Discrete Truncated Powers And Lattice Points In Rational Polytope *

Ren-hong Wang ,[†] Zhi-qiang Xu[‡]

Institute of Mathematical Sciences , Dalian University of Technology, Dalian 116024, China

Abstract

Discrete truncate power is very useful for studying the number of nonnegative integer solutions of linear Diophantine equations. In this paper, some detail information about discrete truncated power is presented. To study the number of integer solutions of linear Diophantine inequations, the generalized truncated power and generalized discrete truncated power are defined and discussed respectively. We use generalized discrete truncated powers and multivariate splines to investigate the lattice points in rational polytopes. In particular, we present the degree and period of multidimensional Ehrhart polynomial.

KeyWords: Discrete Truncated Power, Rational Polytope, Multivariate Spline

1. Introduction

Let M be an $s \times n$ integer matrix with columns $m_1, \cdots, m_n \in \mathbf{Z}^s \setminus \{0\}$ such that the only solution of the equation $M\mathbf{x} = 0, \mathbf{x} \in \mathbf{R}_+^n$ is the zero vector, where $\mathbf{R}_+$ denotes the non-negative real. Consequently, if we denote the cardinality of A by $\#A$ we see that

$$t(\alpha|M) = \#\{\beta \in \mathbf{Z}_+^n : M\beta = \alpha\}, \tag{1}$$

where $\mathbf{Z}_+$ denotes non-negative integer. The $t(\alpha|M)$ is called as discrete truncated power. Discrete truncated powers were first introduced by Dahmen and Micchelli in [3]. In [4], Dahmen and Micchelli showed the piecewise structure of discrete truncated power. Moreover,they exploited the relationship of $t(\alpha|M)$ to multivariate splines and established detailed information about $t(\alpha|M)$. In particular, they reproved and extended certain results of Stanley about magic squares[13]. Following their approach, Jia in [10] solved a conjecture of Stanley about symmetric magic squares. Furthermore, the discrete truncated power is also closely related to some classical problems in combinatorics, such as, the volume of polytope, Frobenius problem, and Dedekind sums etc. Hence, the discrete truncated power is not only important for multivariate spline but also useful for combinatorics.

In this paper, more detail information about $t(\alpha|M)$ is presented. To study the number of integer solutions of linear Diophantine inequations, the generalized truncated powers

*Project is supported by The National Natural Science Foundation of China.

†renhong@dlut.edu.cn

‡zhiqiang@student.dlut.edu.cn

and generalized discrete truncated powers are defined and discussed. In special, we show the piecewise structure of the generalized truncated power and generalized discrete truncated power. The number of lattice points inside a convex lattice polytope in $\mathbf{R}^{\mathbf{n}}$ has been studied intensively by combinatorialist, algebraic geometers, number theorists, Fourier analysts, and differential geometers([8],[12],[14]). Hence, we use generalized discrete truncated power and multivariate spline to investigate the lattice points in rational polytopes and reprove some classical results about lattice points in rational polytopes. Some new results are also presented.

Recall some notations(cf.[10]). We shall adopt the terminology of multiset theory(see [13]). Intuitively, a multiset is a set with possible repeated elements; for instance $\{2, 2, 3, 5, 5\}$. Let A and B be two subsets of $\mathbf{R}^m$ and $c \in \mathbf{R}$. Then $A - B$ is the set of all elements of the form $a - b$, where $a \in A$ and $b \in B$. The sets $A + B$ and cA are defined in a similar way. The set $A \backslash B$ is the complement of B in A. A subset Ω of $\mathbf{R}^m$ is called a cone if $\Omega + \Omega \subseteq \Omega$ and $c\Omega \subseteq \Omega$ for all $c > 0$. If a cone is also an open set, then we call it an open cone. Let Y be an $s \times n$ matrix. Y can be considered as a multiset of elements of $\mathbf{R}^s$. The linear span of Y, denoted by $span(Y)$, is the set $\{\sum_{y \in Y} a_y y : a_y \in \mathbf{R}$ for all $y\}$. The cone spanned by Y, denoted by $cone(Y)$, is the set $\{\sum_{y \in Y} a_y y : a_y \geq 0$ for all $y\}$. Let X be also a multiset of elements of $\mathbf{R}^m$. As usual, the convex hull of Y, denoted by $conv(Y)$, is the set $\{\sum_{y \in Y} a_y y : a_y \geq 0$ for all y and $\sum_{y \in Y} a_y = 1\}$. The generalized cone spanned by (X, Y), denoted by $gcone(X, Y)$, is the set $span(X) + cone(Y)$. The generalized convex hull of (X, Y), denoted by $gconv(X, Y)$, is the set $\{\sum_{x \in X} a_x x + \sum_{y \in Y} a_y y : a_x \in \mathbf{R}, a_y \in \mathbf{R}_+$ for all x, y and $\sum_{x \in X} |a_x| + \sum_{y \in Y} a_y = 1\}$.

We shall use the standard multiindex notation. Specifically, an element $\alpha \in \mathbf{N}^m$ is called an $m-$index, and $|\alpha|$ is called the length of α. Define $z^\alpha := z_1^{\alpha_1} \cdots z_m^{\alpha_m}$ for $z = (z_1, \cdots, z_m) \in \mathbf{C}^m$ and $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbf{N}^m$. Given $k \in \mathbf{Z}$, we denote by $\mathbf{P}_k = \mathbf{P}_k(\mathbf{R}^s)$ the linear space of polynomials of degree $\leq k$. If k is a negative integer, then we interpret $\mathbf{P}_k$ as the trivial linear space $\{0\}$.

2. Multivariate Truncated Powers And Generalized Multivariate Truncated Powers

Let M be an $s \times n$ real matrix. Recall that M is also viewed as the multiset of its column vectors. Throughout this section we assume that the convex hull of M does not contain the origin. The multivariate truncated power $T(\cdot|M)$ associated with M was introduced by Dahemn [2]. The $T(\cdot|M)$ is the distribution given the rule

$$T(\cdot|M) : \phi \mapsto \int_{\mathbf{R}_+^n} \phi(Mu)du, \phi \in \mathcal{D}(\mathbf{R}^s) \quad (2)$$

where $\mathcal{D}(\mathbf{R}^s)$ is the space of test functions on $\mathbf{R}^s$,i.e., the space of all compactly supported and infinitely differentiable functions on $\mathbf{R}^s$. From (2), we see that the support of $T(\cdot|M)$ is $cone(M)$. For more detailed information about $T(\cdot|M)$, the reader is referred to [9],[2].

Let $\mathcal{Y}(M)$ denote the set consisting of those submultisets Y of M for which $M\backslash Y$ does not span $\mathbf{R}^s$. Let $D(M)$ denote the linear space of those infinitely differentiable complex-valued functions f on $\mathbf{R}^s$ which satisfy the following system of linear partial differential equations: $D_Y f = 0, Y \in \mathcal{Y}(M)$. It was proved in [9] and [5] that $D(M) \subseteq \mathbf{P}_{\#M-s}$.

Let the second set $c(M)$ be the union of span$(M\setminus Y)$ where Y runs over $\mathcal{Y}(M)$. Denote by $cone^\circ(M)$ the relative interior of $cone(M)$. According to [4], a connected component of $cone^\circ(M)\setminus c(M)$ is called to be a fundamental $M-$cone. The following theorem describes the piecewise structure of the truncated power (cf [10]).

Theorem 1 *Let M be an $s \times n$ real matrix with $rankM = s \leq n$. Suppose the convex hull of M does not contain* 0. *Then $T(\cdot|M)$ agrees with some homogeneous polynomial of degree $n-s$ in $D(M)$ on each fundamental $M-$cone. Moreover, $T(\cdot|M)$ is continuous and positive on $cone^\circ(M)$.*

Let M_1 and M_2 be $s \times n_1$ and $s \times n_2$ real matrixes respectively . Assume that $gconv(M_1, M_2)$ does not contain the origin. By the hypothesis, it is clear that $n_1 < s$. The generalized multivariate truncated power $GT(\cdot|M_1, M_2)$ associate (M_1, M_2) is the distribution given by the rule

$$GT(\cdot|M_1, M_2) : \phi \mapsto \int_{\mathbf{R}^{n_1}} \int_{\mathbf{R}_+^{n_2}} \phi(M_1u_1 + M_2u_2)du_2du_1, \phi \in \mathcal{D}(\mathbf{R}^s). \quad (3)$$

The generalized multivariate truncated power is useful for studying the lattice points in polytope. Hence, we will study its some properties. According to (3), it is easy for proving:

$$GT(\cdot|M_1, M_2) = \sum_{Y\subseteq M_1} T(\cdot|M_{12}(Y)), \quad (4)$$

where M_{12} is an $s \times (n_1 + n_2)$ matrix and $M_{12} = (M_1, M_2) = (m_1, m_2, \cdots, m_{n_1+n_2})$, $M_{12}(Y) = (\overline{m}_1, \cdots, \overline{m}_{n_1+n_2})$, if $m_i \in Y, \overline{m}_i = -m_i$, otherwise, $\overline{m}_i = m_i$.

By (4) and properties of multivariate truncated power,the following properties of generalized multivariate truncated power can be proved easily. Hence, we omit their proofs.

If M_{12} is an $s \times s$ invertible matrix, $GT(\cdot|M_1, M_2)$ agrees with the function on $\mathbf{R}^s$ which takes value $\frac{1}{det(M_{12})}$ on $gcone(M_1, M_2)$ and 0 elsewhere. Let M_{12} be an $s \times n$ real matrix with $rank(M_{12}) = s \le n$. If $n > s$, then there exists an element y of M_{12} such that $M_{12} \setminus y$ still spans $\mathbf{R}^s$. The following recurrence relation will be very useful:

$$GT(x|M_1, M_2) = \int_0^{\infty} GT(x - sy|M_1, M_2 \setminus y)ds, x \in \mathbf{R}^s, y \in M_2, \tag{5}$$

$$GT(x|M_1, M_2) = \int_{-\infty}^{\infty} GT(x - sy|M_1, M_2 \setminus y)ds, x \in \mathbf{R}^s, y \in M_1. \tag{6}$$

For $y \in M_1$,

$$D_y GT(\cdot|M_1, M_2) = 0. \tag{7}$$

For $y \in M_2$,

$$D_y GT(\cdot|M_1, M_2) = GT(\cdot|M_1, M_2 \setminus y). \tag{8}$$

More generally

$$D_Y GT(\cdot|M_1, M_2) = GT(\cdot|M_1, M_2 \setminus Y), \tag{9}$$

where Y is a submultiset of M_2 and $D_Y := \prod_{y \in Y} D_y$. From (3), we see that the support of $GT(\cdot|M_1, M_2)$ is $gcone(M_1, M_2)$. Denote by $gcone^{\circ}(M_1, M_2)$ the relative interior of $gcone(M_1, M_2)$. Similarly we have

Theorem 2 *Let M_1 and M_2 be $s \times n_1$ and $s \times n_2$ real matrix respectively. Let $M_{12} = (M_1, M_2)$ with rank $M_{12} = s \le n = n_1 + n_2$. Suppose $gconv(M_1, M_2)$ does not contain* 0*. Then $GT(\cdot|M_1, M_2)$ is locally integrable and is a homogeneous function of degree $n - s$. Moreover, $GT(\cdot|M_1, M_2)$ is continuous and positive on $gcone^{\circ}(M_1, M_2)$.*

By (4) and some properties of truncated powers, the theorem can be proved easily.

3. Discrete Truncated Powers

Let $M = \{m_1, \cdots, m_n\}$ be a multiset of integer vectors in $\mathbf{R}^s$ such that $conv(M)$ does not contain the origin. Denote by S the linear space of all sequences on $\mathbf{Z}^m$ over the field $\mathbf{C}$. Given $y \in \mathbf{Z}^s$, the backward difference operator ∇_y is defined by $\nabla_y f := f - f(\cdot - y), f \in S$. More generally, for a multiset Y of integer vectors, we define $\nabla_Y := \prod_{y \in Y} \nabla_y$. The following difference formula was given in [4]: $\nabla_Y t(\cdot|M) = t(\cdot|M \setminus Y), Y \subseteq M$.

Consider the following system of linear partial difference equations for $f \in S : \nabla_Y f = 0,$ for all $Y \in \mathcal{Y}(M)$. The solutions of this difference equations system form a linear subspace of S which is denoted by $\nabla(M)$. Given $\theta = (\theta_1, \cdots, \theta_s) \in (\mathbf{C} \setminus 0)^s$. Consider the set $M_\theta := \{y \in M : \theta^y = 1\}$. Let $A(M) := \{\theta \in (\mathbf{C} \setminus \{0\})^s : span(M_\theta) = \mathbf{R}^s\}$. As pointed out in [6] every $z \in A(M)$ has the form

$$z = (e^{2\pi i \alpha_1^j / |detY|}, \cdots, e^{2\pi i \alpha_s^j / |detY|}) \tag{10}$$

for some $Y \in \mathcal{B}(M)$ and some integer $j, 1 \leq j \leq k = |detY|$, where $\mathcal{B}(M) = \{Y \subseteq M : \#Y = s, span(Y) = \mathbf{R}^s\}$. The vectors $\alpha^1, \cdots, \alpha^k$ are in $\mathbf{Z}^s$ and given by the equations $Y^T\alpha^j = |detY|\mu^j, j = 1, \cdots, k$, where $\mu^1, \cdots, \mu^k$ are lattice points in the parallelepiped determined by Y^T, that is $\{\mu^1, \cdots, \mu^k\} = \{Y^T v : v \in [0,1)^s\} \cap \mathbf{Z}^s$. In [6], Dahmen and Micchelli presented the following theorem.

Theorem 3 *A sequence $f \in \nabla(M)$ if and only if it has the form $f(\alpha) = \sum_{\theta \in A(M)} \theta^\alpha p_\theta(\alpha), \alpha \in \mathbf{Z}^s$, where p_θ is a polynomial in $D(M_\theta)$ for each $\theta \in A(M)$.*

Let M be a multiset of integer $m-$vectors $m_1, \cdots, m_n$. Denote by $[[M]]$the zonotope spanned by $m_1, \cdots, m_n$: $[[M]] := \{\sum_{j=1}^n a_j m_j : 0 \leq a_j \leq 1 \forall j\}$. For a subset Ω of $\mathbf{R}^s$, we set $v(\Omega|M) := \mathbf{Z}^s \cap (\Omega - [[M]])$. Dahmen and Micchelli proved the following important result in[4].

Theorem 4 *Let $M = \{m_1, \cdots, m_n\}$ be a multiset of integer vectors in $\mathbf{R}^s$ such that M spans $\mathbf{R}^s$ and the convex hull of M does not contain the origin. Then for any fundamental $M-$cone Ω, there exists a unique element $f_\Omega(\alpha|M) \in \nabla(M)$ such that $f_\Omega(\alpha|M)$ agrees with $t(\alpha|M)$ on $v(\Omega|M)$. Moreover, $f_\Omega(\cdot|M)$ has the following properties:*

$$f_\Omega(\alpha|M) = (-1)^{n-s} f_\Omega(-\alpha - \sum_{j=1}^n m_j|M), \alpha \in \mathbf{Z}^s. \tag{11}$$

In [11], Jia extended the result to be the following theorem:

Theorem 5 *Let Ω be a nonempty connected subset of $\mathbf{R}^s$, and M a multiset of integer vectors in $\mathbf{R}^s$ such that M spans $\mathbf{R}^s$ and the convex hull of M does not contain the origin. Let g be a sequence on $v(\Omega|M)$ satisfying the condition that for every $Y \in \mathcal{Y}(M)$,*

$$\nabla_Y g(\alpha) = 0, \text{ for all } \alpha \in v(\Omega|M \setminus Y).$$

Then there exists a unique element $f \in \nabla(M)$ such that f agrees with g on $v(\Omega|M)$.

Given $\theta \in (\mathbf{C} \setminus 0)^s$. Let $p_\theta \in \mathbf{P}(\mathbf{R}^s)$. We denote by $\theta^{()}p_\theta$ the sequence given by $\alpha \rightarrow \theta^\alpha p_\theta(\alpha), \alpha \in \mathbf{Z}^s$. Let E denote the linear space of all sequences of the form $f = \sum_\theta \theta^{()} p_\theta$. It is easily seen that every $f \in E$ can be written uniquely in the form $\sum_\theta \theta^{()} p_\theta$. The mapping from E to $\mathbf{P}(\mathbf{R}^s)$ given by $f \rightarrow p_\theta$ is denoted by P_θ, i.e. $P_\theta f = p_\theta$.

By Theorem 3 and 4, $\forall \theta \in A(M), \exists p_\theta(\alpha) \in D(M_\theta)$ such that

$$f_\Omega(\alpha|M) = \sum_{\theta \in A(M)} \theta^\alpha p_\theta(\alpha), \alpha \in \mathbf{Z}^s.$$

According to the definition of P_θ, $P_\theta f_\Omega(\alpha|M) = p_\theta(\alpha)$. Obviously, $e \in A(M)$, where $e = (1, 1, \cdots, 1) \in \mathbf{Z}^s$. In [4], Dahmen and Micchelli proved the following theorem.

Theorem 6 *The leading homogeneous terms of $P_e f_\Omega(\cdot|M)$ agrees on Ω with $T(\cdot|M)$.*

In [10], Jia gave an alternative proof of Theorem 6. Theorem 6 is useful for determining the degree of $f_\Omega(\cdot|M)$. In the following theorem, we give the leading homogenous terms of $P_\theta f_\Omega(\cdot|M), \forall\theta$.

Theorem 7 *The leading part of $P_\theta f_\Omega(\cdot|M)$ agrees with $\prod_{w\in M\setminus M_\theta} \frac{1}{1-\theta^{-w}} T(\cdot|M_\theta)$ on Ω, where $\theta \in A(M)$.*

Proof: The proof proceeds by induction on $\#M_\theta$.

(1)Firstly, by induction on $\#M$ we prove $P_\theta f_\Omega(\cdot|M) = \prod_{w\in M\setminus M_\theta} \frac{1}{1-\theta^{-w}} T(\cdot|M_\theta)$ when $\#M_\theta = s$. When $\#M = s$, $t(\alpha|M) = 1$ if $a = M\beta$ for some $\beta \in \mathbf{N}^s$, and 0 otherwise. Moreover, by [10],the only fundamental $M-$cone is $cone^\circ(M)$ and $\Omega - [[M]] \supseteq \overline{\Omega} = cone(M)$. Observing that $cone(M)$ is the disjoint union of $M[0,1)^s + M\beta, \beta \in \mathbf{N}^s$, we have

$$\sum_{\gamma\in\mathbf{Z}^s\cap M[0,1)^s} t(\alpha-\gamma|M) = 1 \text{ for all } \alpha \in \mathbf{Z}^s \cap cone(M).$$

According to the definition of $A(M)$, $\theta^{\gamma-\alpha} = 1$, when $t(\alpha-\gamma|M) \neq 0$.

Hence, we have

$$\sum_{\gamma\in\mathbf{Z}^s\cap M[0,1)^s} \theta^{\gamma-\alpha} t(\alpha-\gamma|M) = 1, \text{ for all } \alpha \in \mathbf{Z}^s \cap cone(M).$$

Let Λ_θ be an operator on the sequence space S given by

$$\Lambda_\theta f := \sum_{\gamma\in\mathbf{Z}^s\cap M[0,1)^s} (\theta^{-1})^{\cdot-\gamma} f(\cdot-\gamma), f \in S.$$

Since $f_\Omega(\cdot|M)$ agrees with $t(\cdot|M)$ on $v(\Omega|M) \supseteq \mathbf{Z}^s \cap cone(M)$, we have

$$\Lambda_\theta f_\Omega(\alpha) = \Lambda_\theta t(\cdot|M)(\alpha) = 1, \text{ for all } \alpha \in \mathbf{Z}^s \cap cone(M).$$

Since $\#M = s$, according to the definition of $A(M)$, $\#M_\theta = s$. Hence $P_\theta f_\Omega(\cdot|M)$ is a constant sequence. It is easy to see that $P_e \Lambda_\theta f_\Omega(\cdot|M) = \Lambda_e P_\theta f_\Omega(\cdot|M)$. Hence,we have

$$1 = P_e(\Lambda_\theta f_\Omega(\cdot|M)) = \Lambda_e P_\theta f_\Omega(\cdot|M).$$

By [7], $\#\mathbf{Z}^m \cap M[0,1)^s = det|M|$. Since $P_\theta f_\Omega(\cdot|M)$ is a constant sequence , we have

$$1 = \Lambda_e P_\theta f_\Omega(\cdot|M) = |det(M)| P_\theta f_\Omega(\cdot|M).$$

Hence, $P_\theta f_\Omega(\cdot|M) = 1/|det(M)| = 1/|det(M_\theta)|$, which agrees with $T(\cdot|M_\theta)$ on Ω. Because when $\#M = s$, $M \setminus M_\theta = \emptyset$, this completes the proof for the case $\#M = s$. Consider the case $\#M > s$. Obviously, $\forall w \in M\setminus M_\theta, P_\theta \nabla_w f_\Omega(\alpha|M) = P_\theta f_\Omega(\alpha|M) - \theta^{-w} P_\theta f_\Omega(\alpha-w|M)$.

Since $\#M_\theta = s$, $P_\theta f_\Omega(\cdot|M)$ is a constant sequence, i.e. $P_\theta f_\Omega(\cdot|M) = P_\theta f_\Omega(\cdot - w|M)$. Hence, $P_\theta f_\Omega(\alpha|M) = \frac{1}{1-\theta^{-w}} P_\theta \nabla_w f_\Omega(\alpha|M)$, for all $\alpha \in \Omega$. Similarly, we have

$$P_\theta f_\Omega(\alpha|M) = P_\theta \nabla_W f_\Omega(\alpha|M) \prod_{w\in W} \frac{1}{1-\theta^{-w}},$$

where $W = M\backslash M_\theta$. $\nabla_W f.(\alpha|M)$ agrees with $t(\cdot|M\backslash W)$ on $v(\Omega|M\backslash W)$. Since $M\backslash W = M_\theta$ and $P_\theta f_\Omega(\cdot|M_\theta) = T(\cdot|M_\theta)$, we have $P_\theta f_\Omega(\alpha|M) = T(\alpha|M_\theta)\prod_{w\in W} \frac{1}{1-\theta^{-w}}$, $\alpha \in \mathbf{Z}^s \cap \Omega$. This complete the proof for the case $\#M_\theta = s$.

(2) Consider the case $\#M_\theta > s$. Suppose the theorem holds for any M'_θ which satisfies $\#M'_\theta = \#M_\theta \backslash W'$, where W' is a subset of M_θ, and $span(M'_\theta) = \mathbf{R}^s$. Let $\prod_{w\in M\backslash M_\theta} \frac{1}{1-\theta^{-w}} F_\Omega$ be the polynomial in which $F_\Omega \in D(M_\theta)$ and F_Ω agrees with $T(\cdot|M_\theta)$ on Ω. Pick $w \in M_\theta$. If $M \setminus w$ does not span $\mathbf{R}^s$, then both $D_w F_\Omega$ and $\nabla_w f_\Omega(\cdot|M_\theta)$ vanish. If $M \setminus w$ spans $\mathbf{R}^s$, then $D_w F_\Omega$ agrees with $T(\cdot|M_\theta \setminus w)$ on Ω and $\nabla_w f_\Omega(\cdot|\Omega)$ agrees with $t(\cdot M \setminus w)$ on $v(\Omega|M \setminus w)$. By the induction hypothesis,

$$P_\theta(\nabla_w f_\Omega(\cdot|M)) - \prod_{w\in M\backslash M_\theta} \frac{1}{1-\theta^{-w}} D_w F_\Omega \in \mathbf{P}_{\#M_\theta - s - 2}.$$

Since $w \in M_\theta$, $P_\theta \nabla_w(f_\Omega(\cdot|M)) = \nabla_w P_\theta(f_\Omega(\cdot|M))$. Because of $P_\theta f_\Omega(\cdot|M) \in \mathbf{P}_{\#M_\theta - s}$, it is easy to see that

$$D_w(P_\theta f_\Omega(\cdot|M)) - \nabla_w(P_\theta f_\Omega(\cdot|M)) \in \mathbf{P}_{\#M_\theta - s - 2}.$$

Thus we have shown that for every $w \in M_\theta$,

$$\begin{aligned} & D_w(P_\theta f_\Omega(\cdot|M)) - \prod_{w\in M\backslash M_\theta} \frac{1}{1-\theta^{-w}} F_\Omega) \\ = \; & D_w(P_\theta f_\Omega(\cdot|M)) - \nabla_w(P_\theta f_\Omega(\cdot|M)) + \nabla_w(P_\theta f_\Omega(\cdot|M)) - \prod_{w\in M\backslash M_\theta} \frac{1}{1-\theta^{-w}} D_w F_\Omega \in \mathbf{P}_{\#M_\theta - s - 2}. \end{aligned}$$

Since $span M_\theta = \mathbf{R}^s$, we have

$$P_\theta f_\Omega(\cdot|M) - \prod_{w\in M\backslash M_\theta} \frac{1}{1-\theta^{-w}} F_\Omega \in \mathbf{P}_{\#M_\theta - s - 1}.$$

Hence, the leading part of $P_\theta f_\Omega(\cdot|M)$ agrees with $\prod_{w\in M\backslash M_\theta} \frac{1}{1-\theta^{-w}} T(\cdot|M_\theta)$ on Ω. □

In fact, Theorem 7 present the degree of $P_\theta f_\Omega$. By Theorem 7 and properties of multivariate truncated power, when $\Omega \subset cone(M_\theta)$, the degree of $P_\theta f_\Omega$ is $\#M_\theta - s$. If $\Omega \cap cone(M_\theta) = \emptyset$, $P_\theta f_\Omega \equiv 0$. The LEMMA 8.2 in [10] is a key lemma for proving the conjecture of stanley about symmetric magic squares. In fact, by Theorem 7, it is easy for proving the lemma.

4. Generalized Discrete Truncated Powers

Let M_1 and M_2 be $s\times n_1$ and $s\times n_2$ integer matrixes respectively . Assume $gconv(M_1, M_2)$ does not contain the origin. Let

$$gt(\alpha|M_1, M_2) = \#\{\beta = (\beta_1, \beta_2) : M_1\beta_1 + M_2\beta_2 = \alpha, \beta_1 \in \mathbf{Z}^{n_1}, \beta_2 \in \mathbf{Z}_+^{n_2}\}. \qquad (12)$$

$gt(\alpha|M_1, M_2)$ is called to be **generalized discrete truncated power**. Obviously,

$$gt(\alpha|M_1, E_{s\times s}) = \#\{\beta \in Z^{n_1} : M_1\beta \le \alpha\}, \qquad (13)$$

where $E_{s\times s}$ is an $s \times s$ identity matrix,and $gconv(M, E_{s\times s})$ does not contain origin point. Hence, the generalized discrete truncated power present the number of solutions of linear Diophantine inequations. According to (12), we have

$$gt(\cdot|M_1, M_2) = \sum_{Y\subseteq M_1} t(\cdot|M_1(Y) \cup M_2) - \sum_{Y\subseteq M, Y\neq\emptyset} t(\cdot|(M_1 \setminus Y) \cup M_2), \qquad (14)$$

where, $M_1 = (m_1, m_2, \cdots, m_{n_1})$, $M(Y) = (\overline{m}_1, \cdots, \overline{m}_{n_1})$, if $m_i \in Y, \overline{m}_i = -m_i$; otherwise, $\overline{m}_i = m_i$.

Let $M_{12} = M_1 \bigcup M_2$ and the set $\mathcal{Y}(M_1, M_2)$ consisting of those submultisets Y of M_2 for which $M_{12} \setminus Y$ does not span $\mathbf{R}^s$. The set $c(M_1, M_2)$ is the union of $span(M_{12} \setminus Y)$, where Y runs over $\mathcal{Y}(M_1, M_2)$. A connected component of $gcone^\circ(M_1, M_2) \setminus c(M_1, M_2)$, is called a generalized fundamental (M_1, M_2)-cone. According to the definition of generalized discrete truncated power, we have

$$\nabla_y gt(\cdot|M_1, M_2) = 0, \forall y \in M_1, \nabla_y gt(\cdot|M_1, M_2) = gt(\cdot|M_1, M_2 \setminus y), \forall y \in M_2. \qquad (15)$$

Hence, for $Y \in \mathcal{Y}(M_1 \cup M_2)$, if $Y \cap M_1 \neq \emptyset$, then $\nabla_Y gt(\alpha|M_1, M_2) = 0, \alpha \in \mathbf{Z}^s$. If $Y \cap M_1 = \emptyset$, then $\nabla_Y gt(\alpha|M_1, M_2) = 0, \alpha \in \Omega$, where Ω is a generalized fundamental $(M_1, M_2)-$ cone.

Suppose $M_1 = (m_1, \cdots, m_{n_1}), M_2 = (m_1', \cdots, m_{n_2}')$. Let

$$[[M_1, M_2]] := \{\sum_{j=1}^{n_1} a_j m_j + \sum_{j=1}^{n_2} a_j' m_j', -1 \le a_j \le 1, 0 \le a_j' \le 1, \forall j\}.$$

For a set $\Omega \in \mathbf{R}^s$, we set $gv(\Omega|M_1, M_2) := \mathbf{Z}^s \cap (\Omega - [[M_1, M_2]])$.

The following theorem shows the piecewise structure of $gt(\cdot|M_1, M_2)$.

Theorem 8 *Let M_1, M_2 be two multisets of integer vectors in $\mathbf{R}^s$ such that $gconv(M_1, M_2)$ does not contain the origin. Then for any generalized fundamental $(M_1, M_2)-$cone Ω, there exists a unique element $f_\Omega(\alpha|M_1, M_2) \in \nabla(M_{12})$ such that $f_\Omega(\alpha|M_1, M_2)$ agrees with $gt(\alpha|M_1, M_2)$ on $gv(\Omega|M_1, M_2)$, where $M_{12} = M_1 \cup M_2$.*

Let Ω be a generalized fundamental (M_1, M_2)-cone. By Theorem 5 and (15), there exists a unique element $f_\Omega \in \nabla(M_{12})$ such that f_Ω agrees with $gt(\cdot|M_1, M_2)$ on $v(\Omega|M_{12})$. According to the definition of $gt(\cdot|M_1, M_2)$, $\forall Y \subseteq M_1, gt(\cdot|M_1, M_2) \equiv gt(\cdot|M_1(Y), M_2)$. Hence, f_Ω agrees with $gt(\cdot|M_1, M_2)$ on $gv(\Omega|M_1, M_2)$.

The following theorem present the Reciprocity law of $gt(\cdot|M_1, M_2)$. The Reciprocity law of discrete truncated powers in Theorem 4 is a special case of the theorem.

Theorem 9 *If M_2 span $\mathbf{R}^s$, under the condition of Theorem 8,*

$$f_\Omega(\alpha|M_1, M_2) = (-1)^{n_1+n_2-s} f_\Omega(-\alpha - \sum_{m\in M_2} m|M_1, M_2), \alpha \in \mathbf{Z}^s.$$

Proof: According to the definition of $gt(\cdot|M_1, M_2)$ and $t(\cdot|M)$ we have

$$\begin{aligned} gt(\alpha|M_1, M_2) &= \sum_{Y\subseteq M_1} t(\alpha - \sum_{m\in M_1(Y)} m|M_1(Y) \cup M_2) \\ &+ \sum_{Y_1\subseteq M_1} \sum_{Y_2\subseteq M_1(Y_1), Y_2\neq\emptyset} t(\alpha - \sum_{m\in M_1(Y_1)\setminus Y_2} m|(M_1(Y_1)\setminus Y_2) \cup M_2), \alpha \in \mathbf{Z}^s. \end{aligned}$$

By Theorem 4 and Theorem 8, $f_\Omega(\alpha|M_1, M_2)$ agrees with $gt(\alpha|M_1, M_2)$ on $gv(\Omega|M_1, M_2)$ and $f_\Omega(\alpha|M_1 \cup M_2)$ agrees with $t(\alpha|M_1 \cup M_2)$ on $v(\Omega|M_1 \cup M_2)$.

Since $f_\Omega(\cdot|M_1, M_2)$ being polynomial,

$$\begin{aligned} f_\Omega(\alpha|M_1, M_2) &= \sum_{Y\subseteq M_1} f_\Omega(\alpha - \sum_{m\in M_1(Y)} m|M_1(Y) \cup M_2) \\ &+ \sum_{Y_1\subseteq M_1} \sum_{Y_2\subseteq M_1(Y_1), Y_2\neq\emptyset} f_\Omega(\alpha - \sum_{m\in M_1(Y_1)\setminus Y_2} m|(M_1(Y_1)\setminus Y_2) \cup M_2), \alpha \in \mathbf{Z}^s. \end{aligned}$$

Then

$$\begin{aligned} & f_\Omega(-\alpha - \sum_{m\in M_2} m|M_1, M_2) = \\ & \sum_{Y\subseteq M_1} f_\Omega(-\alpha - \sum_{m\in M_1(Y)} m - \sum_{m\in M_2} m|M_1(Y) \cup M_2) \\ + & \sum_{Y_1\subseteq M_1} \sum_{Y_2\subseteq M_1(Y_1), Y_2\neq\emptyset} f_\Omega(-\alpha - \sum_{m\in M_1(Y_1)\setminus Y_2} m - \sum_{m\in M_2} m|(M_1(Y_1)\setminus Y_2) \cup M_2). \end{aligned}$$

By Theorem 4 and the above formulation equals

$$(-1)^{n_1+n_2-s} \sum_{Y\subseteq M_1} f_\Omega(\alpha|M_1(Y)\cup M_2) + \sum_{Y_1\subseteq M_1} \sum_{Y_2\subseteq M_1(Y_1), Y_2\neq\emptyset} (-1)^{n_1+n_2-s-\#Y_2} f_\Omega(\alpha|(M_1(Y_1)\setminus Y_2)\cup M_2).$$

By the definition of $gt(\cdot|M_1, M_2)$ and the Inclusion-Exclusion Principle, the above formulation equals $(-1)^{n_1+n_2-s} f_\Omega(\alpha|M_1, M_2)$. Hence, the theorem holds. □

By (4) and (14), we have

Theorem 10 *Under the condition of Theorem 8, the leading part of* $P_e f_\Omega(\cdot|M_1, M_2)$ *agrees with* $GT(\cdot|M_1, M_2)$ *on* Ω.

5.Lattice Points Counting In Rational Polytope

In general, a quasipolynomial $Q(t)$ is an expression of the form $c_n(t)t^n + \cdots + c_1(t) + c_0(t)$, where $c_0, \cdots, c_n$ are periodic functions in t. The least common multiple of the periods of $c_0, \cdots, c_n$ is called the period of $Q(t)$. A quasipolynomial $Q(\mathbf{t})$ in the $d-$dimensional variable $\mathbf{t} = (t_1, \cdots, t_d)$ is the natural generalization of a quasipolynomial in a 1-dimensional variable: an expression of the form $\sum_{0\leq k_1,\cdots,k_d\leq n} c_{(k_1,\cdots,k_d)} t_1^{k_1} \cdots t_d^{k_d}$, where $c_{(k_1\cdots k_d)} = c_{(k_1\cdots k_d)}(t_1, \cdots, t_d)$ is periodice in $t_1, \cdots, t_d$. The least common multiple of the periods of $c_0, \cdots, c_n$ on t_k is called to be the period of $Q(\mathbf{t})$ on t_k. Recall that two polytopes are combinatorially equivalent if there exists a bijection between their faces that preserves the inclusion relation. Recall some notations(see [1]).

Definition 1 *Let the convex rational polytope* $\mathcal{P}$ *be given by* $\mathcal{P} = \{x \in \mathbf{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$, *with* $\mathbf{A} \in M_{m\times n}(\mathbf{Z}), \mathbf{b} \in \mathbf{Z}^m$. *Here the inequality is understood componentwise. For* $\mathbf{t} \in \mathbf{Z}^m$, *define the vector-dilated polytope* $\mathcal{P}^{(t)}$ *as* $\mathcal{P}^{(\mathbf{t})} = \{\mathbf{x} \in \mathbf{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{t}\}$.

For those $\mathbf{t}$ *for which* $P^{(\mathbf{t})}$ *is combinatorially equivalent to* $\mathcal{P} = \mathcal{P}^{(b)}$, *we define the number of lattice points in the interior and closure of* $\mathcal{P}^{(t)}$ *as*

$$i_{\mathcal{P}}(\mathbf{t}) = \#(\mathcal{P}^{(\mathbf{t})^\circ} \cap \mathbf{Z}^n), j_{\mathcal{P}}(\mathbf{t}) = \#(\overline{\mathcal{P}^{(\mathbf{t})}} \cap \mathbf{Z}^n),$$

respectively.

Definition 2 *Let* $\mathcal{P}$ *be a rational polytope. Write* $\mathcal{P} = \bigcup_{k=1}^r \mathcal{P}_k$, *where* $\mathcal{P}_k$ *are convex rational polytope, say,*

$$\mathcal{P}_k = \{\mathbf{x} \in \mathbf{R}^n : \mathbf{A}_k x \leq \mathbf{b}_k\},$$

with $\mathbf{b}_k \in \mathbf{Z}^{m_k}$. *Given* $\mathbf{t} \in \mathbf{Z}^m$, *where* $m = m_1 + \cdots m_r$, *combine the first* m_1 *components of* $\mathbf{t}$ *in a vector* $\mathbf{t}_1$, *the next* m_2 *components in* $\mathbf{t}_2$, *etc. Define the vector-dilated polytope* $\mathcal{P}^{(\mathbf{t})}$ *as*

$$\mathcal{P}^{(\mathbf{t})} = \bigcup_{k=1}^r \mathcal{P}_k^{(\mathbf{t}_k)}.$$

For those $\mathbf{t}$ *for which* $\mathcal{P}^{(\mathbf{t})}$ *is combinatorially equivalent to* $\mathcal{P}$, *we define as above*

$$i_{\mathcal{P}}(\mathbf{t}) = \#(\mathcal{P}^{(\mathbf{t})^\circ} \cap \mathbf{Z}^n), j_{\mathcal{P}}(\mathbf{t}) = \#(\overline{\mathcal{P}^{(\mathbf{t})}} \cap \mathbf{Z}^n).$$

In [1], M.Beck proved the following.

Theorem 11 *(Matthias Beck [1]) Suppose the rational polytope $\mathcal{P}$ is homeomorphic to an n-manifold. Then $i_{\mathcal{P}}(\mathbf{t})$ and $j_{\mathcal{P}}(\mathbf{t})$ are quasipolynomials in $\mathbf{t} \in \mathbf{Z}^m$, satisfying*

$$i_{\mathcal{P}}(-\mathbf{t}) = (-1)^n j_{\mathcal{P}}(\mathbf{t}).$$

Let the convex rational n-dimensional polytope $\mathcal{P}_{\mathbf{A}}$ be given by

$$\mathcal{P}_{\mathbf{A}} = \{\mathbf{x} \in \mathbf{R}^n : \mathbf{Ax} \le \mathbf{b}\},$$

with $\mathbf{A} \in M_{s\times n}$, $\mathbf{b} \in \mathbf{Z}^s$. Hence, $L(\overline{\mathcal{P}}_{\mathbf{A}}, t) = gt(t\mathbf{b}|\mathbf{A}, E_{s\times s}), L(\mathcal{P}^{\circ}_{\mathbf{A}}, t) = gt(t\mathbf{b}-e|\mathbf{A}, E_{s\times s})$, where $e = (1, \cdots, 1)^T \in \mathbf{Z}^s$. For reasons of simplicity, in the remaind the section, we only consider the rational convex polytope. Because of any rational polytope can be considered as the union of some convex rational polytopes, the similar results can be generalized to any rational polytope.

In fact, if both $\mathbf{t} \in \mathbf{Z}^s$ and $\mathbf{b}$ belong to the same generalized fundamental $(\mathbf{A}, E_{s\times s})-$cone Ω, $\mathcal{P}^{(\mathbf{t})}_{\mathbf{A}}$ is combinatorially equivalent to $\mathcal{P}_{\mathbf{A}}$. Using the properties of $gt(\cdot|\mathbf{A}, E_{s\times s})$, we can prove Theorem 11. Moreover, we have

Theorem 12 *Suppose the rational polytope $\mathcal{P}_{\mathbf{A}}$ is homeomorphic to an $n-$manifold. Then $i_{\mathcal{P}_{\mathbf{A}}}(\mathbf{t})$ and $j_{\mathcal{P}_{\mathbf{A}}}(\mathbf{t})$ are quasipolynomial in $\mathbf{t}$ whose degree is the dimension of $\mathcal{P}_{\mathbf{A}}$ and whose period is less than the least common multiple of $det(Y), Y \in \mathcal{B}(\mathbf{A} \cup E_{s\times s})$.*

Since $\mathbf{t} \in \Omega$, where Ω is a generalized fundamental $(\mathbf{A}, E_{s\times s})-$cone, the leader term of $i_{\mathcal{P}_{\mathbf{A}}}(\mathbf{t})$ is $GT(\mathbf{t}|\mathbf{A}, E_{s\times s})$. So the degree of $i_{\mathcal{P}_{\mathbf{A}}}(\mathbf{t})$ can be presented. The period of $i_{\mathcal{P}_{\mathbf{A}}}(\mathbf{t})$ depends on $\theta^{\mathbf{t}}$. By (10),$\theta^{\mathbf{t}} = e^{2\pi i \sum_{k=1}^{s} t_k \alpha^j_k/|detY|}$. Since $\alpha^j_k \in \mathbf{Z}$, the period of $i_{\mathcal{P}_{\mathbf{A}}}(\mathbf{t})$ is less than the least common multiple of $det(Y), Y \in \mathcal{B}(\mathbf{A} \cup E_{s\times s})$. Because of $j_{\mathcal{P}_{\mathbf{A}}}(\mathbf{t}) = i_{\mathcal{P}_{\mathbf{A}}}(\mathbf{t}-e)$, the same result can be generalized to $j_{\mathcal{P}_{\mathbf{A}}}(\mathbf{t})$.

References

[1] M.Beck,Multidimensional Ehrhart Reciprocity, J. Combin. Theory Ser. A 97, No.1(2002)187-194.

[2] W.Dahmen, On multivariate B-splines, SIAM J. Numer. Anal. 17(1980),179-191.

[3] W.Dahmen, and C. A. Micchelli, Recent progress in multivariate splines, in *Approximation Theory IV*(C. K. Chui, L. L. Schumaker, and J. Ward,Eds.), Academic, New York,(1983)27-121.

[4] W. Dahmen and C.A. Micchelli, The number of solutions to linear diophantine equations and multivariate splines, Trans. Amer. Math. Soc.308(1988)509-532.

[5] W. Dahmen and C.A. Micchelli, Translates of multivariate splines, linear Algebra Appl. 52/53(1983)217-234.

[6] W. Dahmen and C.A. Micchelli, On the solution of certain systems of partial difference equations and linear dependence of translates of box splines, Trans. Amer. Math. Soc. 292(1985)305-320.

[7] W.Dahmen, C.A. Micchelli,On the local linear independence of translate of a box spline, Studia Math. 82(1985)243-263.

[8] V.I. Danilov, The geometry of toric varieties, Russian Math. Surveys 33(1978),97-154.

[9] C. de Boor, K.Höllig, B-splines from parallelepipeds, J. Analyse Math,42(1982/83).

[10] R.Q. Jia, Symmetric Magic Squares and Multivariate Splines,Linear Algebra Appl.,250(1997)69-103.

[11] R.Q. Jia, Multivariate Discrete Splines And Linear Diophantine Equations, Trans. Amer. Math. Soc.304(1993)179-198.

[12] B. Randol, On the Fourier transform of the indicator function of a planar set, Trans. of the AMS(1969)271-278.

[13] R. Stanley, Enumerative Combinctorics, Vol.1, Wadsworth, Belmont, Calif.,1986.

[14] Y.-J. Xu and S.-T. Yau, A sharp estimate of the number of integral points in a tetrahedron, J. Reine Angew. Math. 412(1992)119-219.