Polynomial Interpolation

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1 Lagrange interpolation formula

A very general question in applied mathematics is how to reconstruct a function from incomplete information about it. Suppose $y = f(x)$ is a function about x and we know the value $f(x)$ at $n+1$ distinct points $x_0, \ldots, x_n$, i.e.,

$$y_i \;=\; f(x_i), \quad i = 0, \ldots, n.$$

Polynomial interpolation is to find a polynomial p so that

$$p(x_i) \;=\; f(x_i) \quad i = 0, \ldots, n.$$

Suppose that p is a polynomial with degree m

$$p(x) = a_0 + a_1 x + \cdots + a_m x^m, \quad a_m \neq 0.$$

The polynomial interpolation is to find $a_0, \ldots, a_m$ so that

$$p(x_i) \;=\; f(x_i) \quad i = 0, \ldots, n. \tag{1.1}$$

It is equivalent to solve the following linear equations:

$$\begin{aligned} a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_m x_0^n &= y_0 \\ a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_m x_1^n &= y_1 \\ &\vdots \\ a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_m x_n^n &= y_n. \end{aligned} \tag{1.2}$$

When $n = m$, the linear equations (1.2) is solvable. If we can find a polynomial ℓ_i with degree n satisfying

$$\ell_i(x_i) = 1 \;\text{ and }\; \ell_i(x_j) = 0, i \neq j$$

then the polynomial

$$p(x) \;=\; \sum_{i=0}^{n} y_i \ell_i(x).$$

satisfy (1.1).

In fact

$$\ell_i(x) = \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}.$$

Denote $\omega(x) = (x-x_0)\cdots(x-x_n)$. Then, ℓ_i can be written as

$$\ell_i(x) = \frac{\omega(x)}{(x-x_i)\omega'(x_i)}.$$

Then

$$p(x) = \sum_{i=0}^{n} y_i \frac{\omega(x)}{(x-x_i)\omega'(x_i)}$$

Example 1.1. *Suppose $f(-1) = 2, f(1) = 1$ and $f(2) = 1$. Calculation the Lagrange interpolation formula.*

2 Newton interpolation formula

We hope that write the interpolation polynomial $p_n(x)$ in the form of

$$p_n(x) = a_0 + a_1(x-x_0) + \cdots + a_n(x-x_0)\cdots(x-x_{n-1}).$$

By the calculation, we have

$$p_n(x) = f(x_0) + f(x_0,x_1)(x-x_0) + \cdots + f(x_0,\ldots,x_n)(x-x_0)\cdots(x-x_{n-1}).$$

Here, $f(x_0,\ldots,x_i)$ is called as *i-order divided difference* and is defined as

$$f(x_0,x_1,\ldots,x_i) = \frac{f(x_1,\ldots,x_i) - f(x_0,\ldots,x_{i-1})}{x_i - x_0}.$$

We first introduce some properties about divide difference:

1. $(\lambda \cdot f)(x_0,\ldots,x_n) = \lambda \cdot f(x_0,\ldots,x_n)$.
2. $(f+g)(x_0,\ldots,x_n) = f(x_0,\ldots,x_n) + g(x_0,\ldots,x_n)$.
3. $$\lim_{(x_0,\ldots,x_n)\to(\xi,\ldots,\xi)} f(x_0,\ldots,x_n) = \frac{f^{(n)}(\xi)}{n!}.$$
4. $f(x_0,\ldots,x_n)$ is a symmetric function about $x_0,\ldots,x_n$.
5. $$f(x_0,\ldots,x_n) = \sum_{j=0}^{n} \frac{f(x_j)}{\prod_{k\neq j}(x_j-x_k)}.$$

6.

$$(f\cdot g)(x_0,x_1,\ldots,x_n) = f(x_0)g(x_0,x_1,\ldots,x_n) + f(x_0,x_1)g(x_1,\ldots,x_n) + \cdots + f(x_0,\ldots,x_n)g(x_n).$$

Research Problems:

1. Does the property $f(x_0,\ldots,x_n) = h(x_0+\cdots+x_n)$ for $n \geq 2$ and h a given function guarantee that f is a *polynomial* of degree $\leq n$? (n=2, [1] ; n=3 [2])

2. Extend Leibniz rule and chain rule in calculus to the divide difference.

3 Error bound of the polynomial interpolation

Theorem 3.1. *Suppose the nodes* $x_0, \ldots, x_n \in [a, b]$ *and* $f \in C^{n+1}[a, b]$*. The for any* $x \in [a, b]$*, there is a* $\xi \in (a, b)$ *so that*

$$E(f; x) := f(x) - p_n(x) = \frac{\omega(x)}{(n+1)!} f^{(n+1)}(\xi),$$

where $\omega(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$*.*

We also can use other methods to estimate the error bound. Let $[a, b]$ is a finite interval and $m \in \mathbb{Z}$. If $f, f^{(1)}, \ldots, f^{(m-1)}$ are continuous on $[a, b]$ and $f^{(m)}$ is piecewise continuous and $|f^{(m)}(x)| \leq M_m, x \in [a, b]$, then we say $f \in W^m(M_m; a, b)$.

Now we consider

$$E(f; \alpha) := f(\alpha - p_n(\alpha)).$$

We suppose $[a, b]$ is an interval which contains $\alpha, x_0, \ldots, x_n$.

Theorem 3.2. *Suppose* $m \in \mathbb{Z}$ *and* $1 \leq m \leq n + 1$*. When* $f \in W^m(M_m; a, b)$*, there is a function*

$$K_m(t) = \frac{1}{(m-1)!} E((x - t)_+^{m-1}; \alpha),$$

so that

$$E(f; \alpha) = \int_a^b K_m(t) f^{(m)}(t) dt.$$

References

[1] Aczel, J, "A mean value property of the derivative of quadratic polynomials-without means values and derivatives", Math. Mag. 58, 42-45, 1985.

[2] Bailey, D. F. "A mean-value property of cubic polynomials without mean values", Math. Mag. 65, 123-124, 1992.