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Strong convergence rate of the Euler scheme for SDEs driven by additive rough fractional noises

Chuying Huang[a,*], Xu Wang[b,c]

[a] School of Mathematics and Statistics & FJKLMAA, Fujian Normal University, Fuzhou 350117, China
[b] LSEC, ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
[c] School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

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ABSTRACT

The strong convergence rate of the Euler scheme for stochastic differential equations driven by additive fractional Brownian motions is studied, where the fractional Brownian motion has Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$ and the drift coefficient is not required to be bounded. The Malliavin calculus, the rough path theory and the 2D Young integral are utilized to overcome the difficulties caused by the low regularity of the fractional Brownian motion and the unboundedness of the drift coefficient. The Euler scheme is proved to have strong order $2H$ for the case that the drift coefficient has bounded derivatives up to order three and have strong order $H + \frac{1}{2}$ for linear cases.

1. Introduction

Stochastic differential equations driven by fractional Brownian motions with Hurst parameter $H \in (0, 1)$ are basic models to characterize the randomness phenomena and have various applications in the fields of hydrology (Mandelbrot and Van Ness, 1968), porous media (Cao et al., 2017), oscillators (Hong et al., 2018), explorations (Feng et al., 2020), finance (Hong et al., 2020) and so on. If $H > \frac{1}{2}$, the fractional Brownian motion (fBm) exhibits a long-range dependence property. If $H = \frac{1}{2}$, the fBm is equivalent to the standard Brownian motion so that the increments are independent. If $H < \frac{1}{2}$, the fBm exhibits a short-range dependence property and the regularity of the sample paths is relatively low, in which case we call it rough fractional noise.

In this article, we investigate the numerical approximation for the stochastic differential equation (SDE) driven by an additive rough fractional noise

$$dX_t = a(X_t)dt + \sigma dB_t, \quad t \in (0, T] \tag{1}$$

starting from a deterministic initial value $X_0 \in \mathbb{R}$, where the drift coefficient a is unbounded and $B = \{B_t\}_{t\in[0,T]}$ is the fBm with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

* Corresponding author.
E-mail addresses: huangchuying@fjnu.edu.cn, huangchuying@lsec.cc.ac.cn (C. Huang), wangxu@lsec.cc.ac.cn (X. Wang).

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The main challenges in the convergence analysis on numerical schemes for SDEs in the rough case are twofold. First, the unboundedness of the drift coefficient and the correlation of the increments of the fBm make the interaction of the local errors between the numerical solution and the exact solution more complicated. Second, the covariance kernel of the rough fractional noise can hardly be expressed explicitly due to the low regularity of the noise, which makes it more difficult to obtain the optimal convergence rate of a numerical scheme. These difficulties result in that the numerical analysis in this case is far from well-developed. To deal with the problems mentioned above, we apply the Malliavin calculus, the rough path theory and the 2D Young integral to establish the strong convergence rate of the Euler scheme for (1).

For $n \in \mathbb{N}_+$, denoting $h = \frac{T}{n}$ and $t_k = kh$, we focus on the following continuous interpolation of the Euler scheme

$$Y_t = Y_{t_k} + a(Y_{t_k})(t - t_k) + \sigma\left(B_t - B_{t_k}\right), \quad t \in (t_k, t_{k+1}],\ k = 0, \dots, n-1. \tag{2}$$

Our main result is stated in the following.

Theorem 1.1. *Let $H \in (\frac{1}{3}, \frac{1}{2})$. Assume that $a : \mathbb{R} \to \mathbb{R}$ has bounded derivatives up to order three. Then it holds that*

$$\left(\sup_{t\in[0,T]} \mathbb{E}\left|X_t - Y_t\right|^2\right)^{1/2} \le Ch^{2H},$$

where X solves (1) and Y is given by the Euler scheme (2).

As H tends to $\frac{1}{2}$, the strong convergence rate of the Euler scheme above goes to 1, which is consistent with the classical result on the Euler–Maruyama scheme for SDEs driven by additive standard Brownian motions (Milstein and Tretyakov, 2004, Chapter 1). Moreover, comparing with Bayer et al. (2016), Huang (2023) and Liu and Tindel (2019), Theorem 1.1 reveals that the strong convergence rate of the Euler scheme in the above additive noise case is half order higher than those of the Euler-type schemes in the multiplicative noise case. In particular, if a is linear, the strong convergence rate of the Euler scheme is improved to $H + \frac{1}{2}$ as stated in Corollary 3.1. We would like to mention that the results of Theorem 1.1 and Corollary 3.1 can be extended directly to multi-dimensional cases. If the drift coefficient is bounded but less regular, we refer to Butkovsky et al. (2021) for the optimal strong convergence rate of the Euler scheme in Hölder spaces.

The rest of the article is arranged as follows. In Section 2, some preliminaries for the 2D Young integral and the Malliavin calculus are introduced. In Section 3, the proof of our main result, i.e., Theorem 1.1, is established.

2. 2D Young integral and Malliavin calculus

2.1. 2D Young integral

Let U and W be Banach spaces with norms $\|\cdot\|_U$ and $\|\cdot\|_W$, respectively. Denote by $\mathcal{L}(U, W)$ the set of linear operators from U to W.

Definition 2.1. For fixed $p \ge 1$ and $T > 0$, the p-variation of $f : [0, T] \to U$ on $[s, t] \subseteq [0, T]$ is defined as

$$\|f\|_{p\text{-}var;[s,t]} := \sup_{\mathcal{P}\in\mathcal{D}([s,t])} \left(\sum_{k=0}^{N-1} \left\|f_{t_{k+1}} - f_{t_k}\right\|_U^p\right)^{1/p},$$

where $\mathcal{P} = \{t_k : k = 0, \dots, N,\ s = t_0 < t_1 < \cdots < t_N = t\}$ denotes a partition of $[s, t]$ and $\mathcal{D}([s, t])$ is the set of all such partitions. In addition, we define $C^{p\text{-}var}(U; [0, T]) := \{f : \|f\|_{p\text{-}var;[0,T]} < +\infty\}$.

Definition 2.2. Fix $p \ge 1$ and $T > 0$. For $g : [0, T]^2 \to U$, let

$$g([u_i, u_{i+1}] \times [v_j, v_{j+1}]) := g_{u_{i+1},v_{j+1}} - g_{u_{i+1},v_j} - g_{u_i,v_{j+1}} + g_{u_i,v_j}.$$

The p-variation of g on $[s, t] \times [u, v] \subseteq [0, T]^2$ is defined as

$$\|g\|_{V^p;[s,t]\times[u,v]} := \sup_{\pi\in\mathcal{D}([s,t]\times[u,v])} \left(\sum_{i,j} \left\|g([u_i, u_{i+1}] \times [v_j, v_{j+1}])\right\|_U^p\right)^{1/p},$$

where $\pi = \{(u_i, v_j)\}$ is a partition of $[s, t]\times[u, v]$ and $\mathcal{D}([s, t]\times[u, v])$ denotes the set of grid-like partitions of $[s, t]\times[u, v]$. Moreover, we define $C^{p\text{-}var}(U; [0, T]^2) := \{g : \|g\|_{V^p;[0,T]^2} < +\infty\}$.

Remark 2.1. For $f : [0, T] \to U$, the β-Hölder semi-norm of f on $[s, t] \subseteq [0, T]$ is denoted by

$$\|f\|_{\beta;[s,t]} := \sup_{s\le u<v\le t} \frac{\|f_v - f_u\|_U}{|v - u|^\beta}.$$

If $\|f\|_{\beta;[0,T]} < +\infty$, then we have $f \in C^{1/\beta\text{-}var}(U; [0, T])$. Moreover, if g also satisfies $\|g\|_{\beta;[0,T]} < +\infty$, then the $1/\beta$-variation of the function $fg : (r_1, r_2) \mapsto f_{r_1}g_{r_2}$ defined on $[0, T]^2$ is finite.

Definition 2.3. Assume $f \in C^{p\text{-}var}(U, [0,T]^2)$ and $g \in C^{q\text{-}var}(W, [0,T]^2)$. If $\frac{1}{p} + \frac{1}{q} > 1$, then we say that f and g have complementary regularity.

Lemma 2.1. *(Friz and Victoir, 2011; Towghi, 2002) For $f : [0,T]^2 \to \mathcal{L}(U, W)$ and $g : [0,T]^2 \to U$, the 2D Young integral is defined as*

$$\int_{[0,T]^2} f_{r_1,r_2} \mathrm{d}g_{r_1,r_2} := \lim_{|\pi|\to 0} \sum_{i,j} f_{u_i,v_j} g([u_i, u_{i+1}] \times [v_j, v_{j+1}])$$

if the limit exists. If f and g have complementary regularity, then the integral exists and satisfies

$$\left\| \int_{[0,T]^2} f_{r_1,r_2} \mathrm{d}g_{r_1,r_2} \right\|_W \le C_{p,q} |||f|||_{V^p;[0,T]^2} \|g\|_{V^q;[0,T]^2},$$

where

$$|||f|||_{V^p;[0,T]^2} := \|f_{0,0}\|_{\mathcal{L}(U,W)} + \|f_{0,\cdot}\|_{p\text{-}var;[0,T]} + \|f_{\cdot,0}\|_{p\text{-}var;[0,T]} + \|f\|_{V^p;[0,T]^2}.$$

In particular, the result can also be restricted to $[s,t] \times [u,v] \subseteq [0,T]^2$.

2.2. Malliavin calculus

Definition 2.4. The scalar-valued fractional Brownian motion $B = \{B_t\}_{t\in[0,T]}$ is a continuous centered Gaussian process with $B_0 = 0$ almost surely and the covariance

$$R_{s,t} := \mathbb{E}\big[B_s B_t\big] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \qquad s, t \in [0,T].$$

Here, $H \in (0,1)$ is called the Hurst parameter of B.

Based on Definition 2.4, the regularity of the fBm and its covariance can be obtained.

Lemma 2.2. *(Nualart, 2006, Chapter 5) For $H \in (0,1)$ and $p \ge 1$, there exists a constant $C = C_p$ such that*

$$\sup_{0\le s<t\le T} \frac{\|B_t - B_s\|_{L^p(\Omega)}}{|t-s|^H} \le C.$$

Meanwhile, for any $\beta \in (0,H)$, there exists a nonnegative random variable $G = G_{\beta,T} \in L^p(\Omega)$ for all $p \ge 1$, such that $\|B\|_{\beta;[0,T]} \le G$ almost surely.

Lemma 2.3. *(Friz and Victoir, 2011, Example 1) For $H \in (0, \frac{1}{2}]$, we have $R \in C^{1/2H\text{-}var}(\mathbb{R}; [0,T]^2)$. More precisely, it holds that $\|R\|_{V^{1/2H};[s,t]^2} \le C_H |t-s|^{2H}$.*

According to Lemmas 2.1–2.3, if a function $f : [0,T]^2 \to \mathbb{R}$ has the same regularity as B, i.e., $f \in C^{1/\beta\text{-}var}(\mathbb{R}; [0,T]^2)$ for any $\beta \in (0,H)$, then $\int_{[0,T]^2} f_{r_1,r_2} \mathrm{d}R_{r_1,r_2}$ is well-defined as long as $H \in (\frac{1}{3}, \frac{1}{2})$. Based on the 2D Young integral, we next introduce the Malliavin derivative and its adjoint associated to the fBm with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$.

Define the inner product

$$\langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} \rangle_{\mathscr{H}} := R_{s,t} = \int_{[0,s]\times[0,t]} \mathrm{d}R_{r_1,r_2} = \int_{[0,T]^2} \mathbb{1}_{[0,s]}(r_1)\mathbb{1}_{[0,t]}(r_2)\mathrm{d}R_{r_1,r_2}$$

with $\mathbb{1}_{[0,t]}(\cdot)$ being the indicator function. It defines a Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle_{\mathscr{H}})$, which is the closure of the space of all step functions on $[0,T]$ with respect to $\langle\cdot,\cdot\rangle_{\mathscr{H}}$.

Definition 2.5. Let $F = f(B_{t_1}, \ldots, B_{t_N})$ with $t_1, \ldots, t_N \in [0,T]$ and $f : \mathbb{R}^N \to \mathbb{R}$ be a bounded smooth function with derivatives bounded up to any order. The Malliavin derivative of F is defined by

$$D_{\cdot}F := \sum_{i=1}^N \frac{\partial f}{\partial x_i}(B_{t_1}, \ldots, B_{t_N})\mathbb{1}_{[0,t_i]}(\cdot).$$

For $p \ge 1$, the space $\mathbb{D}^{1,p}$ is the closure of the set of random variables in terms of the norm

$$\|F\|_{\mathbb{D}^{1,p}} := \left(\mathbb{E}\big[|F|^p\big] + \mathbb{E}\big[\|DF\|^p_{\mathscr{H}}\big]\right)^{\frac{1}{p}}.$$

Definition 2.6. Given an $\mathscr{H}$-valued random variable $\varphi \in L^2(\Omega; \mathscr{H})$ satisfying

$$\left|\mathbb{E}\big[\langle \varphi, DF\rangle_{\mathscr{H}}\big]\right| \le C_\varphi \|F\|_{L^2(\Omega)}, \quad F \in \mathbb{D}^{1,2},$$

the adjoint operator δ of the derivative operator D acting on φ is $\delta(\varphi) \in L^2(\Omega;\mathbb{R})$ such that

$$\mathbb{E}\Big[\langle \varphi, DF\rangle_{\mathscr{H}}\Big] = \mathbb{E}\Big[F\delta(\varphi)\Big]$$

for all $F \in \mathbb{D}^{1,2}$. In this case, we say $\varphi \in \mathrm{Dom}(\delta)$. Furthermore, the Skorohod integral of φ with respect to B is defined by $\int_0^T \varphi_t \delta B_t := \delta(\varphi)$. In particular, for $t \in [0,T]$, $\int_0^t \varphi_u \delta B_u := \delta\big(\varphi \mathbb{1}_{[0,t]}\big)$.

The fBm with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$ can be naturally lifted to the rough path almost surely such that the integral $\int_0^T \varphi_t \mathrm{d}B_t$ can be interpreted as the rough integral (Friz and Hairer, 2014; Lyons, 1998). In the sequel, we introduce the transformation formula for the Skorohod integral and the rough integral, which is essential for us in the numerical analysis.

Lemma 2.4. *(Cass and Lim, 2019, Theorems 6.1 and 6.3),(Song and Tindel, 2022, Theorem 3.1) Let $H \in (\frac{1}{3}, \frac{1}{2})$. Assume that the stochastic processes $\Phi, \Psi \in C^{p\text{-}var}(\mathbb{R};[0,T])$ for any $p > \frac{1}{H}$ and φ satisfies $\mathrm{d}\varphi_t = \Phi_t \mathrm{d}B_t + \Psi_t \mathrm{d}t$ in the sense of rough path. Then it holds almost surely that*

$$\int_0^T \varphi_t \mathrm{d}B_t = \int_0^T \varphi_t \delta B_t + H\int_0^T \Phi_s s^{2H-1}\mathrm{d}s + \int_{[0,T]^2} \mathbb{1}_{[0,r_2]}(r_1)\Big[D_{r_1}\varphi_{r_2} - \Phi_{r_2}\Big]\mathrm{d}R_{r_1,r_2}.$$

3. Convergence analysis on the Euler scheme

In this section, we set $h = \frac{T}{n}$ and $t_k = kh$, $k = 0,\dots,n$. For $t \in (t_k, t_{k+1}]$, define $\lfloor t\rfloor := t_k$ and $\lceil t\rceil := t_{k+1}$. Before proving the main results, we give some estimates for the solution of (1) and the covariance of the fBm.

Lemma 3.1. *Assume that the derivative of the drift coefficient a is bounded. Then (1) admits a unique solution satisfying*

$$\mathbb{E}\Big[\sup_{\tau\in[0,T]} |X_\tau|^p\Big] + \mathbb{E}\Big[\|X\|^p_{\beta;[0,T]}\Big] \le C, \quad p \ge 1,\ \beta < H.$$

Proof. Since a has bounded derivative, the existence and uniqueness of the solution to (1) is deduced from a standard argument by the contractive mapping principle. Moreover, based on

$$\begin{aligned}\sup_{\tau\in[0,t]} |X_\tau| &\le |X_0| + \int_0^t \sup_{\tau\in[0,s]} |a(X_\tau)|\mathrm{d}s + \sigma \sup_{\tau\in[0,t]} \big|B_\tau\big| \\ &\le |X_0| + C\int_0^t \Big(1 + \sup_{\tau\in[0,s]} |X_\tau|\Big)\mathrm{d}s + \sigma \sup_{\tau\in[0,T]} \big|B_\tau\big|,\end{aligned}$$

we get

$$\sup_{\tau\in[0,t]} |X_\tau| \le C\Big(1 + \sup_{\tau\in[0,T]} \big|B_\tau\big|\Big) \tag{3}$$

according to Gronwall's inequality, which, together with Lemma 2.2, yields

$$\mathbb{E}\Big[\sup_{\tau\in[0,T]} |X_\tau|^p\Big] \le C, \quad p \ge 1.$$

Moreover, (3) also leads to

$$\begin{aligned}|X_t - X_s| &\le \int_s^t |a(X_\tau)|\mathrm{d}\tau + \sigma\big|B_t - B_s\big| \le C\int_s^t \big(1 + |X_\tau|\big)\mathrm{d}\tau + \sigma\big|B_t - B_s\big| \\ &\le C\Big(1 + \sup_{\tau\in[0,T]} |B_\tau|\Big)|t-s| + \sigma\big|B_t - B_s\big|,\end{aligned}$$

which implies $\mathbb{E}\Big[\|X\|^p_{\beta;[0,T]}\Big] \le C$ for any $p \ge 1$ and $\beta < H$. $\square$

Lemma 3.2. *Let R be the covariance of the fractional Brownian motion B with Hurst parameter $H \in (0, \frac{1}{2})$. Then it holds*

$$\int_0^T\int_0^T \|R\|_{V^{1/2H};[\lfloor t\rfloor,t]\times[\lfloor s\rfloor,s]}\mathrm{d}s\mathrm{d}t \le Ch^{2H+1}, \tag{4}$$

$$\int_0^T \|R\|_{V^{1/2H};[\lfloor t\rfloor,t]^2}\mathrm{d}t + \int_0^T \|R\|_{V^{1/2H};[0,\lfloor t\rfloor]\times[\lfloor t\rfloor,t]}\mathrm{d}t \le Ch^{2H}. \tag{5}$$

Proof. We decompose the double integral in (4) as

$$\begin{aligned}
&\int_0^T\int_0^T \|R\|_{V^{1/2H};[\lfloor t\rfloor,t]\times[\lfloor s\rfloor,s]}\mathrm{d}s\mathrm{d}t\\
&=\int_0^T\int_0^T \|R\|_{V^{1/2H};[\lfloor t\rfloor,t]\times[\lfloor s\rfloor,s]}\mathbb{1}_{[\lfloor t\rfloor,\lceil t\rceil]}(s)\mathrm{d}s\mathrm{d}t+\int_0^T\int_0^T \|R\|_{V^{1/2H};[\lfloor t\rfloor,t]\times[\lfloor s\rfloor,s]}\mathbb{1}_{[0,T]\setminus[\lfloor t\rfloor,\lceil t\rceil]}(s)\mathrm{d}s\mathrm{d}t\\
&=:I_1+I_2.
\end{aligned}$$

By means of Lemma 2.3, we get

$$I_1\le\int_0^T\int_0^T h^{2H}\mathbb{1}_{[\lfloor t\rfloor,\lceil t\rceil]}(s)\mathrm{d}s\mathrm{d}t\le Ch^{2H+1}.$$

For I_2, notice that if $s\notin[\lfloor t\rfloor,\lceil t\rceil]$, then the sets $[\lfloor t\rfloor,\lceil t\rceil]$ and $[\lfloor s\rfloor,\lceil s\rceil]$ are essentially disjoint. We claim that for any two essentially disjoint sets $[a,b]$ and $[c,d]$ with $a<b\le c<d$, the covariance of the increments of the fBm is negative. Indeed, due to $H<1/2$, it holds

$$\begin{aligned}
\mathbb{E}\Big[\big(B_b-B_a\big)\big(B_d-B_c\big)\Big]&=\frac{1}{2}\Big[(d-a)^{2H}-(d-b)^{2H}+(c-b)^{2H}-(c-a)^{2H}\Big]\\
&=H(2H-1)\left(\int_a^b\int_c^d(v-u)^{2H-2}\mathrm{d}v\mathrm{d}u\right)<0.
\end{aligned}$$

It then leads to

$$\begin{aligned}
\|R\|^{1/2H}_{V^{1/2H};[\lfloor t\rfloor,t]\times[\lfloor s\rfloor,s]}&=\sup_\pi\sum_{i,j}\left|R([u_i,u_{i+1}]\times[v_j,v_{j+1}])\right|^{1/2H}\le\sup_\pi\left|\sum_{i,j}R([u_i,u_{i+1}]\times[v_j,v_{j+1}])\right|^{1/2H}\\
&=\left|\mathbb{E}\Big[\big(B_s-B_{\lfloor s\rfloor}\big)\big(B_t-B_{\lfloor t\rfloor}\big)\Big]\right|^{1/2H}
\end{aligned}$$

with $\pi=\{(u_i,v_j)\}$ being a partition of $[\lfloor t\rfloor,t]\times[\lfloor s\rfloor,s]$, which yields

$$\|R\|_{V^{1/2H};[\lfloor t\rfloor,t]\times[\lfloor s\rfloor,s]}\le\left|\mathbb{E}\Big[\big(B_s-B_{\lfloor s\rfloor}\big)\big(B_t-B_{\lfloor t\rfloor}\big)\Big]\right|=H(1-2H)\left(\int_{\lfloor t\rfloor}^t\int_{\lfloor s\rfloor}^s|v-u|^{2H-2}\mathrm{d}v\mathrm{d}u\right).$$

Then we obtain

$$\begin{aligned}
I_2&\le C\int_0^T\int_0^T\int_{\lfloor t\rfloor}^t\int_{\lfloor s\rfloor}^s|v-u|^{2H-2}\mathrm{d}v\mathrm{d}u\mathbb{1}_{[0,T]\setminus[\lfloor t\rfloor,\lceil t\rceil]}(s)\mathrm{d}s\mathrm{d}t\\
&=C\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left(\int_0^{t_i}+\int_{t_{i+1}}^T\right)\int_{t_i}^t\int_{\lfloor s\rfloor}^s|v-u|^{2H-2}\mathrm{d}v\mathrm{d}u\mathrm{d}s\mathrm{d}t\\
&=C\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_u^{t_{i+1}}\left(\int_0^{t_i}+\int_{t_{i+1}}^T\right)\int_{\lfloor s\rfloor}^s|v-u|^{2H-2}\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}u\\
&=C\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}(t_{i+1}-u)\left(\int_0^{t_i}+\int_{t_{i+1}}^T\right)\int_{\lfloor s\rfloor}^s|v-u|^{2H-2}\mathrm{d}v\mathrm{d}s\mathrm{d}u\\
&=C\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}(t_{i+1}-u)\left(\int_0^{t_i}+\int_{t_{i+1}}^T\right)\int_v^{\lceil v\rceil}|v-u|^{2H-2}\mathrm{d}s\mathrm{d}v\mathrm{d}u\\
&\le Ch^2\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left(\int_0^{t_i}+\int_{t_{i+1}}^T\right)|v-u|^{2H-2}\mathrm{d}v\mathrm{d}u\\
&=\frac{Ch^2}{2H(1-2H)}\left[\left(\sum_{i=0}^{n-1}2h^{2H}\right)-2T^{2H}\right]\le Ch^{2H+1},
\end{aligned}$$

which completes the proof of (4). Similarly, the second integral in (5) satisfies

$$\begin{aligned}
\int_0^T\|R\|_{V^{1/2H};[0,\lfloor t\rfloor]\times[\lfloor t\rfloor,t]}\mathrm{d}t&=\int_0^T\left|\mathbb{E}\Big[\big(B_t-B_{\lfloor t\rfloor}\big)\big(B_{\lfloor t\rfloor}-B_0\big)\Big]\right|\mathrm{d}t\\
&=C\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{t_i}^t\int_0^{t_i}|v-u|^{2H-2}\mathrm{d}v\mathrm{d}u\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{u}^{t_{i+1}}\int_0^{t_i}|v-u|^{2H-2}\mathrm{d}v\mathrm{d}t\mathrm{d}u\\
&\le Ch\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_0^{t_i}(u-v)^{2H-2}\mathrm{d}v\mathrm{d}u\le Ch^{2H},
\end{aligned}
$$

which, together with $\int_0^T\|R\|_{V^{1/2H};[\lfloor t\rfloor,t]^2}\mathrm{d}t\le Ch^{2H}$ due to Lemma 2.3, completes the proof. □

Now we are in position to prove Theorem 1.1.

Proof of Theorem 1.1. By (1)–(2), we get

$$
\mathbb{E}\big|X_t-Y_t\big|^2\le C\int_0^t\mathbb{E}\big|a(X_{\lfloor s\rfloor})-a(Y_{\lfloor s\rfloor})\big|^2\mathrm{d}s+C\mathbb{E}\bigg|\int_0^t\big(a(X_s)-a(X_{\lfloor s\rfloor})\big)\mathrm{d}s\bigg|^2.
$$

Taking the supremum with respect to t and using the Lipschitz continuity of a, we get

$$
\sup_{\tau\in[0,t]}\mathbb{E}\big|X_\tau-Y_\tau\big|^2\le C\int_0^t\sup_{\tau\in[0,s]}\mathbb{E}\big|X_\tau-Y_\tau\big|^2\mathrm{d}s+C\sup_{t\in[0,T]}\mathbb{E}\bigg|\int_0^t\big(a(X_s)-a(X_{\lfloor s\rfloor})\big)\mathrm{d}s\bigg|^2.
$$

Based on Gronwall's inequality, to prove the result in Theorem 1.1, it suffices to show

$$
\sup_{t\in[0,T]}\mathbb{E}\bigg|\int_0^t\big(a(X_s)-a(X_{\lfloor s\rfloor})\big)\mathrm{d}s\bigg|^2\le Ch^{4H}. \tag{6}
$$

Based on the definition of the solution to a rough differential equation (Friz and Victoir, 2010, Definition 10.17) and properties of Gaussian processes (Friz and Victoir, 2010, Theorem 15.33), the solution X of (1) is the limit of solutions to ordinary differential equations driven by piecewise linear approximations of B. Since the chain rule holds for ordinary differential equations (Friz and Victoir, 2010, Exercise 3.17), taking the limit produces that $a(X), a'(X)$ solve the rough differential equations

$$
\begin{aligned}
\mathrm{d}a(X_r)&=a'(X_r)a(X_r)\mathrm{d}r+\sigma a'(X_r)\mathrm{d}B_r,\\
\mathrm{d}a'(X_r)&=a''(X_r)a(X_r)\mathrm{d}r+\sigma a''(X_r)\mathrm{d}B_r,
\end{aligned}
$$

respectively. Applying Lemma 2.4 with $\varphi_t=a'(X_t)$, $\Phi_t=\sigma a''(X_t)$ and $\Psi_t=a''(X_t)a(X_t)$, we get

$$
\begin{aligned}
\int_{\lfloor s\rfloor}^s a'(X_r)\mathrm{d}B_r&=\int_{\lfloor s\rfloor}^s a'(X_r)\delta B_r+\sigma H\int_{\lfloor s\rfloor}^s a''(X_r)r^{2H-1}\mathrm{d}r\\
&\quad+\int_{[0,T]^2}\mathbb{1}_{[\lfloor s\rfloor,s]}(r_2)\mathbb{1}_{[0,r_2]}(r_1)\Big[D_{r_1}\big[a'(X_{r_2})\big]-\sigma a''(X_{r_2})\Big]\mathrm{d}R_{r_1,r_2}.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
a(X_s)-a(X_{\lfloor s\rfloor})&=\int_{\lfloor s\rfloor}^s a'(X_r)a(X_r)\mathrm{d}r+\sigma\int_{\lfloor s\rfloor}^s a'(X_r)\mathrm{d}B_r\\
&=\int_{\lfloor s\rfloor}^s a'(X_r)a(X_r)\mathrm{d}r+\sigma\int_{\lfloor s\rfloor}^s a'(X_r)\delta B_r+\sigma^2H\int_{\lfloor s\rfloor}^s a''(X_r)r^{2H-1}\mathrm{d}r\\
&\quad+\sigma\int_{[0,T]^2}\mathbb{1}_{[\lfloor s\rfloor,s]}(r_2)\mathbb{1}_{[0,r_2]}(r_1)\Big[D_{r_1}\big[a'(X_{r_2})\big]-\sigma a''(X_{r_2})\Big]\mathrm{d}R_{r_1,r_2}\\
&=:J_1(s)+J_2(s)+J_3(s)+J_4(s).
\end{aligned}
$$

It follows that

$$
\mathbb{E}\bigg|\int_0^u\big(a(X_s)-a(X_{\lfloor s\rfloor})\big)\mathrm{d}s\bigg|^2\le\sum_{i,j=1}^4\bigg(\mathbb{E}\bigg[\Big(\int_0^u J_i(t)\mathrm{d}t\Big)^2\bigg]\bigg)^{1/2}\bigg(\mathbb{E}\bigg[\Big(\int_0^u J_j(t)\mathrm{d}t\Big)^2\bigg]\bigg)^{1/2}.
$$

It then remains to estimate $\mathbb{E}\bigg[\Big(\int_0^u J_i(t)\mathrm{d}t\Big)^2\bigg]$ for each $i\in\{1,2,3,4\}$.

For J_1, it holds

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_0^u J_1(t)\mathrm{d}t\right)^2\right] &= \mathbb{E}\left[\int_0^u \int_{\lfloor t\rfloor}^t a'(X_r)a(X_r)\mathrm{d}r\mathrm{d}t \int_0^u \int_{\lfloor s\rfloor}^s a'(X_v)a(X_v)\mathrm{d}v\mathrm{d}s\right] \\
&= \mathbb{E}\left[\int_0^u \int_r^{\lceil r\rceil} a'(X_r)a(X_r)\mathrm{d}t\mathrm{d}r \int_0^u \int_v^{\lceil v\rceil} a'(X_v)a(X_v)\mathrm{d}s\mathrm{d}v\right] \\
&\leq h^2 \int_0^u \int_0^u \mathbb{E}\Big[\big|a'(X_r)a(X_r)a'(X_v)a(X_v)\big|\Big]\mathrm{d}v\mathrm{d}r \leq Ch^2
\end{aligned}
$$

according to the facts $\left|a'(X_r)\right| \leq C$ and $\left|a(X_r)\right| \leq C\left(1+\sup_{0\leq t\leq T}|X_t|\right)$ since a has a bounded derivative, as well as the boundedness of the solution proved in Lemma 3.1.

For J_2, based on the formula (1.54) in Nualart (2006, Section 1.3.2), we get

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_0^u J_2(t)\mathrm{d}t\right)^2\right] &= \sigma^2\mathbb{E}\left[\int_0^u \int_{\lfloor t\rfloor}^t a'(X_r)\delta B_r\mathrm{d}t \int_0^u \int_{\lfloor s\rfloor}^s a'(X_v)\delta B_v\mathrm{d}s\right] \\
&= \sigma^2 \int_0^u \int_0^u \mathbb{E}\left[\int_{\lfloor t\rfloor}^t a'(X_r)\delta B_r \int_{\lfloor s\rfloor}^s a'(X_v)\delta B_v\right]\mathrm{d}t\mathrm{d}s \\
&= \sigma^2 \int_0^u \int_0^u \mathbb{E}\left[\int_{[0,T]^2} \mathbb{1}_{[\lfloor t\rfloor,t]}(r_1)\mathbb{1}_{[\lfloor s\rfloor,s]}(r_2)a'(X_{r_1})a'(X_{r_2})\mathrm{d}R_{r_1,r_2}\right]\mathrm{d}t\mathrm{d}s \\
&\quad + \sigma^2 \int_0^u \int_0^u \mathbb{E}\bigg[\int_{[0,T]^2}\int_{[0,T]^2} \mathbb{1}_{[\lfloor t\rfloor,t]}(r_1)\mathbb{1}_{[\lfloor s\rfloor,s]}(r_2)\mathbb{1}_{[0,r_1]}(u_1)\mathbb{1}_{[0,r_2]}(u_2) \\
&\qquad \times D_{u_1}\left[a'(X_{r_1})\right]D_{r_2}\left[a'(X_{u_2})\right]\mathrm{d}R_{u_1,u_2}\mathrm{d}R_{r_1,r_2}\bigg]\mathrm{d}t\mathrm{d}s \\
&=: \sigma^2 A_1 + \sigma^2 A_2.
\end{aligned}
$$

Denote

$$
f : [0,T]^2 \to \mathbb{R}, \quad (r_1, r_2) \mapsto f_{r_1,r_2} := a'(X_{r_1})a'(X_{r_2}).
$$

Due to the fact $H > 1/3$, inspiring by Remark 2.1, we get for any $\beta < H$ that

$$
\begin{aligned}
\|f\|_{V^{1/\beta};[0,T]^2} &= \sup_{\pi\in\mathcal{D}([0,T]^2)} \left(\sum_{i,j}\left|f([u_i,u_{i+1}]\times[v_j,v_{j+1}])\right|^{1/\beta}\right)^{\beta} \\
&= \sup_{\pi\in\mathcal{D}([0,T]^2)} \left(\sum_{i,j}\left|\left(a'(X_{u_{i+1}})-a'(X_{u_i})\right)\left(a'(X_{v_{j+1}})-a'(X_{v_j})\right)\right|^{1/\beta}\right)^{\beta} \\
&\leq C \sup_{\pi\in\mathcal{D}([0,T]^2)} \left(\sum_{i,j}\left|\|X\|^2_{\beta;[0,T]}|u_{i+1}-u_i|^{\beta}|v_{j+1}-v_j|^{\beta}\right|^{1/\beta}\right)^{\beta} \\
&\leq C\|X\|^2_{\beta;[0,T]}T^2.
\end{aligned}
$$

Together with $\|R\|_{V^{1/2H};[0,T]^2} \leq C$ by Lemma 2.3 and $\mathbb{E}\left[\|X\|^p_{\beta;[0,T]}\right] \leq C$ by Lemma 3.1 for any $p \geq 1$, we obtain that f and R have complementary regularity almost surely, which is defined in Definition 2.3. Then Lemma 2.1 and Lemma 3.2 produce

$$
\begin{aligned}
|A_1| &\leq C\int_0^u\int_0^u \mathbb{E}\left[\|f\|_{V^{1/\beta};[\lfloor t\rfloor,t]\times[\lfloor s\rfloor,s]}\right]\|R\|_{V^{1/2H};[\lfloor t\rfloor,t]\times[\lfloor s\rfloor,s]}\mathrm{d}s\mathrm{d}t \\
&\leq C\int_0^u\int_0^u \|R\|_{V^{1/2H};[\lfloor t\rfloor,t]\times[\lfloor s\rfloor,s]}\mathrm{d}s\mathrm{d}t \leq Ch^{2H+1}.
\end{aligned}
$$

Meanwhile, the Malliavin derivative satisfies

$$
D_{u_1}\left[a'(X_{r_1})\right] = a''(X_{r_1})D_{u_1}X_{r_1} = \sigma\,\mathscr{J}_{r_1}\mathscr{J}^{-1}_{u_1}a''(X_{u_1}),
$$

where $\mathscr{J}$ and $\mathscr{J}^{-1}$ solve the linear system (Cass and Lim, 2019)

$$
\begin{cases}
\mathscr{J}_t = 1 + \int_0^t a'(X_s)\mathscr{J}_s\mathrm{d}s, \\
\mathscr{J}^{-1}_t = 1 + \int_0^t \mathscr{J}^{-1}_s a'(X_s)\mathrm{d}s.
\end{cases}
$$

Denote

$$\tilde{f}:[0,T]^2\to\mathbb{R},$$
$$(r_1,r_2)\mapsto \tilde{f}_{r_1,r_2}:=\int_{[0,T]^2}\mathbb{1}_{[0,r_1]}(u_1)\mathbb{1}_{[r_2,T]}(u_2)D_{u_1}\Big[a'(X_{r_1})\Big]D_{r_2}\Big[a'(X_{u_2})\Big]\mathrm{d}R_{u_1,u_2}$$
$$=\sigma^2\int_{[0,r_1]\times[r_2,T]}\mathscr{J}_{r_1}\mathscr{J}_{u_1}^{-1}a''(X_{u_1})\mathscr{J}_{u_2}\mathscr{J}_{r_2}^{-1}a''(X_{r_2})\mathrm{d}R_{u_1,u_2}.$$

Consider the decomposition

$$\begin{aligned}
&\tilde{f}([u_i,u_{i+1}]\times[v_j,v_{j+1}])\\
=\;&\sigma^2\big(\mathscr{J}_{u_{i+1}}-\mathscr{J}_{u_i}\big)\Big(\mathscr{J}_{v_{j+1}}^{-1}a''(X_{v_{j+1}})-\mathscr{J}_{v_j}^{-1}a''(X_{v_j})\Big)\int_{[0,u_i]\times[v_{j+1},T]}\mathscr{J}_{u_1}^{-1}a''(X_{u_1})\mathscr{J}_{u_2}\mathrm{d}R_{u_1,u_2}\\
&-\sigma^2\big(\mathscr{J}_{u_{i+1}}-\mathscr{J}_{u_i}\big)\mathscr{J}_{v_j}^{-1}a''(X_{v_j})\int_{[0,u_i]\times[v_j,v_{j+1}]}\mathscr{J}_{u_1}^{-1}a''(X_{u_1})\mathscr{J}_{u_2}\mathrm{d}R_{u_1,u_2}\\
&+\sigma^2\mathscr{J}_{u_{i+1}}\Big(\mathscr{J}_{v_{j+1}}^{-1}a''(X_{v_{j+1}})-\mathscr{J}_{v_j}^{-1}a''(X_{v_j})\Big)\int_{[u_i,u_{i+1}]\times[v_j,T]}\mathscr{J}_{u_1}^{-1}a''(X_{u_1})\mathscr{J}_{u_2}\mathrm{d}R_{u_1,u_2}\\
&-\sigma^2\mathscr{J}_{u_{i+1}}\mathscr{J}_{v_j}^{-1}a''(X_{v_j})\int_{[u_i,u_{i+1}]\times[v_j,v_{j+1}]}\mathscr{J}_{u_1}^{-1}a''(X_{u_1})\mathscr{J}_{u_2}\mathrm{d}R_{u_1,u_2}.
\end{aligned}$$

Since $\mathbb{E}\Big[\|\mathscr{J}\|_{1;[0,T]}^p\Big]\le C$, $\mathbb{E}\Big[\|\mathscr{J}^{-1}\|_{1;[0,T]}^p\Big]\le C$, $\mathbb{E}\Big[\|X\|_{\beta;[0,T]}^p\Big]\le C$ and $\|R\|_{V^{1/2H};[0,T]^2}\le C$ for any $\beta<H$ and $p\ge1$, together with the fact that a has bounded derivatives up to order three, the functions $\tilde{f}$ and R have complementary regularity almost surely, and $\mathbb{E}\Big[|\!|\!|\tilde{f}|\!|\!|_{V^{1/\beta};[0,T]^2}^p\Big]\le C$. Then we deduce from Lemma 2.1 and Lemma 3.2 that

$$\begin{aligned}
|A_2|&\le C\int_0^u\int_0^u\mathbb{E}\Big[|\!|\!|\tilde{f}|\!|\!|_{V^{1/\beta};[\lfloor t\rfloor,t]\times[\lfloor s\rfloor,s]}\Big]\|R\|_{V^{1/2H};[\lfloor t\rfloor,t]\times[\lfloor s\rfloor,s]}\mathrm{d}s\mathrm{d}t\\
&\le C\int_0^u\int_0^u\|R\|_{V^{1/2H};[\lfloor t\rfloor,t]\times[\lfloor s\rfloor,s]}\mathrm{d}s\mathrm{d}t\le Ch^{2H+1}.
\end{aligned}$$

The above estimates for A_1 and A_2 yield

$$\mathbb{E}\bigg[\Big(\int_0^u J_2(t)\mathrm{d}t\Big)^2\bigg]\le Ch^{2H+1}.$$

For J_3, due to $H>1/3$ and Lemma 3.1, it holds

$$\begin{aligned}
\mathbb{E}\bigg[\Big(\int_0^u J_3(t)\mathrm{d}t\Big)^2\bigg]&=\sigma^4H^2\mathbb{E}\bigg[\int_0^u\int_{\lfloor t\rfloor}^t a''(X_r)r^{2H-1}\mathrm{d}r\mathrm{d}t\int_0^u\int_{\lfloor s\rfloor}^s a''(X_v)v^{2H-1}\mathrm{d}v\mathrm{d}s\bigg]\\
&=\sigma^4H^2\mathbb{E}\bigg[\int_0^u\int_r^{\lceil r\rceil}a''(X_r)r^{2H-1}\mathrm{d}t\mathrm{d}r\int_0^u\int_v^{\lceil v\rceil}a''(X_v)v^{2H-1}\mathrm{d}s\mathrm{d}v\bigg]\\
&\le Ch^2\int_0^u\int_0^u\mathbb{E}\Big[\big|a''(X_r)a''(X_v)\big|\Big]r^{2H-1}v^{2H-1}\mathrm{d}v\mathrm{d}r\le Ch^2.
\end{aligned}$$

For J_4, recall that

$$\begin{aligned}
&\mathbb{E}\bigg[\Big(\int_0^u J_4(t)\mathrm{d}t\Big)^2\bigg]\\
=&\sigma^2\mathbb{E}\bigg[\bigg(\int_0^u\int_{[0,T]^2}\mathbb{1}_{[\lfloor t\rfloor,t]}(r_2)\mathbb{1}_{[0,r_2]}(r_1)\Big[D_{r_1}\Big[a'(X_{r_2})\Big]-\sigma a''(X_{r_2})\Big]\mathrm{d}R_{r_1,r_2}\mathrm{d}t\bigg)\\
&\times\bigg(\int_0^u\int_{[0,T]^2}\mathbb{1}_{[\lfloor s\rfloor,s]}(r_4)\mathbb{1}_{[0,r_4]}(r_3)\Big[D_{r_3}\Big[a'(X_{r_4})\Big]-\sigma a''(X_{r_4})\Big]\mathrm{d}R_{r_3,r_4}\mathrm{d}s\bigg)\bigg].
\end{aligned}$$

Define

$$g:[0,T]^2\to\mathbb{R},\quad (r_1,r_2)\mapsto g_{r_1,r_2}:=\mathbb{1}_{[0,r_2]}(r_1)\Big[D_{r_1}\Big[a'(X_{r_2})\Big]-\sigma a''(X_{r_2})\Big].$$

Due to the boundedness of the third order derivative of a, it follows from Cass and Lim (2019, Section 6) and Lemma 2.3 that g and R have complementary regularity almost surely, and $\mathbb{E}\Big[|\!|\!|g|\!|\!|_{V^{1/\beta};[0,T]^2}^p\Big]\le C$. Using Lemmas 2.1 and 3.2, as well as the formulation

$$\begin{aligned}&\int_0^u \int_{[0,T]^2} \mathbb{1}_{[\lfloor t\rfloor,t]}(r_2)\mathbb{1}_{[0,r_2]}(r_1)\Big[D_{r_1}\big[a'(X_{r_2})\big]-\sigma a''(X_{r_2})\Big]\mathrm{d}R_{r_1,r_2}\mathrm{d}t\\ &=\int_0^u \int_{[0,T]^2} \mathbb{1}_{[\lfloor t\rfloor,t]}(r_2)\mathbb{1}_{[0,\lfloor t\rfloor]}(r_1)g_{r_1,r_2}\mathrm{d}R_{r_1,r_2}\mathrm{d}t+\int_0^u \int_{[0,T]^2} \mathbb{1}_{[\lfloor t\rfloor,t]}(r_2)\mathbb{1}_{[\lfloor t\rfloor,t]}(r_1)g_{r_1,r_2}\mathrm{d}R_{r_1,r_2}\mathrm{d}t,\end{aligned}$$

we get

$$\mathbb{E}\left[\left(\int_0^u J_4(t)\mathrm{d}t\right)^2\right]\le Ch^{4H},$$

which completes the proof. □

Corollary 3.1. *Let $H\in(\frac{1}{3},\frac{1}{2})$. Assume that $a(x)=Ax$ with a constant A. Then it holds*

$$\left(\sup_{t\in[0,T]}\mathbb{E}\big|X_t-Y_t\big|^2\right)^{1/2}\le Ch^{H+\frac{1}{2}},$$

where X solves (1) and Y is given by the Euler scheme (2).

Proof. Since $a(x)=Ax$, the second derivative of a vanishes. Repeating the proof of Theorem 1.1, we have

$$\mathbb{E}\left[\left(\int_0^u J_1(t)\mathrm{d}t\right)^2\right]+\mathbb{E}\left[\left(\int_0^u J_2(t)\mathrm{d}t\right)^2\right]\le Ch^{2H+1}$$

and $J_3=J_4=0$. Then the result is obtained. □

Remark 3.1. In the case of $H>\frac{1}{2}$, the framework for Malliavin calculus holds with R being more regular and we refer to Dai and Xiao (2021), Hong et al. (2020, 2021), Hu et al. (2016), Kloeden et al. (2011) and references therein for the analysis on numerical schemes.

Data availability

No data was used for the research described in the article.

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