{\mathrm{imp}} \in \mathbb{R}^{3\times 3}$, we obtain by the standard argument that

$$\|\boldsymbol{u}\|^2_{\boldsymbol{H}(\mathrm{curl};D)} \leqslant C|(\boldsymbol{F},\boldsymbol{u})_D + \langle\boldsymbol{g}_1\times\boldsymbol{n},\boldsymbol{u}\rangle_{\partial D}|. \tag{9}$$

By the Birman-Solomyak regular decomposition theorem [2], Hiptmair [9, Lemma 2.4], $\boldsymbol{u} \in \boldsymbol{H}(\mathrm{curl}; D)$ can be split as $\boldsymbol{u} = \boldsymbol{u}_s + \nabla\psi$ for some $\boldsymbol{u}_s \in \boldsymbol{H}^1(D)$, $\psi \in H^1(D)$ such that $\|\boldsymbol{u}_s\|_{H^1(D)} + \|\psi\|_{H^1(D)} \leqslant C\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{curl};D)}$.

By integration by parts, $(\boldsymbol{F},\nabla\psi)_D = -(\mathrm{div}\,\boldsymbol{F},\psi)_D + \langle\boldsymbol{F}\cdot\boldsymbol{n},\psi\rangle_{\partial D}$. Thus,

$$\begin{aligned}&(\boldsymbol{F},\boldsymbol{u})_D + \langle\boldsymbol{g}_1\times\boldsymbol{n},\boldsymbol{u}\rangle_{\partial D}\\ =\ & (\boldsymbol{F},\boldsymbol{u}_s)_D - (\mathrm{div}\,\boldsymbol{F},\psi)_D + \langle\boldsymbol{g}_1\times\boldsymbol{n},\boldsymbol{u}_s\rangle_{\partial D} + \langle\boldsymbol{F}\cdot\boldsymbol{n} - \mathrm{div}_{\partial D}(\boldsymbol{g}_1\times\boldsymbol{n}),\psi\rangle_{\partial D}\\ \leqslant\ & C(\|\boldsymbol{F}\|_{H^1(D)'} + \|\mathrm{div}\,\boldsymbol{F}\|_{H^1(D)'} + \|\boldsymbol{g}_1\times\boldsymbol{n}\|_{H^{-1/2}(\partial D)})\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathrm{curl};D)}\\ & + C\|\boldsymbol{F}\cdot\boldsymbol{n} - \mathrm{div}_{\partial D}(\boldsymbol{g}_1\times\boldsymbol{n})\|_{H^{-1/2}(\partial D)}\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathrm{curl};D)}.\end{aligned}$$

This completes the proof by (9).

The following lemma, which shows the H^1 regularity for the $\boldsymbol{H}(\mathrm{curl})$-coercive Maxwell equations with impedance boundary conditions, was essentially proved in Costabel et al. [6, §4.5.d].

Lemma 2 *Let D be a bounded C^2 domain in $\mathbb{R}^3$. Assume that $\mathbb{A}, \mathbb{B} \in \mathbb{R}^{3\times 3}$ are symmetric and strongly positive definite matrices with elements in $C^{0,1}(\bar{D})$ and $\mathbb{L}_{\mathrm{imp}} \in \mathbb{R}^{3\times 3}$ is a symmetric* *and strongly positive definite matrix with elements in* $C^{0,1}(\partial D)$. *Let* $\boldsymbol{u} \in \boldsymbol{H}(\text{curl}; D)$ *be the solution of* (7), (8). *Then for any* $\boldsymbol{F} \in H(\text{div}; D)$, $\boldsymbol{g}_1 \times \boldsymbol{n} \in \boldsymbol{H}^{-1/2}(\text{div}_{\partial D}; \partial D)$, *we have* $\boldsymbol{u} \in \boldsymbol{H}^1(D)$ *and*

$$\|\boldsymbol{u}\|_{H^1(D)} \leqslant C(\|\boldsymbol{u}\|_{\boldsymbol{H}(\text{curl};D)} + \|\text{div}\,\boldsymbol{F}\|_{L^2(D)} + \|\boldsymbol{F}\cdot\boldsymbol{n} - \text{div}_{\partial D}(\boldsymbol{g}_1 \times \boldsymbol{n})\|_{H^{-1/2}(\partial D)}),$$

where the constant C *depends only on the domain* D, *the coefficients* $\mathbb{A}, \mathbb{B}, \mathbb{L}_{\text{imp}}$, *and the wave number* k.

Proof We give a proof by following the argument in [6, §4.5.d] in order to trace the stability estimate in the lemma. By the regular decomposition theorem, $\boldsymbol{u} = \boldsymbol{u}_s + \nabla\psi$ for some $\boldsymbol{u}_s \in \boldsymbol{H}^1(D)$, $\psi \in H^1(D)$ such that

$$\|\boldsymbol{u}_s\|_{H^1(D)} + \|\psi\|_{H^1(D)} \leqslant C\|\boldsymbol{u}\|_{\boldsymbol{H}(\text{curl};D)}. \tag{10}$$

It is clear from (7), (8) that $\psi \in H^1(D)$ satisfies

$$\text{div}\,(\mathbb{B}\nabla\psi) = \text{div}\,\boldsymbol{F} - \text{div}\,(\mathbb{B}\boldsymbol{u}_s) \quad \text{in } D, \tag{11}$$

$$(\mathbb{A}\nabla\times\boldsymbol{u}_s)\times\boldsymbol{n} - \mathbf{i}k(\mathbb{L}_{\text{imp}}(\boldsymbol{n}\times\boldsymbol{u}_s))\times\boldsymbol{n} - \mathbf{i}k(\mathbb{L}_{\text{imp}}(\boldsymbol{n}\times\nabla\psi))\times\boldsymbol{n} = \boldsymbol{g}_1\times\boldsymbol{n} \quad \text{on } \partial D. \tag{12}$$

By taking the surface gradient in (12), since $\text{div}_{\partial D}((\mathbb{A}\nabla\times\boldsymbol{u}_s)\times\boldsymbol{n}) = \nabla\times(\mathbb{A}\nabla\times\boldsymbol{u}_s)\cdot\boldsymbol{n} = (\boldsymbol{F} - \mathbb{B}\boldsymbol{u})\cdot\boldsymbol{n}$ on ∂D by (7), we obtain

$$-\text{div}_{\partial D}((\mathbb{L}_{\text{imp}}(\boldsymbol{n}\times\nabla\psi))\times\boldsymbol{n}) = (\mathbf{i}k)^{-1}\left(\text{div}_{\partial D}(\boldsymbol{g}_1\times\boldsymbol{n}) - (\boldsymbol{F} - \mathbb{B}\boldsymbol{u})\cdot\boldsymbol{n}\right) + \text{div}_{\partial D}((\mathbb{L}_{\text{imp}}(\boldsymbol{n}\times\boldsymbol{u}_s))\times\boldsymbol{n}).$$

Denote by $\boldsymbol{L}^2_{\mathbf{t}}(\partial D) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\partial D)\colon \boldsymbol{v}\cdot\boldsymbol{n} = 0\}$ and $r\colon \boldsymbol{L}^2_{\mathbf{t}}(\partial D) \to \boldsymbol{L}^2_{\mathbf{t}}(\partial D)$ the linear mapping defined by $r(\boldsymbol{v}) = \boldsymbol{n}\times\boldsymbol{v}$. Then $r(\boldsymbol{v}) = \mathbb{G}\boldsymbol{v}$, where for $\boldsymbol{n} = (n_1, n_2, n_3)^{\text{T}}$,

$$\mathbb{G} = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \in \mathbb{R}^{3\times 3}.$$

Obviously, $\mathbb{G}^{\text{T}} = -\mathbb{G}$ and $\mathbb{G}(\nabla_{\partial D}\psi) = \boldsymbol{n}\times\nabla\psi$ on ∂D, where $\nabla_{\partial D}\psi = \boldsymbol{n}\times\nabla\psi\times\boldsymbol{n}$ on ∂D. It follows that

$$\text{div}_{\partial D}((\mathbb{L}_{\text{imp}}(\boldsymbol{n}\times\nabla\psi))\times\boldsymbol{n}) = \text{div}_{\partial D}((\mathbb{G}^{\text{T}}\mathbb{L}_{\text{imp}}\mathbb{G})(\nabla_{\partial D}\psi)).$$

Notice that $\mathbb{G}^{\text{T}}\mathbb{L}_{\text{imp}}\mathbb{G}$ is symmetric and strongly positive definite with elements in $C^{0,1}(\partial D)$ since $\mathbb{L}_{\text{imp}}$ is symmetric and strongly positive definite which satisfies (6). Now from the regularity theory of elliptic equations on smooth surfaces we know that $\psi \in H^{3/2}(\partial D)$ and it satisfies

$$\|\psi\|_{H^{3/2}(\partial D)} \leqslant C(\|\text{div}_{\partial D}(\boldsymbol{g}_1\times\boldsymbol{n}) - (\boldsymbol{F} - \mathbb{B}\boldsymbol{u})\cdot\boldsymbol{n}\|_{H^{-1/2}(\partial D)} + \|\boldsymbol{u}_s\|_{H^{1/2}(\partial D)}).$$

Then using the regularity theorem for the Dirichlet problem of elliptic equations (see, e.g., McLean [13, Theorem 4.18]), we have from (11) that $\psi \in H^2(D)$ and

$$\begin{aligned} &\|\psi\|_{H^2(D)} \\ &\leqslant C(\|\text{div}\,\boldsymbol{F} - \text{div}\,(\mathbb{B}\boldsymbol{u}_s)\|_{L^2(D)} + \|\psi\|_{H^{3/2}(\partial D)}) \\ &\leqslant C(\|\boldsymbol{u}\|_{\boldsymbol{H}(\text{curl};D)} + \|\text{div}\,\boldsymbol{F}\|_{L^2(D)} + \|\text{div}_{\partial D}(\boldsymbol{g}_1\times\boldsymbol{n}) - (\boldsymbol{F} - \mathbb{B}\boldsymbol{u})\cdot\boldsymbol{n}\|_{H^{-1/2}(\partial D)}), \end{aligned}$$

where we have used (10) in the last inequality. The lemma now follows since

$$\|\mathbb{B}\boldsymbol{u}\cdot\boldsymbol{n}\|_{H^{-1/2}(\partial D)} \leqslant C\|\mathbb{B}\boldsymbol{u}\|_{H(\text{div};D)} \leqslant C(\|\boldsymbol{u}\|_{L^2(D)} + \|\text{div}\,\boldsymbol{F}\|_{L^2(D)}).$$

This completes the proof.

Remark 1 We briefly consider the regularity result of elliptic equations on C^2 surfaces used in the proof of Lemma 2. We follow the development in Dziuk and Elliott [7], Bonito et al. [3] for the Laplace-Beltrami equation. A general regularity theory for elliptic systems on compact manifold without boundary can be found in [6, §1.3.b].

Denote $\Gamma = \partial D$. Let $\mathbb{A}(\boldsymbol{x}) \in \mathbb{R}^{3\times 3}$ be a symmetric and strongly positive definite matrix with elements in $C^{0,1}(\Gamma)$ that maps the tangent space $T_{\boldsymbol{x}}\Gamma$ to itself. By the Lax-Milgram lemma and the Poincaré-Friedrichs inequality, for any $f \in H^{-1}(\Gamma)$ with $\langle f, 1\rangle_\Gamma = 0$, the elliptic equation $-\mathrm{div}_\Gamma(\mathbb{A}\nabla_\Gamma \tilde{u}) = f$ on Γ has a unique solution $\tilde{u} \in H^1(\Gamma)/\mathbb{R}$ and $\|\tilde{u}\|_{H^1(\Gamma)} \leqslant C\|f\|_{H^{-1}(\Gamma)}$. To show the H^2 regularity, by the partition of unity, one can consider only one single chart $(V, U, \boldsymbol{\chi})$, where $V \subset \mathbb{R}^2$ is an open connected set, $U \subset \mathbb{R}^3$ is an open set, and $\boldsymbol{\chi}\colon V \to U \cap \Gamma$ is a C^2 isomorphism. For $\boldsymbol{x} \in U \cap \Gamma$, set $\boldsymbol{y} = \boldsymbol{\chi}^{-1}(\boldsymbol{x}) \in V$. It is known (e.g., [3]) that the surface gradient of a function $\tilde{v}\colon \Gamma \to \mathbb{R}$ is

$$\nabla_\Gamma \tilde{v}(\boldsymbol{x}) = D\boldsymbol{\chi}(\boldsymbol{y})\boldsymbol{g}(\boldsymbol{y})^{-1}\nabla v(\boldsymbol{y}), \quad \forall \boldsymbol{y} \in V, \quad v(\boldsymbol{y}) = \tilde{v}(\boldsymbol{\chi}(\boldsymbol{y})),$$

where $D\boldsymbol{\chi} \in \mathbb{R}^{3\times 2}$ is a matrix whose column vectors are $\partial_1\boldsymbol{\chi}(\boldsymbol{y}), \partial_2\boldsymbol{\chi}(\boldsymbol{y})$, and $\boldsymbol{g}(\boldsymbol{y}) = D\boldsymbol{\chi}^{\mathrm{T}}(\boldsymbol{y})\boldsymbol{\chi}(\boldsymbol{y}) \in \mathbb{R}^{2\times 2}$ is the first fundamental form. On the local chart $(V, U, \boldsymbol{\chi})$, the equation can be written for $u = \tilde{u} \circ \boldsymbol{\chi}$ as

$$-\mathrm{div}\,[q(\boldsymbol{y})\boldsymbol{g}(\boldsymbol{y})^{-1}D\boldsymbol{\chi}(\boldsymbol{y})^{\mathrm{T}}\mathbb{A}(\boldsymbol{\chi}(\boldsymbol{y}))D\boldsymbol{\chi}(\boldsymbol{y})\boldsymbol{g}(\boldsymbol{y})^{-1}\nabla u] = q(\boldsymbol{y})f(\boldsymbol{\chi}(\boldsymbol{y})), \quad \forall \boldsymbol{y} \in V,$$

where $q(\boldsymbol{y}) = \sqrt{\det \boldsymbol{g}(\boldsymbol{y})}$. Then by the H^2 interior regularity estimate for elliptic equations (see, e.g., Gilbarg and Trudinger [8, Theorem 8.8]), one can deduce $\tilde{u} \in H^2(\Gamma)$ and $\|\tilde{u}\|_{H^2(\Gamma)} \leqslant C\|f\|_{L^2(\Gamma)}$. Now the interpolation theory of interpolation spaces (e.g., [13, Theorem 3.2]) implies that if $f \in H^{-1/2}(\Gamma)$, then $\tilde{u} \in H^{3/2}(\Gamma)$ and it satisfies $\|\tilde{u}\|_{H^{3/2}(\Gamma)} \leqslant C\|f\|_{H^{-1/2}(\Gamma)}$. This is the regularity result used in Lemma 2.

We will use the impedance space

$$\boldsymbol{H}_{\mathrm{imp}}(\mathrm{curl};\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl};\Omega)\colon \boldsymbol{v}_T \in \boldsymbol{L}^2(\Sigma) \ \text{ on } \Sigma\},$$

which is a Hilbert space under the norm

$$\|\boldsymbol{v}\|^2_{\boldsymbol{H}_{\mathrm{imp}}(\mathrm{curl};\Omega)} = \|\nabla \times \boldsymbol{v}\|^2_{L^2(\Omega)} + \|\boldsymbol{v}\|^2_{L^2(\Omega)} + \|k^{1/2}\boldsymbol{v}\|^2_{L^2(\Sigma)}.$$

It is known (Monk [14, Theorem 4.17]) that there exists a unique weak solution $\boldsymbol{E} \in \boldsymbol{H}_{\mathrm{imp}}(\mathrm{curl};\Omega)$ to (1), (2) such that for any $\boldsymbol{v} \in \boldsymbol{H}_{\mathrm{imp}}(\mathrm{curl};\Omega)$,

$$(\mu^{-1}\nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{v}) - k^2(\varepsilon\boldsymbol{E}, \boldsymbol{v}) - \langle \mathbf{i}k\lambda_{\mathrm{imp}}\boldsymbol{E}_T, \boldsymbol{v}_T\rangle_\Sigma = (\boldsymbol{J}, \boldsymbol{v}) + \langle \boldsymbol{g} \times \boldsymbol{n}, \boldsymbol{v}_T\rangle_\Sigma, \quad (13)$$

where $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$ is the L^2 inner product in $\boldsymbol{L}^2(\Omega)$ or the duality pairing between $\boldsymbol{H}^1(\Omega)$ and $\boldsymbol{H}^1(\Omega)'$. Moreover, there is a constant $C_{\mathrm{stab}} > 0$ which depends on the domain Ω, the coefficients μ, ε, the wave number k, and the impedance λ_{imp} such that

$$\|\boldsymbol{E}\|_{\boldsymbol{H}_{\mathrm{imp}}(\mathrm{curl};\Omega)} \leqslant C_{\mathrm{stab}}(\|\boldsymbol{J}\|_{L^2(\Omega)} + \|\boldsymbol{g} \times \boldsymbol{n}\|_{L^2(\Sigma)}). \quad (14)$$

Denote $\boldsymbol{H}^1(\mathrm{curl};\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)\colon \nabla \times \boldsymbol{v} \in \boldsymbol{H}^1(\Omega)\}$. The following lemma has been essentially proved in [6, Section 4.5.d], Hiptmair et al. [10, Lemma 3.2].

Lemma 3 *Let* $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$, $\boldsymbol{g} \times \boldsymbol{n} \in \boldsymbol{H}^{1/2}(\Sigma)$ *satisfy* $\mathrm{div}\,\boldsymbol{J} = 0$ *in* Ω. *Then the solution of the problem* (13) *satisfies* $\boldsymbol{E} \in \boldsymbol{H}^1(\mathrm{curl};\Omega)$ *and*

$$\|\boldsymbol{E}\|_{\boldsymbol{H}^1(\mathrm{curl};\Omega)} \leqslant C(\|\boldsymbol{J}\|_{L^2(\Omega)} + \|\boldsymbol{g} \times \boldsymbol{n}\|_{H^{1/2}(\Sigma)}), \quad (15)$$

where the constant C *depends on the domain* Ω, *the coefficients* k, μ, ε, *and the impedance* λ_{imp}.

Proof We sketch a proof to trace the stability estimate in the lemma. Let $\boldsymbol{F} = \boldsymbol{J} + (1+k^2)\varepsilon \boldsymbol{E}$. Then div $\boldsymbol{F} = 0$ in Ω and $\|\boldsymbol{F}\cdot\boldsymbol{n}\|_{H^{-1/2}(\Sigma)} \leqslant C\|\boldsymbol{F}\|_{L^2(\Omega)} \leqslant C(\|\boldsymbol{J}\|_{L^2(\Omega)} + \|\boldsymbol{E}\|_{L^2(\Omega)})$. By Lemma 2 and (14), we obtain

$$\begin{aligned}\|\boldsymbol{E}\|_{H^1(\Omega)} &\leqslant C(\|\boldsymbol{E}\|_{\boldsymbol{H}(\mathrm{curl};\Omega)} + \|\mathrm{div}_\Sigma(\boldsymbol{g}\times\boldsymbol{n}) - \boldsymbol{F}\cdot\boldsymbol{n}\|_{H^{-1/2}(\Sigma)})\\ &\leqslant C(\|\boldsymbol{J}\|_{L^2(\Omega)} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{H^{1/2}(\Sigma)}).\end{aligned} \tag{16}$$

To show the estimate for $\|\nabla\times\boldsymbol{E}\|_{H^1(\Omega)}$, we use the argument in Lu et al. [12, Theorem 3.2]. Denote $\boldsymbol{W} = \mu^{-1}\nabla\times\boldsymbol{E}$ and notice that

$$\nabla\times\boldsymbol{W} - k^2\varepsilon\boldsymbol{E} = \boldsymbol{J} \ \text{ in } \Omega, \quad \boldsymbol{W}\times\boldsymbol{n} - \mathbf{i}k\lambda_{\mathrm{imp}}\boldsymbol{E}_T = \boldsymbol{g}\times\boldsymbol{n} \ \text{ on } \Sigma. \tag{17}$$

By the well-known embedding inequality (see, e.g., Amrouche et al. [1, Corollary 2.15])

$$\begin{aligned}&\|\boldsymbol{W}\|_{H^1(\Omega)}\\ \leqslant\ & C(\|\boldsymbol{W}\|_{L^2(\Omega)} + \|\nabla\times\boldsymbol{W}\|_{L^2(\Omega)} + \|\mathrm{div}\,\boldsymbol{W}\|_{L^2(\Omega)} + \|\boldsymbol{W}\times\boldsymbol{n}\|_{H^{1/2}(\Sigma)})\\ \leqslant\ & C(\|\boldsymbol{W}\|_{\boldsymbol{H}(\mathrm{curl};\Omega)} + \|\mathbf{i}k\lambda_{\mathrm{imp}}\boldsymbol{E}_T + \boldsymbol{g}\times\boldsymbol{n}\|_{H^{1/2}(\Sigma)})\\ \leqslant\ & C(\|\boldsymbol{E}\|_{H^1(\Omega)} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{H^{1/2}(\Sigma)})\\ \leqslant\ & C(\|\boldsymbol{J}\|_{L^2(\Omega)} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{H^{1/2}(\Sigma)}),\end{aligned}$$

where we have used (17) in the third inequality and (16) in the last inequality. This completes the proof.

The following theorem is the main result of this paper whose proof will be given in the next section.

Theorem 1 *Let* Ω *be a bounded domain with a* C^2 *boundary* Σ, $\boldsymbol{J}\in\boldsymbol{L}^2(\Omega)$ *with* div $\boldsymbol{J} = 0$ *in* Ω, $\boldsymbol{g}\times\boldsymbol{n}\in\boldsymbol{H}^{1/2}(\Sigma)$, *and* $\boldsymbol{J}\cdot\boldsymbol{n} - \mathrm{div}_\Sigma(\boldsymbol{g}\times\boldsymbol{n})\in H^{1/2}(\Sigma)$. *Then the solution of* (13) *has the regularity* $\boldsymbol{E}\in\boldsymbol{H}^2(\Omega)$ *and*

$$\|\boldsymbol{E}\|_{H^2(\Omega)} \leqslant C_{\mathrm{reg}}(\|\boldsymbol{J}\|_{L^2(\Omega)} + \|\boldsymbol{J}\cdot\boldsymbol{n} - \mathrm{div}_\Sigma(\boldsymbol{g}\times\boldsymbol{n})\|_{H^{1/2}(\Sigma)} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{H^{1/2}(\Sigma)}),$$

where the constant $C_{\mathrm{reg}} > 0$ *depends on the domain* Ω, *the coefficients* k, ε, μ, *and the impedance* λ_{imp}.

We remark that under the additional assumption that Ω is a C^3 domain and star-shaped with respect to a ball, the H^2 regularity of $\boldsymbol{E}$ is proved in [12] using the stabilities estimates in [10] and the argument of regular decomposition of vector fields which is also used in Lemma 2.

3 The Proof of Theorem 1

In this section, we prove Theorem 1 on the H^2 regularity of the solution to (1), (2) by extending the ideas of proving the H^2 regularity for elliptic equations in e.g., [8, Chapter 8] or [13, Chaper 4]. We denote by $B(\boldsymbol{x}, d)$ the ball centered at $\boldsymbol{x}$ with the radius $d > 0$.

We start with the following lemma on the estimates of difference quotients in the negative Sobolev norms.

Lemma 4 *Let $\hat{B} = B(\mathbf{0}, 1)$ be the unit ball in $\mathbb{R}^3$, $\hat{B}_2 = \{\hat{\boldsymbol{x}} \in \hat{B} : \hat{x}_3 > 0\}$, $\hat{\Sigma} = \{\hat{\boldsymbol{x}} \in \hat{B} : \hat{x}_3 = 0\}$, and $\hat{\boldsymbol{n}} = (0, 0, 1)^{\mathrm{T}}$. For any $\delta > 0$, let $(\Delta_l^\delta \boldsymbol{v})(\hat{\boldsymbol{x}}) = (\boldsymbol{v}(\hat{\boldsymbol{x}} + \delta \mathbf{e}_l) - \boldsymbol{v}(\hat{\boldsymbol{x}}))/\delta$, $l = 1, 2$, be the difference quotients of $\boldsymbol{v} \in \boldsymbol{L}^2(\hat{B})$ or $\boldsymbol{v} \in \boldsymbol{L}^2(\hat{\Sigma})$. Then for any fixed $\delta_0 \in (0, 1)$, there exists a constant C depending only on δ_0 such that*

(i) *if $\boldsymbol{W} \in \boldsymbol{L}^2(\hat{B}_2)$ is supported in $B(\mathbf{0}, 1-\delta_0)$, then for any $\delta \in (0, \delta_0/2)$, $\|\Delta_l^\delta \boldsymbol{W}\|_{H^1(\hat{B}_2)'} \leqslant C\|\boldsymbol{W}\|_{L^2(\hat{B}_2)}$;*

(ii) *if $\boldsymbol{w} \in \boldsymbol{H}^{1/2}(\hat{\Sigma})$ is supported in $B(\mathbf{0}, 1-\delta_0) \cap \hat{\Sigma}$, then for any $\delta \in (0, \delta_0/2)$, $\|\Delta_l^\delta(\boldsymbol{w} \times \hat{\boldsymbol{n}})\|_{H^{-1/2}(\partial \hat{B}_2)} \leqslant C\|\boldsymbol{w} \times \hat{\boldsymbol{n}}\|_{H^{1/2}(\hat{\Sigma})}$.*

Proof Let $\phi \in C_0^\infty(\hat{B})$ be the cut-off function such that $0 \leqslant \phi \leqslant 1$ in $\hat{B}$, $\phi = 1$ in $B(\mathbf{0}, 1-\delta_0/2)$, $\phi = 0$ in $\hat{B}\backslash\overline{B(\mathbf{0}, 1-\delta_0/4)}$, and $|\nabla\phi| \leqslant C\delta_0^{-1}$. If $\boldsymbol{W} \in \boldsymbol{L}^2(\hat{B}_2)$ is supported in $B(\mathbf{0}, 1-\delta_0)$, then for $\delta \in (0, \delta_0/2)$, $\Delta_l^\delta \boldsymbol{W}$ is supported in $B(\mathbf{0}, 1-\delta_0/2)$. Thus, for any $\boldsymbol{v} \in \boldsymbol{H}^1(\hat{B}_2)$, by [13, Lemma 4.13],

$$\begin{aligned}(\Delta_l^\delta \boldsymbol{W}, \boldsymbol{v})_{\hat{B}_2} &= (\phi\Delta_l^\delta \boldsymbol{W}, \boldsymbol{v})_{\hat{B}_2} = -(\boldsymbol{W}, \Delta_l^{-\delta}(\phi\boldsymbol{v}))_{\hat{B}_2}\\ &\leqslant \|\boldsymbol{W}\|_{L^2(\hat{B}_2)}\|\partial(\phi\boldsymbol{v})/\partial\hat{x}_l\|_{L^2(\hat{B}_2)}\\ &\leqslant C\|\boldsymbol{W}\|_{L^2(\hat{B}_2)}\|\boldsymbol{v}\|_{H^1(\hat{B}_2)}.\end{aligned}$$

This shows (i). Next, since $\boldsymbol{w} \in \boldsymbol{H}^{1/2}(\hat{\Sigma})$ is supported in $B(\mathbf{0}, 1-\delta_0) \cap \hat{\Sigma}$, by the definition of Sobolev spaces on the boundary (see, e.g., [13, (3.29)]),

$$\|\Delta_l^\delta(\boldsymbol{w} \times \hat{\boldsymbol{n}})\|_{H^s(\partial\hat{B}_2)} = \|\phi\Delta_l^\delta(\boldsymbol{w} \times \hat{\boldsymbol{n}})\|_{H^s(\mathbb{R}^2)}, \quad \forall s \in \mathbb{R}. \tag{18}$$

By [13, Lemma 4.13] and the argument above, we have $\|\phi\Delta_l^\delta(\boldsymbol{w} \times \hat{\boldsymbol{n}})\|_{L^2(\mathbb{R}^2)} \leqslant \|\boldsymbol{w} \times \hat{\boldsymbol{n}}\|_{H^1(\hat{\Sigma})} = \|\boldsymbol{w} \times \hat{\boldsymbol{n}}\|_{H^1(\partial\hat{B}_2)}$, $\|\phi\Delta_l^\delta(\boldsymbol{w} \times \hat{\boldsymbol{n}})\|_{H^{-1}(\mathbb{R}^2)} \leqslant C\|\boldsymbol{w} \times \hat{\boldsymbol{n}}\|_{L^2(\partial\hat{B}_2)}$. On the other hand, by the interpolation of Sobolev spaces (e.g., [13, Theorem B.7]), $(\boldsymbol{L}^2(\partial\hat{B}_2), \boldsymbol{H}^1(\hat{\partial}B_2))_{\frac{1}{2},2} = \boldsymbol{H}^{1/2}(\hat{B}_2)$, $(\boldsymbol{H}^{-1}(\mathbb{R}^2), \boldsymbol{L}^2(\mathbb{R}^2))_{\frac{1}{2},2} = \boldsymbol{H}^{-1/2}(\mathbb{R}^2)$. Thus, by the interpolation property of interpolation spaces [13, Theorem B.2],

$$\|\phi\Delta_l^\delta(\boldsymbol{w} \times \hat{\boldsymbol{n}})\|_{H^{-1/2}(\mathbb{R}^2)} \leqslant C\|\boldsymbol{w} \times \hat{\boldsymbol{n}}\|_{H^{1/2}(\partial\hat{B}_2)} = C\|\boldsymbol{w} \times \hat{\boldsymbol{n}}\|_{H^{1/2}(\hat{\Sigma})}.$$

This completes the proof by (18).

The proof of Theorem 1. Let $\{B(\boldsymbol{y}_i, d_i)\}_{i=1}^m$ be a set of balls with $\boldsymbol{y}_i \in \Sigma$ such that $\mathcal{O}_\Sigma = \cup_{i=1}^m B(\boldsymbol{y}_i, d_i/2)$ covers Σ. Moreover, we assume that there exists a C^2, one-to-one mapping $\boldsymbol{\Phi}_i$: $\hat{B} \to B(\boldsymbol{y}_i, d_i)$ such that $\Omega \cap B(\boldsymbol{y}_i, d_i) = \boldsymbol{\Phi}_i(\hat{B}_2)$ and $\Sigma \cap B(\boldsymbol{y}_i, d_i) = \boldsymbol{\Phi}_i(\hat{\Sigma})$, where $\hat{B}$ is the unit ball in $\mathbb{R}^3$, $\hat{B}_2 = \{\hat{\boldsymbol{x}} \in \hat{B}: \hat{x}_3 > 0\}$ and $\hat{\Sigma} = \{\hat{\boldsymbol{x}} \in \hat{B}: \hat{x}_3 = 0\}$. Denote $\mathcal{O} = \Omega\backslash\bar{\mathcal{O}}_\Sigma$. The proof is divided into two steps.
(i) Interior regularity. We first note that by (1) and Lemma 3, $\boldsymbol{E} \in \boldsymbol{H}^1(\Omega)$ satisfies

$$-\Delta\boldsymbol{E} = \mu\boldsymbol{J} + k^2\mu\varepsilon\boldsymbol{E} \quad \text{in } \Omega.$$

By the interior regularity of elliptic equations (see, e.g., [13, Theorem 4.16]), we know that $\boldsymbol{E} \in \boldsymbol{H}^2(\mathcal{O})$, and

$$\begin{aligned}\|\boldsymbol{E}\|_{H^2(\mathcal{O})} &\leqslant C(\|\boldsymbol{E}\|_{H^1(\Omega)} + \|\mu\boldsymbol{J} + k^2\mu\varepsilon\boldsymbol{E}\|_{L^2(\Omega)})\\ &\leqslant C(\|\boldsymbol{J}\|_{L^2(\Omega)} + \|\boldsymbol{g} \times \boldsymbol{n}\|_{H^{1/2}(\Sigma)}),\end{aligned} \tag{19}$$

where we have used Lemma 3.

(ii) BOUNDARY REGULARITY. Let $B = B(\boldsymbol{y}_i, d_i)$ for some $i = 1, 2, \cdots, m$, and $\boldsymbol{\Phi}$: $\hat{B} \to B$ be the C^2, one-to-one mapping such that $B_2 = \Omega \cap B = \boldsymbol{\Phi}(\hat{B}_2)$ and $\Sigma \cap B = \boldsymbol{\Phi}(\hat{\Sigma})$. By the Piola transform [14, §3.9], we have

$$\nabla = D\boldsymbol{\Phi}^{-\mathrm{T}}\hat{\nabla}, \quad \nabla\cdot = J^{-1}\hat{\nabla}\cdot(JD\boldsymbol{\Phi}^{-1}), \quad \nabla\times = J^{-1}D\boldsymbol{\Phi}\hat{\nabla}\times D\boldsymbol{\Phi}^{\mathrm{T}}, \tag{20}$$

where $D\boldsymbol{\Phi}$ is the gradient matrix and $J = \det(D\boldsymbol{\Phi})$ is the Jacobi determinant. Moreover, the unit normal $\boldsymbol{n}$ to Σ and the surface area $\mathrm{d}s$ of Σ satisfy (see, e.g., Hofmann et al. [11])

$$\boldsymbol{n}\circ\boldsymbol{\Phi} = \frac{D\boldsymbol{\Phi}^{-\mathrm{T}}\hat{\boldsymbol{n}}}{|D\boldsymbol{\Phi}^{-\mathrm{T}}\hat{\boldsymbol{n}}|}, \quad \mathrm{d}s = |J|\,|D\boldsymbol{\Phi}^{-\mathrm{T}}\hat{\boldsymbol{n}}|\,\mathrm{d}\hat{s}, \tag{21}$$

where $\hat{\boldsymbol{n}} = (0, 0, 1)^{\mathrm{T}}$ and $\mathrm{d}\hat{s}$ is the surface area of $\hat{\Sigma}$.

Let $\chi \in C_0^\infty(B)$ be the cut-off function such that $0 \leqslant \chi \leqslant 1$ in B, $\chi = 1$ in $B(\boldsymbol{y}_i, d_i/2)$, and $\chi = 0$ in $B\backslash\overline{B(\boldsymbol{y}_i, 3d_i/4)}$. Denote $\boldsymbol{u} = \chi\boldsymbol{E}$. Then we obtain

$$\nabla\times(\mu^{-1}\nabla\times\boldsymbol{u}) + \varepsilon\boldsymbol{u} = \boldsymbol{F}, \quad \mathrm{div}\,(\varepsilon\boldsymbol{u}) = \nabla\chi\cdot(\varepsilon\boldsymbol{E}) \ \text{ in } B_2, \tag{22}$$

$$(\mu^{-1}\nabla\times\boldsymbol{u})\times\boldsymbol{n} - \mathbf{i}k\lambda_{\mathrm{imp}}\boldsymbol{u}_T = \boldsymbol{g}_1\times\boldsymbol{n} \ \text{ on } \Sigma, \tag{23}$$

where $\boldsymbol{g}_1\times\boldsymbol{n} = \chi\boldsymbol{g}\times\boldsymbol{n} + (\mu^{-1}\nabla\chi\times\boldsymbol{E})\times\boldsymbol{n}$ on Σ, and

$$\begin{cases} \boldsymbol{F} = \boldsymbol{F}_1 + \nabla\times(\mu^{-1}\nabla\chi\times\boldsymbol{E}), \\ \boldsymbol{F}_1 = \chi\boldsymbol{J} + (1+k^2)\varepsilon\chi\boldsymbol{E} + \nabla\chi\times(\mu^{-1}\nabla\times\boldsymbol{E}) \ \text{ in } \Omega. \end{cases} \tag{24}$$

Obviously, $\mathrm{div}\,\boldsymbol{F}_1 = \mathrm{div}\,\boldsymbol{F}$ in Ω. Thus, $\boldsymbol{F}_1 \in H(\mathrm{div};\Omega)$ and

$$\mathrm{div}\,\boldsymbol{F}_1 = \nabla\chi\cdot(\varepsilon\boldsymbol{E}) \ \text{ in } \Omega. \tag{25}$$

Let $\hat{\boldsymbol{v}} = D\boldsymbol{\Phi}^{\mathrm{T}}(\boldsymbol{v}\circ\boldsymbol{\Phi})$ for any function $\boldsymbol{v}$: $B \to \mathbb{C}^3$. By (20), (21), and the following vector identity [11], for any invertible matrix $\mathbb{D} \in \mathbb{C}^{3\times3}$ and $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^3$,

$$\mathbb{D}\boldsymbol{a}\times\mathbb{D}\boldsymbol{b} = \det(\mathbb{D})\mathbb{D}^{-\mathrm{T}}(\boldsymbol{a}\times\boldsymbol{b}), \tag{26}$$

we deduce from (22), (23) that

$$\hat{\nabla}\times(\mu^{-1}\mathbb{A}\hat{\nabla}\times\hat{\boldsymbol{u}}) + \varepsilon\mathbb{A}^{-1}\hat{\boldsymbol{u}} = \mathbb{A}^{-1}\hat{\boldsymbol{F}}, \hat{\nabla}\cdot(\varepsilon\mathbb{A}^{-1}\hat{\boldsymbol{u}}) = \hat{\nabla}\hat{\chi}\cdot(\varepsilon\mathbb{A}^{-1}\hat{\boldsymbol{E}}) \ \text{ in } \hat{B}_2, \tag{27}$$

$$(\mu^{-1}\mathbb{A}\hat{\nabla}\times\hat{\boldsymbol{u}})\times\hat{\boldsymbol{n}} - \mathbf{i}k\hat{\lambda}_{\mathrm{imp}}(\mathbb{A}(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{u}}))\times\hat{\boldsymbol{n}} = \hat{\boldsymbol{g}}_1\times\hat{\boldsymbol{n}} \ \text{ on } \hat{\Sigma}, \tag{28}$$

where $\mathbb{A} = J^{-1}D\boldsymbol{\Phi}^{\mathrm{T}}D\boldsymbol{\Phi}$, $\hat{\chi} = \chi\circ\Phi$, and $\hat{\lambda}_{\mathrm{imp}} = |D\boldsymbol{\Phi}^{-\mathrm{T}}\hat{\boldsymbol{n}}|^{-1}(\lambda_{\mathrm{imp}}\circ\boldsymbol{\Phi})$. By (24) we have

$$\mathbb{A}^{-1}\hat{\boldsymbol{F}} = \mathbb{A}^{-1}\hat{\boldsymbol{F}}_1 + \hat{\nabla}\times(\mu^{-1}\mathbb{A}(\hat{\nabla}\hat{\chi}\times\hat{\boldsymbol{E}})) \ \text{ in } \hat{B}_2. \tag{29}$$

From (25), we deduce by (20) that

$$\hat{\nabla}\cdot(\mathbb{A}^{-1}\hat{\boldsymbol{F}}_1) = \hat{\nabla}\hat{\chi}\cdot(\varepsilon\mathbb{A}^{-1}\hat{\boldsymbol{E}}) \ \text{ in } \hat{B}_2. \tag{30}$$

By (20), (21) we have

$$\hat{\boldsymbol{g}}_1\times\hat{\boldsymbol{n}} = \hat{\chi}\hat{\boldsymbol{g}}\times\hat{\boldsymbol{n}} + \mu^{-1}(\mathbb{A}(\hat{\nabla}\hat{\chi}\times\hat{\boldsymbol{E}}))\times\hat{\boldsymbol{n}}. \tag{31}$$

Now we extend the well-known Nirenberg argument developed for second order elliptic equations to show the H^2 regularity. For any $\delta > 0$ sufficiently small, we consider the difference quotients Δ_l^δ, $l = 1, 2$, defined as

$$\Delta_l^\delta\hat{\boldsymbol{u}}(\hat{\boldsymbol{x}}) = (\hat{\boldsymbol{u}}(\hat{\boldsymbol{x}} + \delta\boldsymbol{e}_l) - \hat{\boldsymbol{u}}(\hat{\boldsymbol{x}}))/\delta.$$

By (27), (28), the difference quotients $\Delta_l^\delta \hat{\boldsymbol{u}}$, $l = 1, 2$, satisfy

$$\hat{\nabla} \times (\mu^{-1}\mathbb{A}_\delta \hat{\nabla} \times (\Delta_l^\delta \hat{\boldsymbol{u}})) + \varepsilon \mathbb{A}_\delta^{-1}(\Delta_l^\delta \hat{\boldsymbol{u}}) = \hat{\boldsymbol{F}}' \quad \text{in } \hat{B}_2, \tag{32}$$

$$(\mu^{-1}\mathbb{A}_\delta \hat{\nabla} \times (\Delta_l^\delta \hat{\boldsymbol{u}})) \times \hat{\boldsymbol{n}} - \mathbf{i}k((\hat{\lambda}_{\text{imp}}\mathbb{A})_\delta (\hat{\boldsymbol{n}} \times (\Delta_l^\delta \hat{\boldsymbol{u}}))) \times \hat{\boldsymbol{n}} = \hat{\boldsymbol{g}}_1' \times \hat{\boldsymbol{n}} \text{ on } \hat{\Sigma}, \tag{33}$$

where $\mathbb{A}_\delta = \mathbb{A}(\hat{\boldsymbol{x}} + \delta \boldsymbol{e}_l)$, $(\hat{\lambda}_{\text{imp}}\mathbb{A})_\delta = (\hat{\lambda}_{\text{imp}}\mathbb{A})(\hat{\boldsymbol{x}} + \delta \boldsymbol{e}_l)$, and

$$\hat{\boldsymbol{F}}' = \Delta_l^\delta(\mathbb{A}^{-1}\hat{\boldsymbol{F}}) - \hat{\nabla} \times (\mu^{-1}(\Delta_l^\delta \mathbb{A})\hat{\nabla} \times \hat{\boldsymbol{u}}) - \varepsilon(\Delta_l^\delta \mathbb{A}^{-1})\hat{\boldsymbol{u}}, \tag{34}$$

$$\hat{\boldsymbol{g}}_1' \times \hat{\boldsymbol{n}} = \Delta_l^\delta(\hat{\boldsymbol{g}}_1 \times \hat{\boldsymbol{n}}) - (\mu^{-1}(\Delta_l^\delta \mathbb{A})\hat{\nabla} \times \hat{\boldsymbol{u}}) \times \hat{\boldsymbol{n}} + \mathbf{i}k(\Delta_l^\delta(\hat{\lambda}_{\text{imp}}\mathbb{A})(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{u}})) \times \hat{\boldsymbol{n}}. \tag{35}$$

Since $\chi = 0$ in $B \backslash \overline{B(\boldsymbol{y}_i, 3d_i/4)}$, there exists a $\delta_0 > 0$ depending on d_i and $\boldsymbol{\Phi}$ such that $\hat{\chi}$ is supported in $B(\boldsymbol{0}, 1 - \delta_0)$. Thus $\hat{\boldsymbol{F}}'$ is supported in $B(\boldsymbol{0}, 1 - \delta_0)$ and $\hat{\boldsymbol{g}}_1' \times \hat{\boldsymbol{n}}$ is supported in $\hat{\Sigma} \cap B(\boldsymbol{0}, 1 - \delta_0)$. Now since $\boldsymbol{\Phi}^{-1}(B(\boldsymbol{y}_i, d_i/2) \cap \Omega) \subset \hat{B}_2$, by Lemmas 2 and 1 we have

$$\begin{aligned} \|\Delta_l^\delta \hat{\boldsymbol{u}}\|_{H^1(\boldsymbol{\Phi}^{-1}(B(\boldsymbol{y}_i, d_i/2)\cap\Omega))} \leqslant{} & C(\|\hat{\boldsymbol{F}}'\|_{H^1(\hat{B}_2)'} + \|\hat{\nabla} \cdot \hat{\boldsymbol{F}}'\|_{L^2(\hat{B}_2)} + \|\hat{\boldsymbol{g}}_1' \times \hat{\boldsymbol{n}}\|_{H^{-1/2}(\hat{\Sigma})}) \\ & + C(\|\hat{\boldsymbol{F}}' \cdot \hat{\boldsymbol{n}} - \operatorname{div}_{\hat{\Sigma}}(\hat{\boldsymbol{g}}_1' \times \hat{\boldsymbol{n}})\|_{H^{-1/2}(\hat{\Sigma})}). \end{aligned} \tag{36}$$

By Lemma 4, (34), (29), and (24) we have

$$\begin{aligned} \|\hat{\boldsymbol{F}}'\|_{H^1(\hat{B}_2)'} &\leqslant C(\|\mathbb{A}^{-1}\hat{\boldsymbol{F}}\|_{L^2(\hat{B}_2)} + \|\hat{\boldsymbol{u}}\|_{H^1(\hat{B}_2)}) \\ &\leqslant C(\|\mathbb{A}^{-1}\hat{\boldsymbol{F}}_1\|_{L^2(\hat{B}_2)} + \|\hat{\boldsymbol{E}}\|_{H^1(\hat{B}_2)}) \\ &\leqslant C(\|\boldsymbol{F}_1\|_{L^2(B_2)} + \|\boldsymbol{E}\|_{H^1(B_2)}) \\ &\leqslant C(\|\boldsymbol{J}\|_{L^2(B_2)} + \|\boldsymbol{E}\|_{H^1(B_2)}). \end{aligned} \tag{37}$$

By (34), (29), and (30), we obtain

$$\begin{aligned} \|\hat{\nabla} \cdot \hat{\boldsymbol{F}}'\|_{L^2(\hat{B}_2)} &\leqslant \|\hat{\nabla} \cdot (\Delta_l^\delta(\mathbb{A}^{-1}\hat{\boldsymbol{F}}_1))\|_{L^2(\hat{B}_2)} + \|\hat{\nabla} \cdot (\varepsilon(\Delta_l^\delta \mathbb{A}^{-1})\hat{\boldsymbol{u}})\|_{L^2(\hat{B}_2)} \\ &= \|\Delta_l^\delta(\hat{\nabla}\hat{\chi} \cdot (\varepsilon \mathbb{A}^{-1}\hat{\boldsymbol{E}}))\|_{L^2(\hat{B}_2)} + \|\hat{\nabla} \cdot (\varepsilon(\Delta_l^\delta \mathbb{A}^{-1})\hat{\boldsymbol{u}})\|_{L^2(\hat{B}_2)} \\ &\leqslant C\|\hat{\boldsymbol{E}}\|_{H^1(\hat{B}_2)}. \end{aligned}$$

This implies

$$\|\hat{\nabla} \cdot \hat{\boldsymbol{F}}'\|_{L^2(\hat{B}_2)} \leqslant C\|\boldsymbol{E}\|_{H^1(B_2)}. \tag{38}$$

By Lemma 4, (35), and the definition $\boldsymbol{g}_1 \times \boldsymbol{n} = \chi \boldsymbol{g} \times \boldsymbol{n} + (\mu^{-1}\nabla\chi \times \boldsymbol{E}) \times \boldsymbol{n}$ on Σ, we have

$$\begin{aligned} \|\hat{\boldsymbol{g}}_1' \times \hat{\boldsymbol{n}}\|_{H^{-1/2}(\hat{\Sigma})} &\leqslant C(\|\hat{\boldsymbol{g}}_1 \times \hat{\boldsymbol{n}}\|_{H^{1/2}(\hat{\Sigma})} + \|\hat{\nabla} \times \hat{\boldsymbol{u}}\|_{L^2(\hat{\Sigma})} + \|\hat{\boldsymbol{u}}\|_{L^2(\hat{\Sigma})}) \\ &\leqslant C(\|\boldsymbol{g}_1 \times \boldsymbol{n}\|_{H^{1/2}(B\cap\Sigma)} + \|\boldsymbol{E}\|_{\boldsymbol{H}^1(\text{curl}; B_2)}) \\ &\leqslant C(\|\boldsymbol{g} \times \boldsymbol{n}\|_{H^{1/2}(B\cap\Sigma)} + \|\boldsymbol{E}\|_{\boldsymbol{H}^1(\text{curl}; B_2)}). \end{aligned} \tag{39}$$

Moreover, using (34), (35), and (29)

$$\begin{aligned} & \hat{\boldsymbol{F}}' \cdot \hat{\boldsymbol{n}} - \operatorname{div}_{\hat{\Sigma}}(\hat{\boldsymbol{g}}_1' \times \hat{\boldsymbol{n}}) \\ ={}& \Delta_l^\delta(\mathbb{A}^{-1}\hat{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}} - \operatorname{div}_{\hat{\Sigma}}(\hat{\boldsymbol{g}}_1 \times \hat{\boldsymbol{n}})) - \varepsilon(\Delta_l^\delta \mathbb{A}^{-1})\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}} \\ & - \mathbf{i}k\operatorname{div}_{\hat{\Sigma}}((\Delta_l^\delta(\hat{\lambda}_{\text{imp}}\mathbb{A})(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{u}})) \times \hat{\boldsymbol{n}}) \\ ={}& \Delta_l^\delta(\mathbb{A}^{-1}\hat{\boldsymbol{F}}_1 \cdot \hat{\boldsymbol{n}} - \operatorname{div}_{\hat{\Sigma}}(\hat{\chi}\hat{\boldsymbol{g}} \times \hat{\boldsymbol{n}})) - \varepsilon(\Delta_l^\delta \mathbb{A}^{-1})\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}} \\ & - \mathbf{i}k\operatorname{div}_{\hat{\Sigma}}((\Delta_l^\delta(\hat{\lambda}_{\text{imp}}\mathbb{A})(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{u}})) \times \hat{\boldsymbol{n}}). \end{aligned}$$

Thus,

$$\begin{aligned}&\|\hat{\boldsymbol{F}}'\cdot\hat{\boldsymbol{n}}-\operatorname{div}_{\hat{\Sigma}}(\hat{\boldsymbol{g}}_1'\times\hat{\boldsymbol{n}})\|_{H^{-1/2}(\hat{\Sigma})}\\ &\leqslant C(\|\mathbb{A}^{-1}\hat{\boldsymbol{F}}_1\cdot\hat{\boldsymbol{n}}-\operatorname{div}_{\hat{\Sigma}}(\hat{\chi}\hat{\boldsymbol{g}}\times\hat{\boldsymbol{n}})\|_{H^{1/2}(\hat{\Sigma})}+\|\hat{\boldsymbol{u}}\|_{H^{1/2}(\hat{\Sigma})}).\end{aligned}$$

For any $\hat{\phi}\in H^{1/2}(\hat{\Sigma})$, by (20), (21), and (26), we have

$$\begin{aligned}\langle\mathbb{A}^{-1}\hat{\boldsymbol{F}}_1\cdot\hat{\boldsymbol{n}}-\operatorname{div}_{\hat{\Sigma}}(\hat{\chi}\hat{\boldsymbol{g}}\times\hat{\boldsymbol{n}}),\hat{\phi}\rangle_{\hat{\Sigma}}&=\langle\mathbb{A}^{-1}\hat{\boldsymbol{F}}_1\cdot\hat{\boldsymbol{n}},\hat{\phi}\rangle_{\hat{\Sigma}}+\langle\hat{\chi}\hat{\boldsymbol{g}}\times\hat{\boldsymbol{n}},\hat{\nabla}\hat{\phi}\rangle_{\hat{\Sigma}}\\ &=\langle\boldsymbol{F}_1\cdot\boldsymbol{n},\phi\rangle_{\Sigma\cap B}+\langle\chi\boldsymbol{g}\times\boldsymbol{n},\nabla\phi\rangle_{\Sigma\cap B}\\ &=\langle\boldsymbol{F}_1\cdot\boldsymbol{n}-\operatorname{div}_{\Sigma}(\chi\boldsymbol{g}\times\boldsymbol{n}),\phi\rangle_{\Sigma\cap B}.\end{aligned}$$

Thus, by (24)

$$\begin{aligned}&\|\mathbb{A}^{-1}\hat{\boldsymbol{F}}_1\cdot\hat{\boldsymbol{n}}-\operatorname{div}_{\hat{\Sigma}}(\hat{\chi}\hat{\boldsymbol{g}}\times\hat{\boldsymbol{n}})\|_{H^{1/2}(\hat{\Sigma})}\\ &\leqslant\|\boldsymbol{F}_1\cdot\boldsymbol{n}-\operatorname{div}_{\Sigma}(\chi\boldsymbol{g}\times\boldsymbol{n})\|_{H^{1/2}(\Sigma\cap B)}\\ &\leqslant C(\|\boldsymbol{J}\cdot\boldsymbol{n}-\operatorname{div}_{\Sigma}(\boldsymbol{g}\times\boldsymbol{n})\|_{H^{1/2}(\Sigma\cap B)}+\|\boldsymbol{E}\|_{\boldsymbol{H}^1(\mathrm{curl};B_2)}+\|\boldsymbol{g}\times\boldsymbol{n}\|_{H^{1/2}(\Sigma\cap B)}).\end{aligned}$$

This yields

$$\begin{aligned}&\|\hat{\boldsymbol{F}}'\cdot\hat{\boldsymbol{n}}-\operatorname{div}_{\hat{\Sigma}}(\hat{\boldsymbol{g}}_1'\times\hat{\boldsymbol{n}})\|_{H^{-1/2}(\hat{\Sigma})}\\ &\leqslant C(\|\boldsymbol{J}\cdot\boldsymbol{n}-\operatorname{div}_{\Sigma}(\boldsymbol{g}\times\boldsymbol{n})\|_{H^{1/2}(\Sigma\cap B)}+\|\boldsymbol{E}\|_{\boldsymbol{H}^1(\mathrm{curl};B_2)}+\|\boldsymbol{g}\times\boldsymbol{n}\|_{H^{1/2}(\Sigma\cap B)}).\end{aligned}$$

Combining (36), (38), (39), and the above estimate, we obtain

$$\begin{aligned}&\|\Delta_l^{\delta}\hat{\boldsymbol{u}}\|_{H^1(\boldsymbol{\Phi}^{-1}(B(\boldsymbol{y}_i,d_i/2)\cap\Omega))}\\ &\leqslant C(\|\boldsymbol{J}\|_{L^2(B_2)}+\|\boldsymbol{J}\cdot\boldsymbol{n}-\operatorname{div}_{\Sigma}(\boldsymbol{g}\times\boldsymbol{n})\|_{H^{1/2}(B\cap\Sigma)}+\|\boldsymbol{g}\times\boldsymbol{n}\|_{H^{1/2}(B\cap\Sigma)}\\ &\quad+\|\boldsymbol{E}\|_{\boldsymbol{H}^1(\mathrm{curl};B_2)}).\end{aligned}$$

This implies, by letting $\delta\to0$,

$$\begin{aligned}&\sum_{l=1}^{2}\sum_{i,j=1}^{3}\left\|\frac{\partial^2\hat{u}_i}{\partial\hat{x}_l\partial\hat{x}_j}\right\|_{L^2(\boldsymbol{\Phi}^{-1}(B(\boldsymbol{y}_i,d_i/2)\cap\Omega))}\\ &\leqslant C(\|\boldsymbol{J}\|_{L^2(B_2)}+\|\boldsymbol{J}\cdot\boldsymbol{n}-\operatorname{div}_{\Sigma}(\boldsymbol{g}\times\boldsymbol{n})\|_{H^{1/2}(B\cap\Sigma)}+\|\boldsymbol{g}\times\boldsymbol{n}\|_{H^{1/2}(B\cap\Sigma)}\\ &\quad+\|\boldsymbol{E}\|_{\boldsymbol{H}^1(\mathrm{curl};B_2)}).\end{aligned}\tag{40}$$

Finally, it follows from (27) that

$$-\hat{\nabla}\cdot(\varepsilon\mathbb{A}^{-1}\hat{\boldsymbol{u}})=-\hat{\nabla}\hat{\chi}\times(\varepsilon\mathbb{A}^{-1}\hat{\boldsymbol{E}})\quad\text{in }\hat{B}_2.$$

Notice that $\mathbb{A}^{-1}=JD\boldsymbol{\Phi}^{-1}D\boldsymbol{\Phi}^{-\mathrm{T}}$ whose elements $(\mathbb{A}^{-1})_{ij}=J\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{a}_j$, $i,j=1,2,3$, where $\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3\in\mathbb{R}^3$ are column vectors of $D\boldsymbol{\Phi}^{-\mathrm{T}}$. Obviously, $(\mathbb{A}^{-1})_{33}=J|\boldsymbol{a}_3|^2\geqslant a_0$ for some constant $a_0>0$. Thus, by differentiating the equation in $\hat{x}_3$ we obtain

$$\left\|\frac{\partial\hat{u}_3^2}{\partial\hat{x}_3^2}\right\|_{L^2(\boldsymbol{\Phi}^{-1}(B(\boldsymbol{y}_i,d_i/2)\cap\Omega))}\leqslant C(\|\hat{\boldsymbol{E}}\|_{H^1(\hat{B}_2)}+\|\hat{\boldsymbol{u}}\|_{H^1(\hat{B}_2)})+C\sum_{l=1}^{2}\sum_{i,j=1}^{3}\left\|\frac{\partial^2\hat{u}_i}{\partial\hat{x}_l\partial\hat{x}_j}\right\|_{L^2(\hat{B}_2)}.$$

Therefore, it follows from (40) that

$$\begin{aligned}\|\hat{\boldsymbol{u}}\|_{H^2(\Phi^{-1}(B(\boldsymbol{y}_i,d_i/2)\cap\Omega))}&\leqslant C(\|\boldsymbol{J}\|_{L^2(B_2)}+\|\boldsymbol{J}\cdot\boldsymbol{n}-\operatorname{div}_{\Sigma}(\boldsymbol{g}\times\boldsymbol{n})\|_{H^{1/2}(B\cap\Sigma)})\\ &\quad+C(\|\boldsymbol{g}\times\boldsymbol{n}\|_{H^{1/2}(B\cap\Sigma)}+\|\boldsymbol{E}\|_{\boldsymbol{H}^1(\mathrm{curl};B_2)}).\end{aligned}$$

Since $B = B(\boldsymbol{y}_i, d_i), i = 1, 2, \cdots, m$, we deduce by Lemma 3 that

$$\begin{aligned}
&\|\boldsymbol{u}\|_{H^2(\mathcal{O}_\Sigma\cap\Omega)}\\
&\leqslant C(\|\boldsymbol{J}\|_{L^2(\Omega)} + \|\boldsymbol{J}\cdot\boldsymbol{n} - \mathrm{div}_\Sigma(\boldsymbol{g}\times\boldsymbol{n})\|_{H^{1/2}(\Sigma)} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{H^{1/2}(\Sigma)} + \|\boldsymbol{E}\|_{\boldsymbol{H}^1(\mathrm{curl};\Omega)})\\
&\leqslant C(\|\boldsymbol{J}\|_{L^2(\Omega)} + \|\boldsymbol{J}\cdot\boldsymbol{n} - \mathrm{div}_\Sigma(\boldsymbol{g}\times\boldsymbol{n})\|_{H^{1/2}(\Sigma)} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{H^{1/2}(\Sigma)}).
\end{aligned} \tag{41}$$

This completes the proof of the theorem by combining (19) and (41).

Remark 2 It is crucial to use Lemma 1 in (36). In fact, by Lemma 2,

$$\begin{aligned}
&\|\Delta_l^\delta\hat{\boldsymbol{u}}\|_{H^1(\boldsymbol{\Phi}^{-1}(B(\boldsymbol{y}_i,d_i/2)\cap\Omega))}\\
&\leqslant C(\|\Delta_l^\delta\hat{u}\|_{\boldsymbol{H}(\mathrm{curl};\hat{B}_2)} + \|\hat{\nabla}\cdot\hat{\boldsymbol{F}}'\|_{L^2(\hat{B}_2)} + \|\hat{\boldsymbol{F}}'\cdot\hat{\boldsymbol{n}} - \mathrm{div}_{\hat{\Sigma}}(\hat{\boldsymbol{g}}_1'\times\hat{\boldsymbol{n}})\|_{H^{-1/2}(\hat{\Sigma})}).
\end{aligned}$$

If one uses the standard argument instead of Lemma 1, one obtains from (32), (33) that

$$\|\Delta_l^\delta\hat{u}\|_{\boldsymbol{H}(\mathrm{curl};\hat{B}_2)} \leqslant C(\|\hat{\boldsymbol{F}}'\|_{L^2(\hat{B}_2)} + \|\hat{\boldsymbol{g}}_1'\times\hat{\boldsymbol{n}}\|_{L^2(\hat{\Sigma})}).$$

This will require the stronger assumption $\boldsymbol{J}\in\boldsymbol{H}^1(\Omega)$ to show the $\boldsymbol{H}^2$ regularity of the solution $\boldsymbol{E}$.

Remark 3 If Ω is a bounded domain with a C^{m+2} boundary Σ, $m\geqslant 0$, the impedance $\lambda_{\mathrm{imp}}\in C^{m,1}(\Sigma)$ with $\lambda_{\mathrm{imp}}\geqslant\lambda_0>0$, $\boldsymbol{J}\in\boldsymbol{H}^m(\Omega)$ with $\mathrm{div}\,\boldsymbol{J}=0$ in Ω, $\boldsymbol{g}\times\boldsymbol{n}\in H^{m+1/2}(\Sigma)$, and $\boldsymbol{J}\cdot\boldsymbol{n}-\mathrm{div}_\Sigma(\boldsymbol{g}\times\boldsymbol{n})\in\boldsymbol{H}^{m+1/2}(\Sigma)$, then similar to the argument for elliptic equations in [8, Theorem 8.10], one can deduce from the proof of Theorem 1 that $\boldsymbol{E}\in\boldsymbol{H}^{m+2}(\Omega)$ and

$$\|\boldsymbol{E}\|_{H^{m+2}(\Omega)} \leqslant C(\|\boldsymbol{J}\|_{H^m(\Omega)} + \|\boldsymbol{J}\cdot\boldsymbol{n} - \mathrm{div}_\Sigma(\boldsymbol{g}\times\boldsymbol{n})\|_{H^{m+1/2}(\Sigma)} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{H^{m+1/2}(\Sigma)}).$$

Remark 4 The assumption that $\boldsymbol{J}\in\boldsymbol{L}^2(\Omega)$, $\boldsymbol{g}\times\boldsymbol{n}\in\boldsymbol{H}^{1/2}(\Sigma)$, $\boldsymbol{J}\cdot\boldsymbol{n}-\mathrm{div}_\Sigma(\boldsymbol{g}\times\boldsymbol{n})\in H^{1/2}(\Sigma)$ in Theorem 1 is minimum. Indeed, if $\boldsymbol{E}\in\boldsymbol{H}^2(\Omega)$, it is obvious that (1), (2) implies $\boldsymbol{J}\in\boldsymbol{L}^2(\Omega)$, $\boldsymbol{g}\times\boldsymbol{n}\in\boldsymbol{H}^{1/2}(\Sigma)$. By taking the surface divergence of (2) and using (1), we obtain

$$(\boldsymbol{J}+k^2\varepsilon\boldsymbol{E})\cdot\boldsymbol{n} - \mathbf{i}k\,\mathrm{div}_\Sigma(\lambda_{\mathrm{imp}}\boldsymbol{E}_T) = \mathrm{div}_\Sigma(\boldsymbol{g}\times\boldsymbol{n}) \quad \text{on } \Sigma.$$

Thus, $\boldsymbol{E}\in\boldsymbol{H}^2(\Omega)$ also implies $\boldsymbol{J}\cdot\boldsymbol{n}-\mathrm{div}_\Sigma(\boldsymbol{g}\times\boldsymbol{n})\in H^{1/2}(\Sigma)$.

Acknowledgements The author is grateful to the referees for the helpful comments that lead to the great improvement of the paper.

Data Availability The paper does not have any additional data to be disclosed.

Compliance with Ethical Standards

Conflict of Interest The author states that there is no conflict of interest.

References

1. Amrouche, C., Bernardi, C., Dauge, M., Girault, V.: Vector potentials in three-dimensional non-smooth domains. Math. Methods Appl. Sci. **21**, 823–864 (1998)
2. Birman, Sh.M., Solomyak, M.Z.: L^2-theory of the Maxwell operator in arbitrary domains. Russ. Math. Sur. **43**, 75–96 (1987)
3. Bonito, A., Demlow, A., Nochetto, R.H.: Finite element methods for the Laplace-Beltrami operator. Handb. Numer. Anal. **21**, 1–103 (2020)

4. Buffa, A., Costabel, M., Sheen, D.: On traces for $\boldsymbol{H}(\mathrm{curl}; \Omega)$ in Lipschitz domains. J. Math. Anal. Appl. **276**, 845–867 (2002)
5. Chen, Z., Li, K., Xiang, X.: A high order unfitted finite element method for time-harmonic Maxwell interface problems. arXiv:2301.08944v1 (2023)
6. Costabel, M., Dauge, M., Nicaise, S.: Corner singularities and analytic regularity for linear elliptic systems. Part I: smooth domains (2010). https://hal.science/hal-00453934v1
7. Dziuk, G., Elliott, C.E.: Finite element methods for surface PDEs. Acta Numer. **22**, 289–396 (2013)
8. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (2001)
9. Hiptmair, R.: Finite elements in computational electromagnetism. Acta Numer. **11**, 237–339 (2002)
10. Hiptmair, R., Moiola, A., Perugia, I.: Stability results for the time-harmonic Maxwell equations with impedance boundary conditions. Math. Models Methods Appl. Sci. **21**, 2263–2287 (2011)
11. Hofmann, S., Mitrea, M., Taylor, M.: Geometric and transformational properties of Lipschitz domains, Semmes-Kenig-Toro domains, and other classes of finite perimeter domains. J. Geom. Anal. **17**, 593–647 (2007)
12. Lu, P., Wang, Y., Xu, X.: Regularity results for the time-harmonic Maxwell equations with impedance boundary condition. arXiv:1804.07856v1 (2018)
13. McLean, W.: Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, Cambridge (2000)
14. Monk, P.: Finite Element Methods for Maxwell's Equations. Oxford University Press, Oxford (2003)
15. Nicaise, S., Tomezyk, J.: The time-harmonic Maxwell equations with impedance boundary conditions in convex polyhedral domains. In: Langer, U., Pauly, D., Repin, S. (eds.) Maxwell's Equations: Analysis and Numerics, pp. 285–340. De Gruyter, Berlin (2019)