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An Inverse Random Source Problem for the Biharmonic Wave Equation*

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Abstract. This paper is concerned with an inverse source problem for the stochastic biharmonic wave equation. The driven source is assumed to be a microlocally isotropic Gaussian random field with its covariance operator being a classical pseudo-differential operator. The well-posedness of the direct problem is examined in the distribution sense, and the regularity of the solution is discussed for the given rough source. For the inverse problem, the strength of the random source, involved in the principal symbol of its covariance operator, is shown to be uniquely determined by a single realization of the magnitude of the wave field averaged over the frequency band with probability one. Numerical experiments are presented to illustrate the validity and effectiveness of the proposed method for the case that the random source is white noise.

Key words. inverse random source problem, biharmonic operator, Gaussian random fields, stochastic differential equations, pseudo-differential operator, principal symbol

MSC codes. 35R30, 35R60, 65M32

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1. Introduction.

As one of the important research subjects in inverse scattering theory, inverse source problems for wave propagation have diverse scientific and industrial applications such as antenna design and synthesis and medical imaging [11]. They have continuously attracted much attention of many researchers. We refer the reader to [4] and the references cited therein for some recent advances on this topic. Meanwhile, the study on boundary value problems for higher-order elliptic operators has generated sustained interest in the mathematics community [7]. The biharmonic operator, which arises from the modeling of elasticity, appears to be a natural candidate for such a study [23, 24]. Recently, scattering problems for biharmonic waves have attracted great attention due to important applications in thin plate elasticity. For instance, they are fundamental to an understanding of wave propagation through very large floating structures, which can be used as artificial breakwaters to control destructive surface waves [26]. The design of platonic diffraction gratings and arrays can be utilized to steer and disperse flexural waves [22]. Ultrabroadband elastic cloaking in thin plates can be achieved by constructing a multilayered concentric coating filled with piecewise constant isotropic elastic material [6]. Compared with inverse problems involving second

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order differential operators, inverse problems for the biharmonic operator are much less studied. The reason is not only the increase of the order, which leads to the failure of the methods developed for the second order equations, but also the fact that the properties of the solutions for the higher-order equations are more sophisticated. Some of the inverse boundary value problems for the biharmonic operator can be found in [8, 12, 20, 25, 27].

In practice, there are many uncertainties caused by, e.g., the unpredictability of the environment, incomplete knowledge of the system, or fine-scale spatial or temporal variations, which cannot be neglected during analysis or simulation. To take account of uncertainties, it would be necessary to introduce random parameters to the mathematical modeling. Stochastic inverse problems are inverse problems that involve randomness. Compared to their deterministic counterparts, stochastic inverse problems are more difficult due to the following two extra challenges: the random parameter is sometimes too rough to exist pointwisely and can only be interpreted as a distribution; and the statistics such as the average and variance of the random parameter need to be reconstructed. Apparently, new methodology needs to be developed for both the direct and inverse problems in stochastic settings.

In this paper, we consider an inverse source problem for the biharmonic wave equation

$$\Delta^2 u - k^4 u = f \quad \text{in } \mathbb{R}^d, \tag{1.1}$$

where $d = 2$ or 3 and $k > 0$ is the wavenumber. The wave field u and its Laplacian Δu are required to satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} (\partial_r u - \mathrm{i}ku) = \lim_{r\to\infty} r^{\frac{d-1}{2}} (\partial_r \Delta u - \mathrm{i}k\Delta u) = 0, \quad r = |x|. \tag{1.2}$$

The source f is assumed to be a microlocally isotropic Gaussian random field of order $-m$ (cf. Definition 2.1) such that its covariance operator is a classical pseudo-differential operator with principal symbol $\mu(x)|\xi|^{-m}$, where μ is called the strength of the random source f. This type of random field belongs to the generalized fractional Gaussian random fields (cf. [17]).

For the white noise case with $m = 0$, the random source can be written as $f = \sqrt{\mu}\dot{W}$, where $\dot{W}$ denotes the white noise. Then the biharmonic wave equation (1.1) is interpreted as a stochastic partial differential equation driven by an additive white noise. We refer the reader to [2] and [3] for the inverse random source problem of the acoustic and elastic wave equations, respectively, where the strength μ is shown to be uniquely determined by the variance of the wave field at multiple frequencies. Recently, the microlocally isotropic Gaussian random field with a general m has been studied widely (cf. [5, 9, 15, 17]). For the case $m \in [d, d + \frac{1}{2})$, it was shown in [15] for both the acoustic and elastic wave equations that the strength μ is uniquely determined almost surely by a single realization of the amplitude of the scattering field averaged over the frequency band. In [17] and [18], these results are extended to rougher sources with $m \in (d - 2, d]$ for the acoustic and electromagnetic wave equations. However, the existing works do not contain the case $m = 0$ for $d = 2, 3$. To the best of our knowledge, little is known about stochastic inverse problems on higher-order wave equations. This is the first study on the inverse random source problem of the biharmonic wave equation.

In this paper, we intend to examine both the direct and inverse source problems for the biharmonic wave equation. Particular attention is paid to the rough source with $m \leq d$. We show that the direct problem is well-posed with $m \in (d-6, d]$ in the distribution sense (cf. Theorem 3.2). This work includes the white noise case $m = 0$ and even rougher cases $m < 0$ for both two- and three-dimensional problems. For the inverse problem, we prove that the strength μ of the random source is uniquely determined almost surely by a single realization of the magnitude of the wave field u averaged over the frequency band (cf. Theorems 4.2 and 4.5), which is the main result of the work and is summarized in the following theorem.

Theorem 1.1. *Let f be a centered microlocally isotropic Gaussian random field of order $-m$ in a bounded domain $D \subset \mathbb{R}^d$ with $m \in (d-6, d]$ and $d = 2, 3$, and $U \subset \mathbb{R}^d$ be a bounded domain that has a positive distance to D, i.e., dist $(D, U) = r_0 > 0$. For any $x \in U$, it holds almost surely that*

$$\lim_{T\to\infty} \frac{1}{T} \int_T^{2T} k^{m+7-d} |u(x;k)|^2 dk = \frac{1}{16(2\pi)^{d-1}} \int_D \frac{1}{|x-\zeta|^{d-1}} \mu(\zeta) d\zeta =: T_d(x).$$

Moreover, the strength μ can be uniquely determined by data $\{T_d(x)\}_{x\in U}$.

The paper is organized as follows. In section 2, we introduce the regularity and kernel functions of microlocally isotropic Gaussian random fields, as well as the fundamental solution to the biharmonic wave equation. Section 3 addresses the well-posedness of the direct problem and the regularity of the solution. Section 4 is devoted to the inverse problem, where the uniqueness is obtained. Numerical experiments are presented in section 5 for the white noise case to illustrate the theoretical results. The paper is concluded with some general remarks and future work in section 6.

2. Preliminaries.

In this section, we introduce some basic properties of microlocally isotropic Gaussian random fields and the fundamental solution to the biharmonic operator wave equation, which are essential for the study of both the direct and inverse problems.

2.1. Microlocally isotropic Gaussian random fields.

Denote by $\mathcal{D}(\mathbb{R}^d)$ the space of test functions, which is $C_0^\infty(\mathbb{R}^d)$ equipped with a convex topology, and by $\mathcal{D}'(\mathbb{R}^d)$ the dual space of $\mathcal{D}(\mathbb{R}^d)$, which is known as the Schwartz distribution space. In the following, we use the notation $\mathcal{D} := \mathcal{D}(\mathbb{R}^d)$ and $\mathcal{D}' := \mathcal{D}'(\mathbb{R}^d)$ for convenience.

For a random field $f \in \mathcal{D}'$, its covariance operator $\mathcal{Q}_f : \mathcal{D} \to \mathcal{D}'$ is defined by

$$\langle \mathcal{Q}_f \varphi, \psi \rangle := \mathbb{E}[\langle f, \varphi \rangle \langle f, \psi \rangle] \quad \forall\ \varphi, \psi \in \mathcal{D},$$

where $\langle \cdot, \cdot \rangle$ is the dual product between $\mathcal{D}$ and $\mathcal{D}'$.

If $\mathcal{Q}_f$ is a classical pseudo-differential operator of order $-m$, then it can be defined through its symbol $\sigma \in S^{-m}(\mathbb{R}^d \times \mathbb{R}^d)$ by

$$(\mathcal{Q}_f \varphi)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{\mathrm{i}x\cdot\xi} \sigma(x,\xi) \hat{\varphi}(\xi) d\xi, \tag{2.1}$$

where

$$S^{-m}(\mathbb{R}^d \times \mathbb{R}^d) := \left\{ a(x,\xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) : |\partial_\xi^\alpha \partial_z^\beta a(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{-m-|\alpha|} \right\}$$

is the space of symbols of order $-m$, and α, β are multi-indices whose length is defined by $|\alpha| := \sum_{j=1}^d \alpha_j$ for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$.

By the Schwartz kernel theorem, the Schwartz kernel $K_f \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ of $\mathcal{Q}_f$ satisfies

$$\langle \mathcal{Q}_f \varphi, \psi \rangle = \langle K_f, \psi \otimes \varphi \rangle \quad \forall\, \varphi, \psi \in \mathcal{D},$$

which implies

$$(\mathcal{Q}_f \varphi)(x) = \int_{\mathbb{R}^d} K_f(x,y)\varphi(y)dy. \tag{2.2}$$

Combining (2.1) and (2.2) yields that K_f can be represented in terms of its symbol σ as an oscillatory integral (cf. [10, 17])

$$K_f(x,y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{\mathrm{i}(x-y)\cdot\xi} \sigma(x,\xi) d\xi = \frac{1}{(2\pi)^d} \hat{\sigma}(x, y-x),$$

where $\hat{\sigma}(x, y-x)$ is the Fourier transform of $\sigma(x,\xi)$ with respect to ξ taking the value at $y-x$.

Definition 2.1. *A Gaussian random field f is said to be microlocally isotropic of order $-m$ in $D \subset \mathbb{R}^d$ if its covariance operator $\mathcal{Q}_f$ is a classical pseudo-differential operator of order $-m$ and its principal symbol has the form $\mu(x)|\xi|^{-m}$ with $\mu \in C_0^\infty(D)$ and $\mu \ge 0$, where μ is called the strength of the random field f.*

The regularity of the random field f is determined by the principal symbol of the pseudo-differential operator $\mathcal{Q}_f$. To investigate the regularity of f, we consider the fractional Gaussian random field (cf. [17, 21]) $\tilde{f} := \sqrt{\mu}(-\Delta)^{-\frac{m}{4}}\dot{W}$. The regularity of $\tilde{f}$ is relatively easy to get since the regularity of the white noise has already been investigated. It is shown in Proposition 2.5 of [17] that $\tilde{f}$ satisfies Assumption 2.3 and has the principal symbol $\mu(x)|\xi|^{-m}$. Consequently, the microlocally isotropic Gaussian random field f has the same regularity as $\tilde{f}$. The result is stated in the following lemma, whose proof can be found in Lemma 2.6 of [17].

Lemma 2.2. *Let f be a microlocally isotropic Gaussian random field of order $-m$ in $D \subset \mathbb{R}^d$.*

(i) *If $m \in (d, d+2)$, then $f \in C^{0,\alpha}(D)$ almost surely for all $\alpha \in (0, \frac{m-d}{2})$.*
(ii) *If $m \in (-\infty, d]$, then $f \in W^{\frac{m-d}{2}-\epsilon,p}(D)$ almost surely for all $\epsilon > 0$ and $p > 1$.*

By the above lemma, if $m \in (d, d+2)$, then f is almost surely Hölder continuous and is relatively smooth. If $m \ge d+2$, f is even smoother. The recovery of smooth random sources can follow approaches developed for the deterministic case, which is not the scope of this work. We mainly focus on the rougher case where f satisfies the following assumption.

Assumption 2.3. Assume that the random source f is a centered microlocally isotropic Gaussian random field of order $-m$ in a bounded domain $D \subset \mathbb{R}^d$ with strength $\mu \in C_0^\infty(D)$, $\mu \ge 0$, and $m \in (d-6, d]$.

According to the relationship between f and $\tilde{f}$, the leading term in the Schwartz kernel of f is the same as the one of $\tilde{f}$. Based on the expression of the kernel of $\tilde{f}$ given in Theorem 3.3 of [21], we have the following explicit expression for the kernel K_f.

Lemma 2.4. *Let f be a microlocally isotropic Gaussian random field of order $-m$ in $D \subset \mathbb{R}^d$ with strength μ. Denote by $H := \frac{m-d}{2}$ the general Hurst parameter.*

(i) *If H is a nonnegative integer, then*

$$K_f(x,y) = C_1(m,d)\mu(x)|x-y|^{2H}\ln|x-y| + r(x,y),$$

where $C_1(m,d) = (-1)^{H+1}2^{-m+1}\pi^{-\frac{d}{2}}/(H!\Gamma(\frac{m}{2}))$ with $\Gamma(\cdot)$ being the Gamma function, and $r(x,y)$ denotes the residual which is more regular than the leading term.

(ii) *If H is not a nonnegative integer and $m > 0$, then*

$$K_f(x,y) = C_2(m,d)\mu(x)|x-y|^{2H} + r(x,y),$$

where $C_2(m,d) = 2^{-m}\pi^{-\frac{d}{2}}\Gamma(-H)/\Gamma(\frac{m}{2})$.

(iii) *If H is not a nonnegative integer and $m \in (-2n-2, -2n)$ with n being a nonnegative integer, then*

$$K_f(x,y) = C_2(m,d)\mu(x)|x-y|^{2H}\left[1 - \sum_{j=0}^{n}|x-y|^{2j}c_j\Delta^j\delta(x-y)\right] + r(x,y),$$

where $c_0 = 1$ and

$$c_j = \frac{A_d}{2^j j! d(d+2)\cdots(d+2j-2)}$$

for $j \geq 1$ with $A_d = 2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$ being the surface area of the unit sphere in $\mathbb{R}^d$, and $\delta(\cdot)$ is the Dirac delta function centered at 0.

(iv) *If H is not a nonnegative integer and $m = -2n$ with n being a nonnegative integer, then*

$$K_f(x,y) = \mu(x)(-\Delta)^n\delta(x-y) + r(x,y).$$

Remark 2.5. In cases (iii) and (iv) of Lemma 2.4, all the partial derivatives for the Dirac delta function should be interpreted as distributions, and hence the kernels K_f in these cases should also be interpreted as distributions (cf. [13]). More precisely, by assuming $r(x,y) \equiv 0$ without loss of generality,, K_f given in (iii) and (iv) satisfies

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_f(x,y)\varphi(x)\psi(y)dxdy\\ &\quad = C_2(m,d)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mu(x)|x-y|^{2H}\left[\varphi(x)\psi(y) - \sum_{j=0}^{n}c_j|x-y|^{2j}\varphi(x)\Delta^j\psi(x)\right]dxdy\end{aligned}$$

and

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_f(x,y)\varphi(x)\psi(y)dxdy = \int_{\mathbb{R}^d}\mu(x)\varphi(x)(-\Delta)^n\psi(x)dx,$$

respectively, for any test functions $\varphi, \psi \in \mathcal{D}$.

2.2. The fundamental solution. Denote by $\Phi(x, y, k)$ the outgoing fundamental solution to the biharmonic wave operator $\mathcal{L} = \Delta^2 - k^4$ such that

$$\Delta^2\Phi(x, y, k) - k^4\Phi(x, y, k) = -\delta(x - y) \quad \text{in } \mathbb{R}^d, \tag{2.3}$$

where δ is the Dirac delta distribution. The expression of Φ can be obtained via two different approaches.

Since the biharmonic wave operator can be written as the product of the Helmholtz and modified Helmholtz operators, i.e., $\mathcal{L} = (\Delta - k^2)(\Delta + k^2)$, the fundamental solution Φ is a linear composition of the fundamental solutions to the Helmholtz equation $(\Delta + k^2)u = 0$ and the modified Helmholtz equation $(\Delta - k^2)u = 0$, respectively. Hence, it can be shown that Φ takes the following form (cf. [25]):

$$\Phi(x, y, k) = \frac{\mathrm{i}}{8k^2}\left(\frac{k}{2\pi|x - y|}\right)^{\frac{d-2}{2}}\left(H^{(1)}_{\frac{d-2}{2}}(k|x - y|) + \frac{2\mathrm{i}}{\pi}K_{\frac{d-2}{2}}(k|x - y|)\right),$$

where $H^{(1)}_\nu$ is the Hankel function of the first kind and order $\nu \in \mathbb{R}$, and

$$K_\nu(z) = \frac{\pi}{2}\mathrm{i}^{\nu+1}H^{(1)}_\nu(\mathrm{i}z), \quad -\pi < \arg z \le \frac{\pi}{2}, \tag{2.4}$$

is the Macdonald function of order $\nu \in \mathbb{R}$. More precisely, we have

$$\Phi(x, y, k) = \begin{cases} \frac{\mathrm{i}}{8k^2}\left(H^{(1)}_0(k|x - y|) - H^{(1)}_0(\mathrm{i}k|x - y|)\right), & d = 2, \\ \frac{1}{8\pi k^2|x-y|}\left(e^{\mathrm{i}k|x-y|} - e^{-k|x-y|}\right), & d = 3, \end{cases} \tag{2.5}$$

where we use the fact $H^{(1)}_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\frac{e^{\mathrm{i}z}}{\mathrm{i}}$.

The fundamental solution Φ may also be derived from the Fourier transform. Let

$$\Phi_k(x) := \mathcal{F}^{-1}\left[\frac{-1}{|\xi|^4 - k^4}\right](x), \tag{2.6}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Taking the Fourier transform of (2.3) gives that $\Phi_k(x - y)$ also satisfies (2.3) and hence $\Phi_k(x - y) = \Phi(x, y, k)$.

3. The direct problem. In this section, we examine the well-posedness of the direct problem (1.1)–(1.2). The basic idea is to derive an equivalent integral equation, which will also be used in the recovery of the strength for the random source.

Define the volume potential

$$\mathcal{H}_k(\phi)(x) := -\int_{\mathbb{R}^d}\Phi(x, y, k)\phi(y)dy = -(\Phi_k * \phi)(x),$$

where $*$ denotes the convolution of Φ_k and ϕ.

Lemma 3.1. *Let B and G be two bounded domains in $\mathbb{R}^d$. The operator $\mathcal{H}_k : H^{-s_1}(B) \to H^{s_2}(G)$ is bounded and satisfies*

$$\|\mathcal{H}_k\|_{\mathcal{L}(H^{-s_1}(B),H^{s_2}(G))} \lesssim \frac{1}{k^{3-s}}$$

for $s := s_1 + s_2 \in (0,3)$ with $s_1, s_2 \geq 0$.

Proof. For any $\phi \in C_0^\infty(B)$ and $\psi \in C_0^\infty(G)$, we still denote by ϕ and ψ the zero extensions to $\mathbb{R}^d \setminus \overline{B}$ and $\mathbb{R}^d \setminus \overline{G}$, respectively. Then

$$\begin{aligned}
\langle \mathcal{H}_k\phi, \psi\rangle = \langle \widehat{\mathcal{H}_k\phi}, \hat{\psi}\rangle &= \int_{\mathbb{R}^d} \frac{1}{|\xi|^4 - k^4}\hat{\phi}(\xi)\hat{\psi}(\xi)d\xi \\
&= \int_{\Omega_1} \frac{(1+|\xi|^2)^{\frac{s}{2}}}{|\xi|^4 - k^4}\widehat{\mathcal{J}^{-s_1}\phi}(\xi)\widehat{\mathcal{J}^{-s_2}\psi}(\xi)d\xi + \int_{\Omega_2} \frac{(1+|\xi|^2)^{\frac{s}{2}}}{|\xi|^4 - k^4}\widehat{\mathcal{J}^{-s_1}\phi}(\xi)\widehat{\mathcal{J}^{-s_2}\psi}(\xi)d\xi \\
&=: \mathscr{A} + \mathscr{B},
\end{aligned}$$

where $\hat{\phi} = \mathcal{F}[\phi]$ is the Fourier transform of ϕ,

$$\begin{aligned}
\Omega_1 &:= \left\{\xi \in \mathbb{R}^d : ||\xi| - k| > \frac{k}{2}\right\} = \left\{\xi \in \mathbb{R}^d : |\xi| > \frac{3k}{2} \text{ or } |\xi| < \frac{k}{2}\right\}, \\
\Omega_2 &:= \left\{\xi \in \mathbb{R}^d : ||\xi| - k| < \frac{k}{2}\right\} = \left\{\xi \in \mathbb{R}^d : \frac{k}{2} < |\xi| < \frac{3k}{2}\right\},
\end{aligned}$$

and $\mathcal{J}^s : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is the Bessel potential of order $s \in \mathbb{R}$ defined by (cf. [19])

$$\mathcal{J}^s\phi := (I - \Delta)^{\frac{s}{2}}\phi = \mathcal{F}^{-1}\left[(1 + |\cdot|^2)^{\frac{s}{2}}\hat{\phi}\right] \quad \forall\, \phi \in \mathcal{S}(\mathbb{R}^d)$$

with $\mathcal{S}(\mathbb{R}^d)$ being the Schwartz space of all rapidly decreasing smooth functions.

For any $s \in (0, \frac{3}{2})$, the term $\mathscr{A}$ satisfies

$$\begin{aligned}
|\mathscr{A}| &\leq \int_{\Omega_1} \frac{(1+|\xi|^2)^{\frac{s}{2}}}{||\xi| - k|(|\xi| + k)(|\xi|^2 + k^2)}|\widehat{\mathcal{J}^{-s_1}\phi}||\widehat{\mathcal{J}^{-s_2}\psi}|d\xi \\
&\lesssim \frac{1}{k}\int_{\{|\xi| > \frac{3k}{2}\}\cup\{|\xi| < \frac{k}{2}\}} \frac{(1+|\xi|^2)^{\frac{s}{2}}}{(|\xi| + k)(|\xi|^2 + k^2)}|\widehat{\mathcal{J}^{-s_1}\phi}||\widehat{\mathcal{J}^{-s_2}\psi}|d\xi \\
&\lesssim \frac{1}{k}\int_{\{|\xi| > \frac{3k}{2}\}} \frac{1}{|\xi|^{3-s}}|\widehat{\mathcal{J}^{-s_1}\phi}||\widehat{\mathcal{J}^{-s_2}\psi}|d\xi + \frac{1}{k}\int_{\{|\xi| < \frac{k}{2}\}} \frac{1}{k^{3-s}}|\widehat{\mathcal{J}^{-s_1}\phi}||\widehat{\mathcal{J}^{-s_2}\psi}|d\xi \\
&\lesssim \frac{1}{k^{4-s}}\|\phi\|_{H^{-s_1}(B)}\|\psi\|_{H^{-s_2}(G)}.
\end{aligned}$$

To estimate the term $\mathscr{B}$, we use the change of variables $\xi^* = (\frac{2k}{|\xi|} - 1)\xi$, which maps the domain $\Omega_{21} := \{\xi : \frac{k}{2} < |\xi| < k\}$ to the domain $\Omega_{22} := \{\xi : k < |\xi| < \frac{3k}{2}\}$ and has the Jacobian $J(\xi) = |\det(\frac{\partial \xi^*}{\partial \xi})| = (\frac{2k}{|\xi|} - 1)^{d-1}$. Then the term $\mathscr{B}$ satisfies

$$
\begin{aligned}
\mathscr{B} &= \int_{\Omega_{21}\cup\Omega_{22}} \frac{(1+|\xi|^2)^{\frac{s}{2}}}{|\xi|^4-k^4}\widehat{\mathcal{J}^{-s_1}\phi}(\xi)\widehat{\mathcal{J}^{-s_2}\psi}(\xi)d\xi \\
&= \int_{\Omega_{21}} \frac{(1+|\xi|^2)^{\frac{s}{2}}}{|\xi|^4-k^4}\widehat{\mathcal{J}^{-s_1}\phi}(\xi)\widehat{\mathcal{J}^{-s_2}\psi}(\xi)d\xi \\
&\quad + \int_{\Omega_{21}} \frac{(1+|\xi^*|^2)^{\frac{s}{2}}}{|\xi^*|^4-k^4}\widehat{\mathcal{J}^{-s_1}\phi}(\xi^*)\widehat{\mathcal{J}^{-s_2}\psi}(\xi^*)J(\xi)d\xi \\
&= \int_{\Omega_{21}} \left[\frac{1}{|\xi|^4-k^4}+\frac{J(\xi)}{|\xi^*|^4-k^4}\right](1+|\xi|^2)^{\frac{s}{2}}\widehat{\mathcal{J}^{-s_1}\phi}(\xi)\widehat{\mathcal{J}^{-s_2}\psi}(\xi)d\xi \\
&\quad + \int_{\Omega_{21}} \frac{J(\xi)}{|\xi^*|^4-k^4}\Big[(1+|\xi^*|^2)^{\frac{s}{2}}\widehat{\mathcal{J}^{-s_1}\phi}(\xi^*)\widehat{\mathcal{J}^{-s_2}\psi}(\xi^*) \\
&\quad +(1+|\xi|^2)^{\frac{s}{2}}\widehat{\mathcal{J}^{-s_1}\phi}(\xi)\widehat{\mathcal{J}^{-s_2}\psi}(\xi)\Big]d\xi \\
&= \int_{\Omega_{21}} \left[\frac{1}{|\xi|^4-k^4}+\frac{J(\xi)}{|\xi^*|^4-k^4}\right](1+|\xi|^2)^{\frac{s}{2}}\widehat{\mathcal{J}^{-s_1}\phi}(\xi)\widehat{\mathcal{J}^{-s_2}\psi}(\xi)d\xi \\
&\quad + \int_{\Omega_{21}} \frac{J(\xi)}{|\xi^*|^4-k^4}\left[(1+|\xi^*|^2)^{\frac{s}{2}}-(1+|\xi|^2)^{\frac{s}{2}}\right]\widehat{\mathcal{J}^{-s_1}\phi}(\xi)\widehat{\mathcal{J}^{-s_2}\psi}(\xi)d\xi \\
&\quad + \int_{\Omega_{21}} \frac{J(\xi)}{|\xi^*|^4-k^4}(1+|\xi^*|^2)^{\frac{s}{2}}\left[\widehat{\mathcal{J}^{-s_1}\phi}(\xi^*)-\widehat{\mathcal{J}^{-s_1}\phi}(\xi)\right]\widehat{\mathcal{J}^{-s_2}\psi}(\xi)d\xi \\
&\quad + \int_{\Omega_{21}} \frac{J(\xi)}{|\xi^*|^4-k^4}(1+|\xi^*|^2)^{\frac{s}{2}}\widehat{\mathcal{J}^{-s_1}\phi}(\xi^*)\left[\widehat{\mathcal{J}^{-s_2}\psi}(\xi^*)-\widehat{\mathcal{J}^{-s_2}\psi}(\xi)\right]d\xi \\
&=: \mathscr{B}_1+\mathscr{B}_2+\mathscr{B}_3+\mathscr{B}_4.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\left|\frac{1}{|\xi|^4-k^4}+\frac{J(\xi)}{|\xi^*|^4-k^4}\right| \\
&\quad = \left|\frac{1}{(|\xi|-k)(|\xi|+k)(|\xi|^2+k^2)}+\frac{2k-|\xi|}{|\xi|(k-|\xi|)(3k-|\xi|)(|\xi|^2-4k|\xi|+5k^2)}\right| \\
&\quad = \left|\frac{2k(3|\xi|^2-6k|\xi|+k^2)}{|\xi|(|\xi|+k)(|\xi|^2+k^2)(3k-|\xi|)(|\xi|^2-4k|\xi|+5k^2)}\right| \lesssim \frac{1}{k^4}
\end{aligned}
$$

if $d=2$, and

$$
\begin{aligned}
&\left|\frac{1}{|\xi|^4-k^4}+\frac{J(\xi)}{|\xi^*|^4-k^4}\right| \\
&\quad = \left|\frac{1}{(|\xi|-k)(|\xi|+k)(|\xi|^2+k^2)}+\frac{(2k-|\xi|)^2}{|\xi|^2(k-|\xi|)(3k-|\xi|)(|\xi|^2-4k|\xi|+5k^2)}\right| \\
&\quad = \left|\frac{-2(|\xi|^4-4k|\xi|^3+5k^2|\xi|^2-2k^3|\xi|-2k^4)}{|\xi|^2(|\xi|+k)(|\xi|^2+k^2)(3k-|\xi|)(|\xi|^2-4k|\xi|+5k^2)}\right| \lesssim \frac{1}{k^4}
\end{aligned}
$$

if $d=3$, which leads to

$$
|\mathscr{B}_1| \lesssim \frac{1}{k^{4-s}}\|\phi\|_{H^{-s_1}(B)}\|\psi\|_{H^{-s_2}(G)}.
$$

For the term $\mathscr{B}_2$, since

$$\begin{aligned}
&\left|\frac{J(\xi)}{|\xi^*|^4-k^4}\left[(1+|\xi^*|^2)^{\frac{s}{2}}-(1+|\xi|^2)^{\frac{s}{2}}\right]\right| \\
&\quad=\left|\frac{(2k-|\xi|)^{d-1}(|\xi^*|^2-|\xi|^2)}{|\xi|^{d-1}(k-|\xi|)(3k-|\xi|)(|\xi|^2-4k|\xi|+5k^2)}s(1+\theta|\xi^*|^2+(1-\theta)|\xi|^2)^{s-1}\right| \\
&\quad=\left|\frac{2(2k-|\xi|)^{d-1}(|\xi^*|+|\xi|)}{|\xi|^{d-1}(3k-|\xi|)(|\xi|^2-4k|\xi|+5k^2)}s(1+\theta|\xi^*|^2+(1-\theta)|\xi|^2)^{s-1}\right| \lesssim \frac{1}{k^{4-s}}
\end{aligned}$$

for some $\theta\in(0,1)$, we then get

$$|\mathscr{B}_2|\lesssim\frac{1}{k^{4-s}}\|\phi\|_{H^{-s_1}(B)}\|\psi\|_{H^{-s_2}(G)}.$$

For the term $\mathscr{B}_3$, it holds that (see Theorem 3.2 of [19])

$$\begin{aligned}
|\mathscr{B}_3|\le&\int_{\Omega_{21}}\left|\frac{J(\xi)(1+|\xi^*|^2)^{\frac{s}{2}}(|\xi^*|-|\xi|)}{|\xi^*|^4-k^4}\right| \\
&\times\left|M(|\nabla\widehat{\mathcal{J}^{-s_1}\phi}|)(\xi^*)+M(|\nabla\widehat{\mathcal{J}^{-s_1}\phi}|)(\xi)\right|\left|\widehat{\mathcal{J}^{-s_2}\psi}(\xi)\right|d\xi \\
=&\int_{\Omega_{21}}\left|\frac{2(2k-|\xi|)^{d-1}(1+|\xi^*|^2)^{\frac{s}{2}}}{|\xi|^{d-1}(3k-|\xi|)(|\xi|^2-4k|\xi|+5k^2)}\right| \\
&\times\left|M(|\nabla\widehat{\mathcal{J}^{-s_1}\phi}|)(\xi^*)+M(|\nabla\widehat{\mathcal{J}^{-s_1}\phi}|)(\xi)\right|\left|\widehat{\mathcal{J}^{-s_2}\psi}(\xi)\right|d\xi \\
\lesssim&\frac{1}{k^{3-s}}\|\phi\|_{H^{-s_1}(B)}\|\psi\|_{H^{-s_2}(G)},
\end{aligned}$$

where $M(f)$ is the Hardy–Littlewood maximal function of f. The term $\mathscr{B}_4$ can be estimated similarly.

Combining the above estimates, we conclude for $s\in(0,3)$ that

$$|\langle\mathcal{H}_k\phi,\psi\rangle|\lesssim\frac{1}{k^{3-s}}\|\phi\|_{H^{-s_1}(B)}\|\psi\|_{H^{-s_2}(G)}\quad\forall\ \phi\in C_0^\infty(B),\ \psi\in C_0^\infty(G).$$

The proof is completed by extending the above result to $\phi\in H^{-s_1}(B)$ and $\psi\in H^{-s_2}(G)$ based on the facts that $C_0^\infty(B)$ is dense in $L^2(B)$ and $H^{-s_1}(B)=\overline{L^2(B)}^{\|\cdot\|_{H^{-s_1}(B)}}$. ■

Theorem 3.2. *Let f satisfy Assumption 2.3. Then the problem (1.1) – (1.2) admits a unique solution*

$$u(x;k)=-\int_D\Phi(x,y,k)f(y)dy \tag{3.1}$$

in the distribution sense such that $u\in W^{\gamma,q}_{loc}(\mathbb{R}^d)$ almost surely for any $q>1$ and $0<\gamma<\min\{\frac{6-d+m}{2},\frac{6-d+m}{2}+(\frac{1}{q}-\frac{1}{2})d\}$.

Proof. The uniqueness can be proved similarly to the deterministic case given in [20]. It then suffices to show the existence and regularity of the solution.

We first prove that the random field u defined in (3.1) is a solution of (1.1)–(1.2) in the distribution sense. In fact, for any test function $v \in \mathcal{D}$, it holds that

$$\langle \Delta^2 u - k^4 u, v\rangle = -\left\langle \int_{\mathbb{R}^d} (\Delta^2 - k^4)\Phi(\cdot, y, k) f(y)dy, v \right\rangle = \left\langle \int_{\mathbb{R}^d} \delta(\cdot - y) f(y)dy, v \right\rangle = \langle f, v\rangle.$$

Hence, $u = \mathcal{H}_k f$ satisfies (1.1) in the distribution sense, where $f \in W^{\frac{m-d}{2}-\epsilon,p}(D)$ with $m \in (d-6, d]$ for any $\epsilon > 0$ and $p > 1$ according to Lemma 2.2. Note that $(\frac{d-m}{2}, 3) \neq \emptyset$ since $m > d-6$ according to Assumption 2.3. Then for any $s_1 \in (\frac{d-m}{2}, 3)$ and $p \geq 2$, the condition $\frac{1}{2} > \frac{1}{p} - \frac{\frac{m-d}{2}-\epsilon+s_1}{d}$ is satisfied, and hence the embedding

$$W^{\frac{m-d}{2}-\epsilon,p}(D) \hookrightarrow H^{-s_1}(D)$$

is continuous according to the Kondrachov embedding theorem.

For any bounded domain $G \subset \mathbb{R}^d$ with a C^1-boundary, it follows from Lemma 3.1 that $\mathcal{H}_k : H^{-s_1}(D) \to H^{s_2}(G)$ is bounded for any positive $s_2 < 3 - s_1 < \frac{6-d+m}{2}$. Choosing $s_1 = \frac{d-m+\epsilon}{2}$ and $s_2 = \frac{6-d+m}{2} - \epsilon$ for a sufficiently small $\epsilon > 0$, then parameters γ and q given in the theorem satisfy $\gamma < s_2$ and $\frac{1}{q} > \frac{1}{2} - \frac{s_2-\gamma}{d}$ such that the embedding

$$H^{s_2}(G) \hookrightarrow W^{\gamma,q}(G)$$

is also continuous. We then conclude that $\mathcal{H}_k$ is bounded from $W^{\frac{m-d}{2}-\epsilon,p}(D)$ to $W^{\gamma,q}(G)$ with $p \geq 2$, and hence $u = \mathcal{H}_k f \in W^{\gamma,q}(G)$, which completes the proof. ∎

It is easy to verify that the solution $u = \mathcal{H}_k f$ obtained above is a linear combination of the solutions to the second order differential equations $\Delta u \pm k^2 u = f$. In fact, we may rewrite the fundamental solution Φ as

$$\Phi(x, y, k) = \frac{1}{2k^2}\Phi_+(x, y, k) - \frac{1}{2k^2}\Phi_-(x, y, k),$$

where

$$\Phi_+(x, y, k) := \frac{\mathrm{i}}{4}\left(\frac{k}{2\pi|x-y|}\right)^{\frac{d-2}{2}} H^{(1)}_{\frac{d-2}{2}}(k|x-y|),$$
$$\Phi_-(x, y, k) := \frac{1}{2\pi}\left(\frac{k}{2\pi|x-y|}\right)^{\frac{d-2}{2}} K_{\frac{d-2}{2}}(k|x-y|)$$

are Green's functions to the second order linear operators $\Delta \pm k^2$ and satisfy

$$\Delta\Phi_\pm(x, y, k) \pm k^2\Phi_\pm(x, y, k) = -\delta(x-y) \quad \text{in } \mathbb{R}^d.$$

Then

$$v^\pm := -\int_{\mathbb{R}^d} \Phi_\pm(x, y, k)\phi(y)dy$$

are solutions of the equations $\Delta v^\pm \pm k^2 v^\pm = f$ such that

$$u = \frac{1}{2k^2}(v^+ - v^-). \tag{3.2}$$

4. The inverse problem. In this section, we study the inverse source problem, which is to determine the strength μ of the random source f based on some proper data of the wave field u. Let $U \subset \mathbb{R}^d$ be the measurement domain, which is bounded and satisfies $\operatorname{dist}(D, U) = r_0 > 0$.

In the following, we begin with the discussion on the three-dimensional problem and then proceed to the more involved two-dimensional problem.

4.1. The three-dimensional case. For the case $d = 3$ and $m \in (-3, 3]$, it follows from Assumption 2.3 and (2.5) that the distributional solution (3.1) has the form

$$u(x;k) = -\frac{1}{8\pi k^2}\int_D \frac{e^{\mathrm{i}k|x-y|} - e^{-k|x-y|}}{|x-y|} f(y)dy. \tag{4.1}$$

Before introducing the framework of the inverse problem for a general order m, we first study a special case with order $m = 0$. Consider a Gaussian random field $f = \sqrt{\mu}\dot{W}$ generated by the white noise $\dot{W}$. It is easy to verify that f is microlocally isotropic of order $m = 0$ with symbol $\sigma(x, \xi) = \mu(x)$. The Itô isometry can be used in this case to recover the strength μ. In fact, by noting that

$$\begin{aligned}\mathbb{E}|u(x;k)|^2 =& \frac{1}{64\pi^2 k^4}\mathbb{E}\left|\int_D \frac{e^{\mathrm{i}k|x-y|} - e^{-k|x-y|}}{|x-y|}\sqrt{\mu(y)}dW(y)\right|^2 \\ =& \frac{1}{64\pi^2 k^4}\int_D \frac{1 - 2\cos(k|x-y|)e^{-k|x-y|} + e^{-2k|x-y|}}{|x-y|^2}\mu(y)dy,\end{aligned}$$

we get the recovery formula

$$\lim_{k\to\infty} k^4 \mathbb{E}|u(x;k)|^2 = \frac{1}{64\pi^2}\int_D \frac{1}{|x-y|^2}\mu(y)dy, \tag{4.2}$$

where μ can be uniquely determined by the integral on the right-hand side for all $x \in U$.

For the general order case, the Itô isometry is not available if $m \neq 0$, and hence some other technique is required to deal with the correlation of the random source f. Moreover, in (4.2), a huge number of realizations is required to approximate the expectation involved in the high frequency data on the left-hand side.

To deduce the recovery result based on the data from a single realization almost surely and get a more efficient recovery result for the general order case, the decay property of the solution with respect to the frequency is needed. According to the linear combination (3.2), the required decay property of the solution u can be obtained based on an analogue of the ergodicity in the frequency domain of v^+ (cf. [15]) and the exponential decay property of v^-, which is stated in the following lemma.

Lemma 4.1. *Let f satisfy Assumption 2.3 with $d = 3$. For $k_1, k_2 \geq 1$, it holds uniformly for $x \in U$ that*

$$|\mathbb{E}[u(x;k_1)\overline{u(x;k_2)}]| \lesssim k_1^{-2}k_2^{-2}\left[(k_1+k_2)^{-m}(1+|k_1-k_2|)^{-M_1} + k_1^{-M_2} + k_2^{-M_2}\right], \tag{4.3}$$

$$|\mathbb{E}\left[u(x;k_1)u(x;k_2)\right]| \lesssim k_1^{-2}k_2^{-2}\left[(k_1+k_2)^{-M_1}(1+|k_1-k_2|)^{-m} + k_1^{-M_2} + k_2^{-M_2}\right], \tag{4.4}$$

where $M_1, M_2 > 0$ are arbitrary integers. In particular, if $k_1 = k_2 = k$, then

$$\mathbb{E}|u(x;k)|^2 = \left[\frac{1}{64\pi^2}\int_D \frac{1}{|x-\zeta|^2}\mu(\zeta)d\zeta\right]k^{-m-4} + O(k^{-m-5}) \quad \text{as} \quad k\to\infty. \tag{4.5}$$

Proof. According to (4.1), we get

$$\begin{aligned}
&\mathbb{E}[u(x;k_1)\overline{u(x;k_2)}] \\
&= \frac{1}{64\pi^2k_1^2k_2^2}\int_D\int_D \frac{e^{\mathrm{i}k_1|x-y|}-e^{-k_1|x-y|}}{|x-y|}\frac{e^{\mathrm{i}k_2|x-z|}-e^{-k_2|x-z|}}{|x-z|}\mathbb{E}[f(y)f(z)]dydz \\
&= \frac{1}{64\pi^2k_1^2k_2^2}\int_D\int_D \frac{e^{\mathrm{i}(k_1|x-y|-k_2|x-z|)}}{|x-y||x-z|}K_f(y,z)dydz \\
&\quad - \frac{1}{64\pi^2k_1^2k_2^2}\int_D\int_D \frac{e^{\mathrm{i}k_1|x-y|-k_2|x-z|}+e^{-k_1|x-y|-\mathrm{i}k_2|x-z|}}{|x-y||x-z|}K_f(y,z)dydz \\
&\quad + \frac{1}{64\pi^2k_1^2k_2^2}\int_D\int_D \frac{e^{-k_1|x-y|-k_2|x-z|}}{|x-y||x-z|}K_f(y,z)dydz \\
&=: \mathrm{I}_1(x;k_1,k_2)+\mathrm{I}_2(x;k_1,k_2)+\mathrm{I}_3(x;k_1,k_2).
\end{aligned}$$

The first term I_1 has been estimated in Lemma A.1 of [16] and satisfies

$$|\mathrm{I}_1(x;k_1,k_2)| \lesssim k_1^{-2}k_2^{-2}(k_1+k_2)^{-m}(1+|k_1-k_2|)^{-M_1} \tag{4.6}$$

and

$$\begin{aligned}
\mathrm{I}_1(x;k,k) &= \frac{1}{64\pi^2k^4}\left[\left(\int_D \frac{1}{|x-\zeta|}\mu(\zeta)d\zeta\right)k^{-m} + O(k^{-m-1})\right] \\
&= \left[\frac{1}{64\pi^2}\int_D \frac{1}{|x-\zeta|^2}\mu(\zeta)d\zeta\right]k^{-m-4} + O(k^{-m-5}),
\end{aligned} \tag{4.7}$$

where $M_1>0$ is an arbitrary integer.

The other two terms can be estimated by utilizing Lemma 2.4 and the exponential decay property of the integrant, i.e., $e^{-k_1|x-y|}\le k_1^{-M_2}$ for any $M_2>0$ since $|x-y|$ is bounded below and above for any $x\in U$ and $y\in D$. Without loss of generality, we only consider the leading term in the kernel function K_f since the residual r is more regular than the corresponding leading term. For $d=3$, we get $m\in(-3,3]$ according to Assumption 2.3. We take the term I_2 as an example, whose estimate is given separately for different cases of m.

(i) The case $m\in(0,3]$. By Lemma 2.4, it holds that

$$K_f(y,z) = \begin{cases} C_1(m,3)\mu(y)\ln|y-z|, & m=3, \\ C_2(m,3)\mu(y)|y-z|^{m-3}, & m\in(0,3), \end{cases}$$

and hence $\int_D\int_D |K_f(y,z)|dydz<\infty$ due to the boundedness of the domain D. Then the term I_2 satisfies

$$\begin{aligned}
|\mathrm{I}_2(x;k_1,k_2)| &\lesssim k_1^{-2}k_2^{-2}\int_D\int_D \frac{e^{-k_2|x-z|}+e^{-k_1|x-y|}}{|x-y||x-z|}|K_f(y,z)|dydz \\
&\lesssim k_1^{-2}k_2^{-2}\left(k_1^{-M_2}+k_2^{-M_2}\right),
\end{aligned}$$

where $M_2>0$ is an arbitrary integer.

(ii) The case $m = 0$. We get from Lemma 2.4(iv) with $n = 0$ that $K_f(y, z) = \mu(y)\delta(y-z)$, which leads to

$$|\mathrm{I}_2(x; k_1, k_2)| = \left| \frac{1}{64\pi^2 k_1^2 k_2^2} \int_D \frac{e^{(\mathrm{i}k_1 - k_2)|x-y|} + e^{(-k_1 - \mathrm{i}k_2)|x-y|}}{|x-y|^2} dy \right| \lesssim k_1^{-2} k_2^{-2} \left(k_1^{-M_2} + k_2^{-M_2} \right)$$

with $M_2 > 0$ being an arbitrary integer.

(iii) The case $m \in (-2, 0)$. Utilizing Lemma 2.4(iii) with $n = 0$ and Remark 2.5, we get

$$\begin{aligned} |\mathrm{I}_2(x; k_1, k_2)| \leq & \frac{|C_2(m,3)|}{64\pi^2 k_1^2 k_2^2} \left| \int_D \int_D \frac{e^{\mathrm{i}k_1|x-y|}}{|x-y|} \left(\frac{e^{-k_2|x-z|}}{|x-z|} - \frac{e^{-k_2|x-y|}}{|x-y|} \right) \mu(y) |y-z|^{m-3} dy dz \right| \\ & + \frac{|C_2(m,3)|}{64\pi^2 k_1^2 k_2^2} \left| \int_D \int_D \frac{e^{-k_1|x-y|}}{|x-y|} \left(\frac{e^{-\mathrm{i}k_2|x-z|}}{|x-z|} - \frac{e^{-\mathrm{i}k_2|x-y|}}{|x-y|} \right) \mu(y) |y-z|^{m-3} dy dz \right|. \end{aligned}$$

Since the estimates are the same for the two terms on the right-hand side of the above inequality, to estimate the term I_2, it suffices to estimate the integral

$$\mathcal{I}(x, y) := \int_D \left(\frac{e^{-k_2|x-z|}}{|x-z|} - \frac{e^{-k_2|x-y|}}{|x-y|} \right) |y-z|^{m-3} dz = \int_{D-\{y\}} \left(F_x(y+\tilde{z}) - F_x(y) \right) |\tilde{z}|^{m-3} d\tilde{z}$$

for $x \in U$ and $y \in D$, where

$$F_x(z) := \frac{e^{-k_2|x-z|}}{|x-z|}.$$

It is clear to note that F_x is smooth in D and its derivatives decay exponentially. Define

$$\tilde{F}_x(y, r) = \frac{1}{A_3} \int_{|\tilde{z}|=r} F_x(y+\tilde{z}) ds(\tilde{z}),$$

where A_3 is the surface area of the unit sphere in $\mathbb{R}^3$ given in Lemma 2.4 and $R^* := \max_{y,z \in D} |y-z|$. We get from (1.1.5) of [13]that

$$\begin{aligned} |\mathcal{I}(x, y)| &= \left| \int_{D-\{y\}} \left(F_x(y+\tilde{z}) - F_x(y) \right) |\tilde{z}|^{m-3} d\tilde{z} \right| \\ &= \left| A_3 \int_0^{R^*} \left(\tilde{F}_x(y, r) - F_x(y) \right) r^{m-1} dr \right| \lesssim k_2^{-M_2}, \end{aligned}$$

where we use the fact $|\tilde{F}_x(y, r) - F_x(y)| \lesssim k_2^{-M_2} r^2$ based on the Pizzetti formula (cf. [13])

$$\tilde{F}_x(y, r) = F_x(y) + \frac{\Delta F_x(y)}{2 \cdot 1! d} r^2 + \cdots + \frac{\Delta^j F_x(y)}{2^j j! d(d+2) \cdots (d+2j-2)} r^{2j} + \cdots \quad \text{as} \quad r \to 0$$

and the exponential decay property of F_x.

(iv) The case $m = -2$. Based on Lemma 2.4(iv) with $n = 1$ and Remark 2.5, it holds that

$$\begin{aligned} |\mathrm{II}(x; k_1, k_2)| &= \frac{1}{64\pi^2 k_1^2 k_2^2} \left| \int_D \mu(y) \left(\frac{e^{\mathrm{i}k_1|x-y|}}{|x-y|} (-\Delta_y) \frac{e^{-k_2|x-y|}}{|x-y|} + \frac{e^{-k_1|x-y|}}{|x-y|} (-\Delta_y) \frac{e^{-\mathrm{i}k_2|x-y|}}{|x-y|} \right) dy \right| \\ &\lesssim k_1^{-2} k_2^{-2} \left(k_1^{-M_2} + k_2^{-M_2} \right), \end{aligned}$$

where we use again the smoothness and exponential decay property of the function $\frac{e^{-k_2|x-y|}}{|x-y|}$ for $x \in U$ and $y \in D$.

(v) The case $m \in (-3, -2)$. This case can be proved through the same procedure used in the case (iii) by applying Lemma 2.4(iii) with $n = 1$ and the Pizzetti formula.

We can now conclude from the above discussions that

$$|\mathrm{I}_2(x; k_1, k_2)| \lesssim k_1^{-2} k_2^{-2} \left(k_1^{-M_2} + k_2^{-M_2} \right). \tag{4.8}$$

By choosing $M_2 > m + 1$, we have from the above estimate that

$$\mathrm{I}_2(x; k, k) = O(k^{-m-5}). \tag{4.9}$$

Following estimates similar to those for the term I_2, we may show that I_3 satisfies

$$|\mathrm{I}_3(x; k_1, k_2)| \lesssim k_1^{-2-M_2} k_2^{-2-M_2} \lesssim k_1^{-2} k_2^{-2} \left(k_1^{-M_2} + k_2^{-M_2} \right) \tag{4.10}$$

and

$$\mathrm{I}_3(x; k, k) = O(k^{-m-5}). \tag{4.11}$$

As a result, the estimate (4.3) is proved by combining (4.6), (4.8), and (4.10), and the formula (4.5) is concluded by using (4.7), (4.9), and (4.11). The proof is completed by noting that (4.4) can be estimated based on the same procedure as the proof of (4.3). ∎

Theorem 4.2. *Let f satisfy Assumption 2.3 with $d = 3$. For any $x \in U$, it holds almost surely that*

$$\lim_{T\to\infty} \frac{1}{T} \int_T^{2T} k^{m+4} |u(x; k)|^2 dk = \frac{1}{64\pi^2} \int_D \frac{1}{|x-\zeta|^2} \mu(\zeta) d\zeta =: T_3(x). \tag{4.12}$$

Moreover, the strength μ can be uniquely recovered by the measurement $\{T_3(x)\}_{x\in U}$.

Proof. If $T_3(x)$ is known for $x \in U$, which is smooth in U, then the strength μ can be uniquely recovered by solving a deconvolution problem (cf. Theorem 1 of [14], Lemma 3.6 of [15], or Theorem 4.4 of [17]).

Next, we prove (4.12). It follows from Lemma 4.1 that

$$\lim_{T\to\infty} \frac{1}{T} \int_T^{2T} k^{m+4} \mathbb{E}|u(x; k)|^2 dk = \frac{1}{64\pi^2} \int_D \frac{1}{|x-\zeta|^2} \mu(\zeta) d\zeta.$$

It then suffices to show that

$$\lim_{T\to\infty} \frac{1}{T} \int_T^{2T} Y(x; k) dk = 0 \tag{4.13}$$

almost surely with

$$\begin{aligned} Y(x; k) :&= k^{m+4} \left(|u(x; k)|^2 - \mathbb{E}|u(x; k)|^2 \right) \\ &= k^{m+4} \left(u_{\mathrm{r}}(x; k)^2 - \mathbb{E}[u_{\mathrm{r}}(x; k)^2] \right) + k^{m+4} \left(u_{\mathrm{i}}(x; k)^2 - \mathbb{E}[u_{\mathrm{i}}(x; k)^2] \right) \end{aligned} \tag{4.14}$$

being a real-valued random process. Here, $u_{\mathrm{r}} := \Re[u]$ and $u_{\mathrm{i}} := \Im[u]$ denote the real and imaginary parts of u, respectively. Note that

$$\mathbb{E}\left|\frac{1}{T}\int_T^{2T} Y(x;k)dk\right|^2 = \frac{1}{T^2}\int_T^{2T}\int_T^{2T} \mathbb{E}[Y(x;k_1)Y(x;k_2)]dk_1dk_2.$$

To show (4.13), we only need to show

$$\lim_{T\to\infty}\frac{1}{T^2}\int_T^{2T}\int_T^{2T} \mathbb{E}[Y(x;k_1)Y(x;k_2)]dk_1dk_2 = 0. \tag{4.15}$$

According to (4.14), we get

$$\begin{aligned}\mathbb{E}[Y(x;k_1)Y(x;k_2)] &= k_1^{m+4}k_2^{m+4}\mathbb{E}\left[\left(u_{\mathrm{r}}(x;k_1)^2 - \mathbb{E}[u_{\mathrm{r}}(x;k_1)^2]\right)\left(u_{\mathrm{r}}(x;k_2)^2 - \mathbb{E}[u_{\mathrm{r}}(x;k_2)^2]\right)\right]\\ &\quad + k_1^{m+4}k_2^{m+4}\mathbb{E}\left[\left(u_{\mathrm{r}}(x;k_1)^2 - \mathbb{E}[u_{\mathrm{r}}(x;k_1)^2]\right)\left(u_{\mathrm{i}}(x;k_2)^2 - \mathbb{E}[u_{\mathrm{i}}(x;k_2)^2]\right)\right]\\ &\quad + k_1^{m+4}k_2^{m+4}\mathbb{E}\left[\left(u_{\mathrm{i}}(x;k_1)^2 - \mathbb{E}[u_{\mathrm{i}}(x;k_1)^2]\right)\left(u_{\mathrm{r}}(x;k_2)^2 - \mathbb{E}[u_{\mathrm{r}}(x;k_2)^2]\right)\right]\\ &\quad + k_1^{m+4}k_2^{m+4}\mathbb{E}\left[\left(u_{\mathrm{i}}(x;k_1)^2 - \mathbb{E}[u_{\mathrm{i}}(x;k_1)^2]\right)\left(u_{\mathrm{i}}(x;k_2)^2 - \mathbb{E}[u_{\mathrm{i}}(x;k_2)^2]\right)\right]\\ &=: \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3 + \mathcal{Y}_4.\end{aligned}$$

It is shown in Lemma 4.2 of [5] that for two real-valued random variables X and Z with (X, Z) being a Gaussian random vector and $\mathbb{E}[X] = \mathbb{E}[Z] = 0$, it holds that

$$\mathbb{E}[(X^2 - \mathbb{E}X^2)(Z^2 - \mathbb{E}Z^2)] = 2(\mathbb{E}[XZ])^2.$$

Note that, for any fixed $x \in U$ and $k > 1$, $u_{\mathrm{r}}(x;k)$ and $u_{\mathrm{i}}(x;k)$ are both real-valued Gaussian random variables. Hence, we obtain

$$\begin{aligned}\mathcal{Y}_1 &= 2k_1^{m+4}k_2^{m+4}(\mathbb{E}[u_{\mathrm{r}}(x;k_1)u_{\mathrm{r}}(x;k_2)])^2\\ &= \frac{1}{2}k_1^{m+4}k_2^{m+4}\left(\Re[\mathbb{E}[u(x;k_1)u(x;k_2)] + \mathbb{E}[u(x;k_1)\overline{u(x;k_2)}]]\right)^2.\end{aligned}$$

The estimate of $\mathcal{Y}_1$ is given below for the two different cases (i) $m > 0$ and (ii) $m \le 0$.

(i) For the case $m > 0$, by choosing $M_1 = m$ in Lemma 4.1, we get

$$\begin{aligned}\mathcal{Y}_1 &\lesssim k_1^{m+4}k_2^{m+4}\left(k_1^{-2}k_2^{-2}\left[(k_1+k_2)^{-m}(1+|k_1-k_2|)^{-m} + k_1^{-M_2} + k_2^{-M_2}\right]\right)^2\\ &\lesssim (1+|k_1-k_2|)^{-2m} + k_1^{-2M_2+m}k_2^m + k_1^m k_2^{-2M_2+m},\end{aligned} \tag{4.16}$$

where we use the fact $k_1^m k_2^m (k_1+k_2)^{-2m} = (\frac{k_1k_2}{(k_1+k_2)^2})^m \le 1$. Note that

$$\mathcal{Y}_{11} := \frac{1}{T^2}\int_T^{2T}\int_T^{2T}(1+|k_1-k_2|)^{-2m}dk_1dk_2 = \frac{2}{T^2}\int_T^{2T}\int_{k_2}^{2T}(1+k_1-k_2)^{-2m}dk_1dk_2.$$

If $m = \frac{1}{2}$,

$$\mathcal{Y}_{11} = \frac{2}{T^2}\int_T^{2T}\ln(1+2T-k_2)dk_2 \le \frac{2}{T}\ln(1+2T).$$

If $m = 1$,

$$\mathcal{Y}_{11} = \frac{2}{T} - \frac{2}{T^2}\ln(1+T).$$

If $m \neq \frac{1}{2}, 1$,

$$\mathcal{Y}_{11} = \frac{2 - 2(1+T)^{2-2m}}{T^2(2-2m)(1-2m)} - \frac{2}{(1-2m)T}.$$

The above estimates lead to

$$\lim_{T\to\infty} \frac{1}{T^2}\int_T^{2T}\int_T^{2T}(1+|k_1-k_2|)^{-2m}dk_1dk_2 = 0 \tag{4.17}$$

for $m > 0$. Moreover, by choosing $M_2 > m$, we have

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T^2}\int_T^{2T}\int_T^{2T} k_1^{-2M_2+m}k_2^m dk_1dk_2 \\
&= \lim_{T\to\infty} \frac{1}{T^2}\frac{(2T)^{-2M_2+m+1} - T^{-2M_2+m+1}}{-2M_2+m+1}\frac{(2T)^{m+1}-T^{m+1}}{m+1} \\
&= \lim_{T\to\infty} \frac{(2^{-2M_2+m+1}-1)(2^{m+1}-1)}{(-2M_2+m+1)(m+1)}T^{-2M_2+2m} = 0,
\end{aligned} \tag{4.18}$$

which, together with (4.16) and (4.17), leads to

$$\lim_{T\to\infty} \frac{1}{T^2}\int_T^{2T}\int_T^{2T}\mathcal{Y}_1 dk_1dk_2 = 0.$$

(ii) For the case $m \leq 0$, an application of Lemma 4.1 yields

$$\begin{aligned}
\mathcal{Y}_1 &\lesssim k_1^{m+4}k_2^{m+4}\left(k_1^{-2}k_2^{-2}\left[(k_1+k_2)^{-m}(1+|k_1-k_2|)^{-M_1} + k_1^{-M_2} + k_2^{-M_2}\right]\right)^2 \\
&\lesssim k_1^m k_2^m (k_1+k_2)^{-2m}(1+|k_1-k_2|)^{-2M_1} + k_1^{-2M_2+m}k_2^m + k_1^m k_2^{-2M_2+m}
\end{aligned} \tag{4.19}$$

due to the fact

$$(k_1+k_2)^{-M_1}(1+|k_1-k_2|)^{-m} \lesssim (1+|k_1-k_2|)^{-M_1}(k_1+k_2)^{-m}.$$

It is easy to obtain

$$\lim_{T\to\infty} \frac{1}{T^2}\int_T^{2T}\int_T^{2T}\left(k_1^{-2M_2+m}k_2^m + k_1^m k_2^{-2M_2+m}\right)dk_1dk_2 = 0 \tag{4.20}$$

for any $M_2 > 0$ according to (4.18). In addition,

$$\begin{aligned}
&\frac{1}{T^2}\int_T^{2T}\int_T^{2T} k_1^m k_2^m (k_1+k_2)^{-2m}(1+|k_1-k_2|)^{-2M_1}dk_1dk_2 \\
&\lesssim \left(\frac{1}{T^2}\int_T^{2T}\int_T^{2T} k_1^{2m}k_2^{2m}(k_1+k_2)^{-4m}dk_1dk_2\right)^{\frac{1}{2}}\left(\frac{1}{T^2}\int_T^{2T}\int_T^{2T}(1+|k_1-k_2|)^{-4M_1}dk_1dk_2\right)^{\frac{1}{2}} \\
&\to 0 \quad \text{as} \quad T\to\infty
\end{aligned}$$

for any $M_1 > 0$ based on (4.17) and the fact

$$\frac{1}{T^2}\int_T^{2T}\int_T^{2T} k_1^{2m}k_2^{2m}(k_1+k_2)^{-4m}dk_1dk_2 \lesssim \frac{1}{T^2}\int_T^{2T}\int_T^{2T}\left(k_1^{-2m}k_2^{2m}+k_1^{2m}k_2^{-2m}\right)dk_1dk_2 \lesssim 1.$$

The above estimate together with (4.19) and (4.20) also gives rise to

$$\lim_{T\to\infty}\frac{1}{T^2}\int_T^{2T}\int_T^{2T}\mathcal{Y}_1 dk_1dk_2 = 0.$$

The terms $\mathcal{Y}_2$, $\mathcal{Y}_3$, and $\mathcal{Y}_4$ can be estimated similarly. The details are omitted for brevity. Combining these estimates yields (4.15) and completes the proof. ■

4.2. The two-dimensional case. For the case $d = 2$ and $m \in (-4, 2]$, we obtain from Assumption 2.3 and (2.5) that the distributional solution given in (3.1) takes the form

$$u(x;k) = -\frac{\mathrm{i}}{8k^2}\int_D \left(H_0^{(1)}(k|x-y|) - H_0^{(1)}(\mathrm{i}k|x-y|)\right) f(y)dy, \tag{4.21}$$

where the Hankel function $H_0^{(1)}$ has the following asymptotic expansion (cf. [1]):

$$H_0^{(1)}(z) = \sum_{j=0}^{\infty} a_j z^{-(j+\frac{1}{2})}e^{\mathrm{i}z} \quad \text{as} \quad |z|\to\infty$$

with $a_0 = \sqrt{\frac{2}{\pi}}e^{-\frac{\mathrm{i}\pi}{4}}$ and $a_j = \sqrt{\frac{2}{\pi}}(\frac{\mathrm{i}}{8})^j(\prod_{l=1}^{j}(2l-1)^2/j!)e^{-\frac{\mathrm{i}\pi}{4}}$ for $j \geq 1$.

To recover the strength μ of f, compared to the three-dimensional case, an additional truncation technique is needed in the two-dimensional case. Define the truncated functions

$$H_{0,N}^{(1)}(z) := \sum_{j=0}^{N} a_j z^{-(j+\frac{1}{2})}e^{\mathrm{i}z},$$

$$\Phi_N(x,y,k) := \frac{\mathrm{i}}{8k^2}\left(H_{0,N}^{(1)}(k|x-y|) - H_{0,N}^{(1)}(\mathrm{i}k|x-y|)\right).$$

For any $N \in \mathbb{N}$, a simple calculation yields

$$\begin{aligned}\Phi(x,y,k) - \Phi_N(x,y,k) &= \frac{\mathrm{i}}{8k^2}\sum_{j=N+1}^{\infty} a_j\left((k|x-y|)^{-(j+\frac{1}{2})}e^{\mathrm{i}k|x-y|} - (\mathrm{i}k|x-y|)^{-(j+\frac{1}{2})}e^{-k|x-y|}\right)\\ &= O\left(\frac{1}{k^2(k|x-y|)^{N+\frac{3}{2}}}\right) \quad \text{as} \quad k|x-y|\to\infty. \end{aligned}\tag{4.22}$$

Using the truncated fundamental solution Φ_N, we define the truncated solution

$$\begin{aligned}u_N(x;k) &:= -\int_D \Phi_N(x,y,k)f(y)dy\\ &= -\frac{\mathrm{i}}{8k^2}\int_D\left(H_{0,N}^{(1)}(k|x-y|) - H_{0,N}^{(1)}(\mathrm{i}k|x-y|)\right)f(y)dy\\ &= -\frac{\mathrm{i}}{8k^2}\sum_{j=0}^{N} a_j\int_D\left((k|x-y|)^{-(j+\frac{1}{2})}e^{\mathrm{i}k|x-y|} - (\mathrm{i}k|x-y|)^{-(j+\frac{1}{2})}e^{-k|x-y|}\right)f(y)dy.\end{aligned}$$

To ensure that the contribution of the residual $u - u_N$ is negligible in the measurement such that we only need to investigate the contribution of u_N, we choose $N = 3$ and give the estimate of the residual as follows.

Lemma 4.3. *Let f satisfy Assumption 2.3 with $d = 2$. For $k \gg 1$ and $x \in U$, the following estimate holds almost surely:*

$$|u(x;k) - u_3(x;k)| \lesssim \begin{cases} k^{-\frac{7}{2}}, & m \in (-4, 0], \\ k^{-\frac{11}{2}}, & m \in (0, 2]. \end{cases}$$

Proof. According to (4.22), for $y \in D$ and $x \in U$ with $\operatorname{dist}(D, U) = r_0 > 0$, we get

$$\begin{aligned} |\Phi(x,y,k) - \Phi_3(x,y,k)| &= O\left(k^{-\frac{13}{2}}|x-y|^{-\frac{9}{2}}\right), \\ |\partial_{y_i}\Phi(x,y,k) - \partial_{y_i}\Phi_3(x,y,k)| &= O\left(k^{-\frac{11}{2}}|x-y|^{-\frac{9}{2}}\right), \\ |\partial^2_{y_iy_j}\Phi(x,y,k) - \partial^2_{y_iy_j}\Phi_3(x,y,k)| &= O\left(k^{-\frac{9}{2}}|x-y|^{-\frac{9}{2}}\right), \\ |\partial^3_{y_iy_jy_l}\Phi(x,y,k) - \partial^3_{y_iy_jy_l}\Phi_3(x,y,k)| &= O\left(k^{-\frac{7}{2}}|x-y|^{-\frac{9}{2}}\right). \end{aligned}$$

If $m \in (-4, 0]$, then $f \in W^{\frac{m-d}{2}-\epsilon,p}(D) \subset W^{-3,p}(D)$ for any $p > 1$ according to Lemma 2.2. We then get

$$|u(x;k) - u_3(x;k)| \le \|\Phi(x,\cdot,k) - \Phi_3(x,\cdot,k)\|_{W^{3,q}(D)}\|f\|_{W^{-3,p}(D)} \lesssim k^{-\frac{7}{2}}$$

with q satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

If $m \in (0, 2]$, then $f \in W^{\frac{m-d}{2}-\epsilon,p}(D) \subset W^{-1,p}(D)$ for any $p > 1$, and hence

$$|u(x;k) - u_3(x;k)| \le \|\Phi(x,\cdot,k) - \Phi_3(x,\cdot,k)\|_{W^{1,q}(D)}\|f\|_{W^{-1,p}(D)} \lesssim k^{-\frac{11}{2}},$$

which completes the proof. ∎

Similarly, we need to show the asymptotic independence of the truncated solution u_3.

Lemma 4.4. *Let f satisfy Assumption 2.3. For $k_1, k_2 \ge 1, x \in U$, it holds uniformly that*

$$|\mathbb{E}[u_3(x;k_1)\overline{u_3(x;k_2)}]| \lesssim k_1^{-\frac{5}{2}}k_2^{-\frac{5}{2}}\left[(k_1+k_2)^{-m}(1+|k_1-k_2|)^{-M_1} + k_1^{-M_2} + k_2^{-M_2}\right], \tag{4.23}$$

$$|\mathbb{E}[u_3(x;k_1)u_3(x;k_2)]| \lesssim k_1^{-\frac{5}{2}}k_2^{-\frac{5}{2}}\left[(k_1+k_2)^{-M_1}(1+|k_1-k_2|)^{-m} + k_1^{-M_2} + k_2^{-M_2}\right], \tag{4.24}$$

where $M_1, M_2 > 0$ are arbitrary integers. In particular, if $k_1 = k_2 = k$, then

$$\mathbb{E}|u_3(x;k)|^2 = \left[\frac{1}{32\pi}\int_D \frac{1}{|x-\zeta|}\mu(\zeta)d\zeta\right]k^{-m-5} + O(k^{-m-6}). \tag{4.25}$$

Proof. The truncated solution u_3 at two different frequencies k_1 and k_2 satisfies

$$
\begin{aligned}
&\mathbb{E}[u_3(x;k_1)\overline{u_3(x;k_2)}]\\
&= \frac{1}{64k_1^2k_2^2}\sum_{j,l=0}^{3} a_j\overline{a_l}\int_D\int_D \left((k_1|x-y|)^{-(j+\frac{1}{2})}e^{\mathrm{i}k_1|x-y|} - (\mathrm{i}k_1|x-y|)^{-(j+\frac{1}{2})}e^{-k_1|x-y|}\right)\\
&\quad\times\left((k_2|x-z|)^{-(l+\frac{1}{2})}e^{-\mathrm{i}k_2|x-z|} - (-\mathrm{i}k_2|x-z|)^{-(l+\frac{1}{2})}e^{-k_2|x-z|}\right)\mathbb{E}[f(y)f(z)]dydz\\
&= \frac{1}{64k_1^2k_2^2}\sum_{j,l=0}^{3}\frac{a_j\overline{a_l}}{k_1^{j+\frac{1}{2}}k_2^{l+\frac{1}{2}}}\int_D\int_D\frac{e^{\mathrm{i}(k_1|x-y|-k_2|x-z|)}}{|x-y|^{j+\frac{1}{2}}|x-z|^{l+\frac{1}{2}}}\mathbb{E}[f(y)f(z)]dydz\\
&\quad-\frac{1}{64k_1^2k_2^2}\sum_{j,l=0}^{3}\frac{a_j\overline{a_l}}{k_1^{j+\frac{1}{2}}(-\mathrm{i}k_2)^{l+\frac{1}{2}}}\int_D\int_D\frac{e^{\mathrm{i}k_1|x-y|-k_2|x-z|}}{|x-y|^{j+\frac{1}{2}}|x-z|^{l+\frac{1}{2}}}\mathbb{E}[f(y)f(z)]dydz\\
&\quad-\frac{1}{64k_1^2k_2^2}\sum_{j,l=0}^{3}\frac{a_j\overline{a_l}}{(\mathrm{i}k_1)^{j+\frac{1}{2}}k_2^{l+\frac{1}{2}}}\int_D\int_D\frac{e^{-k_1|x-y|-\mathrm{i}k_2|x-z|}}{|x-y|^{j+\frac{1}{2}}|x-z|^{l+\frac{1}{2}}}\mathbb{E}[f(y)f(z)]dydz\\
&\quad+\frac{1}{64k_1^2k_2^2}\sum_{j,l=0}^{3}\frac{a_j\overline{a_l}}{(\mathrm{i}k_1)^{j+\frac{1}{2}}(-\mathrm{i}k_2)^{l+\frac{1}{2}}}\int_D\int_D\frac{e^{-k_1|x-y|-k_2|x-z|}}{|x-y|^{j+\frac{1}{2}}|x-z|^{l+\frac{1}{2}}}\mathbb{E}[f(y)f(z)]dydz\\
&=: \mathrm{J}_1(x;k_1,k_2)+\mathrm{J}_2(x;k_1,k_2)+\mathrm{J}_3(x;k_1,k_2)+\mathrm{J}_4(x;k_1,k_2).
\end{aligned}
$$

For the term J_1, we have from Lemma A. 1. of [16] that

$$|\mathrm{J}_1(x;k_1,k_2)| \lesssim k_1^{-\frac{5}{2}}k_2^{-\frac{5}{2}}(k_1+k_2)^{-m}(1+|k_1-k_2|)^{-M_1}$$

and

$$
\begin{aligned}
\mathrm{J}_1(x;k,k) &= \frac{|a_0|^2}{64k^5}\int_D\int_D\frac{e^{\mathrm{i}(k|x-y|-k|x-z|)}}{|x-y|^{\frac{1}{2}}|x-z|^{\frac{1}{2}}}\mathbb{E}[f(y)f(z)]dydz\\
&\quad+\frac{1}{64k^4}\sum_{\substack{j,l=0\\ j\text{ or }l\neq 0}}^{3}\frac{a_j\overline{a_l}}{k^{j+l+1}}\int_D\int_D\frac{e^{\mathrm{i}(k|x-y|-k|x-z|)}}{|x-y|^{j+\frac{1}{2}}|x-z|^{l+\frac{1}{2}}}\mathbb{E}[f(y)f(z)]dydz\\
&= \frac{1}{32\pi}\left[\int_D\frac{1}{|x-\zeta|}\mu(\zeta)d\zeta\right]k^{-m-5}+O(k^{-m-6}),
\end{aligned}
$$

where $M_1>0$ is an arbitrary integer.

Similar to the three-dimensional case, the other three terms can be estimated by taking advantage of the exponential decay of the integrands. We then obtain

$$|\mathrm{J}_2(x;k_1,k_2)+\mathrm{J}_3(x;k_1,k_2)+\mathrm{J}_4(x;k_1,k_2)| \lesssim k_1^{-\frac{5}{2}}k_2^{-\frac{5}{2}}\left(k_1^{-M_2}+k_2^{-M_2}\right)$$

for any $M_2>0$ and

$$\mathrm{J}_2(x;k,k)+\mathrm{J}_3(x;k,k)+\mathrm{J}_4(x;k,k)=O(k^{-m-6})$$

by choosing $M_2>m+1$.

The estimates above lead to (4.23) and (4.25). The proof of (4.24) is to combine a similar proof of (4.23) and Corollary 5.4 of [16]. ∎

Based on the estimates for the truncated solution u_3, the unique recovery of the strength can be obtained by a single realization of the wave field u in the almost surely sense, which is stated in the following theorem.

Theorem 4.5. *Let f satisfy Assumption 2.3. For any $x \in U$, it holds almost surely that*

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{2T} k^{m+5}|u(x;k)|^2 dk = \frac{1}{32\pi}\int_D \frac{1}{|x-\zeta|}\mu(\zeta)d\zeta =: T_2(x), \tag{4.26}$$

and the strength μ can be uniquely determined by the measurement $\{T_2(x)\}_{x\in U}$.

Proof. We first mention that, similar to the three-dimensional case, the unique recovery of the strength μ from the data $\{T_2(x)\}_{x\in U}$ follows directly from Lemma 3.6 of [15] or Theorem 4.4 of [17].

Next, we aim to prove (4.26). Using (4.25) in Lemma 4.4, we get for $x \in U$ that

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{2T} k^{m+5}\mathbb{E}|u_3(x;k)|^2 dk = \frac{1}{32\pi}\int_D \frac{1}{|x-\zeta|}\mu(\zeta)d\zeta. \tag{4.27}$$

First we show that

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{2T} k^{m+5}|u_3(x;k)|^2 dk = \frac{1}{32\pi}\int_D \frac{1}{|x-\zeta|}\mu(\zeta)d\zeta \tag{4.28}$$

in the almost surely sense. In fact, following the same procedure as the proof of (4.13) in Theorem 4.2 and utilizing Lemma 4.4, we have almost surely that

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{2T} k^{m+5}\left(|u_3(x;k)|^2 - \mathbb{E}|u_3(x;k)|^2\right) dk = 0,$$

which, together with (4.27), leads to (4.28).

Note that

$$\begin{aligned}\frac{1}{T}\int_T^{2T} k^{m+5}|u(x;k)|^2 dk &= \frac{1}{T}\int_T^{2T} k^{m+5}|u_3(x;k)|^2 dk \\ &+ \frac{1}{T}\int_T^{2T} k^{m+5}|u(x;k)-u_3(x;k)|^2 dk + \frac{2}{T}\int_T^{2T} k^{m+5}\Re\left[\overline{u_3(x;k)}(u(x;k)-u_3(x;k))\right] dk,\end{aligned}$$

where

$$\begin{aligned}&\frac{2}{T}\int_T^{2T} k^{m+5}\Re\left[\overline{u_3(x;k)}(u(x;k)-u_3(x;k))\right] dk \\ &\lesssim \left[\frac{1}{T}\int_T^{2T} k^{m+5}|u_3(x;k)|^2 dk\right]^{\frac{1}{2}}\left[\frac{1}{T}\int_T^{2T} k^{m+5}|u(x;k)-u_3(x;k)|^2 dk\right]^{\frac{1}{2}}.\end{aligned}$$

As a result, to prove (4.26), it suffices to show

$$\lim_{T\to\infty}\frac{1}{T}\int_T^{2T} k^{m+5}|u(x;k)-u_3(x;k)|^2dk=0.$$

For the case $m\in(-4,0]$, according to Lemma 4.3, it holds that

$$\frac{1}{T}\int_T^{2T} k^{m+5}|u(x;k)-u_3(x;k)|^2dk\lesssim\frac{1}{T}\int_T^{2T} k^{m-2}dk\to 0\quad\text{as } T\to\infty.$$

For the case $m\in(0,2]$, an application of Lemma 4.3 leads to

$$\frac{1}{T}\int_T^{2T} k^{m+5}|u(x;k)-u_3(x;k)|^2dk\lesssim\frac{1}{T}\int_T^{2T} k^{m-6}dk\to 0\quad\text{as } T\to\infty,$$

which completes the proof. ■

5. Numerical experiments. In this section, we present some numerical experiments to demonstrate the validity and effectiveness of the proposed method. Specifically, we consider the case $d=2$ and $m=0$, where $f=\sqrt{\mu}\dot{W}$.

5.1. The reconstruction formula. By the Itô isometry, the covariance operator $\mathcal{Q}_f$ is given explicitly by

$$\langle\mathcal{Q}_f\varphi,\psi\rangle=\mathbb{E}[\langle f,\varphi\rangle\langle f,\psi\rangle]=\mathbb{E}\left[\int_D\varphi(x)\sqrt{\mu(x)}dW(x)\int_D\psi(y)\sqrt{\mu(y)}dW(y)\right]=\langle\mu\varphi,\psi\rangle$$

for any $\varphi,\psi\in\mathcal{D}$, which implies

$$(\mathcal{Q}_f\varphi)(x)=\mu(x)\varphi(x)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}e^{\mathrm{i}x\cdot\xi}\mu(x)\hat{\varphi}(\xi)d\xi.$$

Hence, the symbol $\sigma(x,\xi)$ of the pseudo-differential operator $\mathcal{Q}_f$ depends only on x in the form $\sigma(x,\xi)=\mu(x)$ with μ being the strength of the source f.

Consequently, when using the second moment of u to recover the strength μ, the wavenumber k is not required to be sufficiently large for the white noise case. More precisely, according to the expression of the solution given in (4.21), we get

$$u(x;k)=-\frac{\mathrm{i}}{8k^2}\int_D\left(H_0^{(1)}(k|x-y|)-H_0^{(1)}(\mathrm{i}k|x-y|)\right)\sqrt{\mu(y)}dW(y),\tag{5.1}$$

which leads to

$$64k^4\mathbb{E}|u(x;k)|^2=\int_D\left|H_0^{(1)}(k|x-y|)-H_0^{(1)}(\mathrm{i}k|x-y|)\right|^2\mu(y)dy.\tag{5.2}$$

Since the symbol $\sigma(x,\xi)$ for the white noise case has only the principal symbol term $\mu(x)$, to recover the strength μ numerically, we may use (5.2) directly without taking the limit $k\to\infty$. However, the numerical solution is rather unstable if one uses the numerical integration of (5.2) directly to recover the strength μ due to the fast decay of its singular values. We refer the

reader to [2] for the investigation of numerical instability for the Helmholtz equation, whose solution has a similar form as (5.1).

Assuming in addition that the phased data $\{u(x,k)\}_{x\in U}$ is available, we utilize a modified integral equation to handle the instability which arises from the ill-posedness of the integral equation (5.2). Note that $H_0^{(1)} = J_0 + \mathrm{i}Y_0$ with J_0 and Y_0 being the real-valued Bessel functions of the first kind and the second kind, respectively, and the function

$$\mathrm{i}H_0^{(1)}(\mathrm{i}k|x-y|) = \frac{2}{\pi}K_0(k|x-y|)$$

obtained by (2.4) with $d = 2$ is also real-valued. We then split the solution u into its real and imaginary parts as follows:

$$\begin{aligned}
\Re[u(x;k)] &= \frac{1}{8k^2}\int_D \left(Y_0(k|x-y|) + \mathrm{i}H_0^{(1)}(\mathrm{i}k|x-y|)\right)\sqrt{\mu(y)}dW(y),\\
\Im[u(x;k)] &= -\frac{1}{8k^2}\int_D J_0(k|x-y|)\sqrt{\mu(y)}dW(y),
\end{aligned}$$

and use the modified integral equation

$$\begin{aligned}
&64k^4\mathbb{E}\left[(\Re[u(x;k)])^2 - (\Im[u(x;k)])^2\right]\\
&= \int_D \left[\left(Y_0(k|x-y|) + \mathrm{i}H_0^{(1)}(\mathrm{i}k|x-y|)\right)^2 - (J_0(k|x-y|))^2\right]\mu(y)dy\\
&=: \int_D G(x-y)\mu(y)dy \qquad (5.3)
\end{aligned}$$

to reconstruct the strength μ.

5.2. The synthetic data. The direct problem is solved numerically to generate the synthetic data. In the experiments, we choose a square domain $D := [-1,1]\times[-1,1]$ for the support and the measurement domain U, which is specified in the next subsection, such that $\mathrm{dist}(D,U) > 0$. For square domains D and U, we define two index sets

$$\begin{aligned}
\mathcal{T}_U &:= \{\boldsymbol{i} = (i_1,i_2) : i_l = 0,\dots,N_U,\ l = 1,2\},\\
\mathcal{T}_D &:= \{\boldsymbol{j} = (j_1,j_2) : j_l = 0,\dots,N_D,\ l = 1,2\}
\end{aligned}$$

with $N_U = 40$ and $N_D = 20$, and define two sets of discrete points

$$\begin{aligned}
\{x_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\mathcal{T}_U} &:= \left\{x_{\boldsymbol{i}} = (x_{i_1}^{(1)}, x_{i_2}^{(2)})^\top \in U : x_{\boldsymbol{i}} = x_{(0,0)} + (i_1\delta x, i_2\delta x)^\top\right\}_{\boldsymbol{i}\in\mathcal{T}_U},\\
\{y_{\boldsymbol{j}}\}_{\boldsymbol{j}\in\mathcal{T}_D} &:= \left\{y_{\boldsymbol{j}} = (y_{j_1}^{(1)}, y_{j_2}^{(2)})^\top \in D : y_{\boldsymbol{j}} = y_{(0,0)} + (j_1\delta y, j_2\delta y)^\top\right\}_{\boldsymbol{i}\in\mathcal{T}_D},
\end{aligned}$$

where $\delta x = 1/N_U$ and $\delta y = 1/N_D$. The synthetic data is generated at the discrete points $\{x_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\mathcal{T}_U}$, and the solution $u(x_{\boldsymbol{i}};k)$ is approximated through the numerical quadrature of the Itô integral (5.1) by

$$u(x_{\boldsymbol{i}};k) \approx u_{\text{num}}(x_{\boldsymbol{i}},\omega,k) := \frac{1}{8\mathrm{i}k^2}\sum_{\boldsymbol{j}\in\mathcal{T}_D}\left(H_0^{(1)}(k|x_{\boldsymbol{i}}-y_{\boldsymbol{j}}|) - H_0^{(1)}(\mathrm{i}k|x_{\boldsymbol{i}}-y_{\boldsymbol{j}}|)\right)\sqrt{\mu(y_{\boldsymbol{j}})}\delta_{\boldsymbol{j}}W,$$

where $\delta_{\boldsymbol{j}}W := \int_{I_{\boldsymbol{j}}} dW(y) \overset{d}{=} \sqrt{|I_{\boldsymbol{j}}|}\xi_{\boldsymbol{j}}$. Here, the notation $A \overset{d}{=} B$ means that A and B have the same distributions, $\{\xi_{\boldsymbol{j}}\}_{\boldsymbol{j}\in\mathcal{T}_D}$ is a set of independent and identically distributed normal random variables, $I_{\boldsymbol{j}} = [j_1\delta y,(j_1+1)\delta y]\times[j_2\delta y,(j_2+1)\delta y]$ is a square with side length δy, and $|I_{\boldsymbol{j}}|$ is the area of $I_{\boldsymbol{j}}$.

5.3. The numerical method. According to (5.3), we define the measurement

$$\mathcal{M}(x,k) = 64k^4\mathbb{E}\left[(\Re[u(x;k)])^2 - (\Im[u(x;k)])^2\right], \quad x\in U.$$

Then its evaluation at the discrete points $\{x_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\mathcal{T}_U}$ can be approximated by

$$\mathcal{M}(x_{\boldsymbol{i}},k) \approx \sum_{\boldsymbol{j}\in\mathcal{T}_D}|I_{\boldsymbol{j}}|G(x_{\boldsymbol{i}}-y_{\boldsymbol{j}})\mu(y_{\boldsymbol{j}}).$$

In the numerical experiments, the measurement is taken as

$$\mathcal{M}_{\text{num}}(x_{\boldsymbol{i}},k) := 64k^4\frac{1}{P}\sum_{\omega=1}^{P}\left[(\Re[u_{\text{num}}(x_{\boldsymbol{i}},\omega,k)]^2 - (\Im[u_{\text{num}}(x_{\boldsymbol{i}},\omega,k)]^2)\right],$$

where $P = 1000$ denotes the number of sample paths used to approximate the expectation involved in $\mathcal{M}(x_{\boldsymbol{i}},k)$. Then the strength μ at the discrete points $\{y_{\boldsymbol{j}}\}_{\boldsymbol{j}\in\mathcal{T}_D}$ can be numerically recovered through the formula

$$\mathcal{M}_{\text{num}}(x_{\boldsymbol{i}},k) = \sum_{\boldsymbol{j}\in\mathcal{T}_D}|I_{\boldsymbol{j}}|G(x_{\boldsymbol{i}}-y_{\boldsymbol{j}})\mu(y_{\boldsymbol{j}}). \tag{5.4}$$

Next we introduce the regularized Kaczmarz algorithm, which is an iterative method with two loops. To enhance the stability and get more accurate reconstructions, we choose $N = 4$ measurement domains

$$U_1 = [1.5,2.5]\times[1.5,2.5], \qquad U_2 = [1.5,2.5]\times[-2.5,-1.5],$$
$$U_3 = [-2.5,-1.5]\times[-2.5,-1.5], \qquad U_4 = [-2.5,-1.5]\times[1.5,2.5].$$

For each domain U_n, $n = 1,\dots,N$, according to (5.4), we get a linear system

$$b_n = A_n q, \quad n = 1,\dots,N,$$

where b_n is the discrete measurement vector with components $\mathcal{M}_{\text{num}}(x_{\boldsymbol{i}},k)$ for $x_{\boldsymbol{i}}\in U_n$, A_n is the matrix generated by $G(x_{\boldsymbol{i}}-y_{\boldsymbol{j}})$, and q is the unknown vector consisting of $\mu(y_{\boldsymbol{j}})$. The inner loop of the Kaczmarz algorithm is formed by taking iterations with respect to the index n. The outer loop with respect to the index $l = 1,\dots,L$ is used to ensure the convergence of the method as $L\to\infty$ (cf. [3]). Given an initial guess $q^0 = 0$, for each $l\in\mathbb{N}_+$, the regularized Kaczmarz algorithm reads

$$\begin{cases} q_0 = q^l, \\ q_n = q_{n-1} + A_n^\top(\gamma I + A_n A_n^\top)^{-1}(b_n - A_n q_{n-1}), \quad n = 1, \dots, N, \\ q^{l+1} = q_N, \end{cases}$$

where $\gamma > 0$ is the regularization parameter.

5.4. The numerical examples. We present two numerical examples to illustrate the validity and effectiveness of the proposed method.

Example 1: Reconstruct the strength function given by

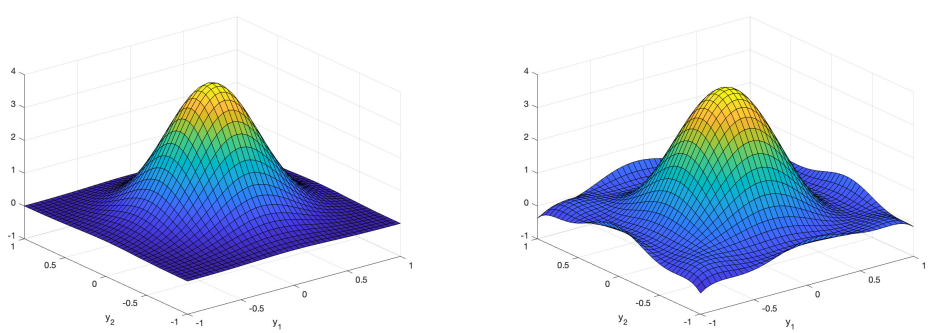
$$\mu(y_1, y_2) = 4e^{-4(y_1^2+y_2^2)}, \quad y = (y_1, y_2)^\top \in D.$$

The exact strength μ is plotted in Figure 1(a). We choose the iteration number of the outer loop $L = 6$ and the regularization parameter $\gamma = 10^{-7}$. Figure 1(b) plots the reconstructed strength by using a single frequency $k = 2$. It can be seen that the bump of the exact strength is well reconstructed by using data with only one frequency.

Example 2: Reconstruct the strength function give by $\mu(y_1, y_2) = \tilde{\mu}(3y_1, 3y_2)$, where

$$\tilde{\mu}(y_1, y_2) = 0.3(1 - y_1)^2 e^{-y_1^2-(y_2+1)^2} - (0.2y_1 - y_1^3 - y_2^5)e^{-y_1^2-y_2^2} - 0.03e^{-(y_1+1)^2-y_2^2}.$$

The exact strength μ is plotted in Figure 2(a). This example is oscillatory and contains more Fourier modes than Example 1. It is expected that the multifrequency data is needed for such an example. To incorporate the data with multiple frequencies, one more outer loop is added to the Kaczmarz algorithm, and this loop is taken with respect to the wavenumber k. We choose $L = 6$ for the intermediate loop and $\gamma = 10^{-5}$ for the regularization parameter. As a comparison, Figure 2(b) shows the reconstruction at a single frequency $k = 2$. Clearly, it is insufficient to reconstruct all the details. Figure 2(c) plots the reconstruction by using multifrequency data at $k = 1 : 3$. The improvement of the reconstruction is obvious, and some details of the true strength are already recovered. Figure 2(d) shows the reconstruction by using multifrequency data at $k = 1 : 5$. It can be seen that almost all the details of the exact strength are recovered.



(a) (b)

Figure 1. *Example* 1: (a) *the exact strength;* (b) *the reconstructed strength at a single frequency* $k = 2$.

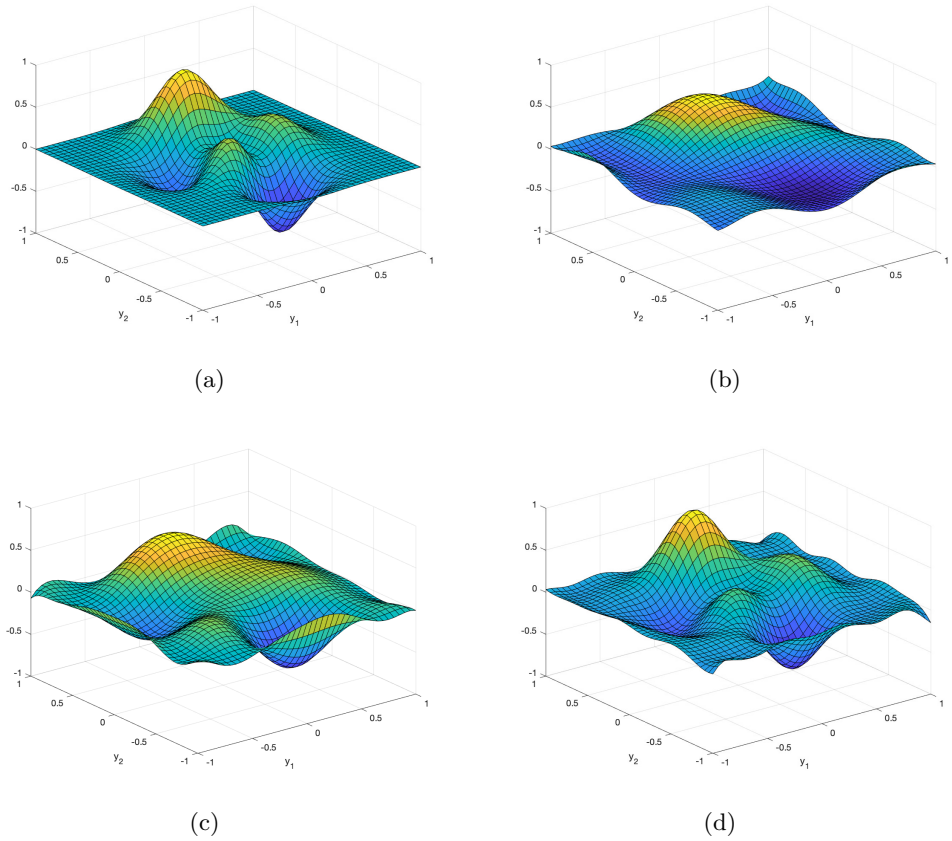


Figure 2. *Example* 2: (a) *the exact strength;* (b) *the reconstructed strength at a single frequency* $k = 2$; (c) *the reconstructed strength by using multiple frequencies* $k = 1 : 3$; (d) *the reconstructed strength by using multiple frequencies* $k = 1 : 5$.

6. Conclusion.

We have studied the direct and inverse source problems for the stochastic biharmonic wave equation. The well-posedness of the direct problem is obtained in the distribution sense. For the inverse problem, we show that a single realization of the magnitude of the wave field averaged over the frequency band is enough to uniquely determine the strength of the random source. Numerical experiments are presented for the white noise model to demonstrate the effectiveness of the proposed method.

The inverse source problem is linear, which makes it possible to get an explicit integral expression of the wave field by using the fundamental solution. For the inverse random medium or potential problem, it is nonlinear and the present method is no longer applicable. It is open for the inverse random potential or medium problem of the biharmonic wave equation. We hope to be able to report the progress on these problems elsewhere in the future.

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