# Convergence theory of Subspace Correction Methods for Singular and Nearly Singular System of Equations

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# Contents

# Some basic equations in CFD

- Non-Newtonian model equations

$$
\begin{aligned}
\mathrm{RE}\frac{D\mathbf{u}}{Dt} - \eta_s \Delta \mathbf{u} + \nabla p - \mathrm{div}\tau &= \mathbf{f} \\
-\mathrm{div}\mathbf{u} &= 0 \\
\tau + \mathrm{Wi}\frac{\delta_F \tau}{\delta_F t} &= \eta_p(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)
\end{aligned}
$$

where $\delta_F \tau / \delta_F t$ is the Upper convected maxwell derivative,

$$
\frac{\delta_F \tau}{\delta_F t} = \frac{\partial \tau}{\partial t} + \mathbf{u} \cdot \nabla \tau - \tau \nabla \mathbf{u} - (\nabla \mathbf{u})^T \tau
$$

# Some basic equations in CFD

The Weissenberg number **Wi** is the measure of elasticity. For values larger than $10 - 15$, conventional algorithms are known to start to fail. For the industrial applications, the values of the Weissenberg number easily reaches 100 and it is considered that this may not be overcome - SIAM News (2004).

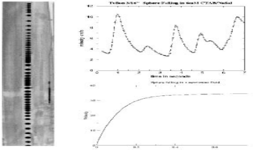
Is the model bad?

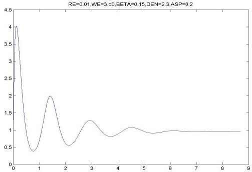
We believe that the answer is NO!

- At least, Discrete Model can be designed so that the resulting system is well-posed (Lee, Xu and Zhang (2010)).
- The model can be solved by Multigrid Method (Lee, Xu and Zhang (2010) - Could not publish it due to the singular point for P4-P3).

# Falling Sphere through cylinder

Collage of video image showing the descent of a $3/16$ inch-diameter teflon sphere (Left). (Right Top) Velocity profile of the sphere through CTAB/NASAL in time. (Right bottom) Velocity profile of the sphere through Newtonian fluids (Numerical simulation). - Belmonte, Jayaraman (Pritchard Lap in Penn State Univ.) Lee)






## Remark

*The model can be solved by Multigrid Method (Lee and Zhang (2010) - Could not publish it due to the nonconvergence in mesh refinement).*

# Fast Solution based on Augmented Lagrangian Uzawa Method I

The time-dependent Stokes equation can be reformulated in the following equivalent form for any $\mathbf{r} \geq 0$ : Find $\mathbf{u}$ and $p$ such that

$$\begin{pmatrix} \frac{\mathrm{RE}}{\Delta t} I - \eta_s \Delta_h - \mathbf{r}\nabla \mathrm{div} & \nabla \\ -\mathrm{div} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}.$$

An appropriate scailing, we may consider the following equation:

$$\begin{pmatrix} A & B^* \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} I - \rho^2 \Delta_h - \kappa^2 \nabla \mathrm{div} & \nabla \\ -\mathrm{div} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}.$$

The application of the Uzawa method for the reformulated system would read

$$\begin{aligned} A\mathbf{u}^{\ell+1} + B^* p^{\ell} &= \mathbf{f} \\ p^{\ell+1} &= p^{\ell} + \omega B u^{\ell+1}, \end{aligned}$$

# Fast Solution based on Augmented Lagrangian Uzawa Method II

The application of the Uzawa method for the reformulated system would read

$$\begin{aligned} A\mathbf{u}^{\ell+1} + B^* p^\ell &= \mathbf{f} \\ p^{\ell+1} &= p^\ell + \omega B u^{\ell+1}, \end{aligned}$$

where $\omega$ should be chosen so that

$$0 < \omega < 2/\rho(S_{\kappa^2}), \text{ with } S_{\kappa^2} = BA^{-1}B^*$$

and $A = I - \rho^2 \Delta_h - \kappa^2 \nabla \mathrm{div}$. It can be shown that by (Sherman-Morrison-Woodbury formula)

$$\rho(S_{\kappa^2}) \leq \frac{1}{\kappa^2}$$

and for $0 < \omega < 2\kappa^2$, the Uzawa method converges.

# Fast Solution based on Augmented Lagrangian Uzawa Method III

The following estimate holds true :

$$\|p - p^\ell\|_0 \leq \frac{1}{(1+\mu_0\kappa^2)^\ell}\|p - p^0\|_0$$

$$\|\mathbf{u} - \mathbf{u}^\ell\|_A \leq \frac{1}{\sqrt{\kappa^2}}\frac{1}{(1+\mu_0\kappa^2)^\ell}\|p - p^0\|_0.$$

## Remark

**The method converges within one iteration if** $\kappa^2 \gg 1$.
*The cost of the one iteration is to solve the following system:*

$$A\mathbf{u} = (I - \rho^2\Delta_h - \kappa^2\nabla\mathrm{div})\mathbf{u} = \widetilde{\mathbf{f}}.$$

*It is an example of nearly singular problem.*

# Method of Subspace Corrections (Xu and Zikatanov 02, J.AMS)

- Variational Problem : Find $u \in V$ such that

$$a(u, v) = \langle f, v \rangle, \quad \forall v \in V.$$

- Space decomposition:

$$V = \sum_{i=1}^{J} V_i.$$

- Approximate subspace problems : $a_i \approx a$ on $V_i \times V_i$ and $T_i \approx P_i$:

$$a_i(T_i v, v_i) = a(v, v_i), \quad v \in V, v_i \in V_i.$$

with $T_i = P_i$ if $a_i = a$.

# Method of Subspace Corrections (Xu and Zikatanov 02, J.AMS)

- **Algorithm MSC :** Let $u^0 \in V$ be given.

  **for** $\ell = 1, 2, \cdots,$

  $u_0^{\ell-1} = u^{\ell-1}.$

  **for** $i = 1, ..., J$

  $u_i^\ell = u_{i-1}^{\ell-1} + T_i e_i,$

  where $e_i \in V_i$ is such that $a(e_i, v_i) = \langle f, v_i \rangle - a(u_{i-1}^{\ell-1}, v_i)$

  **endfor**

  $u^\ell = u_J^\ell.$

  **endfor**

# MSC: Error Transfer Operator

- From the relation that $u - u_i^\ell = (I - T_i)(u - u_{i-1}^{\ell-1})$, we obtain

$$u - u^\ell = E_J(u - u^{\ell-1}) = \cdots = E_J^\ell(u - u^0),$$

where

$$E_J = (I - T_J)(I - T_{j-1}) \cdots (I - T_1).$$

- Convergence : $\|E_J\| = \|E_J\|_a < 1$?

# MSC: Error Transfer Operator

## Theorem (Xu and Zikatanov 2002, J.AMS)

$$\|E_J\|^2 = \|(I - T_J)\cdots(I - T_1)\|^2 = 1 - \frac{1}{K}$$

*where*

$$K = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^{J} (\bar{T}_i^{-1}(v_i + T_i^* w_i), (v_i + T_i^* w_i)),$$

*where* $\bar{T}_i = T_i + T_i^* - T_i^* T_i$ *and* $w_i = \sum_{j=i+1}^{J} v_j$.

## Remark

$\bar{T}_i$ *is positive definite on* $V_i \Rightarrow K > 0$, *which means* $\|E_J\| < 1$. *Therefore, we obtain the convergence.*

# MSC: Error Transfer Operator

## Lemma

*For the case when $T_i = P_i$, i.e., exact subspace solvers. The convergence rate identity is given by*

$$\|E_J\|^2 = \|(I - P_J) \cdots (I - P_1)\|^2 = 1 - \frac{1}{K},$$

*where*

$$K = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^{J} \|P_i \sum_{j=i} v_i\|^2$$

# On Assumptions

(H1) $\mathcal{R}(T_i) = V_i$ and $T_i : V_i \mapsto V_i$ is isomorphic for each $i = 1 : J$.
This is implied by the following inf-sup condition:

$$\inf_{u_i \in V_i} \sup_{v_i \in V_i} \frac{a_i(u_i, v_i)}{\|u_i\|\|v_i\|} = \inf_{v_i \in V_i} \sup_{u_i \in V_i} \frac{a_i(u_i, v_i)}{\|u_i\|\|v_i\|} > 0.$$

(H2) There exists $\omega \in (0, 2)$ such that
$(T_i v_i, T_i v_i) \leq \omega(T_i v_i, v_i), \quad \forall v_i \in V_i$ for each $i = 1 : J$.

This is deduced from the requirement that

$$\|I - T_i\| \leq 1 \quad \Leftrightarrow \quad (T_i v_i, T_i v_i) \leq 2(T_i v_i, v_i), \quad \forall v_i \in V_i.$$

We note that If $I - T_i = -I$ for all $i = 1 : J$, then

$$\|E_J\| = \|(-I)^J\| = \|I\| = 1.$$

# On Assumptions (H2)

We consider to solve $Ax = f$ with $J = 1$, and the matrix $A$ is decomposed into

$$A = D - L - L^t,$$

where $D$ is the diagonal of $A$ and $-L$ is the strictly lower triangular of $A$.
By choosing $a_i = D$, the MSC corresponds to the simple Jacobi relaxation with $T = D^{-1}A$. It is then easy to show that

$$\begin{aligned} \|x\|_A^2 - \|(I-T)x\|_A^2 &= ((D^{-t} + D^{-1} - D^{-t}AD^{-1})Ax, Ax), \quad x \in \mathbb{R}^n \\ &= (D^{-t}(D + D^t - A)D^{-1}Ax, Ax), \quad x \in \mathbb{R}^n \end{aligned}$$

$$\|I - T\|_A < 1 \quad \Leftrightarrow \quad D + D^t - A > 0.$$

# On Assumptions (H2)

Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

For the Jacobi method, we obtain that

$$D + D^t - A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

This is only semidefinite. We can not guarantee the convergence.
In fact, for $v = (1, 1, 1)^t$,
we have

$$(I - T)v = v.$$

# MSC for Singular Equations

Convergence studies of classic iterative methods : H. Keller (1965)
Consider to solve the following system of equations

$$Ax = f,$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric and positive semi-definite or singular.
References : Marek and Szyld (2004) and Dax ('90), Berman and Plemmons (1994)

# MSC for Singular Equations

**Theorem (Keller, 1965)**

*Assume that the splitting $A = R - G$ satisfies the following two properties :*

(a) *$R$ is invertible on $\mathbb{R}^n$*

(b) *$R + R^T - A$ is positive difinite on $\mathbb{R}^n$*

*Then the following iterates converges :*

$$u^\ell = u^{\ell-1} + R^{-1}(f - Au^{\ell-1}), \quad \ell = 1, 2 \cdots$$

**Remark**

**New observation** : *For the convergence result, $R$ may not need be invertible on $\mathbb{R}^n$ and (b) can also be weakened! (Lee,Wu,Xu and Zikatanov, SIMAX (2006))*

# MSC for Singular Equations

For $a(\cdot,\cdot)$ is singular, i.e., symmetric and positive semi-definite, consider the problem : Find $u$ such that

$$a(u,v) = \langle f, v\rangle, \quad \forall v \in V.$$

Denote

$$\begin{aligned} \mathcal{N} &= \{v \in V : a(v,w) = 0, \quad w \in V\} \quad \text{and} && (1) \\ \mathcal{N}_i &= \{v_i \in V_i : a(v_i, w_i) = 0, \quad w_i \in V_i\}. && (2) \end{aligned}$$

- Energy norm convergence :

$$|E_J|_a = \sup_{v \in \mathcal{N}^\perp} \frac{|E_J v|_a}{|v|_a} < 1?$$

# MSC for Singular Equations : Assumptions

(H1) Local subspace problems are solvable :
$a(v_i, v_i) \gtrsim \|v_i\|^2_{V_i/\mathcal{N}_i}, \quad \forall v_i \in V_i.$

(H2) There exists $\omega \in (0, 2)$ such that
$a(T_i v_i, T_i v_i) \leq \omega a(T_i v_i, v_i), \quad \forall v_i \in V_i.$

(H3) The following holds true $a(T_i v_i, T_i v_i) \gtrsim a(v_i, v_i), \quad \forall v_i \in V_i.$

- Assumption (H1) is only needed for the infinite dimensional case
- Assumptions (H2) and (H3) are trivially true for $T_i = P_i$.
- Assumptions (H2) and (H3) are necessary and sufficient to local energy norm convergence (Lee, Wu, Xu and Zikatanov, SIMAX (2006))
- Assumptions (H1), (H2) and (H3) are optimal

# On Assumption (H3)

Consider an iterative solution method to solve $Au = f$, where $f \in \mathcal{R}(A)$ and

$$A = \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}.$$

given by

$$u^\ell = u^{\ell-1} + BA(u - u^{\ell-1}) = u^{\ell-1} + T(u - u^{\ell-1}),$$

where

$$B = \begin{pmatrix} 1 & 2/3 \\ -1/3 & 0 \end{pmatrix}.$$

It is easy to see that $\mathcal{R}(T) = \mathcal{N}(A)$. This means that

$$|(I - T)u|_A^2 = |u|_A^2, \quad \forall u \in V.$$

# The convergence rate identity

## Theorem (Lee,Wu,Xu,Zikatanov 2005)

$$|E_J|_a^2 = |(I - T_J)\cdots(I - T_1)|_a^2 = 1 - \frac{1}{K}.$$

*where* $K = \sup_{|v|_a=1, v\in\mathcal{N}^\perp} K(v)$ *with*

$$K(v) = \inf_{c\in\mathcal{N}} \inf_{\sum_i v_i = v + c} \sum_{i=1}^{J} (\bar{T}_i^\dagger(v_i + T_i^* w_i), (v_i + T_i^* w_i))_a,$$

*where* $\bar{T}_i = T_i + T_i^* - T_i^* T_i$ *and* $w_i = \sum_{j=i+1}^{J} v_j$.

## Remark

$\bar{T}_i^\dagger = (\bar{T}_i^2)^\times \bar{T}_i$, *where* $(\bar{T}_i^2)^\times$ *is the Moore-Penrose generalized inverse of* $\bar{T}_i^2$. *(H.W. Engl, M. Hanke and A. Neubauer (1996))*

# Exact subspace solvers $T_i = P_i$

$$K = \sup_{v\in\mathcal{N}^\perp} \inf_{c\in\mathcal{N}} \inf_{\sum_i v_i = v+c} \frac{\sum_{i=1}^J |P_i(\sum_{j=i}^J v_j)|_a^2}{|v|_a^2}, \quad v_i \in V_i.$$

$$K = \sup_{v\in\mathcal{N}^\perp} \inf_{c\in\mathcal{N}} \inf_{\sum_i v_i = v+c} \frac{\sum_{i=1}^J |P_i(\sum_{j=i}^J v_j)|_a^2}{|v|_a^2}, \quad v_i \in V_i.$$

In case the decomposition is unique (e.g., Gauss-Seidel method), we have

$$K = \sup_{v\in\mathcal{N}^\perp} \inf_{c\in\mathcal{N}} \frac{\sum_{i=1}^J |P_i(\sum_{j=i}^J v_j)|_a^2}{|v|_a^2}, \quad v_i \in V_i.$$

## Remark

*Another interpretation of the efficiency of MG method can be found in many redundant representations for any given $v \in V$.*

# Example 1

We consider to solve

$$Au = f,$$

where $A$ is symmetric and positive semi-definite with the positive diagonal. The Gauss-Seidel method based on the matrix splitting

$$A = D - L - L^t,$$

where $D$ is the diagonal and $-L$ is the lower triangular of $A$ and $L^t$ is the transpose of $L$. Note that $P_i = (Ae_i, \cdot)e_i/(Ae_i, e_i)$. The direct application of the convergence rate identity leads to

$$|E|_A^2 = 1 - \frac{1}{K},$$

where

$$E = (I - (D - L)^{-1}A) \quad \text{and} \quad K = 1 + \sup_{v \in \mathcal{N}^\perp} \inf_{c \in \mathcal{N}} \frac{(LD^{-1}L^t(v+c),(v+c))}{(v,v)_A}$$

# Example 1 continues

Note that

$$
\begin{aligned}
|E|_A^2 &= \sup_{v \in \mathcal{N}^\perp} \frac{(AEv, Ev)}{(v, v)_A} && (3)\\
&= \sup_{v \in \mathcal{N}^\perp} \frac{((I - BA)v, (I - BA)v)_A}{(v, v)_A} && (4)\\
&= \sup_{v \in \mathcal{N}^\perp} \frac{((I - BA)^*(I - BA)v, v)_A}{(v, v)_A}, && (5)
\end{aligned}
$$

where $(I - BA)^* = I - B^t A$ is the adjoint operator of $I - BA$ with respect to the semi-inner product $(\cdot, \cdot)_A$. Note that we have the relation that

$$(I - BA)^*(I - BA) = I - (A + S)^{-1} A,$$

where $S = LD^{-1} L^t$.

Therefore, we have

$$|E|_A^2 = 1 - \inf_{v \in \mathcal{N}^\perp} \frac{((A+S)^{-1}Av, v)_A}{(v,v)_A} = 1 - \frac{1}{K}.$$

Now, we set $M = A^{1/2}(A+S)^{-1}AA^{1/2}$, to obtain

$$\begin{aligned} K &= \left( \inf_{v \in \mathcal{N}^\perp} \frac{((A+S)^{-1}Av, v)_A}{(v,v)_A} \right)^{-1} = \sup_{v \in \mathcal{N}^\perp} \frac{(v,v)_A}{((A+S)^{-1}Av, v)_A} \\ &= \sup_{v \in \mathcal{N}^\perp} \frac{(v,v)_A}{(A^{-1/2}MA^{-1/2}v, v)_A} \\ &= \sup_{v \in \mathcal{N}^\perp} \frac{(A^{-1/2}MA^{-1/2}v, v)_A}{(A^{-1/2}MA^{-1/2}v, A^{-1/2}MA^{-1/2}v)_A} \\ &= \sup_{w \in \mathcal{N}^\perp} \inf_{c \in \mathcal{N}} \frac{((A+S)(w+c), (w+c))}{(w,w)_A} \\ &= 1 + \sup_{v \in \mathcal{N}^\perp} \inf_{c \in \mathcal{N}} \frac{(S(v+c), (v+c))}{(v,v)_A}. \end{aligned}$$

# Example 2

Assume $\Omega$ is the polygonal domain and consider

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

The finite element discretization via the piecewise linear conforming element leads to
Find $u_h \in V_h$ such that

$$(u_h, v_h)_1 = (f, v_h)_0, \quad \forall v_h \in V_h.$$

# Example 2 continues

(i) Assume that there exists a set of nested sequence of quasi-uniform triangles $\mathcal{T}_k = \{T_k^i\}_{i=1}$ of the mesh size $h_k$, with $h_k = \gamma^k$ and $\gamma \in (0,1)$ for $k = 1, \cdots, J$.

(ii) Associated with each $\mathcal{T}_k$, the finite element space of continuous piecewise linear functions $V_k$ satisfies the following trivial relations.

$$V_1 \subset \cdots \subset V_J = V_h.$$

(iii) We consider the following space decompositions except for $k = 1 : J$,

$$V = V_h = \sum_{k=1}^{J} V_k = \sum_{k=1}^{J} \sum_{i=1}^{n_k} V_k^i$$

**Remark**

*Each space $V_k$ contains the null space $\mathrm{span}\{1\}$. Choose the usual basis function $\phi_k^i$ for each space and set $V_k^i = \mathrm{span}\{\phi_k^i\}$.*

# Example 2 continues

**Theorem (Lee,Wu,Xu and Zikatanov, 2005)**

*The multigrid method for the above neumann problem converges at a rate independent of mesh size and the number of levels.*

Sketch of Proof: The convergence rate $|E_J|_a$ is given by

$$|E_J|_a^2 = 1 - \frac{1}{K},$$

where

$$K = \sup_{v \in \mathcal{N}^\perp} \inf_{c \in \mathcal{N}} \inf_{\sum_{k=1}^J \sum_{i=1}^{n_k} v_k^i = v + c} \frac{\sum_{k=1}^J \sum_{i=1}^{n_k} |P_k^i (\sum_{(l,j) \geq (k,i)} v_l^j)|_a^2}{(v, v)_a}. \tag{6}$$

Main conclusion: The null space should be contained in each subspace.
This can be seen in the following required estimate:

$$\sum_{k=1}^{J} |(Q_k - Q_{k-1})\mathbf{v}|_1^2 \lesssim |\mathbf{v}|_1^2.$$

# Nearly singular problems

Find $u \in V$ such that

$$Au = (A_s + \epsilon A_p)u = f,$$

- $V$ is the finite dimensional Hilbert space
- $A_s$ is symmetric and positive semidefinite
- $A_p$ is symmetric and positive definite
- $\epsilon$ is a positive parameter such that $\epsilon \ll 1$.

# Examples of Nearly Singular Problems

- Discretizations of the simple partial differential equations

$$-\Delta u + \epsilon u = f, \quad \text{in} \quad \Omega,$$

with the Neumann boundary condition: $\mathbf{n} \cdot \nabla u = 0$ on $\partial\Omega$.

- Discretizations of nearly incompressible elasticity problem

$$-\nabla(\mathrm{div} u) - (1 - 2\nu)\Delta u = f \quad \text{in} \quad \Omega,$$

where the Poisson ratio $\nu \approx 1/2$.

- Finite element Discretizations of H(div) and H(curl) systems

$$G^* G u + \epsilon u = f,$$

where $G^*$ is the adjoint operator of $G$, $G = \mathbf{curl}$ or $G = \mathrm{div}$.

# Difficulty in the solutions

A simple example and considering the Gauss-Seidel method as the solver. The simple nearly singular system of equations is as follow.

$$Au = (A_s + \epsilon A_p)u = \left[ \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] u = f.$$

| $\epsilon$ | 1 | $10^{-1}$ | $10^{-2}$ | $10^{-3}$ | $10^{-4}$ | $10^{-5}$ | 0 |
|---|---|---|---|---|---|---|---|
| # of iterations | 14 | 94 | 823 | 7427 | 66556 | 588770 | 2 |

Table: The number of iterations to obtain the energy norm error $\|u - u^\ell\|_A < 10^{-6}$ for various values of $\epsilon$

# What is happening ?

The energy norm convergence rate of the Gauss-Seidel method for the system of equation is given as follows,

$$\|E\|_A^2 = 1 - \frac{1}{K}, \tag{7}$$

where

$$K = 1 + \sup_{v=(v_1,v_2,v_3)^t \in \mathbb{R}^3} \frac{(1+\epsilon)^{-1} v_2^2 + (2+\epsilon)^{-1} v_3^2}{(Av, v)}. \tag{8}$$

Choose $v$ from the null space of $A_s$, namely, the eigenvector of $A$ that corresponds to the eigenvalue $\epsilon$, say, $v = (1,1,1)^t$ to obtain that

$$K \geq 1 + \frac{(1+\epsilon)^{-1} + (2+\epsilon)^{-1}}{3\epsilon} \to \infty \quad \text{as} \quad \epsilon \to 0. \tag{9}$$

# Remedy : Augmented System

Consider to solve $Au = (A_s + \epsilon I)u = f$, where $A_s \in \mathbb{R}^{n\times n}$ is singular matrix with $\mathcal{N}$ being the null space.

- Choose a number of vectors $\phi_i$'s so that

$$\mathcal{N} \subset \text{span}\{\phi_1, \cdots, \phi_m\} = W \subset V.$$

- Formulate the Augmented matrix system with an operator $\Phi = [\phi_1 \cdots \phi_m] : W \mapsto V$,

$$\mathcal{A}\mathbf{u} = \begin{pmatrix} \Phi^t A \Phi & \Phi^t A \\ A\Phi & A \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \Phi^t f \\ f \end{pmatrix} = \mathbf{f}.$$

# On Augmented System

- The Augmented system is singular
- The range and null spaces of $\mathcal{A}$ can be completely characterized as follows:

$$\mathcal{R}(\mathcal{A}) = \left\{ \begin{pmatrix} \Phi^t v \\ v \end{pmatrix} : v \in V \right\} \text{ and } \mathcal{N}(\mathcal{A}) = \left\{ \begin{pmatrix} c \\ -\Phi c \end{pmatrix} : c \in W \right\}. \tag{10}$$

- There exist infinitely many solutions to the equation. However, if $\mathbf{u}$ is a solution to the Augmented system of equations given as

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

then the solution $u$ to the original system $Au = f$ can be recovered uniquely as

$$u = \Phi u_1 + u_2.$$

# Solving Augmented System

Solving the Augmented matrix system by the block Gauss-Seidel method with blocks

- $\Phi^t A \Phi$
- each diagonal of $A$

is equivalent to two grid methods with

- one Gauss-Seidel smoothing for $A$ on $V$
- exact subspace solve on the space $W$.

(Xu (1992), M. Griebel (1994) and Lee,Wu,Xu and Zikatanov (2003))

# Convergence Study of Block Gauss-Seidel for Augmented System

Assume $W = \mathrm{span}\{\xi\}$ and $f \in \mathcal{R}(A_s)$. We obtained the following result:

**Theorem (Lee,Wu,Xu and Zikatanov, 2003)**

*The block Gauss-Seidel method for the system $\mathcal{A}$ has the following energy norm convergence rate:*

$$\delta_{\mathcal{A}}^2 = 1 - \frac{1}{K(\mathcal{A})},$$

*with $A_s = D - L - L^t$,*

$$K(\mathcal{A}) \rightarrow K(A_s) = 1 + \sup_{v \in \mathcal{N}^\perp} \inf_{c \in \mathcal{N}} \frac{(LD^{-1}L^t(v+c),(v+c))}{(v,v)_{A_s}} \quad as \quad \epsilon \rightarrow 0.$$

# Sketch of Proof

The Augmented system for the equation $(A_s + \epsilon I)u = f$ is given by

$$\mathcal{A}\mathbf{u} = \begin{pmatrix} \xi^t A\xi & \xi^t A \\ A\xi & A \end{pmatrix} \mathbf{u} = \mathbf{f}, \tag{11}$$

where $\mathbf{f} = ((\xi, f), f)^t = (0, f)^t$.
We decompose the matrices $A$ and $A_s$ into the following:

$$A = D_\epsilon - L - L^t,$$

and

$$A_s = D - L - L^t.$$

Define $S_\epsilon$ and $S$ by

$$S_\epsilon = LD_\epsilon^{-1}L^t \quad \text{and} \quad S = LD^{-1}L^t$$

respectively.

# Proof continues

Since $D_\epsilon = D + \epsilon I$, $\alpha(\epsilon)$ such that for any $v \in V$,

$$\alpha(\epsilon)(Sv, v) \le (S_\epsilon v, v) \le (Sv, v), \quad \text{and} \quad \lim_{\epsilon \to 0} \alpha(\epsilon) = 1. \qquad (12)$$

A decomposition of the augmented matrix $\mathcal{A}$ is given as follows:

$$\mathcal{A} = \mathcal{D} - \mathcal{L} - \mathcal{L}^t, \quad \text{and} \quad \mathcal{S} = \mathcal{L}\mathcal{D}^{-1}\mathcal{L}^t.$$

The energy norm convergence rate of Gauss-Seidel method for the augmented system is given as follows:

$$\delta_{\mathcal{A}}^2 = 1 - \frac{1}{K(\mathcal{A})},$$

with

$$K(\mathcal{A}) = 1 + \sup_{\mathbf{v} \in \mathcal{N}(\mathcal{A})^\perp} \inf_{\mathbf{c} \in \mathcal{N}(\mathcal{A})} \frac{(\mathcal{S}(\mathbf{v}+\mathbf{c}), (\mathbf{v}+\mathbf{c}))}{(\mathbf{v}, \mathbf{v})_{\mathcal{A}}}.$$

# Proof continues

Note that a simple calculation yields

$$K(\mathcal{A}) = 1 + \sup_{\mathbf{v}=((\xi,v),v^t)^t \in \mathcal{N}(\mathcal{A})^\perp} \inf_{\lambda \in \mathbb{R}} \frac{(S_\epsilon(v+\lambda\xi),(v+\lambda\xi)) + \|P_\xi(v+\lambda\xi)\|_A^2}{\|\mathbf{v}\|_{\mathcal{A}}^2}, \tag{13}$$

where $P_\xi$ is the $A$-orthogonal projection on $\mathcal{N}(A_s)$, namely

$$P_\xi = \xi(\xi^t A \xi)^{-1}\xi^t A.$$

Now, for a given $v \in V$, we consider the following orthogonal decomposition:

$$v = v_a + \gamma\xi,$$

where $v_a \in \mathcal{N}(A_s)^\perp$ and $\gamma \in \mathbb{R}$. With this decomposition, $\|\mathbf{v}\|_{\mathcal{A}}^2$ can be written as

$$\|\mathbf{v}\|_{\mathcal{A}}^2 = \|v + (\xi, v)\xi\|_A^2 = \|v_a\|_A^2 + \epsilon^2[\gamma(1+\|\xi\|^2)]^2\|\xi\|^2. \tag{14}$$

# Proof continues

For $v = v_a + \gamma\xi \in V$, we have

$$\inf_{\lambda \in \mathbb{R}} (S_\epsilon(v + \lambda\xi), (v + \lambda\xi)) + \|P_\xi(v + \lambda\xi)\|_A^2$$
$$= \inf_{\lambda \in \mathbb{R}} (S_\epsilon(v_a + \lambda\xi), (v_a + \lambda\xi)) + \|P_\xi(v_a + \lambda\xi)\|_A^2.$$

- (Lower bound)

$$K(\mathcal{A}) \geq 1 + \sup_{v_a \in \mathcal{N}(A_s)^\perp} \inf_{\lambda \in \mathbb{R}} \frac{(S_\epsilon(v_a + \lambda\xi), (v_a + \lambda\xi))}{\|v_a\|_A^2} \quad (15)$$
$$\geq 1 + \sup_{v_a \in \mathcal{N}(A_s)^\perp} \inf_{\lambda \in \mathbb{R}} \frac{\alpha(\epsilon)(S(v_a + \lambda\xi), (v_a + \lambda\xi))}{\|v_a\|_{A_s}^2},$$

# Proof continues

- (Upper bound) We see that $K(\mathcal{A})$ can be written as follows:

$$
\begin{aligned}
K(\mathcal{A}) &= 1 + \sup_{\mathbf{v}\in\chi} \inf_{\lambda\in\mathbb{R}} \frac{(S_\epsilon(v+\lambda\xi),(v+\lambda\xi)) + \|P_\xi(v+\lambda\xi)\|_A^2}{\|v_a\|_A^2 + \epsilon^2|\gamma + (\xi, v)|^2\|\xi\|^2} \\
&\le 1 + \sup_{\mathbf{v}\in\chi} \inf_{\lambda\in\mathbb{R}} \frac{(S_\epsilon(v+\lambda\xi),(v+\lambda\xi)) + \|P_\xi(v+\lambda\xi)\|_A^2}{\|v\|_A^2}, \\
&\le 1 + \sup_{v_a\in\mathcal{N}(A_s)^\perp} \inf_{\lambda\in\mathbb{R}} \frac{(S_\epsilon(v_a+\lambda\xi),(v_a+\lambda\xi)) + \|P_\xi(v_a+\lambda\xi)\|}{\|v_a\|_{A_s}^2}
\end{aligned}
$$

Finally, it is easy to see that

$$
\|P_\xi(v_a+\lambda\xi)\|_A^2 = \epsilon^2\|P_\xi(v_a+\lambda\xi)\|^2. \tag{16}
$$

As a result,

$$
K(\mathcal{A}) \le K(A_s) + \sup_{v_a\in\mathcal{N}(A_s)^\perp} \inf_{\lambda\in\mathbb{R}} \frac{\epsilon^2\|P_\xi(v_a+\lambda\xi)\|^2}{\|v_a\|_{A_s}^2}. \tag{17}
$$

Taking the limit $\epsilon \to 0$, we complete the proof.

# General Convergence Analysis of MSC for Nearly Singular Systems

For $f \in V^*$, find $u \in V$ such that

$$a(u, v) = a_s(u, v) + \epsilon a_p(u, v) = \langle f, v \rangle, \quad \forall u, v \in V, \tag{18}$$

where

(i) The constant $\epsilon$ is positive.

(ii) $a_s$ is symmetric and semi-definite.

(iii) $a_p$ is symmetric and positive-definite.

(iv) $a$ is the inner product $(\cdot, \cdot)$ on $V$.

$$\mathcal{N} = \{\mathbf{v} \in V : a_s(v, w) = 0, \quad \forall w \in V\}$$

$\mathcal{N}^\perp$ is the orthogonal complement of $\mathcal{N}$ with respect to the inner product on $V$.

# On Assumptions

A0 There exist a number of closed subspaces $V_k$ with $k = 1, \cdots, J$ such that

$$V = \sum_{k=1}^{J} V_k.$$

A1 The null space $\mathcal{N}$ can be represented by sum of elements in subspaces, namely,

$$\mathcal{N} = \sum_{k=1}^{J} (V_k \cap \mathcal{N}) = \sum_{k=1}^{J} \mathcal{N}_k.$$

## Remark

$$\mathcal{N} = \sum_{k=0}^{J} V_i \cap \mathcal{N}, \quad V_i = \mathrm{span}\{e_i\} \; for \; i = 1 : n, \mathcal{N} \subset V_0 = W$$

# Notations

We introduce several orthogonal projections:
$P_k : V \mapsto V_k$ by

$$a(P_k v, v_k) = a(v, v_k), \quad \forall v \in V, v_k \in V_k \tag{19}$$

and $P_{k,s} : V \mapsto V_k$ and $P_{k,p} : V \mapsto V_k$ by

$$\begin{aligned} a_s(P_{k,s} v, v_k) &= a_s(v, v_k), \quad \forall v \in V, v_k \in V_k \\ a_p(P_{k,p} v, v_k) &= a_p(v, v_k), \quad \forall v \in V, v_k \in V_k. \end{aligned}$$

# Cauchy Inequality

## Lemma

*For each* $1 \leq k \leq J$*, the following holds true:*

$$\|P_k u\|_a^2 \lesssim \|P_{k,s} u\|_{a_s}^2 + \epsilon \|P_{k,p} u\|_{a_p}^2, \quad \forall u \in V.$$

*For any* $u \in V$*,*

$$\begin{aligned} a(u, P_k u) &= a_s(u, P_k u) + \epsilon a_p(u, P_k u) \\ &= a_s(P_{k,s} u, P_k u) + \epsilon a_p(P_{k,p} u, P_k u) \\ &\leq \frac{1}{2}\left(a_s(P_k u, P_k u) + a_s(P_{k,s} u, P_{k,s} u)\right) \\ &+ \frac{\epsilon}{2}\left(a_p(P_k u, P_k u) + a_p(P_{k,p} u, P_{k,p} u)\right). \end{aligned}$$

# Main result

## Theorem

*Under the assumptions* **A0** *and* **A1**, *the energy norm of the error transfer operator* $E_J = (I - P_J)\cdots(I - P_1)$ *is given by the following:*

$$\|E_J\|_a^2 = 1 - \frac{1}{K}, \tag{20}$$

$$K \lesssim \sup_{v_s \in \mathcal{N}^\perp} \inf_{\sum_{k=1}^J v_{k,s} = v_s} \left( \frac{\sum_{k=1}^J \left| P_{k,s} \sum_{j \geq k} v_{j,s} \right|_{a_s}^2}{(v_s, v_s)_{a_s}} + \frac{\sum_{k=1}^J \left\| P_{k,p} \sum_{j \geq k} v_{j,s} \right\|_{a_p}^2}{(v_s, v_s)_{a_p}} \right)$$

$$+ \sup_{v_c \in \mathcal{N}} \inf_{\sum_{k=1}^J v_{k,c} = v_c} \frac{\sum_{k=1}^J \left\| P_{k,p} \sum_{j \geq k} v_{j,c} \right\|_{a_p}^2}{(v_c, v_c)_{a_p}},$$

*where* $v_{k,s}$*'s belong to* $V_k$ *and* $v_{k,c}$*'s are in* $\mathcal{N}$ *for each* $k = 1, \cdots, J$.

# Sketch of Proof

Note that

$$K = \sup_{v\in V} \inf_{\sum_{i=1}^J v_k = v} \frac{\sum_{k=1}^J \|P_k \sum_{j\geq k} v_j\|_a^2}{\|v\|_a^2}.$$

For any given $v = v_s + v_c \in V$, with $v_s \in \mathcal{N}^\perp$ and $v_c \in \mathcal{N}$, due to (A1),

$$v = \sum_{k=1}^J v_{k,s} + \sum_{k=1}^J v_{k,c} = v_s + v_c \tag{21}$$

where $v_{k,s} \in V_k, v_{k,c} \in \mathcal{N}_k$.

$$\begin{aligned} \|v\|^2 &= (v_s + v_c, v_s + v_c)_{a_s} + \epsilon(v_s + v_c, v_s + v_c)_{a_p} \\ &= \|v_s\|_{a_s}^2 + \epsilon\|v\|_{a_p}^2. \end{aligned}$$

$$\left\|P_k \sum_{j\geq k} v_j\right\|_a^2 \leq \left\|P_{k,s} \sum_{j\geq k} v_{k,s} + v_{k,c}\right\|_{a_s}^2 + \epsilon \left\|P_{k,p} \sum_{j\geq k} v_{k,s} + v_{k,c}\right\|_{a_p}^2$$

# Sketch of Proof

$$\left\|P_k \sum_{j\geq k} v_j\right\|_a^2 \lesssim \left\|P_{k,s} \sum_{j\geq k} v_{k,s}\right\|_{a_s}^2 + \epsilon \left\|P_{k,p} \sum_{j\geq k} v_{k,s}\right\|_{a_p}^2 + \epsilon \left\|P_{k,p} \sum_{j\geq k} v_{k,c}\right\|_{a_p}^2 .$$

Note that the inequality holds for arbitrary decompositions $\{v_{k,s}\}_{k=1}^J$ and $\{v_{k,c}\}_{k=1}^J$ that constitute $v_s$ and $v_c$ respectively.

$$\inf_{v=\sum_{k=1}^J v_k} \sum_{k=1}^J \left\|P_k \sum_{j\geq k} v_j\right\|_a^2 \lesssim \inf_{\sum_{k=1}^J v_{k,s}=v_s} \left( \sum_{k=1}^J \left\|P_{k,s} \sum_{j\geq k} v_{k,s}\right\|_{a_s}^2 + \epsilon \sum_{k=1}^J \left\|P_{k,p} \sum_{j\geq k} v_{k,s}\right\|_{a_p}^2 \right) + \inf_{v_c=\sum_{k=1}^J v_{k,c}} \epsilon \sum_{k=1}^J \left\|P_{k,p} \sum_{j\geq k} v_{k,c}\right\|_{a_p}^2 .$$

# Applications

We consider to solve the following problem:

$$-\Delta u + \epsilon u = f, \quad \text{in} \quad \Omega,$$

$$\mathbf{n} \cdot \nabla u = 0, \quad \text{on} \quad \partial\Omega.$$

The variational problem can be given as follows: Find $u \in H^1(\Omega)$ such that

$$a(u, v) = a_s(u, v) + \epsilon\, a_p(u, v) = (f, v), \quad \forall v \in H^1(\Omega),$$

where

$$a_s(u, v) = (u, v)_1 \quad \text{and} \quad a_p(u, v) = (u, v)_0$$

## Remark

$$\mathcal{N} = \mathrm{span}\{1\}.$$

# Toward parameter independent convergence

(i) Assume that there exists a set of nested sequence of quasi-uniform triangles $\mathcal{T}_k = \{T_k^i\}_{i=1}$ of the mesh size $h_k$, with $h_k = \gamma^k$ and $\gamma \in (0,1)$ for $k = 1, \cdots, J$.

(ii) Associated with each $\mathcal{T}_k$, the finite element space of continuous piecewise linear functions $V_k$ satisfies the following trivial relations.

$$V_1 \subset \cdots \subset V_J = V_h.$$

## Remark

*Note that each space $V_k$ contains the null space $\mathrm{span}\{1\}$. Choose the usual basis function $\phi_k^i$ for each space and set $V_k^i = \mathrm{span}\{\phi_k^i\}$ so that*

$$V_k = \mathrm{span}\left\{\phi_k^1, \cdots, \phi_k^{n_k}\right\} = \sum_{i=1}^{n_k} V_k^i.$$

# Toward parameter independent convergence

- (Space decomposition) Set $n_1 = 1, \mathcal{N} \subseteq V_1$, i.e., $V_1^1 = V_1$. We then decompose $V$ in the following manner:

$$V = \sum_{k=1}^{J} \sum_{i=1}^{n_k} V_k^i.$$

- (Subspace correction) We apply the exact local solvers for each $V_k^i$.

**Theorem**

$$\|E\|^2 = \|(I - T_1) \cdots (I - T_J)\|^2 = \|\Pi_{k=1}^{J} \Pi_{l=1}^{n_k} (I - P_k^l)\|^2 < \delta < 1, \quad (22)$$

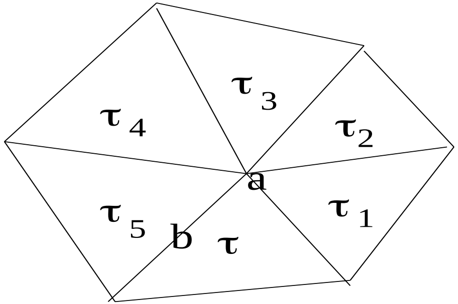
*where $\delta$ is bounded uniformly with respect to the parameter $\epsilon$, the number of levels and the mesh size.*

# Other Applications

- The point Gauss-Seidel method is well-known to be inefficient for the system of equations like $H(\mathrm{div})$ or $H(\mathbf{curl})$ - Why?.
- The Nearly incompressible linear elasticity equations:

$$\mathbf{u} - \rho^2 \nabla \mathrm{div} \mathbf{u} - \kappa^2 \Delta \mathbf{u} = \mathbf{f}.$$

Bigger subspace definition $V_k^\ell$ is necessary for $\mathbf{A}_1$. Thus, the block Gauss-Seidel method should be used.



# K-Way CNcut

K-Way CNcut is basically to solve

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ c \end{pmatrix}, \tag{23}$$

where $A$ is $D^{-1/2}(D - W)D^{-1/2}$ is the normalized Graph Laplacian and $B$ is the long thin matrix, which imposes the constraint.
The Augmented Lagrangian Uzawa requires us to solve

$$(A + rB^T B)u = g. \tag{24}$$

## Remark

*We note that $B^T B$ is diagonal matrix with one or zero in the diagonal. But $A$ is not SPD. But, we can still apply the technique if $N(A) \cap N(B^T B) = \{0\}$. Furthermore $A + rB^T B$ is M-matrix.*

# Concluding Remarks

- Optimal assumptions for the convergence of classic iterative method have been identified.
- Convergence rate identity for MSC for Singular Equations is obtained under the optimal assumptions.
- The difficulty in solving Nearly singular equations has been identified and remedied by introducing Augmented matrix system.
- A convergence rate estimate for MSC for Nearly Singular Equations is obtained under a new abstract assumption, namely, the near null space can be represented by the local subspaces.
- A simple application is demonstrated to show the use of abstract convergence theory.

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