Section 07. Multilevel Iterative Methods

Domain Decomposition Method

Revisiting two-level additive Schwarz method:

$$B_{\text{as},2} := I_0 A_0^{-1} I_0^T + \sum_{i=1}^{n} I_i A_i^{-1} I_i^T \quad \text{and} \quad \kappa(B_{\text{as},2} A) \lesssim 1 + \beta^{-1}$$

Problem settings:

- Let $\mathscr{V} = H_0^1(\Omega)$, $\Omega = \bigcup_{j=1}^{J} \Omega_j$, and $\mathscr{V}_j := \{v \in \mathscr{V} : \operatorname{supp} v \subset \hat{\Omega}_j\} \subset \mathscr{V}$.
- Define finite-dimensional space $V_0 \subset \mathscr{V}$ (coarse space) on a quasi-uniform mesh $H := \operatorname{diam}(\Omega_j)$.
- This way, we have a two-level space decomposition $\mathscr{V} = V_0 + \mathscr{V}_1 + \cdots + \mathscr{V}_J$.

Solution method:

- Applying the SSC method based on the above space decomposition
- Using exact solvers on the coarse space as well as on each sub-domain
- This defines an abstract multiplicative Schwarz method (two-level DDM)

Q: How fast does this abstract two-level domain decomposition method converge?

Space Decomposition for DDM

A technical tool (partition of unity):

We define a partition of unity function using $\theta_j \in C^1(\overline{\Omega})$ $(j = 1, \ldots, J)$ such that

(1) $0 \leq \theta_j \leq 1$ and $\sum_{j=1}^{J} \theta_j = 1$;

(2) $\operatorname{supp} \theta_j \subset \hat{\Omega}_j$;

(3) $\max |\nabla \theta_j| \leq C_\beta / H$, where C_β depends on the relative overlap size β.

Space decomposition:

This way, for any function $v \in \mathscr{V}$, we can obtain its decomposition

$$v = v_0 + v_1 + \cdots + v_J,$$

where

$$v_0 \in V_0 \quad \text{and} \quad v_j := \theta_j (v - v_0) \in \mathscr{V}_j, \;\; j = 1, \ldots, J.$$

Convergence Analysis of DDM

We can show the method converges uniformly by applying the XZ identity for SSC.

Based on the above decomposition, we have $\sum_{j=1}^{J} v_j = v - v_0$ and

$$\sum_{j=0}^{J}\Big|\Pi_j \sum_{i=j+1}^{J} v_i\Big|_1^2 = \sum_{j=0}^{J}\Big|\Pi_j \sum_{i=j+1}^{J} \theta_i(v-v_0)\Big|_1^2 = \Big|\Pi_0(v-v_0)\Big|_1^2 + \sum_{j=1}^{J}\Big|\Pi_j \sum_{i=j+1}^{J} \theta_i(v-v_0)\Big|_1^2.$$

Since Π_j's : $\mathscr{V} \mapsto \mathscr{V}_j$ $(j = 1, \ldots, J)$ are $\mathcal{A}$-projections, we have $|\Pi_j w|_1 \le |w|_1$. Hence it is easy to see that

$$\begin{aligned}
\Big|\Pi_j \sum_{i=j+1}^{J} \theta_i(v-v_0)\Big|_1^2 &= \Big|\Pi_j \sum_{i=j+1}^{J} \theta_i(v-v_0)\Big|_{1,\hat{\Omega}_j}^2 \le \Big|\sum_{i=j+1}^{J} \theta_i(v-v_0)\Big|_{1,\hat{\Omega}_j}^2 \\
&\le \Big\|\Big(\sum_{i>j}\theta_i\Big)\nabla(v-v_0)\Big\|_{0,\hat{\Omega}_j}^2 + \Big\|\nabla\Big(\sum_{i>j}\theta_i\Big)(v-v_0)\Big\|_{0,\hat{\Omega}_j}^2 \\
&\le |v-v_0|_{1,\hat{\Omega}_j}^2 + C_\beta^2 H^{-2}\|v-v_0\|_{0,\hat{\Omega}_j}^2.
\end{aligned}$$

Convergence Analysis of DDM, Continued

By summing up all the terms, we have

$$\begin{aligned}\sum_{j=0}^{J}\Big|\varPi_j\sum_{i=j+1}^{J}v_i\Big|_1^2 &\le \big|v-v_0\big|_1^2+\sum_{j=1}^{J}\big|v-v_0\big|_{1,\hat{\Omega}_j}^2+C_\beta^2H^{-2}\sum_{j=1}^{J}\big\|v-v_0\big\|_{0,\hat{\Omega}_j}^2\\ &\lesssim \big|v-v_0\big|_1^2+C_\beta^2H^{-2}\big\|v-v_0\big\|_0^2,\end{aligned}$$

where the hidden constant depends on the maximal number of overlaps in domain decomposition.

Because v_0 could be any function in V_0, in view of the simultaneous estimate, we obtain

$$\sum_{j=0}^{J}\Big|\varPi_j\sum_{i=j+1}^{J}v_i\Big|_1^2\lesssim|v|_1^2.$$

Proposition (Uniform convergence of two-level DDM)

The abstract two-level DDM with finite-dimensional coarse space correction converges uniformly.

Nested Space Decomposition

Nested meshes:

- Consider a sequence of nested meshes $\mathcal{M}_l$ $(l = 0, \ldots, L)$ generated from an initial mesh $\mathcal{M}_0$ by uniform regular refinements.
- Meshsize h_l of $\mathcal{M}_l$ is proportional to γ^l with $\gamma \in (0, 1)$. Clearly,

$$h_0 > h_1 > h_2 > \cdots > h_L =: h.$$

- For example, usually we have $h_l = (1/2)^{l+1}$ $(l = 0, 1, \ldots, L)$.

Nested spaces:

- Define continuous piecewise linear finite element spaces

$$V_l := \left\{ v \in \mathscr{V} \,:\, v|_\tau \in \mathcal{P}_1(\tau), \ \forall \tau \in \mathcal{M}_l \right\}.$$

- This way, we build a nested subspaces

$$V_0 \subset V_1 \subset \cdots \subset V_L =: V \subset \mathscr{V} = H_0^1(\Omega).$$

HB Space Decomposition

Basis functions on nested spaces:

- The set of interior grid points on the l-th level is denoted as

$$x_{l,i} \in \mathring{G}(\mathcal{M}_l), \quad i = 1, \ldots, n_l.$$

- The subspace V_l is assigned with a nodal basis $\{\phi_{l,i}\}_{i=1}^{n_l}$, where $n_l := \left|\mathring{G}(\mathcal{M}_l)\right|$.
- The space V_l can be further decomposed as the sum of the one-dimensional subspaces spanned with the nodal basis $V_{l,i} := \mathrm{span}\{\phi_{l,i}\}$ $(i = 1, \ldots, n_l)$.

Hierarchical spaces:

- We define

$$W_l := \left\{v \in V_l \;:\; v(x) = 0,\ \forall\, x \in \mathring{G}(\mathcal{M}_{l-1})\right\}. \tag{51}$$

- We obtain a multilevel space decomposition

$$V = W_0 \oplus W_1 \oplus \cdots \oplus W_L. \tag{52}$$

The decomposition (52) is a direct sum and there is no redundancy in this decomposition.

Hierarchical Basis

A natural telescope expansion:

- Let $\mathcal{J}_l : V \mapsto V_l$ be the canonical interpolation operator and define $\mathcal{J}_{-1} := 0$.
- It is easy to check that

$$W_l = (\mathcal{J}_l - \mathcal{J}_{l-1})V = (\mathcal{I} - \mathcal{J}_{l-1})V_l, \quad l = 0, \ldots, L.$$

- A telescope sum can be written as

$$v = \textstyle\sum_{l=0}^{L} (\mathcal{J}_l - \mathcal{J}_{l-1})v, \quad v \in V.$$

Hierarchical basis functions on nested spaces:

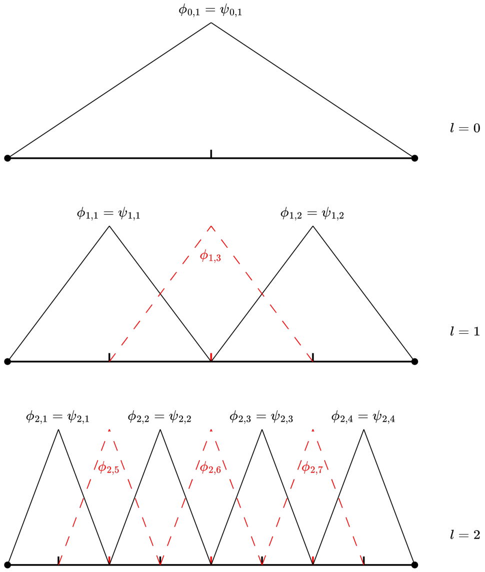
- For level $l = 0, \ldots, L$, we define a nodal basis function

$$\psi_{l,i}(x) = \phi_{l,i}(x), \quad \text{for} \;\; x_{l,i} \in \mathring{G}(\mathcal{M}_l) \backslash \mathring{G}(\mathcal{M}_{l-1}) \;\; \text{and} \;\; i = 1, \ldots, m_l := n_l - n_{l-1}.$$

  Apparently, $\sum_{l=0}^{L} m_l = n_L = N$.
- This gives the so-called hierarchical basis:

$$\{\psi_{l,i}(x) \,:\, i = 1, \ldots, m_l, \;\; l = 0, \ldots, L\}.$$

Regular and Hierarchical Bases in 1D



Notations:

- l: Index of levels
- $\mathcal{M}_l$: Mesh on level l
- V_l: Finite element space on level l
- $\phi_{l,i}$: Finite element basis functions on level l
- $\psi_{l,i}$: Hierarchical basis functions
- W_l: $\mathrm{span}\{\psi_{l,i}, i = 1, \ldots, m_l\}$
- $V = W_0 \oplus W_1 \oplus \cdots \oplus W_L$

Telescope Expansions

Reminder: We have introduced a telescope sum for $v \in V$ such that $v = \sum_{l=0}^{L} (\mathcal{J}_l - \mathcal{J}_{l-1})v$.

- We have defined that

$$\begin{cases} \mathcal{A}_l : V_l \mapsto V_l & (\mathcal{A}_l u_l, v_l) = a[u_l, v_l], \quad \forall\, u_l, v_l \in V_l; \\ \mathcal{Q}_l : L^2 \mapsto V_l & (\mathcal{Q}_l u, v_l) = (u, v_l), \quad \forall\, v_l \in V_l; \\ \mathit{\Pi}_l : \mathscr{V} \mapsto V_l & (\mathit{\Pi}_l u, v_l) = a[u, v_l], \quad \forall\, v_l \in V_l. \end{cases}$$

- We introduce notation $i \wedge j := \min(i, j)$ and notice that

$$\mathcal{Q}_i \mathcal{Q}_j = \mathcal{Q}_{i\wedge j}, \quad \mathit{\Pi}_i \mathit{\Pi}_j = \mathit{\Pi}_{i\wedge j}, \tag{53}$$

  and

$$(\mathcal{Q}_i - \mathcal{Q}_{i-1})(\mathcal{Q}_j - \mathcal{Q}_{j-1}) = (\mathit{\Pi}_i - \mathit{\Pi}_{i-1})(\mathit{\Pi}_j - \mathit{\Pi}_{j-1}) = 0, \quad \forall\, i \neq j. \tag{54}$$

- If we define $\mathcal{Q}_{-1} = \mathit{\Pi}_{-1} = 0$, we have the following possible decompositions

$$v = \sum_{l=0}^{L} (\mathcal{Q}_l - \mathcal{Q}_{l-1})v = \sum_{l=0}^{L} (\mathit{\Pi}_l - \mathit{\Pi}_{l-1})v. \tag{55}$$

Hierarchical Basis Preconditioner

HB Preconditioner:

Apply the Richardson iteration as the subspace solvers, i.e.,

$$\mathcal{S}_{l,i}\mathcal{Q}_{l,i}v = h_l^{2-d}\left(\mathcal{Q}_{l,i}v, \psi_{l,i}\right)\psi_{l,i} = h_l^{2-d}\left(v, \psi_{l,i}\right)\psi_{l,i}.$$

The PSC method based on the space decomposition (52) can then be written

$$\mathcal{B}_{\mathrm{HB}}r = \sum_{j=1}^{N}\mathcal{S}_j\mathcal{Q}_j r = \sum_{l=0}^{L}\left(h_l^{2-d}\sum_{i=1}^{m_l}(r, \psi_{l,i})\psi_{l,i}\right). \tag{56}$$

This is the well-known hierarchical basis (HB) preconditioner [Yserentant 1986].

The HB preconditioner is easy to implement. And it is very efficient in 1D and 2D; however, it is not optimal in general, particularly in 3D.

Stability of Interpolation Operators

Lemma (H^1-stability of interpolation)

We have
$$\left\|(\mathcal{J}_l - \mathcal{J}_{l-1})v\right\|_0^2 + h_l^2\left|\mathcal{J}_l v\right|_1^2 \lesssim c_d(l)\, h_l^2\, |v|_1^2 \qquad \forall v \in V,$$
where $c_1(l) \equiv 1$, $c_2(l) = L - l$, and $c_3(l) = \gamma^{l-L}$.

Sketch of the proof:

Using the interpolation error estimate, we have
$$\|(\mathcal{J}_l - \mathcal{J}_{l-1})v\|_0 = \|\mathcal{J}_l v - \mathcal{J}_{l-1}\mathcal{J}_l v\|_0 \lesssim h_l |\mathcal{J}_l v|_1.$$

Let $\tau \in \mathcal{M}_l$ and $v_\tau := |\tau|^{-1}\int_\tau v\,dx$ be the average of v on τ. Using the standard scaling argument for $|\cdot|_{1,\tau}$, the discrete Sobolev inequality, and the Poincaré inequality, we can obtain that
$$\begin{aligned}|\mathcal{J}_l v|_{1,\tau} = |\mathcal{J}_l v - v_\tau|_{1,\tau} &\lesssim h^{\frac{d}{2}-1}\|\mathcal{J}_l v - v_\tau\|_{\infty,\tau} \\ &\leq h^{\frac{d}{2}-1}\|v - v_\tau\|_{\infty,\tau} \lesssim C_d\|v - v_\tau\|_{1,\tau} \lesssim C_d|v|_{1,\tau}.\end{aligned}$$

The desired result follows by summing up terms on all elements in $\mathcal{M}_l$. $\square$

HB Finite Element Method

We can also employ HB as a FE basis ...

- The above stability lemma suggests that, if $w \in W_l$ for any $0 \le l \le L$, we have

$$c_d^{-1}(l) h_l^{-2}(w, w) \lesssim a[w, w].$$

- Compare this with the general Poincaré inequality:

$$\|w\|_0^2 \lesssim \|\nabla w\|_0^2 = a[w, w].$$

- Furthermore, from the inverse inequality, we always have

$$a[w, w] = |w|_1^2 \lesssim h_l^{-2} \|w\|_0^2 = h_l^{-2}(w, w).$$

The operator $\mathcal{A}_l^{\mathrm{HB}}$ (using HB basis) is "well-conditioned" up to a constant $c_d(l)$.

However, the basis is not completely local! Not popular as a discretization method.

Compared with the regular finite element method, the HB method is more solver-friendly.

Convergence Analysis of HB Method

The HB method is a PSC preconditioner and can be analyzed in the framework of MSC.

Lemma (Inner product between two levels)

If $i \le j$, we have

$$a[u,v] \lesssim \gamma^{\frac{j-i}{2}} h_j^{-1} |u|_1 \|v\|_0, \quad \forall\, u \in V_i, v \in V_j.$$

Lemma (Strengthened Cauchy–Schwarz inequality for interpolations)

If $u, v \in V$, let $u_i := (\mathcal{J}_i - \mathcal{J}_{i-1})u$, and $v_j := (\mathcal{J}_j - \mathcal{J}_{j-1})v$, then we have

$$a[u_i, v_j] \lesssim \gamma^{\frac{|i-j|}{2}} \big\|u_i\big\|_{\mathcal{A}} \big\|v_j\big\|_{\mathcal{A}}.$$

Theorem (Convergence of HB preconditioner)

The multilevel PSC preconditioner $\mathcal{B}_{\rm HB}$ defined in (56) satisfies $\kappa(\mathcal{B}_{\rm HB}\mathcal{A}) \lesssim C_d(h)$, where

$$C_1(h) \equiv 1, \quad C_2(h) = |\log h|^2, \quad C_3(h) = h^{-1}.$$

Sketch of the Convergence Proof

Part 1: $K_1 \lesssim C_d(h)$

- Apply a decomposition

$$v = \sum_{l=0}^{L} v_l := \sum_{l=0}^{L} (\mathcal{J}_l - \mathcal{J}_{l-1})v,$$

where $(\mathcal{J}_l - \mathcal{J}_{l-1})v \in W_l$ and $\mathcal{J}_{-1} = 0$.

- The inverse estimate and stability of interpolation ($\mathcal{J}_l = \Pi_l$ in 1D) yield

$$\sum_{l=0}^{L} \|v_l\|_{\mathcal{A}}^2 \lesssim \sum_{l=0}^{L} h_l^{-2} \|v_l\|_0^2 \lesssim C_d(h) \|v\|_{\mathcal{A}}^2. \tag{57}$$

- On the other hand, from the smoothing effect, we know

$$\hat{\omega}_0 = \min_{l} \rho_l \lambda_{\min}(\mathcal{S}_l) \cong 1.$$

Sketch of the Convergence Proof, Continued

Part 2: $K_2 \lesssim 1$

- Let $\mathcal{T}_j = \mathcal{S}_j \mathcal{A}_j \mathit{\Pi}_j$ and the subspace smoother $\mathcal{S}_j : V_j \mapsto V_j$ satisfies

$$\left\| \mathcal{S}_j \mathcal{A}_j v \right\|_0^2 \lesssim \rho(\mathcal{A}_j)^{-1} \big(\mathcal{A}_j v, v\big), \quad \forall\, v \in V_j.$$

- The inner product between two levels reads

$$(u_i, \mathcal{T}_j v)_{\mathcal{A}} = a[u_i, \mathcal{T}_j v] \lesssim \gamma^{\frac{j-i}{2}} h_j^{-1} \|u_i\|_{\mathcal{A}} \|\mathcal{T}_j v\|_0.$$

- We also notice $\|\mathcal{T}_j v\|_0 = \|\mathcal{S}_j \mathcal{A}_j \mathit{\Pi}_j v\|_0 \lesssim h_j \|\mathcal{A}_j^{1/2} \mathit{\Pi}_j v\|_0 \le h_j \|\mathit{\Pi}_j v\|_{\mathcal{A}} \le h_j \|v\|_{\mathcal{A}}$.
- If $i < j$, we have

$$(u_i, \mathcal{T}_j v)_{\mathcal{A}} \lesssim \gamma^{\frac{j-i}{2}} \|u_i\|_{\mathcal{A}} \|v\|_{\mathcal{A}}, \quad \forall\, u_i \in V_i, v \in V. \tag{58}$$

- For $0 \le i, j \le L$, we have the following strengthened Cauchy–Schwarz inequality

$$(\mathcal{T}_i u, \mathcal{T}_j v)_{\mathcal{A}} \lesssim \gamma^{\frac{|j-i|}{4}} (\mathcal{T}_i u, u)_{\mathcal{A}}^{\frac{1}{2}} (\mathcal{T}_j v, v)_{\mathcal{A}}^{\frac{1}{2}}, \quad \forall\, u, v \in V. \tag{59}$$

Some Remarks on HB Preconditioner

If $v \in V$, Lemma 57 yields

$$
\begin{aligned}
|v|_1^2 &= \sum_{l,m} \big(\nabla(\mathcal{J}_l - \mathcal{J}_{l-1})v, \nabla(\mathcal{J}_m - \mathcal{J}_{m-1})v\big) \\
&\lesssim \sum_{l,m} \gamma^{\frac{|l-m|}{2}} \big\|(\mathcal{J}_l - \mathcal{J}_{l-1})v\big\|_1 \big\|(\mathcal{J}_m - \mathcal{J}_{m-1})v\big\|_1 \ \lesssim \ \sum_{l} \big\|(\mathcal{J}_l - \mathcal{J}_{l-1})v\big\|_1^2 .
\end{aligned}
$$

Define an operator $\mathcal{H} : V \mapsto V$ such that

$$
(\mathcal{H}v, w) := \sum_{l=0}^{L} \sum_{x_i \in \mathring{G}(\mathcal{M}_l) \setminus \mathring{G}(\mathcal{M}_{l-1})} h_l^{d-2} \Big(\big(\mathcal{J}_l v - \mathcal{J}_{l-1} v\big)(x_i), \ \big(\mathcal{J}_l w - \mathcal{J}_{l-1} w\big)(x_i) \Big).
$$

$$
\Longrightarrow \quad (\mathcal{H}v, v) = \sum_{l=0}^{L} \sum_{x_i \in \mathring{G}(\mathcal{M}_l) \setminus \mathring{G}(\mathcal{M}_{l-1})} h_l^{d-2} \Big| \big(\mathcal{J}_l v - \mathcal{J}_{l-1} v\big)(x_i) \Big|^2, \quad \forall\, v \in V.
$$

The operator $\mathcal{H}$ is the inverse of the HB preconditioner, i.e., $\mathcal{H} = \mathcal{B}_{\mathrm{HB}}^{-1}$ and (57) reads

$$
\|v\|_{\mathcal{A}}^2 \lesssim (\mathcal{H}v, v) = \sum_{l=0}^{L} h_l^{-2} \|(\mathcal{J}_l - \mathcal{J}_{l-1})v\|_0^2 \lesssim C_d(h) \|v\|_{\mathcal{A}}^2 .
$$

BPX Preconditioner

An alternative space decomposition

Along with the hierarchical basis decomposition, we obtained another space decomposition

$$V = \sum_{l=0}^{L} V_l = \sum_{l=0}^{L} \sum_{i=1}^{n_l} V_{l,i}, \tag{60}$$

which contains a lot of "redundancy". Heuristically, one might want to avoid such redundancy.

> In order to obtain optimal convergence rate, these extra subspaces are not redundant at all.

The BPX method

Using the decomposition (60), we can construct multilevel subspace correction methods. Among them, the most prominent (multilevel) example of PSC methods is the BPX preconditioner:

$$\mathcal{B} = \sum_{j=1}^{J} \mathcal{S}_j \mathcal{Q}_j, \quad \text{with} \ \ J = \sum_{l=0}^{L} n_l, \tag{61}$$

which is computationally more appealing and converges uniformly.

Convergence of BPX Method

HB and BPX preconditioners belong to the class of multilevel nodal basis preconditioners.

Lemma (Difference between two levels of L^2-projections)

For any $v \in V$, we have

$$\left|(\mathcal{Q}_l - \mathcal{Q}_{l-1})v\right|_1 \cong h_l^{-1} \left\|(\mathcal{Q}_l - \mathcal{Q}_{l-1})v\right\|_0.$$

Lemma (Strengthened Cauchy–Schwarz inequality for L^2-projections)

If $u, v \in V$, let $u_i := (\mathcal{Q}_i - \mathcal{Q}_{i-1})u$, and $v_j := (\mathcal{Q}_j - \mathcal{Q}_{j-1})v$, then we have

$$a[u_i, v_j] \lesssim \gamma^{\frac{|i-j|}{2}} \|u_i\|_{\mathcal{A}} \|v_j\|_{\mathcal{A}}.$$

Lemma (Norm equivalence)

For any $v \in V$, we have

$$\sum_{l=0}^{L} \left\|(\mathcal{Q}_l - \mathcal{Q}_{l-1})v\right\|_1^2 \cong \|v\|_1^2.$$

Proof of Norm Equivalence

(1) Since $\mathcal{Q}_l$ is a L^2-projection, we have

$$\|\mathcal{Q}_l v\|_0 \leq \|v\|_0, \quad \forall\, v \in L^2(\Omega).$$

According to the weighted estimate for L^2-projection, we obtain

$$\|\mathcal{Q}_l v\|_1 \leq \|v\|_1, \quad \forall\, v \in V.$$

By space interpolation, we have, for any $\sigma \in (0, \frac{1}{2})$, that

$$\|\mathcal{Q}_l v\|_\sigma \leq \|v\|_\sigma, \quad \forall\, v \in V.$$

(2) Let $\alpha \in (\frac{1}{2}, 1)$. If $\mathit{\Pi}_l : V \mapsto V_l$ is H^1-projection, the finite element theory gives

$$\big\|v - \mathit{\Pi}_l v\big\|_{1-\alpha} \lesssim h_l^\alpha \big\|v\big\|_1, \quad \forall\, v \in V.$$

Let $v_i := (\mathit{\Pi}_i - \mathit{\Pi}_{i-1})v$. Note that $\rho_l = \rho(\mathcal{A}_l) \cong h_l^{-2}$. With the inverse inequality, we have

$$\big\|(\mathcal{Q}_l - \mathcal{Q}_{l-1})v_i\big\|_1^2 \lesssim h_l^{-2\alpha}\big\|(\mathcal{Q}_l - \mathcal{Q}_{l-1})v_i\big\|_{1-\alpha}^2 \lesssim h_l^{-2\alpha}\big\|v_i\big\|_{1-\alpha}^2$$
$$\lesssim h_l^{-2\alpha} h_i^{2\alpha}\big\|v_i\big\|_1^2 \cong \rho_l^\alpha h_i^{2\alpha}\big\|v_i\big\|_1^2.$$

Proof of Norm Equivalence, Continued

(3) Using the above inequality and the Cauchy–Schwarz inequality, we can derive that

$$\sum_l \sum_{i,j} \big(\nabla(\mathcal{Q}_l - \mathcal{Q}_{l-1})v_i, \nabla(\mathcal{Q}_l - \mathcal{Q}_{l-1})v_j\big) = \sum_{i,j} \sum_{l=1}^{i\wedge j} \big(\nabla(\mathcal{Q}_l - \mathcal{Q}_{l-1})v_i, \nabla(\mathcal{Q}_l - \mathcal{Q}_{l-1})v_j\big)$$

$$\lesssim \sum_{i,j} \sum_{l=1}^{i\wedge j} \rho_l^\alpha h_i^\alpha h_j^\alpha \|v_i\|_1 \|v_j\|_1 \lesssim \sum_{i,j} \rho_{i\wedge j}^\alpha h_i^\alpha h_j^\alpha \|v_i\|_1 \|v_j\|_1 \lesssim \sum_{i,j} \gamma^{\alpha|i-j|} \|v_i\|_1 \|v_j\|_1.$$

We can show that $\sum_{i,j} \gamma^{\alpha|i-j|} \|v_i\|_1 \|v_j\|_1 \lesssim \sum_i \|v_i\|_1^2 \lesssim \|v\|_1^2$, which gives

$$\sum_l \big\|(\mathcal{Q}_l - \mathcal{Q}_{l-1})v\big\|_1^2 \lesssim \|v\|_1^2.$$

(4) On the other hand, using the SCS inequality, we obtain

$$\begin{aligned} |v|_1^2 &= \sum_{l,m} \big(\nabla(\mathcal{Q}_l - \mathcal{Q}_{l-1})v, \nabla(\mathcal{Q}_m - \mathcal{Q}_{m-1})v\big) \\ &\lesssim \sum_{l,m} \gamma^{|l-m|} \big\|(\mathcal{Q}_l - \mathcal{Q}_{l-1})v\big\|_1 \big\|(\mathcal{Q}_m - \mathcal{Q}_{m-1})v\big\|_1 \lesssim \sum_l \big\|(\mathcal{Q}_l - \mathcal{Q}_{l-1})v\big\|_1^2. \end{aligned}$$

PSC and BPX

- Apply exact subspace solver on each one-dimensional subspace:

$$\mathcal{S}_l^0 v := \sum_{i=1}^{n_l} (\mathcal{A}\phi_{l,i}, \phi_{l,i})^{-1} (v, \phi_{l,i})\, \phi_{l,i} = \sum_{i=1}^{n_l} (\nabla\phi_{l,i}, \nabla\phi_{l,i})^{-1} (v, \phi_{l,i})\, \phi_{l,i}.$$

- Consider a uniform refinement, we can use an approximation of $\mathcal{S}_l^0$:

$$\mathcal{S}_l v := \sum_{i=1}^{n_l} h_l^{2-d} (v, \phi_{l,i})\, \phi_{l,i} \quad (\approx \mathcal{S}_l^0 v).$$

  Note that it is just the Richardson method with $\omega = h_l^{2-d}$ on level l.

- Apparently, we have

$$(\mathcal{S}_l v, v) = h_l^{2-d} (\vec{v}, \vec{v}) = h_l^2 (v, v).$$

- The corresponding PSC method yields the well-known BPX preconditioner

$$\mathcal{B} = \sum_{l=0}^{L} \mathcal{S}_l \mathcal{Q}_l = \sum_{l=0}^{L} \mathcal{I}_l \mathcal{S}_l \mathcal{Q}_l = \sum_{l=0}^{L} \mathcal{I}_l \mathcal{S}_l \mathcal{I}_l^T, \quad (\mathcal{B}v, v) = \sum_{l=0}^{L} h_l^2 (\mathcal{Q}_l v, v).$$

Uniform Convergence of BPX

Theorem (Uniform convergence of BPX)

The BPX preconditioner is uniformly convergent, i.e., $\kappa(\mathcal{B}\mathcal{A}) \lesssim 1$.

Sketch of the proof:

Let $v = \sum_{l=0}^{L} v_l := \sum_{l=0}^{L}(\mathcal{Q}_l - \mathcal{Q}_{l-1})v$, where $\mathcal{Q}_{-1} = 0$. Then we can obtain that

$$(\mathcal{A}v, v) \cong \sum_{l=0}^{L} \left|(\mathcal{Q}_l - \mathcal{Q}_{l-1})v\right|_1^2 \cong \sum_{l=0}^{L} h_l^{-2} \|(\mathcal{Q}_l - \mathcal{Q}_{l-1})v\|_0^2 = \left(\sum_{l=0}^{L} h_l^{-2}(\mathcal{Q}_l - \mathcal{Q}_{l-1})v,\, v\right).$$

Define $\tilde{\mathcal{A}} := \sum_{l=0}^{L} h_l^{-2}(\mathcal{Q}_l - \mathcal{Q}_{l-1})$. Apparently, $(\mathcal{A}v, v) \cong (\tilde{\mathcal{A}}v, v)$, $\forall v \in V$. We can verify that

$$\tilde{\mathcal{A}}^{-1} = \sum_{l=0}^{L} h_l^2(\mathcal{Q}_l - \mathcal{Q}_{l-1}), \quad (\tilde{\mathcal{A}}^{-1}v, v) = h_L^2(\mathcal{Q}_L v, v) + \sum_{l=0}^{L-1}(1-\gamma^2)h_l^2(\mathcal{Q}_l v, v).$$

Namely, $(\tilde{\mathcal{A}}^{-1}v, v) \cong (\mathcal{B}v, v)$. That is to say, $(\mathcal{A}v, v) \cong (\tilde{\mathcal{A}}v, v) \cong (\mathcal{B}^{-1}v, v)$. $\square$

Matrix Form of BPX

- Revisit the matrix representation of transfer operators:

$$\mathcal{I}_l : V_l \mapsto V \quad \Longrightarrow \quad \underline{\mathcal{I}_l} = P_l.$$

- We can obtain the matrix representation of the BPX preconditioner:

$$\underline{\mathcal{B}}\,\underline{u} = \underline{\mathcal{B}u} = \sum_{l=0}^{L} \underline{\mathcal{I}_l}\,\underline{\mathcal{S}_l}\,\underline{\mathcal{Q}_l}\,\underline{u} = \sum_{l=0}^{L} P_l\,(h_l^{2-d} M_l)\,(M_l^{-1} P_l^T M)\,\underline{u} = \sum_{l=0}^{L} h_l^{2-d} P_l\, P_l^T M\,\underline{u}.$$

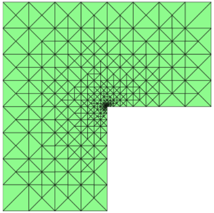
- The matrix form of the BPX preconditioner

$$B := \underline{\mathcal{B}}\,M^{-1} = \sum_{l=0}^{L} h_l^{2-d} P_l\, P_l^T.$$

- To improve efficiency, we often use prolongations between two consecutive levels to obtain P_l.
- We can also use the Jacobi method for the corresponding expanded system to implement BPX.

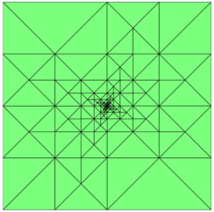
Some Numerical Results for PSC

Solving AFEM for Poisson's equation with PCG:



DOF	9628	13339	18648	26097	36528
HB	31 (0.37)	31 (0.48)	31 (0.67)	36 (0.98)	36 (1.42)
BPX	23 (0.48)	23 (0.71)	22 (0.92)	25 (1.48)	25 (2.32)
THB	18 (0.35)	19 (0.50)	18 (0.71)	20 (0.87)	20 (1.40)
LOHB	10 (0.21)	10 (0.29)	10 (0.37)	11 (0.54)	11 (0.79)

Poisson's equation on a L-shaped domain. Preconditioned CG method with zero initial guess and stopping tolerance is 1e-6.



DOF	7095	9708	13726	19821	28956
HB	26 (0.26)	25 (0.40)	27 (0.62)	27 (1.0)	33 (1.51)
BPX	24 (0.48)	24 (0.60)	24 (0.85)	24 (1.48)	27 (1.93)
THB	19 (0.29)	19 (0.42)	19 (0.56)	19 (0.89)	21 (1.26)
LOHB	10 (0.17)	10 (0.23)	10 (0.32)	10 (0.59)	10 (0.68)

Jump coefficient problem on the unit square. Preconditioned CG method with zero initial guess and stopping tolerance is 1e-6.

More details of the methods and test examples can be found in [Chen and Zhang 2010].

Geometric Multigrid Method

Algorithm (One iteration of MG V-cycle)

Assume that $\mathcal{B}_{l-1} : V_{l-1} \mapsto V_{l-1}$ is defined and the coarsest level solver $\mathcal{B}_0 = \mathcal{A}_0^{-1}$ is exact. Let v_l be the initial guess on each level, i.e., $v_L = u^{(0)}$ and $v_l = 0$ for $0 < l < L$. Do the following steps:

1. Pre-smoothing: For $k = 1, 2, \ldots, m_1$, compute
$$v_l \leftarrow v_l + \mathcal{S}_l\big(r_l - \mathcal{A}_l v_l\big);$$
2. Coarse grid correction: Find an approximate solution $e_{l-1} \in V_{l-1}$ of the residual equation on level $l-1$, i.e., $\mathcal{A}_{l-1} e_{l-1} = \mathcal{Q}_{l,l-1}\big(r_l - \mathcal{A}_l v_l\big)$, by an iterative method:
$$e_{l-1} \leftarrow \mathcal{B}_{l-1}\mathcal{Q}_{l,l-1}(r_l - \mathcal{A}_l v_l), \quad v_l \leftarrow v_l + \mathcal{I}_{l-1,l} e_{l-1};$$
3. Post-smoothing: For $k = 1, 2, \ldots, m_2$, compute
$$v_l \leftarrow v_l + \mathcal{S}_l^T\big(r_l - \mathcal{A}_l v_l\big).$$

This algorithm is the so-called V(m_1,m_2)-cycle.

Space Decomposition for GMG

Another space decomposition

- The $V(1,1)$-GMG method (with one G-S iteration as pre-smoothing and one backward G-S as post-smoothing) is actually SSC based on the following multilevel space decomposition

$$V = \sum_{j=1}^{J} \tilde{V}_j = \sum_{l=L:-1:1} \sum_{i=1:n_l} V_{l,i} + V_0 + \sum_{l=1:L} \sum_{i=n_l:-1:1} V_{l,i}.$$

Error transfer operator

- The error transfer operator of V-cycle on the l-th level can be written as

$$\mathcal{E}_l := \mathcal{I} - \mathcal{B}_l \mathcal{A}_l = \big(\mathcal{I} - \mathcal{S}_l^T \mathcal{A}_l\big)\big(\mathcal{I} - \mathcal{B}_{l-1}\mathcal{A}_{l-1}\mathit{\Pi}_{l-1}\big)\big(\mathcal{I} - \mathcal{S}_l \mathcal{A}_l\big),$$

  where $\mathit{\Pi}_{l-1}$ is the Ritz-projection from V to V_{l-1}.

- By applying this recursively, we obtain the error transfer operator for the MG V-cycle:

$$\mathcal{E}_L = \mathcal{I} - \mathcal{B}_L \mathcal{A}_L \mathit{\Pi}_L = \big(\mathcal{I} - \mathcal{S}_L^T \mathcal{A}_L\big) \cdots \big(\mathcal{I} - \mathcal{S}_1^T \mathcal{A}_1\big)\big(\mathcal{I} - \mathit{\Pi}_0\big)\big(\mathcal{I} - \mathcal{S}_1 \mathcal{A}_1\big) \cdots \big(\mathcal{I} - \mathcal{S}_L \mathcal{A}_L\big).$$

Matrix Representation of GMG

By definition, we have

$$(\mathcal{A}_l u_l, v_l) = (\mathcal{A} u_l, v_l), \quad \forall\, u_l, v_l \in V_l.$$

Hence,

$$(\mathcal{A}_l u_l, v_l) = (\mathcal{A} u_l, v_l) = (\mathcal{A}\mathcal{I}_l u_l, \mathcal{I}_l v_l) = (\mathcal{I}_l^T \mathcal{A}\mathcal{I}_l u_l, v_l), \quad \forall\, u_l, v_l \in V_l.$$

It is easy to see that

$$\mathcal{A}_l = \mathcal{I}_l^T \mathcal{A}\mathcal{I}_l \quad \Longrightarrow \quad \underline{\mathcal{A}_l} = \underline{\mathcal{I}_l^T \mathcal{A}\mathcal{I}_l} = \underline{\mathcal{I}_l^T}\, \underline{\mathcal{A}}\, \underline{\mathcal{I}_l}.$$

This, in turn, give the inter-grid transformations:

$$\hat{\mathcal{A}}_l = M_l\, \underline{\mathcal{A}_l} = M_l\, \underline{\mathcal{I}_l^T}\, \underline{\mathcal{A}}\, \underline{\mathcal{I}_l} = M_l\, \underline{\mathcal{Q}_l}\, M^{-1} \hat{\mathcal{A}}\, \underline{\mathcal{I}_l} = \underline{\mathcal{I}_l}^T \hat{\mathcal{A}}\, \underline{\mathcal{I}_l}, \quad 0 \le l < L.$$

Hence we get the matrix form ($0 \le l < L$) of the coarse level operator:

$$\hat{\mathcal{A}}_l = P_l^T \hat{\mathcal{A}}\, P_l.$$

Remark: As before, we can also use relations between two consecutive levels to obtain coarse problems.

Convergence Analysis of GMG

- Denote the canonical interpolation operators from V to V_l as $\mathcal{J}_l$. For any function $v \in V$,

$$(\mathcal{J}_l v)(x) = \sum_{i=1}^{n_l} v(x_i^l)\, \phi_i^l(x), \quad l = 0, \ldots, L.$$

- Let $\mathcal{J}_{-1} v := 0$, $v_0 := \mathcal{J}_0 v$, and $v_l := (\mathcal{J}_l - \mathcal{J}_{l-1}) v$, $l = 1, \ldots, L$. Using the interpolants in multilevel spaces, we can write

$$v = \mathcal{J}_L v = \sum_{l=0}^{L} (\mathcal{J}_l - \mathcal{J}_{l-1}) v = \sum_{l=0}^{L} v_l.$$

- We also have

$$v = \sum_{l=0}^{L} v_l = \sum_{l=0}^{L} \sum_{i=1}^{n_l} v(x_i^l)\, \phi_i^l(x) =: \sum_{l=0}^{L} \sum_{i=1}^{n_l} v_{l,i}.$$

- It is easy to check that

$$(\mathcal{I} - \mathcal{J}_l) v = \sum_{k=l+1}^{L} v_k = \sum_{k=l+1}^{L} \sum_{j=1}^{n_k} v_{k,j}.$$

Convergence Analysis of GMG, Continued

- To estimate the convergence rate of V-cycle (exact subspace solver), we need to estimate

$$c_1 := \sup_{|v|_1=1} \inf_{\sum_{l,i} v_{l,i}=v} \sum_{l=0}^{L}\sum_{i=1}^{n_l} \Big| \Pi_{l,i} \sum_{(k,j)\geq(l,i)} v_{k,j} \Big|_1^2 .$$

- We now define and estimate

$$c_1(v) := \sum_{l=0}^{L}\sum_{i=1}^{n_l} \Big| \Pi_{l,i} \Big(\sum_{j=i}^{n_l} v_{l,j} + \sum_{k=l+1}^{L}\sum_{j=1}^{n_k} v_{k,j} \Big) \Big|_1^2 .$$

- Note that $\Pi_{l,i} : V \mapsto V_{l,i}$ is the $(\cdot,\cdot)_{\mathcal{A}}$-projection. For 1D problems, it is easy to see that $\Pi_l = \mathcal{J}_l$.
- This leads to the following identity

$$\Pi_{l,i}(\mathcal{I} - \mathcal{J}_l) = 0, \quad \forall\, 1 \leq i \leq n_l,\ 0 \leq l \leq L.$$

- Furthermore, we also have $\Pi_{l,i}(\sum_{j\geq i} v_{l,j}) = \Pi_{l,i}\big(v_{l,i} + v_{l,i+1}\big)$.

Uniform Convergence of GMG

Using these properties, we have

$$
\begin{aligned}
c_1(v) &= \sum_{l=0}^{L}\sum_{i=1}^{n_l}\left|\Pi_{l,i}\big(v_{l,i}+v_{l,i+1}\big)+\Pi_{l,i}(\mathcal{I}-\mathcal{J}_l)v\right|_1^2 \\
&= \sum_{l=0}^{L}\sum_{i=1}^{n_l}\left|\Pi_{l,i}\big(v_{l,i}+v_{l,i+1}\big)\right|_1^2 \\
&\lesssim \sum_{l=0}^{L}\sum_{i=1}^{n_l}|v_{l,i}|_1^2 \\
&\leq \sum_{l=0}^{L} h_l^{-2}\left\|(\mathcal{J}_l-\mathcal{J}_{l-1})v\right\|_0^2 \\
&\lesssim \sum_{l=0}^{L}|v_l|_1^2 \;=\; |v|_1^2.
\end{aligned}
$$

This estimate shows the convergence rate of GMG (in 1D) is uniformly bounded.