A COMPANION TECHNICAL REPORT OF "TIGHTNESS AND EQUIVALENCE OF SEMIDEFINITE RELAXATIONS FOR MIMO DETECTION"

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1. Proof of Theorem 3.3 in [1]. In this section, we prove Theorem 3.3 in [1]: the following SDR

$$\begin{aligned}
&\min_{\boldsymbol{t},\boldsymbol{y},\mathbf{Y}} && \langle \hat{\mathbf{Q}}, \mathbf{Y}\rangle + 2\hat{\boldsymbol{c}}^\mathsf{T}\boldsymbol{y} \\
\text{(ESDR-}\mathbf{Y}\text{)} \quad &\text{s.t.} && \mathcal{Y}(i) = \sum_{j=1}^{M} t_{i,j}\mathbf{K}_j,\ \sum_{j=1}^{M} t_{i,j} = 1,\ i = 1,2,\ldots,n, \\
& && t_{i,j} \geq 0,\ j = 1,2,\ldots,M,\ i = 1,2,\ldots,n, \\
& && \mathbf{Y} \succeq \boldsymbol{y}\boldsymbol{y}^\mathsf{T},
\end{aligned}$$

is tight if and only if there exist $\boldsymbol{\lambda} \in \mathbb{R}^{2n}$, $\boldsymbol{\mu} \in \mathbb{R}^n$, and $\boldsymbol{g} \in \mathbb{R}^{2n}$ that satisfy

$$\hat{\mathbf{H}}^\mathsf{T}\hat{\boldsymbol{v}} = \boldsymbol{g} + (\boldsymbol{\Lambda} + \mathbf{M})\boldsymbol{y}^*, \tag{1.1}$$

$$\langle \boldsymbol{\Gamma}_i, \mathbf{K}_{u_i}\rangle \geq \langle \boldsymbol{\Gamma}_i, \mathbf{K}_j\rangle,\ j = 1,2,\ldots,M,\ i = 1,2,\ldots,n, \tag{1.2}$$

and

$$\hat{\mathbf{Q}} + \boldsymbol{\Lambda} + \mathbf{M} \succeq 0, \tag{1.3}$$

where

$$\boldsymbol{\Lambda} = \mathrm{Diag}(\boldsymbol{\lambda}),\ \mathbf{M} = \begin{bmatrix} \mathbf{0} & \mathrm{Diag}(\boldsymbol{\mu}) \\ \mathrm{Diag}(\boldsymbol{\mu}) & \mathbf{0} \end{bmatrix},\ \text{and } \boldsymbol{\Gamma}_i = \begin{bmatrix} 0 & g_i & g_{n+i} \\ g_i & \lambda_i & \mu_i \\ g_{n+i} & \mu_i & \lambda_{n+i} \end{bmatrix}. \tag{1.4}$$

We first rewrite (ESDR-$\mathbf{Y}$) in the following equivalent form:

$$\begin{aligned}
&\min_{\boldsymbol{t},\tilde{\mathbf{Y}}} && \langle \tilde{\mathbf{C}}, \tilde{\mathbf{Y}}\rangle \\
&\text{s.t.} && \tilde{Y}_{1,1} = 1, \\
& && \tilde{\mathcal{Y}}(i) = \sum_{j=1}^{M} t_{i,j}\mathbf{K}_j,\ \sum_{j=1}^{M} t_{i,j} = 1,\ i = 1,2,\ldots,n, \\
& && t_{i,j} \geq 0,\ j = 1,2,\ldots,M,\ i = 1,2,\ldots,n, \\
& && \tilde{\mathbf{Y}} \succeq 0,
\end{aligned} \tag{1.5}$$

where

$$\tilde{\mathbf{C}} = \begin{bmatrix} 0 & \hat{\boldsymbol{c}}^\mathsf{T} \\ \hat{\boldsymbol{c}} & \hat{\mathbf{Q}} \end{bmatrix},\ \tilde{\mathbf{Y}} = \begin{bmatrix} 1 & \boldsymbol{y}^\mathsf{T} \\ \boldsymbol{y} & \mathbf{Y} \end{bmatrix},\ \text{and } \tilde{\mathcal{Y}}(i) = \begin{bmatrix} 1 & \tilde{Y}_{1,i+1} & \tilde{Y}_{1,n+i+1} \\ \tilde{Y}_{i+1,1} & \tilde{Y}_{i+1,i+1} & \tilde{Y}_{i+1,n+i+1} \\ \tilde{Y}_{n+i+1,1} & \tilde{Y}_{n+i+1,i+1} & \tilde{Y}_{n+i+1,n+i+1} \end{bmatrix}.$$

Consider the Lagrange dual function

$$L(\tau, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{g}, \boldsymbol{\alpha}) = \inf_{\boldsymbol{t}, \tilde{\mathbf{Y}}} \langle \tilde{\mathbf{C}}, \tilde{\mathbf{Y}} \rangle + \tau(1 - \tilde{Y}_{1,1}) + \sum_{i=1}^{n} \langle \boldsymbol{\Gamma}_i, \tilde{\mathcal{Y}}(i) - \sum_{j=1}^{M} t_{i,j} \mathbf{K}_j \rangle + \sum_{i=1}^{n} \alpha_i \Big(\sum_{j=1}^{M} t_{i,j} - 1 \Big),$$

where $\tau \in \mathbb{R}$, $\boldsymbol{\lambda} \in \mathbb{R}^{2n}$, $\boldsymbol{\mu} \in \mathbb{R}^n$, $\boldsymbol{g} \in \mathbb{R}^{2n}$, and $\boldsymbol{\alpha} \in \mathbb{R}^n$ are dual variables and $\boldsymbol{\Gamma}_i$ is defined in (1.4). Since both the nonnegative orthant and the PSD cone are self-dual, the dual constraints are

$$\begin{bmatrix} -\tau & (\hat{\boldsymbol{c}} + \boldsymbol{g})^\mathsf{T} \\ \hat{\boldsymbol{c}} + \boldsymbol{g} & \hat{\mathbf{Q}} + \boldsymbol{\Lambda} + \mathbf{M} \end{bmatrix} \succeq 0, \tag{1.6}$$

$$\alpha_i \geq \langle \boldsymbol{\Gamma}_i, \mathbf{K}_j \rangle, \ i = 1, 2, \dots, n, \ j = 1, 2, \dots, M, \tag{1.7}$$

where $\boldsymbol{\Lambda}$ and $\mathbf{M}$ are given by (1.4). Moreover, the complementary slackness conditions are

$$\left\langle \begin{bmatrix} -\tau & (\hat{\boldsymbol{c}} + \boldsymbol{g})^\mathsf{T} \\ \hat{\boldsymbol{c}} + \boldsymbol{g} & \hat{\mathbf{Q}} + \boldsymbol{\Lambda} + \mathbf{M} \end{bmatrix}, \begin{bmatrix} 1 & \boldsymbol{y}^\mathsf{T} \\ \boldsymbol{y} & \mathbf{Y} \end{bmatrix} \right\rangle = 0, \tag{1.8}$$

$$t_{i,j}(\alpha_i - \langle \boldsymbol{\Gamma}_i, \mathbf{K}_j \rangle) = 0, \ i = 1, 2, \dots, n, \ j = 1, 2, \dots, M. \tag{1.9}$$

Note that the primal constraints in (1.5) and (1.6)–(1.9) are the KKT conditions of (1.5) and provide necessary and sufficient conditions for optimality.

We first show that the conditions (1.1)–(1.3) are necessary. Define $\boldsymbol{t}^* \in \{0, 1\}^{Mn}$ by

$$t^*_{i,u_i} = 1, \ t^*_{i,j} = 0, \ j \neq u_i, \ i = 1, 2, \dots, n.$$

When (ESDR-$\mathbf{Y}$) is tight, by definition,

$$\boldsymbol{t} = \boldsymbol{t}^*, \ \boldsymbol{y} = \boldsymbol{y}^*, \text{ and } \mathbf{Y} = \boldsymbol{y}^*(\boldsymbol{y}^*)^\mathsf{T} \tag{1.10}$$

are optimal solutions of (ESDR-$\mathbf{Y}$). Let $(\tau^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*, \boldsymbol{g}^*, \boldsymbol{\alpha}^*)$ be a dual optimal solution and hence it satisfies the KKT conditions together with $(\boldsymbol{t}^*, \boldsymbol{y}^*, \mathbf{Y}^*)$. Also, we define $\boldsymbol{\Lambda}^*$, $\mathbf{M}^*$, and $\boldsymbol{\Gamma}_i^*$ accordingly as in (1.4). From (1.6), we obtain $\hat{\mathbf{Q}} + \boldsymbol{\Lambda}^* + \mathbf{M}^* \succeq 0$. Combining (1.6), (1.8), and (1.10) shows that

$$\begin{bmatrix} -\tau^* & (\hat{\boldsymbol{c}} + \boldsymbol{g}^*)^\mathsf{T} \\ \hat{\boldsymbol{c}} + \boldsymbol{g}^* & \hat{\mathbf{Q}} + \boldsymbol{\Lambda}^* + \mathbf{M}^* \end{bmatrix} \begin{bmatrix} 1 \\ \boldsymbol{y}^* \end{bmatrix} = 0.$$

This leads to

$$\hat{\boldsymbol{c}} + \boldsymbol{g}^* + (\hat{\mathbf{Q}} + \boldsymbol{\Lambda}^* + \mathbf{M}^*)\boldsymbol{y}^* = 0 \ \Leftrightarrow \ \boldsymbol{g}^* + (\boldsymbol{\Lambda}^* + \mathbf{M}^*)\boldsymbol{y}^* = -\hat{\mathbf{Q}}\boldsymbol{y}^* - \hat{\boldsymbol{c}} = \hat{\mathbf{H}}^\mathsf{T}\hat{\boldsymbol{v}},$$

where we use the facts that $\hat{\mathbf{Q}} = \hat{\mathbf{H}}^\mathsf{T}\hat{\mathbf{H}}$ and $\hat{\boldsymbol{c}} = -\hat{\mathbf{H}}^\mathsf{T}(\hat{\mathbf{H}}\boldsymbol{y}^* + \hat{\boldsymbol{v}}) = -\hat{\mathbf{Q}}\boldsymbol{y}^* - \hat{\mathbf{H}}^\mathsf{T}\hat{\boldsymbol{v}}$. Furthermore, combining (1.9) and (1.10) shows that

$$\alpha_i^* = \langle \boldsymbol{\Gamma}_i^*, \mathbf{K}_{u_i} \rangle, \ i = 1, 2, \dots, n.$$

Together with (1.7), this implies that

$$\langle \boldsymbol{\Gamma}_i^*, \mathbf{K}_{u_i} \rangle \geq \langle \boldsymbol{\Gamma}_i^*, \mathbf{K}_j \rangle,\ j = 1, 2, \ldots, M,\ i = 1, 2, \ldots, n.$$

Hence, $\boldsymbol{\lambda}^*$, $\boldsymbol{\mu}^*$, and $\boldsymbol{g}^*$ satisfy (1.1)–(1.3).

Now we show that the conditions are also sufficient. To this end, we need to construct $(\tau^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*, \boldsymbol{g}^*, \boldsymbol{\alpha}^*)$ such that these dual variables and $(\boldsymbol{t}^*, \boldsymbol{y}^*, \mathbf{Y}^*)$ jointly satisfy the KKT conditions. Assume that $\check{\boldsymbol{\lambda}}$, $\check{\boldsymbol{\mu}}$, and $\check{\boldsymbol{g}}$ satisfy the conditions (1.1)–(1.3). We let

$$\boldsymbol{\lambda}^* = \check{\boldsymbol{\lambda}},\ \boldsymbol{\mu}^* = \check{\boldsymbol{\mu}},\ \boldsymbol{g}^* = \check{\boldsymbol{g}},$$

and

$$\tau^* = (\hat{\boldsymbol{c}} + \check{\boldsymbol{g}})^\mathsf{T} \boldsymbol{y}^*,\ \alpha_i^* = \langle \check{\boldsymbol{\Gamma}}_i, \mathbf{K}_{u_i} \rangle,\ i = 1, 2, \ldots, n,$$

where $\check{\boldsymbol{\Gamma}}_i$ is defined as in (1.4). By direct computation and using (1.1)–(1.3), we can show that

$$\begin{bmatrix} -\tau^* & (\hat{\boldsymbol{c}} + \boldsymbol{g}^*)^\mathsf{T} \\ \hat{\boldsymbol{c}} + \boldsymbol{g}^* & \hat{\mathbf{Q}} + \boldsymbol{\Lambda}^* + \mathbf{M}^* \end{bmatrix} = \begin{bmatrix} (\boldsymbol{y}^*)^\mathsf{T} \\ -\mathbf{I}_{2n} \end{bmatrix} (\hat{\mathbf{Q}} + \boldsymbol{\Lambda}^* + \mathbf{M}^*) \begin{bmatrix} \boldsymbol{y}^* & -\mathbf{I}_{2n} \end{bmatrix} \succeq 0,$$
$$\left\langle \begin{bmatrix} -\tau^* & (\hat{\boldsymbol{c}} + \boldsymbol{g}^*)^\mathsf{T} \\ \hat{\boldsymbol{c}} + \boldsymbol{g}^* & \hat{\mathbf{Q}} + \boldsymbol{\Lambda}^* + \mathbf{M}^* \end{bmatrix}, \begin{bmatrix} 1 & (\boldsymbol{y}^*)^\mathsf{T} \\ \boldsymbol{y}^* & \mathbf{Y}^* \end{bmatrix} \right\rangle = 0,$$

and

$$\alpha_i^* \geq \langle \boldsymbol{\Gamma}_i^*, \mathbf{K}_j \rangle,$$
$$t_{i,j}^* (\alpha_i^* - \langle \boldsymbol{\Gamma}_i^*, \mathbf{K}_j \rangle) = 0,\ i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, M.$$

Therefore, $(\boldsymbol{t}^*, \boldsymbol{y}^*, \mathbf{Y}^*)$ and $(\tau^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*, \boldsymbol{g}^*, \boldsymbol{\alpha}^*)$ satisfy the KKT conditions and this further implies that $(\boldsymbol{t}^*, \boldsymbol{y}^*, \mathbf{Y}^*)$ is an optimal solution of (ESDR-**Y**). The proof is now complete.

2. Proof of Theorem 3.9 in [1]. In this section, we prove Theorem 3.9 in [1]: when M-PSK is used with $M \geq 4$ and the assumptions in [1, Corollary 3.8] hold, for the following SDR

$$\text{(ESDR1-}\mathbf{T}\text{)} \qquad \begin{aligned} \min_{\boldsymbol{t}, \mathbf{T}} \quad & \langle \bar{\mathbf{Q}}, \mathbf{T} \rangle + 2 \bar{\boldsymbol{c}}^\mathsf{T} \boldsymbol{t} \\ \text{s.t.} \quad & t_{i,j} \geq 0,\ \sum_{j=1}^{M} t_{i,j} = 1,\ j = 1, 2, \ldots, M,\ i = 1, 2, \ldots, n, \\ & \mathrm{diag}(\mathbf{T}_{i,i}) = \boldsymbol{t}_i,\ i = 1, 2, \ldots, n, \\ & \mathbf{T} \succeq \boldsymbol{t}\boldsymbol{t}^\mathsf{T}, \end{aligned}$$

we have

$$\mathbf{Prob}\big(\text{(ESDR1-}\mathbf{T}\text{) is tight}\big) \leq \left(\frac{2}{M} \right)^n. \tag{2.1}$$

Recall that M is required to be a power of two. For $M = 4$, (2.1) follows directly from [1, Corollary 3.8]. For $M \geq 8$, our goal is to prove that the tightness of (ESDR1-**T**) implies

$$-\frac{2\pi}{M} \leq \arg(z_i) \leq \frac{2\pi}{M},\ i = 1, 2, \ldots, n, \tag{2.2}$$

where $z_i = (x_i^*)^\dagger(\mathbf{H}^\dagger \boldsymbol{v})_i$. If this is true, we immediately get (2.1) by using the same argument as in [1, Corollary 3.8].

To begin with, note that in [1, Corollary 3.7] the expression of $\boldsymbol{\gamma}_i$ depends on u_i, the index of the i-th transmitted symbol. The following lemma shows that we can set $u_i = 1$ without loss of generality.

LEMMA 2.1. *Suppose that M-PSK is used and $M \geq 4$. If (ESDR1-$\mathbf{T}$) is tight, then there exist $\boldsymbol{\alpha} \in \mathbb{R}^n$ and $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \ldots, \boldsymbol{\gamma}_n \in \mathbb{R}^M$ that satisfy*

$$\gamma_{i,j} = \begin{cases} -2\mathrm{Re}(s_j^\dagger z_i) + \alpha_i, & \text{if } j \neq 1, \\ 2\mathrm{Re}(z_i) - \alpha_i, & \text{if } j = 1, \end{cases} \; j = 1, 2, \ldots, M, \; i = 1, 2, \ldots, n, \tag{2.3}$$

and

$$\boldsymbol{w}^\mathsf{T} \mathrm{Diag}(\boldsymbol{\gamma}_i) \boldsymbol{w} \geq 0, \;\; i = 1, 2, \ldots, n, \tag{2.4}$$

for any $\boldsymbol{w} \in \mathbb{R}^M$ such that $\boldsymbol{w}^\mathsf{T} \boldsymbol{s}_R = \boldsymbol{w}^\mathsf{T} \boldsymbol{s}_I = 0$.

Proof. First, recall that in the M-PSK setting we have

$$s_j = e^{\mathbf{i}\theta_j} \text{ and } \theta_j = \frac{(j-1)2\pi}{M}, \; j = 1, 2, \ldots, M.$$

Hence, it holds that $s_j^\dagger = s_j^{-1}$ and $s_k^\dagger s_j = s_{j-k+1}$, where the subscript is understood under congruence modulo M.

Now fix $i \in \{1, 2, \ldots, n\}$. When $u_i = 1$, we have $x_i^* = 1$ and $z_i = (\mathbf{H}^\dagger \boldsymbol{v})_i$. In this case, (2.3) is identical to that in [1, Corollary 3.7]. In other cases, suppose that $\check{\boldsymbol{\gamma}}_i$ and $\check{\alpha}_i$ satisfy the conditions in [1, Corollary 3.7]. Consider the permutation matrix

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{e}_{u_i} & \boldsymbol{e}_{u_i+1} & \ldots & \boldsymbol{e}_M & \boldsymbol{e}_1 & \ldots & \boldsymbol{e}_{u_i-1} \end{bmatrix}^\mathsf{T},$$

where $\boldsymbol{e}_i$ is the i-th standard basis in $\mathbb{R}^M$. We will show that $\boldsymbol{\gamma}_i = \mathbf{A}\check{\boldsymbol{\gamma}}_i$ and $\alpha_i = \check{\alpha}_i$ satisfy the conditions (2.3) and (2.4). Indeed, we have

$$\gamma_{i,1} = \check{\gamma}_{i,u_i} = 2\mathrm{Re}[s_{u_i}^\dagger (\mathbf{H}^\dagger \boldsymbol{v})_i] - \check{\alpha}_i = 2\mathrm{Re}(z_i) - \alpha_i,$$

and

$$\gamma_{i,j} = -2\mathrm{Re}[s_{u_i+j-1}^\dagger (\mathbf{H}^\dagger \boldsymbol{v})_i] + \check{\alpha}_i = -2\mathrm{Re}(s_j^\dagger z_i) + \alpha_i, \; j \neq 1.$$

Moreover, since $(\mathbf{A}\boldsymbol{s}_R)_k = \cos(\theta_k + \theta_{u_i}) = \cos(\theta_{u_i}) s_{R,k} - \sin(\theta_{u_i}) s_{I,k}$, we have

$$\mathbf{A}\boldsymbol{s}_R = \cos(\theta_{u_i})\boldsymbol{s}_R - \sin(\theta_{u_i})\boldsymbol{s}_I.$$

Similarly, we also have

$$\mathbf{A}\boldsymbol{s}_I = \sin(\theta_{u_i})\boldsymbol{s}_R + \cos(\theta_{u_i})\boldsymbol{s}_I.$$

Therefore, for any $\boldsymbol{w}$ that is orthogonal to $\boldsymbol{s}_R$ and $\boldsymbol{s}_I$, $\mathbf{A}^\mathsf{T}\boldsymbol{w}$ is also orthogonal to both two vectors. This further implies that $\boldsymbol{w}^\mathsf{T}\mathrm{Diag}(\boldsymbol{\gamma}_i)\boldsymbol{w} = \boldsymbol{w}^\mathsf{T}\mathbf{A}\mathrm{Diag}(\check{\boldsymbol{\gamma}}_i)\mathbf{A}^\mathsf{T}\boldsymbol{w} \geq 0$, which means $\boldsymbol{\gamma}_i$ also satisfies (2.4). ∎

Now define $\hat{z}_i = \mathrm{Re}(z_i)$, $\hat{z}_{n+i} = \mathrm{Im}(z_i)$, and

$$\begin{aligned} \mathbf{D}_1 &= \mathrm{Diag}([s_{R,1}, -s_{R,2}, \ldots, -s_{R,M}]^\mathsf{T}), \\ \mathbf{D}_2 &= \mathrm{Diag}([s_{I,1}, -s_{I,2}, \ldots, -s_{I,M}]^\mathsf{T}), \\ \mathbf{E} &= \mathrm{Diag}([-1, 1, \ldots, 1]^\mathsf{T}). \end{aligned}$$

By using (2.3) and the equality $\mathrm{Re}(s_j^\dagger z_i) = s_{R,j}\hat{z}_i + s_{I,j}\hat{z}_{n+i}$, we can write the conditions in Lemma 2.1 explicitly as

$$\exists\ \alpha_i \in \mathbb{R} \text{ s.t. } \boldsymbol{w}^\top(2\hat{z}_i\mathbf{D}_1 + 2\hat{z}_{n+i}\mathbf{D}_2 + \alpha_i\mathbf{E})\boldsymbol{w} \geq 0, \tag{2.5}$$

for any $\boldsymbol{w} \in \mathbb{R}^M$ such that $\boldsymbol{w}^\top\boldsymbol{s}_R = \boldsymbol{w}^\top\boldsymbol{s}_I = 0$. Our proof is based on the following observation: if we further constrain $\boldsymbol{w}$ to satisfy $\boldsymbol{w}^\top\mathbf{E}\boldsymbol{w} = 0$, then (2.5) becomes

$$\hat{z}_i\boldsymbol{w}^\top\mathbf{D}_1\boldsymbol{w} + \hat{z}_{n+i}\boldsymbol{w}^\top\mathbf{D}_2\boldsymbol{w} \geq 0, \tag{2.6}$$

which is an inequality on $\hat{z}_i$ and $\hat{z}_{n+i}$ independent of α_i. Hence, our idea is to derive (2.2) from (2.6) by choosing $\boldsymbol{w}$ properly in the set

$$\mathcal{D} = \{\boldsymbol{w} \in \mathbb{R}^M :\ \boldsymbol{w}^\top\boldsymbol{s}_R = 0,\ \boldsymbol{w}^\top\boldsymbol{s}_I = 0,\ \text{and } \boldsymbol{w}^\top\mathbf{E}\boldsymbol{w} = 0\}.$$

For a non-zero vector $\boldsymbol{w} \in \mathcal{D}$, note that $\boldsymbol{w}^\top\mathbf{E}\boldsymbol{w} = 0$ implies $w_1^2 = \sum_{j=2}^n w_j^2 > 0$. Therefore, we can assume $\boldsymbol{w} = [1, \tilde{\boldsymbol{w}}^\top]^\top$ where $\tilde{\boldsymbol{w}} \in \mathbb{R}^{M-1}$ and $\tilde{\boldsymbol{w}}^\top\tilde{\boldsymbol{w}} = 1$ without loss of generality. And instead of working on $\mathcal{D}$ directly, we will deal with the set

$$\tilde{\mathcal{D}} = \{\tilde{\boldsymbol{w}} \in \mathbb{R}^{M-1} :\ \tilde{\boldsymbol{w}}^\top\tilde{\boldsymbol{s}}_R = -1,\ \tilde{\boldsymbol{w}}^\top\tilde{\boldsymbol{s}}_I = 0,\ \text{and } \tilde{\boldsymbol{w}}^\top\tilde{\boldsymbol{w}} = 1\}$$

where $\tilde{\boldsymbol{s}}_R = [s_{R,2}, s_{R,3}, \dots, s_{R,M}]^\top$ and $\tilde{\boldsymbol{s}}_I = [s_{I,2}, s_{I,3}, \dots, s_{I,M}]^\top$. Accordingly, (2.6) now becomes

$$\hat{z}_i(1 - \tilde{\boldsymbol{w}}^\top\mathrm{Diag}(\tilde{\boldsymbol{s}}_R)\tilde{\boldsymbol{w}}) - \hat{z}_{n+i}\tilde{\boldsymbol{w}}^\top\mathrm{Diag}(\tilde{\boldsymbol{s}}_I)\tilde{\boldsymbol{w}} \geq 0. \tag{2.7}$$

Since $s_{R,j} = \cos(\theta_j) < 1$ for $j = 2, 3, \dots, M$, for any $\tilde{\boldsymbol{w}} \in \tilde{\mathcal{D}}$ we have

$$1 - \tilde{\boldsymbol{w}}^\top\mathrm{Diag}(\tilde{\boldsymbol{s}}_R)\tilde{\boldsymbol{w}} = \tilde{\boldsymbol{w}}^\top(\mathbf{I}_{M-1} - \mathrm{Diag}(\tilde{\boldsymbol{s}}_R))\tilde{\boldsymbol{w}} > 0.$$

Note that (2.2) is also equivalent to $\hat{z}_i - |\hat{z}_{n+i}|\cot\left(\frac{2\pi}{M}\right) \geq 0$. Hence, to derive (2.2) from (2.7), we only need to show that

$$\max_{\tilde{\boldsymbol{w}} \in \tilde{\mathcal{D}}} \frac{\tilde{\boldsymbol{w}}^\top\mathrm{Diag}(\tilde{\boldsymbol{s}}_I)\tilde{\boldsymbol{w}}}{1 - \tilde{\boldsymbol{w}}^\top\mathrm{Diag}(\tilde{\boldsymbol{s}}_R)\tilde{\boldsymbol{w}}} \geq \cot\left(\frac{2\pi}{M}\right) \tag{2.8}$$

and

$$\min_{\tilde{\boldsymbol{w}} \in \tilde{\mathcal{D}}} \frac{\tilde{\boldsymbol{w}}^\top\mathrm{Diag}(\tilde{\boldsymbol{s}}_I)\tilde{\boldsymbol{w}}}{1 - \tilde{\boldsymbol{w}}^\top\mathrm{Diag}(\tilde{\boldsymbol{s}}_R)\tilde{\boldsymbol{w}}} \leq -\cot\left(\frac{2\pi}{M}\right). \tag{2.9}$$

In fact, it is sufficient to prove one of them. To see this, for any $\tilde{\boldsymbol{w}} \in \tilde{\mathcal{D}}$, we can define $\tilde{\boldsymbol{w}}' \in \mathbb{R}^{M-1}$ by

$$\tilde{\boldsymbol{w}}'_j = \tilde{\boldsymbol{w}}_{M-j},\ j = 1, 2, \dots, M-1.$$

It is easy to verify that $\tilde{\boldsymbol{w}}'$ is still in $\tilde{D}$. Moreover, it also satisfies

$$(\tilde{\boldsymbol{w}}')^\top\mathrm{Diag}(\tilde{\boldsymbol{s}}_R)\tilde{\boldsymbol{w}}' = \tilde{\boldsymbol{w}}^\top\mathrm{Diag}(\tilde{\boldsymbol{s}}_R)\tilde{\boldsymbol{w}} \text{ and } (\tilde{\boldsymbol{w}}')^\top\mathrm{Diag}(\tilde{\boldsymbol{s}}_I)\tilde{\boldsymbol{w}}' = -\tilde{\boldsymbol{w}}^\top\mathrm{Diag}(\tilde{\boldsymbol{s}}_I)\tilde{\boldsymbol{w}},$$

which implies that (2.8) is equivalent to (2.9).

Now we are ready to prove that (2.8) holds for $M \geq 8$ and being a multiple of four. When $M = 8$, we can let

$$\tilde{\boldsymbol{w}} = [-\frac{\sqrt{2}+4}{6}, \frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{-\sqrt{2}+2}{6}, \frac{\sqrt{2}-2}{6}, 0, -\frac{\sqrt{2}}{6}]^{\mathsf{T}}.$$

Direct computation shows that $\tilde{\boldsymbol{w}} \in \tilde{D}$. Furthermore, we have

$$\frac{\tilde{\boldsymbol{w}}^{\mathsf{T}}\mathrm{Diag}(\tilde{\boldsymbol{s}}_I)\tilde{\boldsymbol{w}}}{1-\tilde{\boldsymbol{w}}^{\mathsf{T}}\mathrm{Diag}(\tilde{\boldsymbol{s}}_R)\tilde{\boldsymbol{w}}} = \frac{16\sqrt{2}+27}{35} > \cot\left(\frac{2\pi}{8}\right) = 1,$$

which proves (2.8) for $M = 8$.

For the remaining cases where $M \geq 16$, we only need to find $\tilde{\boldsymbol{w}} \in \tilde{D}$ that satisfies

$$\begin{aligned}
& \frac{\tilde{\boldsymbol{w}}^{\mathsf{T}}\mathrm{Diag}(\tilde{\boldsymbol{s}}_I)\tilde{\boldsymbol{w}}}{1-\tilde{\boldsymbol{w}}^{\mathsf{T}}\mathrm{Diag}(\tilde{\boldsymbol{s}}_R)\tilde{\boldsymbol{w}}} \geq \cot\left(\frac{2\pi}{M}\right) \\
\Leftrightarrow\quad & \tilde{\boldsymbol{w}}^{\mathsf{T}}\left(\cos\left(\frac{2\pi}{M}\right)\mathrm{Diag}(\tilde{\boldsymbol{s}}_R) + \sin\left(\frac{2\pi}{M}\right)\mathrm{Diag}(\tilde{\boldsymbol{s}}_I)\right)\tilde{\boldsymbol{w}} \geq \cos\left(\frac{2\pi}{M}\right) \\
\Leftarrow\quad & \tilde{w}_1^2 - (1-\tilde{w}_1^2) \geq \cos\left(\frac{2\pi}{M}\right) \qquad (2.10) \\
\Leftrightarrow\quad & |\tilde{w}_1| \geq \cos\left(\frac{\pi}{M}\right), \qquad (2.11)
\end{aligned}$$

where to derive (2.10) we used the facts that $\tilde{s}_{R,1} = \cos(2\pi/M)$, $\tilde{s}_{I,1} = \sin(2\pi/M)$, and $\tilde{\boldsymbol{w}}^{\mathsf{T}}\tilde{\boldsymbol{w}} = 1$. Now we show that the projection of $-\boldsymbol{e}_1 = [-1, 0, 0, \ldots, 0]^{\mathsf{T}}$ on $\tilde{D}$ satisfies (2.11). Indeed, since $\tilde{\boldsymbol{s}}_R$ is orthogonal to $\tilde{\boldsymbol{s}}_I$, the projection is given by

$$\tilde{\boldsymbol{w}} = -\frac{1}{\|\tilde{\boldsymbol{s}}_R\|_2^2}\tilde{\boldsymbol{s}}_R + \sqrt{\frac{1-1/\|\tilde{\boldsymbol{s}}_R\|_2^2}{1-\tilde{s}_{R,1}^2/\|\tilde{\boldsymbol{s}}_R\|_2^2 - \tilde{s}_{I,1}^2/\|\tilde{\boldsymbol{s}}_I\|_2^2}}\left(-\boldsymbol{e}_1 + \tilde{s}_{R,1}\frac{\tilde{\boldsymbol{s}}_R}{\|\tilde{\boldsymbol{s}}_R\|_2^2} + \tilde{s}_{I,1}\frac{\tilde{\boldsymbol{s}}_I}{\|\tilde{\boldsymbol{s}}_I\|_2^2}\right),$$

and we have

$$\begin{aligned}
-\tilde{w}_1 &= (-\boldsymbol{e}_1)^{\mathsf{T}}\tilde{\boldsymbol{w}} \\
&= \frac{1}{\frac{M}{2}-1}\cos\left(\frac{2\pi}{M}\right) + \sqrt{1-\frac{1}{\frac{M}{2}-1}}\sqrt{1-\frac{1}{\frac{M}{2}-1}\cos^2\left(\frac{2\pi}{M}\right) - \frac{1}{\frac{M}{2}}\sin^2\left(\frac{2\pi}{M}\right)} \\
&\geq \frac{1}{\frac{M}{2}-1}\cos\left(\frac{2\pi}{M}\right) + 1 - \frac{1}{\frac{M}{2}-1} \\
&= 1 - \frac{4}{\frac{M}{2}-1}\cos^2\left(\frac{\pi}{2M}\right)\left(2\sin^2\left(\frac{\pi}{2M}\right)\right) \\
&\geq 1 - 2\sin^2\left(\frac{\pi}{2M}\right) = \cos\left(\frac{\pi}{M}\right),
\end{aligned}$$

where we used $M > 10$ to get the last inequality.

To sum up, we showed that (2.8) and (2.9) are true when $M \geq 8$. Combined with (2.7), this enables us to to derive (2.2) and hence completes the proof.

REFERENCES

[1] R. Jiang, Y.-F. Liu, C. Bao, and B. Jiang, *Tightness and equivalence of semidefinite relaxations for MIMO detection*, preprint, https://arxiv.org/abs/2102.04586, 2021.