Interpolation

September 22, 2009

1 Optimization of the Lagrange Operator

In this section, we concern with polynomial interpolation in one variable. The standard interval $[-1, 1]$ is adopted, and n is the number of interpolation nodes. For nodes $x_1, \ldots, x_n$ arranged so that $-1 \leq x_1 < x_2 < \cdots < x_n \leq 1$, we define

$$\begin{aligned} \ell_j(x) &= \prod_{i=1, i\neq j}^{n} (x - x_i)/(x_j - x_i), \\ Lf &= \sum_{j=1}^{n} f(x_j)\ell_j, \quad \Lambda = \sum_{j=1}^{n} |\ell_j|. \end{aligned}$$

For a linear transformation, T, acting between normed linear spaces, the *norm* of T is defined by the equation

$$\|T\| = \sup\{\|Tf\| : \|f\| \leq 1\}.$$

A linear map P acting on a linear space is called a *projection* if $P^2 = P$.

Theorem 1.1 (Kharshiladze-Lozinski). *Every linear projection $P : C[-1, 1] \to \Pi_{n-1}$ satisfies the inequality*

$$\|P\| \;\geq\; \frac{2}{\pi^2} \log(n-1) - \frac{1}{2}.$$

Since the Lagrange operator L, defined earlier, is a projection, it obeys the inequality in the theorem above. However, a stronger results is available.

Theorem 1.2 (Erdös, Brutman). *The Lagrange operator L obeys the inequality*

$$\|L\| \geq \frac{2}{\pi} \log n + 0.5212.$$

The Chebyshev polynomial is defined by $T_n(x) = \cos(n \cos^{-1} x)$. An easy calculation shows that its zeros are the points $\cos((2j-1)\pi/2n)$. These points can be used for interpolation and are often referred to as the *Chebyshev nodes.*

Theorem 1.3. *If L is the Lagrange interpolation operator for the Chebyshev nodes then*

$$0.9625 < \|L\| - \frac{2}{\pi}\log n < 1.$$

The Lebesgue function Λ has a relative maximum on every interval between adjacent nodes. There are two other local maxima at -1 and 1 if these points are not nodes. In 1932, Bernstein published an important paper, in which he conjectured the following theorem: If the interpolation problem is standardized so that the endpoints of the interval are nodes, then the minimum norm of the interpolation operator is achieved when and only when the local maxima in the Lebesgure function are all equal. This conjecture resisted all efforts to prove it for 44 years. In 1976, Theodore Kilgore succeeded in establishing Bernstein's conjecture and announced his result.

2 Discussion of interpolation

We shall be concerned with real-valued functions defined on a domain X. In the domain X a set of n distinct points is given:

$$\{x_1, \ldots, x_n\}.$$

These points are called *nodes*. For each node x_i, an ordinate $\lambda_i \in \mathbb{R}$ is given. The problem of *interpolation* is to find a suitable function $F : X \to \mathbb{R}$ that takes these prescribed n values. Suppose that U is this vector space and that a basis for U is $\{u_1, \ldots, u_n\}$. The function F that we seek have the form

$$F \;=\; \sum_{j=1}^{n} c_j u_j.$$

When the interpolation conditions are imposed on F, we obtain

$$\lambda_i = F(x_i) = \sum_{j=1}^{n} c_j u_j(x_i), \quad 1 \le i \le n.$$

The matrix

$$[u_i(x_j)]$$

is called the *interpolation matrix.* In order that our problem be solvable for any choice of ordinate λ_i, it is necessary and sufficient that the interpolation matrix be nonsingular.

Theorem 2.1. *Let U be an n-dimensional linear space of functions on X. Let $x_1, \ldots, x_n$ be n distinct nodes in X. In order that U be capable of interpolating arbitrary data at the nodes it is necessary and sufficient that zero data be interpolated only be the zero-element in U.*

An n-dimensional vector space U of functions on a domain X is said to be a *Haar space* if the only element of U which has more than $n-1$ roots in X is the zero element.

Theorem 2.2. *Let U have the basis $\{u_1, \dots, u_n\}$. These properties are equivalent:*

a. U is a Haar space

b. $\det(u_i(x_j)) \neq 0$ for any set of distinct points $x_1, \dots, x_n$ in X.

Any basis for a Haar space is called a *Chebyshev system*. Here are some examples of Chebyshev systems on $\mathbb{R}$:

1. $1, x, \dots, x^n$
2. $e^{\lambda_1 x}, \dots, e^{\lambda_n x} \qquad \lambda_1 < \dots < \lambda_n$.

Here are some Chebyshev systems on $(0, \infty)$

3. $x^{\lambda_1}, x^{\lambda_2} \dots, x^{\lambda_n} \qquad \lambda_1 < \dots < \lambda_n$
4. $(x+\lambda_1)^{-1}, \dots, (x+\lambda_n)^{-1} \qquad 0 \leq \lambda_1 < \dots < \lambda_n$.

Are there any Chebyshev systems of continuous functions on $\mathbb{R}^2$ and on the higher-dimensional Euclidean spaces? No, there is an immediate and absolute barrier:

Theorem 2.3. *On $\mathbb{R}^2, \mathbb{R}^3, \dots$ there are no Haar subspaces of continuous functions except one-dimensional ones.*

3 Bivariate Polynomial Interpolation

We set

$$\mathbb{P} := \mathrm{span}\{p_1(x,y), \dots, p_k(x,y)\}$$

where $p_1, \dots, p_k$ are linear independent. We suppose $q_1, \dots, q_k$ are distinct points in $\mathbb{R}^2$. The bivariate polynomial interpolation is to find a $p \in \mathbb{P}$ so that

$$p(q_i) \;=\; f(q_i), i = 1, \dots, k. \tag{3.1}$$

The Haar theorem implies that we need choose the points $q_1, \dots, q_k$ so that the solution of (3.1) is unique. We call these kind points as *poised nodes*.

Lemma 3.1. *The points $\{q_1, \dots, q_k\}$ are poised nodes for $\mathbb{P}$ if and only if they do not lie in an algebraic curve.*

Theorem 3.1 (Bezout's Theorem)**.** *If two curves of degree m and n have more than mn distinct points in common, then they have a common component.*

Theorem 3.2. *Suppose $q_1, \dots, q_k$ are poised nodes for $\mathbb{P}_n$ and $\ell(x,y)$ is an irreducible algebraic curve with degree $l = 1$ or $l = 2$ Suppose $\ell(q_j) \neq 0, j = 1, \dots, k$ and $p_1, \dots, p_{(n+3)l-1}$ are any $(n+3)l-1$ points in ℓ. Then $q_1, \dots, q_k, p_1, \dots, p_{(n+3)l-1}$ points are poised nodes for $\mathbb{P}_{n+l}$.*

According to Theorem 3.2, one can design algorithms for constructing posed nodes.

Remark 3.1. *Some stuff in this lecture is from* Ward Cheney and W. Light, A course in approximation theory, 2003.