

Weierstrass Approximation Theorem and Approximation By Convolution

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1 Weierstrass Approximation Theorem

Theorem 1.1. *(The Weierstrass Approximation Theorem) Let $f \in C[a, b]$. For any $\varepsilon > 0$ there is a polynomial P so that*

$$\max_{a \leq x \leq b} |P(x) - f(x)| \leq \varepsilon.$$

The proof is based on Bernstein's method: constructing

$$P(x) = \sum_{k=0}^n f(k/n) \binom{n}{k} x^k (1-x)^{n-k}.$$

Remark 1.1. *The bases $B_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}$ is widely used in the Bézier curve.*

2 Approximation By Convolution

What is Weierstrass's way to prove the theorem? He uses the approximation by convolution. For two functions defined on \mathbb{R} , their convolution is defined formally by the equation

$$(f * g)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy.$$

Theorem 2.1. *If f and g belong to $L^1(\mathbb{R})$, then so does $f * g$, and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.*

Exercise 2.1.

$$f * g = g * f, \quad (f * g) * h = f * (g * h).$$

A *Dirac sequence* is a sequence of functions K_n , which satisfies the following conditions:

1. $K_n \geq 0$ for all n .
2. $\int_{\mathbb{R}} K_n(x) dx = 1$ for all n .
3. For every $\varepsilon > 0$ and $\delta > 0$, there is a n_0 , such that for all $n \geq n_0$ we have

$$\int_{|x| \geq \delta} K_n(x) dx < \varepsilon.$$

Theorem 2.2. For each bounded function f in $C(\mathbb{R})$, we have $\{K_n * f\}$ uniform converges to f on compact sets in \mathbb{R} .

Now, we can use Theorem 2.2 to prove Weierstrass Approximation Theorem. We define the *Laudau* sequence as follows:

$$K_n(x) = \begin{cases} \frac{1}{c_n}(1-x^2)^n & -1 \leq x \leq 1 \\ 0 & |x| \geq 1 \end{cases}$$

c_n is a constant chosen so that $\int_{\mathbb{R}} K_n(x) dx = 1$. Then the following two facts imply Weierstrass Theorem:

1. $\{K_n\}$ is a dirac sequence
2. $K_n * f$ is a polynomial.

3 Fourier Series

In this section, we shall consider the function f with period 2π . Similar with the definition of Dirac sequence on \mathbb{R} , we can define the *Dirac sequence* on the interval $[\pi, -\pi]$.

We define *Dirichlet* kernel

$$D_n := \frac{1}{2\pi} \sum_{k=-n}^n \exp(ikx).$$

Then

$$S_{f,n} := D_n * f = \sum_{k=-n}^n c_k \exp(ikx),$$

where

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-ikt) f(t) dt.$$

Then D_n is *not* a Dirac sequence.

We set

$$K_n := \frac{1}{n}(D_0 + \cdots + D_{n-1}).$$

A simple calculation to show

$$K_n(x) = \frac{1}{2\pi n} \frac{\sin^2(nx/2)}{\sin^2(x/2)}.$$

Then $\{K_n\}$ is a Dirac sequence. Then we have

Theorem 3.1 (Fejer-Cesaro Theorem). *If f is a continuous function with period 2π , then*

$$\frac{1}{n}(S_{f,0} + \cdots + S_{f,n-1})$$

converges uniformly to f on compact sets in \mathbb{R} .

4 Heat kernel on the real line

A *Dirac family* is a set of functions $K_t(\cdot) : t > 0$, which satisfies the following conditions:

1. $K_t \geq 0$ for all $t \geq 0$.
2. $\int_{\mathbb{R}} K_t(x) dx = 1$ for all $t > 0$.
3. For every $\varepsilon > 0$ and $\delta > 0$, there is a $0 < t_0$, such that for all $0 < t < t_0$ we have

$$\int_{|x| \geq \delta} K_t(x) dx < \varepsilon.$$

Theorem 4.1. *For each bounded function f in $C(\mathbb{R})$, we have $\{K_t * f\}$ uniform converges to f on compact sets in \mathbb{R} with $t \rightarrow 0$.*

For $t > 0$, set

$$K_t(x) := K(t, x) := \frac{1}{(4\pi t)^{1/2}} \exp(-x^2/4t).$$

Then $\{K_t\}$ is a Dirac family. The function $(t, x) \mapsto K(t, x)$ is called as *heat kernel* on \mathbb{R} .

We now consider the *heat operator*. We define \mathbf{H} on $\mathbb{R}^+ \times \mathbb{R}$ as

$$\mathbf{H}_{t,x} = -\left(\frac{\partial}{\partial x}\right)^2 + \frac{\partial}{\partial t}.$$

Theorem 4.2. *The heat kernel K satisfy the heat equation, i.e., $\mathbf{H}K = 0$.*

Corollary 4.1. *Suppose f is a bounded function in $C(\mathbb{R})$. Set*

$$F(t, x) = (K_t * f)(x).$$

Then $\mathbf{H}F = 0$, i.e., F satisfies the heat equation.

5 Heat kernel on the circle

We will consider the periodic function satisfying the heat equation. We would like to find a Dirac family $\{K_t(\cdot)\}$ satisfying

$$K_t(x + 2\pi) = K_t(x)$$

and the following three conditions:

1. For all $t > 0$, we have $K_t \geq 0$.
2. For all $t > 0$, $\int_0^{2\pi} K_t(x) dx = 1$.
3. For every $\varepsilon > 0$ and $\delta > 0$, there is a $0 < t_0$, such that for all $0 < t < t_0$ we have

$$\int_{\delta \leq |x| \leq \pi} K_t(x) dx < \varepsilon.$$

4. $\mathbf{H}K = 0$.

We define

$$K^{\mathbf{S}}(t, x) = \sum_{n \in \mathbb{Z}} K(t, x + 2n\pi).$$

Theorem 5.1. $K^{\mathbf{S}}$ is a periodic function Dirac family, which satisfies the heat equation

$$\mathbf{H}K^{\mathbf{S}} = 0.$$