A Robust Algorithm for Finding the Real Intersections of Three Quadric Surfaces

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Abstract

By Bezout's theorem, three quadric surfaces may have infinitely intersections, but have at most eight isolated intersections. In this paper, we present an efficient and robust algorithm to obtain the isolated and the connected components of the real intersections of three quadric surfaces. Moreover, the conditions under which the intersections are finite and infinite are thoroughly investigated. Furthermore, our method can be used to find the number of isolated real intersections.

Key Words quadrics, intersection

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1 Introduction

Quadric surfaces, or surfaces of degree two, are widely used in mechanical CAD/CAM and computer graphics. There is plenty of literature on the intersection of two quadric surfaces [12, 13, 4, 16, 17, 3, 10]. Finding the intersection points of three quadrics is also an important subject in CAGD [2]. Throughout the paper, every quadric surface discussed is assumed to be defined by the zero set of a quadratic form X^TAX in 3D real affine space, where $X = (x, y, z, 1)^T$ is a 4D column vector and A is a 4×4 real symmetric matrix. The set of intersection points of three quadrics $X^TAX = 0$, $X^TBX = 0$ and $X^TCX = 0$ comprises all the points in 3D real affine space that satisfy the three equations. The following questions are of the fundamental importance in dealing with the intersection of three quadrics.

(1) Under what conditions do the three surfaces have real intersections?

(2) Under what conditions is the number of real intersections finite?

(3) How to count the real intersections if the number of real intersections are finite?

(4) How to count the isolated real intersections when the number of real intersections is infinite?

(5) How to find the real intersections including isolated and the connected components efficiently and robustly?

The goal of this paper is to answer all the questions above.

Traditionally, the polynomial system representing the intersection is solved by using Sylvester resultants successively. In [2], a more efficient method is proposed to compute the intersection of three quadrics using Macaulay's multivariate resultant. The idea works as follows. First, by using multivariate resultant, an univariate intersection polynomial equation $P = 0$ of degree at most eight is derived. After solving this equation, for each real root of $P = 0$, a system of three polynomial equations in two variables is obtained . The solution of the system of three polynomial equations in two variables involves the use of Sylvester resultant and GCD of the pairwise resultants. All these resultant calculations are expensive ones especially the calculation of the multivariate resultant. In addition, the floating point GCD computation is highly unstable. Another drawback of this method is that in some cases, the multivariate resultant may be identically equal to zero, even if there are only finitely intersection points. In this case, the method fails to produce anything.

Example 1 *Let* $a_1x+b_1y+c_1z=0$*,* $(a_1x+b_1y+c_1z)(a_2x+b_2y+c_2z)+a_2x+b_2y+c_2z=0$*, and* $(a_1x+b_1y+c_1z)(a_3x+b_3y+c_3z)+a_3x+b_3y+c_3z=0$ *be equations of three quadric surfaces. It is easy to see that the system has exactly one real root (0,0,0) if the coefficients are generic. The rank of the matrix of the coefficients of* y^2*,* yz*, and* z^2 *is 2.* x*-homogenizing the system yields*

$$\begin{aligned} a_1xw_x+b_1y+c_1z &= 0, \\ (a_1xw_x+b_1y+c_1z)(a_2xw_x+b_2y+c_2z)+w_x(a_2xw_x+b_2y+c_2z) &= 0, \\ (a_1xw_x+b_1y+c_1z)(a_3xw_x+b_3y+c_3z)+w_x(a_3xw_x+b_3y+c_3z) &= 0, \end{aligned} \tag{1}$$

where w_x *is the homogenizing variable. It is easy to see that the line defined by* $w_x=0$ *and* $b_1y+c_1z=0$ *represents a set of nontrivial solutions of (1). So the multivariate resultant for the* x*-homogenized system is identically equal to zero. Similarly, the multivariate resultants for the* y *and* z*-homogenized systems are identically equal to zero. In this case, a generic linear transformation of the variables yields a system of the same structure. Thus this system cannot be solved by using multivariate resultant.*

Another obvious approach to computing the intersection points of three quadrics is to use Levin's [12] or enhanced Levin's method[17] to get a parametric equation of the intersection curve of two of the three quadrics and then substitute the parametric equation into the remaining quadric. The roots of the resulting equation correspond to the points of the intersection of three quadrics. But the parametric equation of the intersection curve may contain a square root. Therefore, the degree of the resulting equation can be higher than 8. For example, assume that the intersection curve of two quadratics is $x(u) = (a_1(u) + \sqrt{s(u)}b_1(u))/p(u), y(u) = (a_2(u) + \sqrt{s(u)}b_2(u))/p(u), z(u) = (a_3(u) + \sqrt{s(u)}b_3(u))/p(u)$, where $s(u)$ is a quartic polynomial, $a_i(u), 1 \leq i \leq 3$ is a cubic polynomial and $b_i(u), 1 \leq i \leq 3$ and $p(u)$ are both linear polynomials. Substituting the equation into $z^2 = 0$, we have, $a_3^4(u) + b_3^4(u)s(u) + 2a_3^2(u)b_3^2(u)s(u) - 4s(u)a_3^2(u)b_3^2(u) = 0$. The resulting equation is of degree 12. Moreover, for any real root u_0 of the resulting equation, we have to determine whether $s(u_0) \geq 0$. Hence, the number of real roots of the resulting equation does not agree with the number of real quadric intersections.

Based on Levin's method of computing the intersection curve of two quadrics [12, 13], in this paper, we present a more efficient and robust method for finding all of the real quadric intersection points including isolated points and connected components. Assume that three quadrics $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are given. Using Levin's method, we can transform the problem of finding the intersection points of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, to finding the intersection points of parametric curves $E(u,v) = 0$ and $F(u,v) = 0$, where E and F are polynomials of degree at most 6 with degree at most 2 in v and at most 4 in u. Then an univariate polynomial of degree at most eight in u can be derived from those 2 equations by using resultant. Under a minor restriction, the number of real roots of the univariate polynomial agrees with the number of real quadric intersections. Hence, by using Sturm Sequence or the results from [18], we can count the real quadric intersections. So, (1) and (3) can be answered. Based on a detailed analysis of the system $E = F = 0$, (2), (4), and (5) can be answered.

2 Preliminaries

We assume that each quadric surface $\mathcal{A}$ is given as the solution set in $\mathbf{R}^3$ of a quadratic implicit equation $X^TAX = 0$, where $X = (x, y, z, 1)^T$ is a 4D column vector and A is a 4×4 real symmetric matrix. We use the same notation A for a quadratic implicit equation and its associated matrix; the corresponding quadric surface is denoted as $\mathcal{A}$. Let three quadric surfaces be defined by $\mathcal{A} : X^TAX = 0, \mathcal{B} : X^TBX = 0$, and $\mathcal{C} : X^TCX = 0$. For brevity, similar to the notations used in [12], the intersection curve of two quadrics $\mathcal{A}, \mathcal{B}$ and the intersections of three quadrics $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ will be referred to as $QSIC(A, B)$ and $TQSI(A, B, C)$ (Three Quadric Surfaces Intersection) respectively. For a given set S, throughout the rest of the paper , $\#S$ denotes the cardinality of S.

We will refer to the 3×3 upper left submatrices of A, B, and C, denoted by A_u, B_u, and C_u , as the principal submatrices of A, B, and C respectively. The determinant of such a matrix is called the principal subdeterminant.

Given two distinct quadrics $\mathcal{A} : X^TAX = 0$ and $\mathcal{B} : X^TBX = 0$, the pencil generated by $\mathcal{A}$ and $\mathcal{B}$ is the set $\mathcal{R}(\lambda)$ with equations $X^T(A + \lambda B)X = 0, \lambda \in \mathbf{R}$.

When considering the intersection curve of two quadric surfaces, Levin proved that there exits a so called parameterization surface, a simple ruled surface, in the pencil of two quadrics [12]. This idea is very useful for parameterizing $QSIC$ and has inspired several subsequent papers on its applications and improvement [17, 11, 10]. Levin's method is based on the following key result:

Theorem 1 [12] *The pencil generated by two distinct quadric surfaces contains at least one simple ruled quadrics, i.e., a plane, pair of planes, hyperbolic or parabolic cylinder, or a hyperbolic paraboloid.*

A more concise proof of Theorem 1 is given in [17].

One important property of the simple ruled quadrics in Theorem 1 is that their principal subdeterminants are zero(cf.[12]). So to find a simple ruled quadric in the pencil generated by A and B, we need to compute the type of B and of the quadrics $R(\lambda) = A - \lambda B$, where λ is a solution of $det\ R_u(\lambda) = 0.$(cf.[12]) (By [13], when $det\ R_u(\lambda) \equiv 0$, the problem can be easily solved by considering the roots of $det\ R(\lambda) = 0.$) By Theorem 1, one of these quadrics is simple ruled. Let $\mathcal{R}$ be such a quadric and assume that $\mathcal{R}$ and $\mathcal{B}$ are distinct (otherwise, we simply swap $\mathcal{A}$ and $\mathcal{B}$).

By [12] and [13], the simple ruled quadric $\mathcal{R}$ can be reduced, by a series of affine transformations $\mathcal{T}$, to an appropriate normal form given in Table 1. (To remove two layers of nested radicals from Levin's algorithm, L. Dupont etc. suggest using Gauss' method for the reduction of the simple ruled quadric $\mathcal{R}$.cf.[3]) The parameterization of the normal form is also described in Table 1. The parameterization is denoted by $\mathbf{q}(u, v)$. Substituting $\mathbf{q}(u, v)$ for X in $X^T BX = 0$ yields

$$E : e_2(u)v^2 + e_1(u)v + e_0(u) = 0. \quad (2)$$

The respective degrees of the polynomials e_0, e_1 and e_2 are also given in Table 1.

Table 1. Parameterizations of normal simple ruled quadrics

quadric	canonical equation	parameterization	degree of e_2, e_1, e_0
simple plane	x=0	$\mathbf{q}(u,v) = [0, u, v, 1]$	0,1,2
double plane	$x^2 = 0$	$\mathbf{q}(u,v) = [0, u, v, 1]$	0,1,2
parallel planes	$x^2 = 1$	$\mathbf{q}(u,v) = [\pm 1, u, v, 1]$	0,1,2
intersecting planes	$x^2 - y^2 = 0$	$\mathbf{q}(u,v) = [u, \pm u, v, 1]$	0,1,2
hyperbolic paraboloid	$z - xy = 0$	$\mathbf{q}(u,v) = [u, v, uv, 1]$	2,2,2
parabolic cylinder	$x^2 - y = 0$	$\mathbf{q}(u,v) = [u, u^2, v, 1]$	0,2,4
hyperbolic cylinder	$xy - 1 = 0$	$\mathbf{q}(u,v) = [u, 1/u, v/u, 1]$	0,2,4

It is easy to see that when $\mathbf{q}(u,v)$ is substituted into $X^T C X = 0$ for X, we shall obtain an equation with exactly the same form as E:

$$F : f_2(u)v^2 + f_1(u)v + f_0(u) = 0. \qquad (3)$$

This reduction from a system of three equations to a system of two equations in the form of (2) is crucial to our method. This new system is much easier to analyze and yet it still carries all the information about $TQSI(A, B, C)$. We shall analyze the system $E = F = 0$ in the next section.

3 The Morphology of TQSI

For given quadrics $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$, let's consider

$$\begin{aligned} E &: e_2(u)v^2 + e_1(u)v + e_0(u) = 0, \\ F &: f_2(u)v^2 + f_1(u)v + f_0(u) = 0, \end{aligned} \qquad (4)$$

where E and F are defined in (2) and (3).

In general, each real root of system (4), (u_0, v_0), corresponds to exactly one real point in $TQSI(A, B, C)$, i.e. $\mathcal{T}^{-1}\mathbf{q}(u_0, v_0)$, since $\mathcal{T}$ is an invertible transformation. Moreover, if $\mathcal{R}$ isn't a double plane, the multiplicity of a root (u_0, v_0) of $E = 0, F = 0$ agrees with the multiplicity of $\mathcal{T}^{-1}\mathbf{q}(u_0, v_0)$ in $TQSI(A, B, C)$. When $\mathcal{R}$ is a double plane, the double multiplicity of a root (u_0, v_0) of $E = 0, F = 0$ agrees with the multiplicity of $\mathcal{T}^{-1}\mathbf{q}(u_0, v_0)$ in $TQSI(A, B, C)$. But when $\mathcal{R}$ is a hyperbolic cylinder, the real roots of the form $(0, v_0)$ correspond to real points of $TQSI(A, B, C)$ at infinity. Hence, throughout the rest of the paper, when $\mathcal{R}$ is a hyperbolic cylinder, we will discard $u = 0$ from the lists of real roots of relevant polynomial equations without further declaration. Throughout the rest of the paper, by abusing our notation, we denote $\mathcal{T}^{-1}\mathbf{q}(u, v)$ simply by $\mathbf{q}(u, v)$, whenever it does not cause any confusion.

By Table 1, the degrees of E and F in u and v are two and two respectively if $\mathbf{q}(u, v)$ is not a representation of a parabolic cylinder nor a hyperbolic cylinder. In this case, by multi-homogeneous Bezout theorem [15], system (4) has at most 8 isolated roots in the complex 2-projective space $P_1 \times P_1$. (The 2-homogeneous Bezout number for the system is the coefficient of $\alpha_1\alpha_2$ in $(2\alpha_1 + 2\alpha_2)^2$.) In particular, when $q(u, v)$ is a hyperbolic paraboloid, the degree of E and F is 4. In this case, the intersection points of E and F are infinite in complex project space.But the infinite intersection points only occur in infinite. Moreover, by multi-homogeneous Bezout theorem, the isolated intersection points in complex project space is at most 8. In case $\mathbf{q}(u, v)$ is a representation of a parabolic cylinder or a hyperbolic cylinder, system (4) can be easily reduced to an equation in u of degree less than or equal to 8. Thus in any case, no extraneous intersection points are introduced through this transformation from the intersection of 3 surfaces to the intersection of two curves, except at infinity of some projective spaces.

If E or F is independent of v, then the problem becomes very simple. So from now on, we assume both E and F depend on v unless otherwise mentioned. Let $G(u) = res(E, F, v)$, the Sylvester resultant of E and F with respect to v. For example,

$$G = \begin{vmatrix} e_2 & e_1 & e_0 & 0 \\ 0 & e_2 & e_1 & e_0 \\ f_2 & f_1 & f_0 & 0 \\ 0 & f_2 & f_1 & f_0 \end{vmatrix}$$

when $e_2(u) \not\equiv 0$ and $f_2(u) \not\equiv 0$. It is easy to see that $deg\, G \leq 8$. If for $u_0 \in \mathbf{R}$, $e_i(u_0) = 0$ and $f_i(u_0) = 0, 0 \leq i \leq 2$, then $E(u_0, v) \equiv 0$ and $F(u_0, v) \equiv 0$. Hence, in this case, $TQSI(A, B, C)$ contains infinite real intersections. To deal with the case, we let $g(u) = \sum_{i=0}^{2} e_i^2(u) + \sum_{i=0}^{2} f_i^2(u)$. Obviously $g(u_0) = 0$ implies $e_i(u_0) = 0$ and $f_i(u_0) = 0, 0 \leq i \leq 2$.

Hence, when $g(u_0) = 0$, the parametric curve $\mathbf{q}(u_0, v)$ is a subset of $TQSI(A, B, C)$. If $g(u) \equiv 0$, then the parametric surface $\mathbf{q}(u, v)$ is a subset of $TQSI(A, B, C)$. For any $X \in TQSI(A, B)$, X belongs to the surface defined by the matrix $A + \lambda B$. So $X \in \mathbf{q}(u, v)$. Thus $\mathbf{q}(u, v) = TQSI(A, B, C)$. So we have the following

Theorem 2 *If $g(u)$ is not identically zero and $g(u) = 0$ has a real root u_0, then $TQSI(A, B, C)$ contains the curve $q(u_0, v)$. If $g(u) \equiv 0$, then $TQSI(A, B, C)$ is the parametric surface $\mathbf{q}(u, v)$.*

If $g(u) = 0$ has a real root u_0, then $TQSI(A, B, C)$ contains the curve $\mathcal{T}^{-1}\mathbf{q}(u_0, v)$ (here, we need to distinguish $\mathcal{T}^{-1}\mathbf{q}(u_0, v)$ and $\mathbf{q}(u_0, v)$). By table 1, $q(u_0, v)$ is a generating line of the simple ruled quadric $\mathcal{R}$.

When $g(u) = 0$ has no real root, if $G \equiv 0$, $TQSI(A, B, C)$ may still have infinitely many points. Hence, The condition that $g(u) = 0$ has real roots is only sufficient for

$TQSI(A,B,C)$ to contain infinitely many points. The next theorem can be used to check if the number of the real points in TQSI(A,B,C) is finite.

Theorem 3 *Suppose $G \not\equiv 0$. Then # TQSI(A,B,C) is finite if and only if $g(u) = 0$ has no real roots.*

Proof. By theorem 2, if # TQSI(A,B,C) is finite, then $g(u) = 0$ must not have any real roots. Now let us assume that $g(u) = 0$ has no real roots. Let $u_1, ..., u_k$ be the real roots of $G = 0$, where $0 \leq k \leq 8$. Since $g(u_i) \neq 0$, for each u_i, the system $E(u_i, v) = F(u_i, v) = 0$ has at most 2 real roots, $i = 1, ..., k$. Thus # TQSI(A,B,C) is finite. (By Bezout theorem, $\#TQSI(A,B,C) \leq 8$.) □

But when $G \equiv 0$, $TQSI(A,B,C)$ may have infinite or finite points. We will analysis the case in detail. Let $D(u,v) = GCD\ (E,F)$. By Proposition 1 of [1, §3.6], $G \equiv 0$ if and only if the degree of $D(u,v)$ in v is positive. Thus we have the following

Theorem 4 *If $G \equiv 0$, then $TQSI(A,B,C)$ is the union of set of the points corresponding to the real roots of $D(u,v) = 0$ and the set of the points corresponding to the real roots of $E/D = F/D = 0$.*

Remark 1 *If $G \equiv 0$, then the degree of $D(u,v)$ in v is positive. So it can only be 1 or 2. If the degree of $D(u,v)$ in v is 1, then $D = 0$ defines the (real) curve(s) with equation $v = \frac{-d_0(u)}{d_1(u)}$, where $D(u,v) = d_1(u)v + d_0(u)$. If the degree of $D(u,v)$ in v is 2, then we can factor $D(u,v)$ into the form: $D(u,v) = H(u)\tilde{D}(u,v)$, where $\tilde{D}(u,v)$ has no factor depending only on u. If $H(u_i) = 0$ for some real u_i, $i = 1, \cdots, k$, then $D(u,v) = 0$ defines the curves $u = u_i$, $i = 1, \cdots, k$. The handling of $\tilde{D}(u,v) = 0$ is the same as that of*

$D(u,v)=0$ with $H(u)=0$ having no real roots. Now let's assume that $H(u)=0$ has no real roots. Let $Dis(u)$ be the discriminant of $\tilde{D}(u,v)$, where u is regarded as constant. If $Dis(u)\equiv 0$, then $\tilde{D}(u,v)=0$ and therefore $D(u,v)=0$ defines the curve(s) with equation $v=\frac{-d_1(u)}{2d_2(u)}$, where $D(u,v)=d_2(u)v^2+d_1(u)v+d_0(u)$. If $Dis(u_0)>0$ for some real u_0, then $Dis(u)\geq 0$ for $u\in[a,b]$ for some $a<b$. Thus $D(u,v)=0$ defines the curve(s) with equation $v=\frac{-d_1(u)\pm\sqrt{Dis(u)}}{2d_2(u)}$. If $Dis(u)<0$ on $(a,u_0)\cup(u_0,b)$, where $a<b$, and $Dis(u_0)=0$, then $D(u,v)=0$ has an isolated real root $(u_0,\frac{-d_1(u_0)}{2d_2(u_0)})$, provided $d_2(u_0)\neq 0$. Since $Dis(u_0)=(d_1(u_0))^2-4d_2(u_0)d_0(u_0)=0$ and $d_2(u_0)=0$ would imply $H(u_0)=0$, so $d_2(u_0)\neq 0$.

Note that when the degree of $D(u,v)$ in v is 2, E/D and F/D are independent of v. Thus the system $E/D=0$ and $F/D=0$ can not have real roots, since otherwise E/D and F/D would have a nonconstant common factor $h(u)$ which contradicts the fact that $D(u,v)$ is the GCD of E and F.

The following examples demonstrate a variety of cases in Remark 1.

Example 2 *$A=y+x^2=0, B=z^2+1-y=0$, and $C=2z^2+2-2y=0$. A can be parameterized as $x=u, y=-u^2, z=v$. Then $E=v^2+1+u^2$ and $F=2v^2+2+2u^2$. Obviously, $G\equiv 0$ and $D(u,v)=v^2+1+u^2$. Since $Dis(u)=-4-4u^2<0$, $TQSI(A,B,C)=\emptyset$.*

Example 3 *$A=y+x^2=0, B=z^2+y-xy=0$, and $C=2z^2+2y-2xy=0$. A can be parameterized as $x=u, y=-u^2, z=v$. Then $E=v^2-u^2+u^3$ and $F=2v^2-2u^2+2u^3$. Obviously, $G\equiv 0$, $D(u,v)=v^2-u^2+u^3$ and $Dis(u)=-4u^2+4u^3$. It is easy to see that* *on* $(-\infty, 0) \cup (0, 1)$ $Dis(u) < 0$*,* $Dis(0) = 0$*, and* $Dis(u) > 0$ *on* $(1, \infty)$*. So* $\{(0, 0, 0)\}$ *is the only isolated point in* $TQSI(A, B, C)$ *and* $v = \pm 2u\sqrt{u-1}$ *for* $u \geq 1$ *corresponds to curves in* $TQSI(A, B, C)$*.*

Example 4 $A = y + x^2 = 0$*,* $B = z(1 + y) = 0$*, and* $C = z^2 + z(1 + y) = 0$*.* A *can be parameterized as* $x = u, y = -u^2, z = v$*. Then* $E = v(1 - u^2)$ *and* $F = v^2 + v(1 - u^2)$*. Obviously,* $G \equiv 0$ *and* $D(u, v) = v$*. So* $TQSI(A, B, C)$ *contains the curve defined by* $z = 0$ *and* $y = -x^2$*.* $E/D = 0$ *and* $F/D = 0$ *correspond to two isolated points* $(\pm 1, -1, 0)$*.*

Remark 2 *Assume that* $G \not\equiv 0$ *and* $g \not\equiv 0$ *and* $u_1, ..., u_k$ *are the real roots of* $g = 0$*. Then* g *is divisible by* $u - u_i$*. So there exist integers* n_i *so that* $g(u) = (u - u_1)^{n_1} \cdots (u - u_k)^{n_k} \tilde{g}(u)$*, where* $\tilde{g}(u_i) \neq 0$*. Since* $g(u) = \sum_{i=0}^{2} e_i^2(u) + \sum_{i=0}^{2} f_i^2(u)$*, it is easy to see that* n_i*'s are even numbers. Let* $\tilde{E} = E / \prod(u - u_i)^{n_i/2}$*,* $\tilde{F} = F / \prod(u - u_i)^{n_i/2}$*, and* $\tilde{G}$ *be the resultant of* $\tilde{E}$ *and* $\tilde{F}$*. Then* $\tilde{g} = 0$ *has no real roots and* $\tilde{G} \not\equiv 0$*. Thus* $\tilde{E} = \tilde{F} = 0$ *has only finitely real roots. It is easy to see that the real roots of* $\tilde{E} = \tilde{F} = 0$ *corresponds to points in* $TQSI(A, B, C)$*.*

Combining the discussions of Remarks 1 and 2 we have the following

Theorem 5 *If* $G \not\equiv 0$ *and* $g \not\equiv 0$*, then the number of isolated points in* $TQSI(A, B, C)$ *is equal to the number of real solutions of* $\tilde{E} = \tilde{F} = 0$*. If* $G \equiv 0$ *and* $g \not\equiv 0$*, then the number of isolated points in* $TQSI(A, B, C)$ *is equal to the number of real solutions of* $\tilde{E} = \tilde{F} = 0$ *plus the number of isolated solutions of* $D(u, v) = 0$*.*

Furthermore, we have

Theorem 6 *If $TQSI(A,B,C)$ has infinite points, then number of isolated points in $TQSI(A,B,C)$ is less than or equal to 4.*

proof: Since $TQSI(A,B,C)$ has infinitely many real points, by Theorem 3, either $G \equiv 0$ or $g(u) = 0$ has real roots. When $g(u) = 0$ has real roots but $G \not\equiv 0$, by Theorem 5, we only need to consider the number of real roots of $\tilde{E} = 0, \tilde{F} = 0$. In case $e_2 \equiv 0$ and $f_2 \equiv 0$, the degree of $res(\tilde{E}, \tilde{F}, v)$ is less than or equal to 4 and thus the conclusion follows.

If $e_2 \not\equiv 0$ or $f_2 \not\equiv 0$, by Table 1, $g(u) = 0$ has real roots can only happen when $\mathbf{q}(u,v)$ is a parameterization of hyperbolic paraboloid. In this case, the degrees of $\tilde{E}$ and $\tilde{F}$ in u are no more than 1. Write $\tilde{E} = uQ_1(v) + Q_2(v)$ and $\tilde{F} = uQ_3(v) + Q_4(v)$, where $Q_i(v)$ is a polynomial with degree in v no more than 2. Obviously, the degree of $res(\tilde{E}, \tilde{F}, u)$ is less than or equal to 4. Since $\tilde{E}$ and $\tilde{F}$ are linear in u, the number of real roots of system $\tilde{E} = 0, \tilde{F} = 0$ is less than or equal to 4.

When $G \equiv 0$ and $g(u) = 0$ has no real roots, let us consider $\widehat{E} = \frac{E}{D}$ and $\widehat{F} = \frac{F}{D}$, where $D(u,v)$ is a GCD of E and F with positive degree in v. If the degree of $D(u,v)$ in v is 1, then the real roots of $D(u,v) = 0$ correspond to connected components of $TQSI(A,B,C)$. So we only need to consider $\widehat{E} = \widehat{e}_1(u)v + \widehat{e}_0(u) = 0$, and $\widehat{F} = \widehat{f}_1(u)v + \widehat{f}_0(u) = 0$. By looking at the degrees of $e_i(u)$ listed in Table 1, it is easy to prove that the degree of $res(\widehat{E}, \widehat{F}, v)$, i.e. $\widehat{e}_1(u)\widehat{f}_0(u) - \widehat{f}_1(u)\widehat{e}_0(u)$, is less than or equal to 4. So the number of real roots of system $\tilde{E} = 0,\ \tilde{F} = 0$ is less than or equal to 4.

Finally, let us look at the case where the degree of $D(u,v)$ in v is 2. By remark 1, we only need to consider $H(u) = 0$ and $Dis(u)$. Since $H(u) = 0$ implies $g(u) = 0$ and that case has been discussed, it suffices to consider the case where $H(u) = 0$ has no real roots.

Since there are infinitely many intersection points, $Dis(u) \equiv 0$ or $Dis(u_0) > 0$ for some real u_0. If $Dis(u) \equiv 0$, $TQSI(A, B, C)$ contains no isolated real points. If $Dis(u_0) > 0$ for some real u_0, the number of isolated real points in $TQSI(A, B, C)$ is the same as the number of real roots of $Dis(u) = 0$ at which $Dis(u)$ has local minimums. Since the degree of $Dis(u)$ is at most 4, the number of isolated real roots of $D(u, v) = 0$ is less than or equal to 1. □

Remark 3 *It is easy to check whether $G \equiv 0$. By using Sturm Sequence, we can detect whether there exits $u_0 \in \mathbf{R}$ such that $g(u_0) = 0$. When $TQSI(A, B, C)$ has infinitely many real points, Remarks 1 and 2 can be used to calculate the connected components of $TQSI(A, B, C)$. The results of [16, 17] can also be used to calculate and classify the morphology of the connected components of $TQSI(A, B, C)$.*

4 Counting and Computing the Real Points in $TQSI(A, B, C)$

Let $G(u) = res(E, F, v)$. Obviously, the degree of $G(u)$ is no more than 8. In this section, we assume that $G(u) \not\equiv 0$ and $g(u)$ has no real roots, since otherwise we can reduce the problem to a new problem with $G(u) \not\equiv 0$ and $g(u)$ has no real roots, using the results from theorems 4, 5 and remarks 1 and 2. Hence, by Theorems 3, the number of real points in $TQSI(A, B, C)$ is finite. It is to see that system (4) can be reduced to :

$$\begin{aligned} e_2(u)v^2 + e_1(u)v + e_0(u) &= 0, \\ (-f_2(u)e_1(u) + e_2(u)f_1(u))v + f_0(u)e_2(u) - e_0(u)f_2(u) &= 0. \end{aligned} \tag{5}$$

It is not hard to see that if $-f_2(u_0)e_1(u_0) + e_2(u_0)f_1(u_0) = 0$ and $f_0(u_0)e_2(u_0) -$

$e_0(u_0)f_2(u_0) = 0$, then the four columns (rows) of $H = \begin{bmatrix} e_2(u_0) & e_1(u_0) & e_0(u_0) & 0 \\ 0 & e_2(u_0) & e_1(u_0) & e_0(u_0) \\ f_2(u_0) & f_1(u_0) & f_0(u_0) & 0 \\ 0 & f_2(u_0) & f_1(u_0) & f_0(u_0) \end{bmatrix}$ are linearly dependent. Let $s(u) = (-f_2(u)e_1(u)+e_2(u)f_1(u))^2+(f_0(u)e_2(u)-e_0(u)f_2(u))^2$. Then $s(u_0) = 0$ implies $G(u_0) = det(H) = 0$. In this case, (5), and therefore (4), may have 0, 1, or 2 real solutions counting multiplicity. So, it is important to see whether $s(u) = 0$ has a real root. We have

Theorem 7 *When $s(u) = 0$ has no real roots, the number of real roots of $G(u) = 0$ agrees with the number of real roots of polynomial system (4) counting multiplicity.*

Proof. Let u_0 be such that $G(u_0) = 0$ with multiplicity m. Let $S(v) = e_2(u_0)F - f_2(u_0)E$, and consider

$$S = (-f_2(u_0)e_1(u_0) + e_2(u_0)f_1(u_0))v + f_0(u_0)e_2(u_0) - e_0(u_0)f_2(u_0) = 0. \qquad (6)$$

If $-f_2(u_0)e_1(u_0) + e_2(u_0)f_1(u_0) \neq 0$, then (6) is linear and therefore (4) has exactly one solution. Since $G(u) = (f_0(u)e_2(u)-e_0(u)f_2(u))^2-(-f_2(u)e_1(u)+e_2(u)f_1(u))(e_1(u)f_0(u)-e_0(u)f_1(u))$, $-f_2(u_0)e_1(u_0)+e_2(u_0)f_1(u_0) = 0$ would imply that $f_0(u_0)e_2(u_0)-e_0(u_0)f_2(u_0) = 0$ and thus $s(u_0) = 0$. So $-f_2(u_0)e_1(u_0) + e_2(u_0)f_1(u_0) \neq 0$. Therefore, each root of $G(u) = 0$ corresponds to exactly one solution of polynomial system (4)with multiplicity m. Hence, the number of real roots of $G(u) = 0$ agrees with the number of real roots of polynomial system (4) counting multiplicity. □

By using the method of [18] or Sturm Sequence, we can obtain the number of real roots of $G(u)$. Hence, when $s(u) = 0$ has no real roots, we can obtain the number of real points in $TQSI(\mathcal{A}, \mathcal{B}, \mathcal{C})$. In case $s(u)$ has real roots, one can compute $q(u) = GCD(-f_2(u)e_1(u) +$

$e_2(u)f_1(u), f_0(u)e_2(u) - e_0(u)f_2(u))$. Note $q(u)$ is a polynomial of degree ≤ 4. Hence, we can find the exact real roots of equation $q(u) = 0$. Denote them as $u_1, \cdots, u_m$, where $m \leq 4$. Denote the number of common real roots of $e_2(u_i)v^2 + e_1(u_i)v + e_0(u_i) = 0$ and $f_2(u_i)v^2 + f_1(u_i)v + f_0(u_i) = 0$ as n_i. Then we have

Corollary 1 *The number of real roots of polynomial system (4) is equal to* $\sum_{i=1}^{m} n_i + M$, *where* M *is the number of real roots of* $G(u)/\prod_{i=1}^{m}(u-u_i)^{k_i} = 0$, k_i *is the largest integer such that* $G(u)$ *is divisible by* $(u-u_i)^{k_i}$, n_i *is the number of common real roots of* $e_2(u_i)v^2 + e_1(u_i)v + e_0(u_i) = 0$ *and* $f_2(u_i)v^2 + f_1(u_i)v + f_0(u_i) = 0$, *where* u_i *is the real root of* $q(u) = GCD(-f_2(u)e_1(u) + e_2(u)f_1(u), f_0(u)e_2(u) - e_0(u)f_2(u)) = 0, 1 \leq i \leq m$.

Example 5 *Let* $A = z - xy$, $B = z^2 - 1$, *and* $C = yz + xz + x^2 - y$. *Then* $\mathcal{A}$ *can be parameterized as* $x = u$, $y = v$, *and* $z = uv$. *Thus* $E = u^2v^2 - 1$, $F = uv^2 + (u^2-1)v + u^2$, $G = u^4(u^4 - u^2 + 2 + 2u)$, *and* $s(u) = u^4(u^2-1)^2 + (u^4+u)^2$. *The common real roots of* $G(u) = 0$ *and* $s(u) = 0$ *are* $u = 0, 1$. $u = 0$ *corresponds to no real roots, while* $u = -1$ *corresponds to* $v = \pm 1$. *The number of real roots of polynomial system (4) is equal to 2.*

In Section 3 and 4,we obtained some sufficient conditions in terms of the roots of $s(u)$, $g(u)$ and $G(u)$ of $\#TQSI$.We summarize these necessary and sufficient conditions for different cases in the following theorem.

Theorem 8 *Let* $G(u) = res(E, F, v), g(u) = \sum_{i=0}^{2} e_i^2(u) + \sum_{i=0}^{2} f_i^2(u)$ *and* $s(u) = (-f_2(u)e_1(u) + e_2(u)f_1(u))^2 + (f_0(u)e_2(u) - e_0(u)f_2(u))^2$. *Then*

(1) When $G \not\equiv 0$, $\#TQSI(A, B, C)$ *is finite if and only if* $g(u) = 0$ *has no real roots;*

(2) When $G \equiv 0$, $\#TQSI(A,B,C)$ *may have finite or infinite many points. The detail analysis is presented in Remark 1;*

(3) When $g(u)$ *have real roots,* $\#TQSI(A,B,C)$ *is infinite;*

(4)When $TQSI(A,B,C)$ *has infinite points, then number of isolated points in* $TQSI(A,B,C)$ *is less than or equal to 4;*

(5)When $G \not\equiv 0$ *and* $g(u)$ *has no real roots,*$\#TQSI(A,B,C)$ *equal to* $\sum_{i=1}^{m} n_i + M$. *The* n_i *and* M *is defined in Corollary 1.*

By using the Theorem and Corollary above, we shall give an algorithm for counting the real intersections of TQSI. Moreover, by analysing the multiplicity of $G(u) = 0$, it is not difficult to construct a similar algorithm to present the multiplicity of the intersection points in TQSI. A similar algorithm used to calculate TQSI instead of counting the real intersections of TQSI will also be given.

Algorithm 1

Input: *Three distinct quadrics* $\mathcal{A} : X^T AX = 0, \mathcal{B} : X^T BX = 0$ *and* $\mathcal{C} : X^T CX = 0$.

Output: M, *the number of isolated real points in* $TQSI(A,B,C)$; I, *a flag with values "True" or "False" to indicate whether* $TQSI(A,B,C)$ *has connected components.*

- **Begin**

Step 0. *M:=0; I:=False.*

Step 1. *Generate*

$$E : e_2(u)v^2 + e_1(u)v + e_0(u) = 0,$$

$$F: f_2(u)v^2 + f_1(u)v + f_0(u) = 0. \quad (7)$$

Step 2. *Compute* $G = res(E, F, v)$

If $G \equiv 0$ **Do**

go to Step 3.

End If

go to Step 4.

Step 3. *(This step is used to handle the situation where* $G \equiv 0$*.)*

Compute $D(u,v) = GCD(E,F)$*. Use Remark 1 to determine* n *and* N*, the number of isolated and the total number of real roots of* $D(u,v) = 0$ *respectively*

M:=M+n

If $N = \infty$ **Do**

I:= True

End If

$E := \frac{E}{D}$*;* $F := \frac{F}{D}$

Go to Step 5

Step 4. *(This step is used to handle the situation where* $G \not\equiv 0$*.)*

If *there exits* $u_0 \in \mathbf{R}$*, such that* $\sum_{i=0}^{2} e_i^2(u_0) + \sum_{i=0}^{2} f_i^2(u_0) = 0$ **Do**

I := True

Use remark 2 to reduce E *and* F

End If

Go to Step 5

Step 5. *Find* m, *the number of real roots of* $(-f_2(u)e_1(u) + e_2(u)f_1(u))^2 + (f_0(u)e_2(u) - e_0(u)f_2(u))^2 = 0$

If $m = 0$ **Do**

go to Step 6

End If

go to Step 7

Step 6. *Find* n, *the number of real roots of* $G = 0$

M:=M + n

STOP

Step 7. *Compute* $q(u) = GCD(-f_2(u)e_1(u) + e_2(u)f_1(u), f_0(u)e_2(u) - e_0(u)f_2(u))$.

Find the real roots of $q(u) = 0$, *denoted as* $u_1, \cdots, u_m$, $m \leq 4$

Find n_i, *the number of common real roots of* $e_2(u_i)v^2 + e_1(u_i)v + e_0(u_i) = 0$ *and* $f_2(u_i)v^2 + f_1(u_i)v + f_0(u_i) = 0$

$M := N + \sum_{i=1}^{m} n_i$, *where* N *is the number of real roots of* $G(u)/\prod_{i=1}^{m}(u - u_i)^{k_i} = 0$ *and* k_i *denotes the maximal integer such that* G *is divisible by* $(u - u_i)^{k_i}$.

End

Remark 4 *(1) Step 3 requires the determination of the sign of* $Dis(u)$. *Since the degree of* $Dis(u)$ *is less than or equal to* 4, *one can find the exact real roots of* $Dis(u) = 0$. *Denote* *them as* $u_1, \cdots, u_m, m \leq 4$*. By testing the sign of* $Dis(u)$ *between two adjacent real roots, we can determine whether* $Dis(u) < 0, \forall u \in \mathbf{R}$.

(2) In the algorithm, to determine the numbers of real roots of polynomials, one often uses Sturm Sequence. The calculation of the Sturm Sequence requires polynomial division. To avoid polynomial division, we can use the explicit criterion for the determination of the numbers of real roots of polynomial (cf[18]).

Our main algorithm is the following.

Algorithm 2

Input: *Three distinct quadrics* $\mathcal{A} : X^T AX = 0, \mathcal{B} : X^T BX = 0$ *and* $\mathcal{C} : X^T CX = 0$.

Output: *TQSI: Parameterizations of the curves and isolated real points in* $TQSI(A, B, C)$.

Begin

Step 0. $TQSI := \emptyset$

Step 1. *Generate*

$$E : e_2(u)v^2 + e_1(u)v + e_0(u) = 0,$$

$$F : f_2(u)v^2 + f_1(u)v + f_0(u) = 0. \tag{8}$$

If $\frac{E}{F} = constant$ **Do**

D:= E

Go to ♠

End If *Go to Step 2.*

Step 2. *Compute* $G = res(E, F, v)$.

If $G \equiv 0$ **Do**

go to Step 3

End If

go to Step 4.

Step 3. *(This step is used to handle the situation where* $G \equiv 0$.*)*

Compute $D(u, v) = GCD(E, F)$

♠ *Use Remark 1 to generate a parameterization* $\mathbf{p}(t)$ *of the curve(s) and the isolated points* $\{p_1, \cdots, p_k\}$ *in* $TQSI(A, B, C)$ *corresponding to* $D(u, v) = 0$ *(this set could be empty)*

$TQSI := TQSI \cup \{p(t)\} \cup \{p_1, \cdots, p_k\}$

$E := \frac{E}{D}$; $F := \frac{F}{D}$

Go to Step 5

Step 4. *(This step is used to handle the situation where* $G \not\equiv 0$.*)*

If *there exits* $u_0 \in \mathbf{R}$, *such that* $\sum_{i=0}^{2} e_i^2(u_0) + \sum_{i=0}^{2} f_i^2(u_0) = 0$ **Do**

Compute $s(u) = GCD(e_2, e_1, e_0, f_2, f_1, f_0)$

Generate a parameterization $\mathbf{p}(t)$ *of the curve in* $QSIC(A, B)$ *corresponding to* $s(u) = 0$.

$TQSI := TQSI \cup \{p(t)\}$

$E := \frac{E}{s}$; $F := \frac{F}{s}$.

Calculate $G = res(E, F, v)$

End If

Go to Step 5.

Step 5. *Find the real roots of $G(u) = 0$, by numerical method [14]. Denote them as $u_1, \cdots, u_m$.*

Denote common real roots of $e_2(u_i)v^2 + e_1(u_i)v + e_0(u_i) = 0, f_2(u_i)v^2 + f_1(u_i)v + f_0(u_i) = 0$ as $v_{i,1}, \cdots, v_{i,k_i}, k_i \le 2$.

$TQSI := TQSI \cup \{\mathbf{q}(u_i, v_{i,j}), i = 1, \cdots, m, j = 1, \cdots, k_i\}$.

End

Remark 5 *(1) By Theorem 6 and Remark 1, when $TQSI(A, B, C)$ has infinite real points, to find the isolated real points , we only need to solve a equation with degree ≤ 4.*

(2) In Step 5, the common real roots of $e_2(u_i)v^2+e_1(u_i)v+e_0(u_i) = 0, f_2(u_i)v^2+f_1(u_i)v+f_0(u_i) = 0$ need to be calculated. When $e_2(u_i) \ne 0$, the system can be reduced to $e_2(u_i)v^2 + e_1(u_i)v + e_0(u_i) = 0, (e_2(u_i)f_1(u_i) - e_1(u_i)f_2(u_i))v + e_2(u_i)f_0(u_i) - e_0(u_i)f_2(u_i) = 0$. Hence, when $e_2(u_i)f_1(u_i) - e_1(u_i)f_2(u_i) \ne 0$, the common real root is $-\frac{e_2(u_i)f_0(u_i)-e_0(u_i)f_2(u_i)}{e_2(u_i)f_1(u_i)-e_1(u_i)f_2(u_i)}$.

(3) If $G \not\equiv 0$ and $TQSI(A, B, C)$ has finite points, GCD computation is not needed.

We shall use some examples to demonstrate the major steps of our algorithm in a variety of cases.

Example 6 $\mathcal{A}$:$x^2+y^2+z^2-1 = 0$; $\mathcal{B}$:$x^2+(y-0.5)^2-0.25 = 0$; $\mathcal{C}$:$y^2+xz+x^2+2x-1 = 0$; *$\mathcal{R} = \mathcal{A} - \mathcal{B}$ can be parameterized as $(x, y, z)^t = (v, 1-u^2, u)^t$. Then we have*

$$E : v^2 + (1-u^2)^2 + u^2 - 1 = 0; F : v^2 + v(2+u) + (1-u^2)^2 - 1 = 0.$$

$G = res(E,F,v) = -4u^2 - 4u^3 + 4u^4 + 4u^5 + u^6$. *By Sturm Theorem, the number of real roots of* $(2+u)^2 + u^4$ *is 0. Hence, the number of real points in* $TQSI(A,B,C)$ *is equal to the number of real roots of* $G(u) = 0$. *By Sturm's Theorem, the number of real roots of* $G(u) = 0$ *is equal to* 3. *Hence, the number of real points in* $TQSI(A,B,C)$ *is equal to* 3. *By solving* $G(u) = 0$, *the three real points in* $TQSI(A,B,C)$ *are* $(0,1,0), (0.489856, 0.39973, -0.77473)$, *and* $(0.304295, 0.103257, 0.946965)$ *respectively (see Fig.1).*

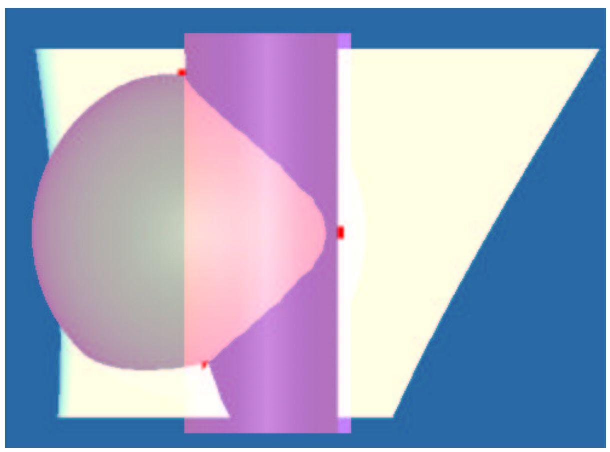

Fig.1.

Example 7 $A = x^2 + y^2 - 1$; $B = 2x^2 + z^2 - 1$; $C = x^2 + z^2 - 1$. *The pencil generated by* A *and* B *contains a hyperbolic cylinder* $\mathcal{R} : 2A - B = 2y^2 - z^2 - 1 = 0$. $\mathcal{R}$ *can be parameterized as* $(\frac{v}{u}, \frac{1}{2\sqrt{2}}(u + \frac{1}{u}), \frac{1}{2}(u - \frac{1}{u}))$. *We have*

$$E : v^2 + \frac{1}{8}(u^4 + 2u^2 + 1) - u^2 = 0; F : v^2 + \frac{1}{4}(u^4 - 2u^2 + 1) - u^2 = 0.$$

Obviously, $q(u) = \frac{u^4}{8} - \frac{3}{4}u^2 + \frac{1}{8}$ *and* $G = ((4 - 24u^2 + 4u^4))^2/1024$. *Solving* $q(u) = 0$ *yields* $u_1 = 1 + \sqrt{2}, u_2 = 1 - \sqrt{2}, u_3 = -1 + \sqrt{2}, u_4 = -1 - \sqrt{2}$. G *is divisible by* $(u - u_i)^2, i = 1, \cdots, 4$. *Substituting* u_i *into* E, *we find that* $TQSI(A,B,C)$ *has four* *different real points. By solving* $G(u) = 0$*, the four real points in* $TQSI(A,B,C)$ *are* $(0,1,-1),(0,-1,1),(0,1,1)$*, and* $(0,-1,-1)$ *respectively (See Fig.2).*

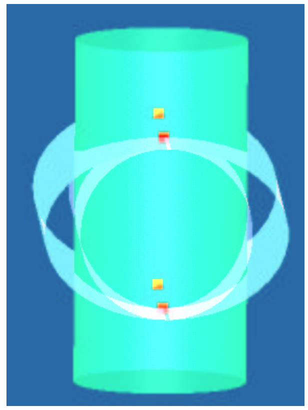

Fig.2.

Example 8 $A = x^2 - y^2 - z = 0$*,* $B = y^2 + z^2 - x = 0$*, and* $C = xz + z^2 - x^2 - x + yz = 0$*.* A *is a hyperbolic paraboloid and can be parameterization as* $((u+v)/2, (u-v)/2, uv)$*. We have,*

$E : (u^2+1/4)v^2+v(-u/2-1/2)+u^2/4-u/2 = 0; F : (u^2-1/4)v^2+v(u^2-u/2-1/2)-u^2/4-u/2 = 0;$

$$G = res(E,F,v) = 1/256((32u^2 + 64u^3 + 16u^4 - 48u^6 - 192u^7 + 128u^8)).$$

By Sturm Theorem, the number of real roots of $((u^2+1/4)(u^2-u/2-1/2)+(u/2+1/2)(u^2-1/4))^2+((u^2+1/4)(-u^2/4-u/2)-(u^2/4-u/2)(u^2-1/4))^2 = 0$ *is 0. By Sturm Theorem, the number of real roots of* $G = 0$ *is equal to* 3*. Hence,* $TQSI(A,B,C)$ *have three different real points. By solving* $G(u) = 0$*, the three points in* $TQSI(A,B,C)$ *are* $(0,0,0),(1,0,1)$*, and* $(1.0828, 0.5216, 0.9004)$ *respectively (See Fig.3).*

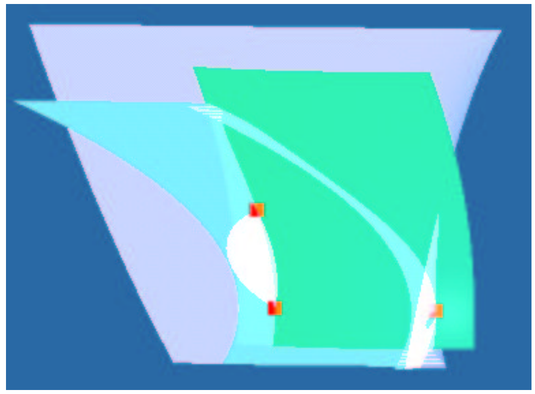

Fig.3.

Example 9 *$A = y + xz = 0, B = y^2 + yx + z = 0,$ and $C = y^2 + yz + z = 0.$ A can be parameterized as $x = u$, $y = -uv$, and $z = v$. Then*

$$E : (u^2 - u)v^2 + u = 0; F : u^2v^2 - u^2v + u = 0.$$

By Sturm theorem, $(u^2 - u)^2 + u^2 + (u^2)^2 + (u^2)^2 + u^2 = 0$ has real roots. Hence, $TQSI(A, B, C)$ have infinite real points. Obviously, $u = GCD(u^2 - u, u, u^2, u)$. Let $E := E/u, F := F/u$. $G = res(E, F, v) = u^3 - u^2 + 1$. has a real root. The number of isolated real points in $TQSI(A, B, C)$ is equal to 1.

By Algorithm 2, the parameterization of the curve in $TQSI(A, B, C)$ is $(t, 0, 0)$. By solving $u^3 - u^2 + 1 = 0$, the isolated real point is $(-0.7548, 0.5698, -0.7549)$ (See Fig.4).

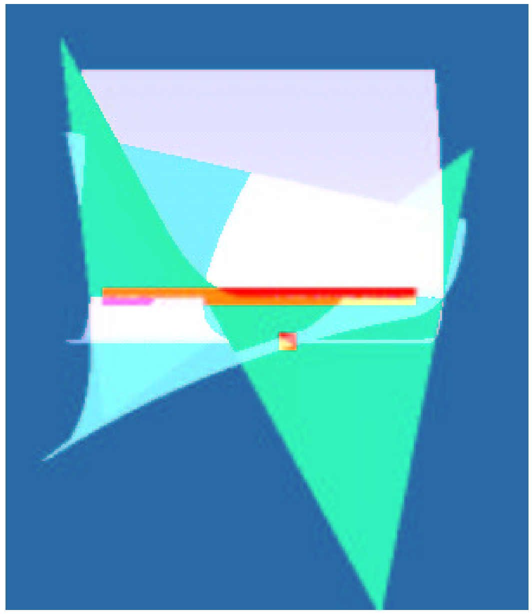

Fig.4.

We implemented our algorithms and performed numerical experiments on a computer with Intel CPU 1.7G and 256M memory. The performance of the algorithm is gratifying. For 200 3-quadric random configurations involving 532 intersections, 532 intersections are found. The total CPU time is 51S. For the intersection point X_0, we use $d(X_0) = \max(|X_0 A X_0^T|, |X_0 B X_0^T|, |X_0 C X_0^T|)$ to measure whether X_0 is accurate. Among 532 intersections, the numbers of intersections satisfying $d(X_0) > 10^{-7}$ and $d(X_0) > 10^{-6}$ are 21 and 5 respectively.

5 Summary

To overcome some weakness of existing algorithms, we present an efficient and robust method for computing the intersections of three quadratic surfaces. Numerical experiments showed that our method is significantly more stable than the existing algorithms. This method can also be used to count the number of isolated intersections of three quadratic surfaces even when there are infinitely many intersections. Moreover, the conditions under which the intersections are finite and infinite are thoroughly investigated.

It seems to be impossible to obtain the intersections without error. But in CAD field, a special class of quadrics, **natural quadrics** –planes, spheres, circular cones, and right circular cylinders, are frequently used. There is plenty of literature which discusses the computation of the intersection curve of two natural quadrics by geometric approach [5, 6, 7, 8, 9]. A future research on how to obtain the intersections of three natural quadrics without error by geometric approach is desired. In [3], the authors showed how to obtain parametric intersection curves that are near-optimal in the number and depth of radicals involved. Improving the accuracy of TQSI by using an idea similar to the one proposed in [3] is also desired.

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