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Positivity-preserving symplectic methods for the stochastic Lotka–Volterra predator-prey model

Jialin Hong[1,2] · Lihai Ji[3] · Xu Wang[1,2] · Jingjing Zhang[4]

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Abstract
In this paper, positivity-preserving symplectic numerical approximations are investigated for the $2d$-dimensional stochastic Lotka–Volterra predator-prey model driven by multiplicative noises, which plays an important role in ecosystem. The model is shown to possess both a unique positive solution and a stochastic symplectic geometric structure, and hence can be interpreted as a stochastic Hamiltonian system. To inherit the intrinsic biological characteristic of the original system, a class of stochastic Runge–Kutta methods is presented, which is proved to preserve positivity of the numerical solution and possess the discrete stochastic symplectic geometric structure as well. Uniform boundedness of both the exact solution and the numerical one are obtained, which are crucial to derive the conditions for convergence order one in the $\mathbb{L}^1(\Omega)$-

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✉ Xu Wang
wangxu@lsec.cc.ac.cn

Jialin Hong
hjl@lsec.cc.ac.cn

Lihai Ji
jilihai@lsec.cc.ac.cn

Jingjing Zhang
jjzhang06@outlook.com

1 Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

2 School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

3 Institute of Applied Physics and Computational Mathematics, Beijing 100094, China

4 School of Science, East China Jiaotong University, Nanchang 330013, China

norm. Numerical examples illustrate the stability and structure-preserving property of the proposed methods over long time.

Keywords Stochastic Lotka–Volterra predator-prey model · Positivity · Stochastic symplecticity · Structure-preserving methods · Convergence order conditions

Mathematics Subject Classification 37M15 · 60H35 · 65C30

1 Introduction

The study of dynamical relationship between population systems with random factors is attracting more and more attention in ecology, where the randomness is usually caused by the unpredictability of the environments and the incomplete knowledge of the ecological system.

In this paper, we consider the following $2d$-dimensional stochastic Lotka–Volterra (LV) predator-prey model driven by multiplicative noises in the Itô sense:

$$\begin{aligned} dX(t) &= \overline{X}(t)\left[\left(-\Gamma^{(2)}Y(t) + \eta^{(2)}\right)dt + \Sigma^{(2)}dW(t)\right], \\ dY(t) &= \overline{Y}(t)\left[\left(\Gamma^{(1)}X(t) - \eta^{(1)}\right)dt + \Sigma^{(1)}dW(t)\right], \end{aligned} \tag{1.1}$$

where $X(t) = (x_1(t), \cdots, x_d(t))^\top$ and $Y(t) = (y_1(t), \cdots, y_d(t))^\top$ denote the population densities of $d \in \mathbb{N}_+$ species of the prey and the predator at time t, respectively, with a deterministic and positive initial value $(X(0)^\top, Y(0)^\top)^\top \in \mathbb{R}^{2d}_+$. For simplicity, we denote $X_0 := X(0)$, $Y_0 := Y(0)$, and use the notation $(X, Y) := (X^\top, Y^\top)^\top \in \mathbb{R}^{2d}$ for any $X, Y \in \mathbb{R}^d$. For any vector $V = (v_1, \cdots, v_d)^\top \in \mathbb{R}^d$, we denote

$$\overline{V} = \mathrm{diag}\{v_1, \cdots, v_d\} \in \mathbb{R}^{d\times d}.$$

For any vector or matrix V, by using the notation $V > 0$, we mean all the entries in V are positive. The vector $\eta^{(1)} = (\eta_1^{(1)}, \cdots, \eta_d^{(1)})^\top > 0$ in (1.1) is the natural death rate of the d species of the predator in the absence of food, and $\eta^{(2)} = (\eta_1^{(2)}, \cdots, \eta_d^{(2)}) > 0$ is the natural growth rate of the d species of the prey in the absence of predation. The matrix $\Gamma^{(1)} = [\gamma_{ij}^{(1)}]_{d\times d} > 0$ denotes the rate of conversion of the prey into the reproduction of the predator and $\Gamma^{(2)} = [\gamma_{ij}^{(2)}]_{d\times d} > 0$ denotes the death rate per encounter of the prey due to predation. Moreover, $W = (W_1, \cdots, W_m)^\top$ is an m-dimensional standard Wiener process with $\{W_i\}_{i=1}^m$ being m independent one-dimensional standard Wiener processes defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and matrices $\Sigma^{(l)} = [\sigma_{ij}^{(l)}]_{d\times m}$, $l = 1, 2$, measure the size of the noises.

Cooperation, predator-prey, and competition are three main interactions among species in ecosystems, in which, competition is one of the most effective interactions. Such interactions occur when two or more species compete for the same resource such as food, shelter, nesting sites, etc. The classical deterministic LV model is one of the most important models of competitive interactions. However, in practical circumstances, randomness is common and inevitable in biological populations (cf. [1,6]). In the past decades, besides deterministic models, stochastic ecology models have gained increasing attention to depict more realistically ecosystems. In recent years, the stochastic LV model was studied extensively and fruitful results were obtained (cf. [4,11,14–16,19] and references therein). More precisely, [4] investigated the existence, uniqueness and non-extinction property for the solution of the stochastic LV model. By using the Fokker–Planck equation, [15,16] analyzed the long–time behavior of densities of the distributions of the solutions of the stochastic LV model and proved that the densities converge in $\mathbb{L}^1$ to an invariant density or converge weakly to a singular measure. The permanence of two-dimensional stochastic LV model was explored in [11], which revealed that the noises play an essential role in the permanence and characterize the systems being permanent or not. Recently, [14] derived the sufficient conditions for the coexistence and exclusion of a stochastic competitive LV model and established the convergence in distribution of positive solutions of the model.

In the nature, the population densities of the prey X and the predator Y should both be positive. Therefore, the first primary goal of this work is to present the positivity and analyze the global well-posedness and uniform boundedness of the solution to stochastic LV model (1.1). By utilizing stopping time technique, it is proved that there exists a unique positive solution to (1.1) (see Theorem 2.1), which is uniformly bounded

$$\sup_{t\in[0,T]} \mathbb{E}\left[\sum_{i=1}^{d}\big(p_i x_i(t) + q_i y_i(t)\big)\right]^p \le C$$

with positive constants p_i, q_i and C (see Proposition 2.1). Moreover, we derive an innovative and undiscovered geometric structure to (1.1) in the almost surely sense

$$\mathrm{d}Z(t) \wedge K(Z(t))\mathrm{d}Z(t) = \mathrm{d}Z_0 \wedge K(Z_0)\mathrm{d}Z_0 \tag{1.2}$$

with $Z(t) = (X(t), Y(t))$, $Z_0 = (X_0, Y_0)$ and a skew-symmetric matrix $K = [k_{ij}]_{2d\times 2d}$ defined in (2.10). Here, the exterior product

$$\mathrm{d}Z \wedge K(Z)\mathrm{d}Z = \sum_{i,j=1,\cdots,2d} k_{ij}\mathrm{d}z_i \wedge \mathrm{d}z_j,$$

and $\mathrm{d}z_i \wedge \mathrm{d}z_j$ of two differential 1-forms $\mathrm{d}z_i$ and $\mathrm{d}z_j$ defines a differential 2-form on $\mathbb{R}^{2d}$:

$$(\mathrm{d}z_i \wedge \mathrm{d}z_j)(\xi,\zeta) = \det\begin{bmatrix} \mathrm{d}z_i(\xi) & \mathrm{d}z_j(\xi) \\ \mathrm{d}z_i(\zeta) & \mathrm{d}z_j(\zeta)\end{bmatrix} \quad \forall\, \xi,\zeta \in \mathbb{R}^{2d},$$

which is the oriented area of the parallelogram generated by $(\xi_i,\xi_j)^\top$ and $(\zeta_i,\zeta_j)^\top$ with ξ_i being the ith entry in ξ. We call (1.2) the stochastic symplectic conservation law for the model (1.1) (see Theorem 2.2). To the best of our knowledge, there has been no work in the literature which studies positivity-preserving symplectic numerical methods for the stochastic LV model.

For the stochastic LV model (1.1), an important property is that if the solution (X, Y) is initially positive, then it remains positive for $t > 0$. The violation of this positivity-preserving property may result in unphysical numerical solutions or cause blow-ups of the numerical algorithm. Thus, it is important to design numerical methods that can preserve the intrinsic properties of the original system, due to their superiority in long-time simulation and stability (cf. [2,3,7,8,18]). Motivated by this issue, the second main goal of this work is to construct and analyze numerical methods that inherit the positivity, uniform boundedness of the solution and stochastic symplecticity for stochastic LV model (1.1). To this end, utilizing an auxiliary function, we introduce a general class of stochastic Runge–Kutta type methods for this model, and derive the symplectic conditions on coefficients for the methods to preserve the stochastic symplectic structure. Moreover, for the case $\Sigma^{(2)} = 0$, the pth moment of the numerical solution is uniformly bounded (see Theorem 3.3)

$$\sup_{n=1,\cdots,N} \mathbb{E}\left[|X_n|^p + |Y_n|^p\right] \leq C.$$

Furthermore, the first-order conditions of the proposed methods in the $\mathbb{L}^1(\Omega)$-norm are given based on the uniform boundedness of both the exact solution and the numerical one. Finally, several numerical experiments show the favorable performance of the proposed numerical methods.

The paper is organized as follows. In Sect. 2, the positivity and the uniform boundedness of the solution to (1.1) are proved. The symplectic geometric structure is established for (1.1), which is preserved by the phase flow of the model. In Sect. 3, a class of structure-preserving methods is proposed and our main results are stated: in Sect. 3.1 we give some conditions to guarantee that a given stochastic Runge–Kutta type method is symplectic; in Sect. 3.2, we get the unique existence, positivity and uniform boundedness of the numerical solution of the general class of stochastic Runge–Kutta type methods; Sect. 3.3 is devoted to obtaining the convergence order condition of the stochastic Runge–Kutta type methods. Finally, numerical experiments are performed in Sect. 4 to testify the effectiveness of the proposed methods.

2 Internal properties of the stochastic LV model

This section is devoted to studying some properties of the stochastic LV model (1.1) in Itô sense, including the well-posedness, positivity, uniform boundedness and stochastic symplecticity of its solution. Throughout this paper, C will be used to denote generic positive constants, which may be different from line to line, and the notation $a \lesssim b$ means $a \leq Cb$ for some positive constant C. When it is necessary, we will use the notation $C(\cdot)$ to indicate the dependence on some parameters.

2.1 Well-posedness and positivity of the solution

The global well-posedness and positivity of the solution to (1.1) are stated in the following theorem.

Theorem 2.1 *For any deterministic initial value* $(X_0, Y_0) \in \mathbb{R}_+^{2d}$*, the system* (1.1) *admits a unique solution* $(X(t), Y(t))$*. Moreover,* $(X(t), Y(t)) \in \mathbb{R}_+^{2d}$ *for all* $t \geq 0$*.*

Proof The local well-posedness of (1.1) is ensured by the local Lipschitz continuity of the coefficients of the stochastic system (cf. [12]). To show the global well-posedness and positivity of the solution, we first denote the explosion time by

$$\tau_e := \inf\{t > 0 |\, |x_i(t)| = \infty \text{ or } |y_i(t)| = \infty \text{ for some } i = 1, \cdots, d\},$$

before which the solution starting from (X_0, Y_0) does not blow up. Since the initial value $(X_0, Y_0) = (x_{0,1}, \cdots, x_{0,d}, y_{0,1}, \cdots, y_{0,d})^\top \in \mathbb{R}_+^{2d}$, there exists some $k_0 \in \mathbb{N}_+$ such that $x_{0,i}, y_{0,i} \in [\frac{1}{k_0}, k_0]$ for all $i = 1, \cdots, d$. For any integer $k \geq k_0$, we define the stopping time

$$\tau_k := \inf\left\{t \in [0, \tau_e) \middle| \, x_i(t) \notin \left(\frac{1}{k}, k\right) \text{ or } y_i(t) \notin \left(\frac{1}{k}, k\right) \text{ for some } i = 1, \cdots, d\right\}.$$

It then suffices to show that $\tau_\infty := \lim_{k\to\infty} \tau_k = \infty$ almost surely.

Motivated by [10], we assume by contradiction that $\tau_\infty < \infty$ with positive probability. More precisely, there exist some constants $T_0 > 0$, $\varepsilon \in (0, 1)$ and $k_1 > k_0$ such that $\mathbb{P}(\tau_k \leq T_0) > \varepsilon$ for all $k \geq k_1$. We introduce the generator $\mathscr{L}$ of (1.1):

$$\begin{aligned}
\mathscr{L} :=& \overline{X}\left(-\Gamma^{(2)}Y + \eta^{(2)}\right) \cdot \nabla_X + \overline{Y}\left(\Gamma^{(1)}X - \eta^{(1)}\right) \cdot \nabla_Y + \frac{1}{2}\left(\overline{X}\Sigma^{(2)}\right)\left(\overline{X}\Sigma^{(2)}\right)^\top : \nabla_X^2 \\
&+ \left(\overline{X}\Sigma^{(2)}\right)\left(\overline{Y}\Sigma^{(1)}\right)^\top : \nabla_X \nabla_Y + \frac{1}{2}\left(\overline{Y}\Sigma^{(1)}\right)\left(\overline{Y}\Sigma^{(1)}\right)^\top : \nabla_Y^2 \\
=& \sum_{i=1}^d x_i \left(-\sum_{j=1}^d \gamma_{ij}^{(2)} y_j + \eta_i^{(2)}\right)\frac{\partial}{\partial x_i} + \sum_{i=1}^d y_i \left(\sum_{j=1}^d \gamma_{ij}^{(1)} x_j - \eta_i^{(1)}\right)\frac{\partial}{\partial y_i} \\
&+ \frac{1}{2}\sum_{i,j=1}^d \sum_{l=1}^m x_i \sigma_{il}^{(2)} \sigma_{jl}^{(2)} x_j \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j=1}^d \sum_{l=1}^m x_i \sigma_{il}^{(2)} \sigma_{jl}^{(1)} y_j \frac{\partial^2}{\partial x_i \partial y_j} \\
&+ \frac{1}{2}\sum_{i,j=1}^d \sum_{l=1}^m y_i \sigma_{il}^{(1)} \sigma_{jl}^{(1)} y_j \frac{\partial^2}{\partial y_i \partial y_j},
\end{aligned} \tag{2.1}$$

where $X = (x_1, \cdots, x_d)^\top$, $Y = (y_1, \cdots, y_d)^\top$, and refer to [17] for more details about generators for stochastic differential equations. For coefficient matrices $\Gamma^{(1)}, \Gamma^{(2)} > 0$ and any given scalars $p_i > 0$, $i = 1, \cdots, d$, we can always find positive scalars

$$q_j := \min_{i=1,\cdots,d} p_i \frac{\gamma_{ij}^{(2)}}{\gamma_{ji}^{(1)}}, \quad j = 1, \cdots, d \tag{2.2}$$

such that

$$q_j \gamma_{ji}^{(1)} - p_i \gamma_{ij}^{(2)} \le 0, \quad \forall\, i, j = 1, \cdots, d. \tag{2.3}$$

Define an auxiliary function $V : \mathbb{R}_+^{2d} \to \mathbb{R}_+$ by

$$V(X, Y) := \sum_{i=1}^{d} \left(V_i^{(1)}(x_i) + V_i^{(2)}(y_i) \right)$$

with

$$\begin{aligned} V_i^{(1)}(x_i) &:= p_i x_i - e_i \left(1 - \ln \left(\frac{e_i}{p_i} \right) \right) - e_i \ln(x_i) \ge 0 \quad \text{for } x_i > 0, \\ V_i^{(2)}(y_i) &:= q_i y_i - f_i \left(1 - \ln \left(\frac{f_i}{q_i} \right) \right) - f_i \ln(y_i) \ge 0 \quad \text{for } y_i > 0, \end{aligned}$$

where $p_i, e_i, f_i \in \mathbb{R}_+$ are parameters to be determined and q_i are determined by p_i in (2.2) such that $\mathscr{L}V(X, Y)$ is uniformly bounded. Based on the definition of V, we get

$$\begin{aligned} \mathscr{L}V(X, Y) =& \sum_{i=1}^{d} x_i \left(-\sum_{j=1}^{d} \gamma_{ij}^{(2)} y_j + \eta_i^{(2)} \right) \left(p_i - \frac{e_i}{x_i} \right) \\ &+ \sum_{i=1}^{d} y_i \left(\sum_{j=1}^{d} \gamma_{ij}^{(1)} x_j - \eta_i^{(1)} \right) \left(q_i - \frac{f_i}{y_i} \right) \\ &+ \frac{1}{2} \sum_{i=1}^{d} \sum_{l=1}^{m} \left(\sigma_{il}^{(2)} \right)^2 e_i + \frac{1}{2} \sum_{i=1}^{d} \sum_{l=1}^{m} \left(\sigma_{il}^{(1)} \right)^2 f_i \\ =& \sum_{i,j=1}^{d} x_i \left(q_j \gamma_{ji}^{(1)} - p_i \gamma_{ij}^{(2)} \right) y_j + \sum_{i=1}^{d} x_i \left(p_i \eta_i^{(2)} - \sum_{j=1}^{d} f_j \gamma_{ji}^{(1)} \right) \\ &- \sum_{j=1}^{d} y_j \left(q_j \eta_j^{(1)} - \sum_{i=1}^{d} e_i \gamma_{ij}^{(2)} \right) + \sum_{i=1}^{d} \left(f_i \eta_i^{(1)} - e_i \eta_i^{(2)} \right) \\ &+ \frac{1}{2} \sum_{i=1}^{d} \sum_{l=1}^{m} \left(\sigma_{il}^{(2)} \right)^2 e_i + \frac{1}{2} \sum_{i=1}^{d} \sum_{l=1}^{m} \left(\sigma_{il}^{(1)} \right)^2 f_i. \end{aligned}$$

Parameters p_i, e_i, f_i can be chosen as positive real numbers such that

$$\begin{aligned} p_i \eta_i^{(2)} - \sum_{j=1}^{d} f_j \gamma_{ji}^{(1)} = 0, \\ q_j \eta_j^{(1)} - \sum_{i=1}^{d} e_i \gamma_{ij}^{(2)} = 0, \end{aligned} \tag{2.4}$$

which, together with (2.3), leads to

$$\begin{aligned} \mathscr{L}V(X,Y) \leq & \sum_{i=1}^{d} \left(f_i \eta_i^{(1)} - e_i \eta_i^{(2)} \right) + \frac{1}{2} \sum_{i=1}^{d} \sum_{l=1}^{m} \left(\sigma_{il}^{(2)} \right)^2 e_i + \frac{1}{2} \sum_{i=1}^{d} \sum_{l=1}^{m} \left(\sigma_{il}^{(1)} \right)^2 f_i \\ =: & \, C^* \end{aligned}$$

for any $(X, Y) \in \mathbb{R}_+^{2d}$. Denote $\hat{\tau}_k := \min\{\tau_k, T_0\}$. For any $t \in (0, \hat{\tau}_k)$, it holds $x_i(t), y_i(t) \in (\frac{1}{k}, k)$ for any $i = 1, \cdots, d$ according to the definition of τ_k and hence $\mathscr{L}V(X(t), Y(t)) \leq C^*$. Then Itô's formula applied to $\mathbb{E}V(X(\hat{\tau}_k), Y(\hat{\tau}_k))$ yields

$$\begin{aligned} \mathbb{E}V(X(\hat{\tau}_k), Y(\hat{\tau}_k)) = & V(X_0, Y_0) + \mathbb{E} \int_0^{\hat{\tau}_k} \mathscr{L}V(X(t), Y(t)) dt \\ \leq & V(X_0, Y_0) + C^* T_0 < \infty. \end{aligned}$$

Note that

$$\frac{\partial V}{\partial x_i} = p_i - \frac{e_i}{x_i}, \quad \frac{\partial V}{\partial y_i} = q_i - \frac{f_i}{y_i}.$$

As a result, V has a global minimum at $(X^*, Y^*) = (\frac{e_1}{p_1}, \cdots, \frac{e_d}{p_d}, \frac{f_1}{q_1}, \cdots, \frac{f_d}{q_d})^\top$ with $V(X^*, Y^*) = 0$. According to the definition of τ_k, there exists an entry in either $X(\tau_k)$ or $Y(\tau_k)$ that reaches the boundary of interval $\left(\frac{1}{k}, k\right)$. Thus, for any $k \geq k_1$ and any sample $\omega \in \{\tau_k \leq T_0\}$, we obtain

$$V(X(\tau_k, \omega), Y(\tau_k, \omega)) \geq \min_{\substack{l=1,2 \\ i=1,\cdots,d}} \left\{ V_i^{(l)} \left(\frac{1}{k} \right), V_i^{(l)}(k) \right\}.$$

We conclude from the above that

$$\begin{aligned} V(X_0, Y_0) + C^* T_0 \geq & \mathbb{E}V(X(\hat{\tau}_k), Y(\hat{\tau}_k)) \\ \geq & \mathbb{E}\left[V(X(\tau_k), Y(\tau_k)) \mathbf{1}_{\{\tau_k \leq T_0\}} \right] \\ \geq & \varepsilon \min_{\substack{l=1,2 \\ i=1,\cdots,d}} \left\{ V_i^{(l)} \left(\frac{1}{k} \right), V_i^{(l)}(k) \right\} \to \infty \end{aligned}$$

as $k \to \infty$, which is a contradiction with $V(X_0, Y_0) + C^* T_0 < \infty$.

Hence, $\tau_\infty = \infty$, which completes the proof. □

2.2 Uniform boundedness of the solution

Based on the positivity of the solution of the stochastic LV model (1.1), we now establish the uniform boundedness of the solution.

Proposition 2.1 *For any $(X_0, Y_0) \in \mathbb{R}_+^{2d}$ and $p \geq 1$, the pth moment of the solution $(X(t), Y(t))$ with $t \in [0, T]$ is uniformly bounded. More precisely, there exists a positive constant $C = C(X_0, Y_0, \Gamma^{(l)}, \eta^{(l)}, \Sigma^{(l)}, p, T)$ with $l = 1, 2$ such that*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\sum_{i=1}^{d}\big(p_i x_i(t) + q_i y_i(t)\big)\right]^p \leq C,$$

where p_i and q_i are positive constants determined by conditions (2.2)–(2.4).

Proof Define an auxiliary functional

$$F_p(t, X, Y) = e^t \left[\sum_{i=1}^{d}(p_i x_i + q_i y_i)\right]^p$$

on the domain $[0, T] \times \mathbb{R}_+^{2d}$ with $X = (x_1, \cdots, x_d)^\top$ and $Y = (y_1, \cdots, y_d)^\top$, which satisfies $F_p(t, X(t), Y(t)) > 0$ for any $t \geq 0$ according to Theorem 2.1.

Moreover, based on the definition of operator $\mathscr{L}$ defined in (2.1), we get

$$\begin{aligned}
\mathscr{L}F_p(t, X, Y) &= \sum_{i=1}^{d} p_i x_i \left(-\sum_{j=1}^{d}\gamma_{ij}^{(2)} y_j + \eta_i^{(2)}\right) p e^t \left[\sum_{k=1}^{d}(p_k x_k + q_k y_k)\right]^{p-1} \\
&\quad + \sum_{i=1}^{d} q_i y_i \left(\sum_{j=1}^{d}\gamma_{ij}^{(1)} x_j - \eta_i^{(1)}\right) p e^t \left[\sum_{k=1}^{d}(p_k x_k + q_k y_k)\right]^{p-1} \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{l=1}^{r} p_i p_j x_i \sigma_{il}^{(2)}\sigma_{jl}^{(2)} x_j p(p-1)e^t \left[\sum_{k=1}^{d}(p_k x_k + q_k y_k)\right]^{p-2} \\
&\quad + \sum_{i,j=1}^{d}\sum_{l=1}^{r} p_i q_j x_i \sigma_{il}^{(2)}\sigma_{jl}^{(1)} y_j p(p-1)e^t \left[\sum_{k=1}^{d}(p_k x_k + q_k y_k)\right]^{p-2} \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{l=1}^{r} q_i q_j y_i \sigma_{il}^{(1)}\sigma_{jl}^{(1)} y_j p(p-1)e^t \left[\sum_{k=1}^{d}(p_k x_k + q_k y_k)\right]^{p-2} \\
&\leq \left[\sum_{i=1}^{d}\left(\eta_i^{(2)} p_i x_i - \eta_i^{(1)} q_i y_i\right)\right] p e^t \left[\sum_{k=1}^{d}(p_k x_k + q_k y_k)\right]^{p-1}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\left[\sum_{i,j=1,\cdots,d}\left(p_i p_j x_i x_j + 2p_i q_j x_i y_j + q_i q_j y_i y_j\right)\right] \\
&\max_{\substack{l,r=1,2\\ i,j=1,\cdots,d}}\left[\Sigma^{(l)}\left(\Sigma^{(r)}\right)^{\top}\right]_{ij} p(p-1)e^t\left[\sum_{k=1}^{d}(p_k x_k + q_k y_k)\right]^{p-2} \\
&\lesssim p^2 F_p(t, X, Y),
\end{aligned}
$$

where we used condition (2.3) and the facts

$$
\eta_i^{(2)} p_i x_i - \eta_i^{(1)} q_i y_i \le \eta_i^{(2)}(p_i x_i + q_i y_i)
$$

and

$$
\sum_{i,j=1,\cdots,d}\left(p_i p_j x_i x_j + 2p_i q_j x_i y_j + q_i q_j y_i y_j\right) = \left[\sum_{i=1}^{d}(p_i x_i + q_i y_i)\right]^2.
$$

Then Itô's formula applied to $F_p(t, X(t), Y(t))$ yields

$$
\begin{aligned}
\mathbb{E}[F_p(t, X(t), Y(t))] = F_p(0, X_0, Y_0) + \mathbb{E}\int_0^t \frac{\partial}{\partial s} F_p(s, X(s), Y(s))ds \\
+\mathbb{E}\int_0^t \mathscr{L} F_p(s, X(s), Y(s))ds \\
\le F_p(0, X_0, Y_0) + \left(1 + Cp^2\right)\int_0^t \mathbb{E}[F_p(s, x(s), y(s))]ds.
\end{aligned}
$$

By Gronwall's lemma, it holds

$$
\mathbb{E}[F_p(t, x(t), y(t))] \le F_p(0, X_0, Y_0)e^{(1+Cp^2)t}.
$$

Multiplying both sides of the above inequality by e^{-t}, we get

$$
\sup_{t\in[0,T]} \mathbb{E}\left[\sum_{i=1}^{d}(p_i x_i(t) + q_i y_i(t))\right]^p \le F_p(0, X_0, Y_0)e^{Cp^2 T},
$$

which completes the proof. □

2.3 Stochastic symplecticity

In this subsection, we investigate the geometric structure of the stochastic LV model (1.1). To this end, we rewrite (1.1) equivalently as a stochastic differential equation in the Stratonovich sense:

$$
\begin{aligned}
dX(t) &= \overline{X}(t)\left(-\Gamma^{(2)}Y(t) + \eta^{(2)} - \frac{1}{2}\Lambda^{(2)}\right)dt + \overline{X}(t) \circ \Sigma^{(2)} dW(t),\\
dY(t) &= \overline{Y}(t)\left(\Gamma^{(1)}X(t) - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)}\right)dt + \overline{Y}(t) \circ \Sigma^{(1)} dW(t),
\end{aligned} \tag{2.5}
$$

where '$\circ$' means that the stochastic integral holds in the Stratonovich sense, and

$$
\Lambda^{(l)} = \left(\lambda_1^{(l)}, \cdots, \lambda_d^{(l)}\right)^\top \quad \text{with} \quad \lambda_i^{(l)} = \sum_{r=1}^{m} \left(\sigma_{ir}^{(l)}\right)^2, \quad i = 1, \cdots, d,\ l = 1, 2.
$$

Before giving the geometric structure of (2.5), we first consider stochastic differential equations in the following form

$$
dZ(t) = K^{-1}(Z(t))\nabla_Z H_0(Z(t))dt + K^{-1}(Z(t))\nabla_Z H_1(Z(t)) \circ dW(t) \tag{2.6}
$$

with a general invertible and skew-symmetric matrix K depending on Z and general Hamiltonian functions H_0 and H_1.

Theorem 2.2 *Assume that* $K = [k_{ij}]_{2d\times 2d}$ *is a skew-symmetric matrix satisfying*

$$
\frac{\partial k_{js}}{\partial z_i} + \frac{\partial k_{ij}}{\partial z_s} + \frac{\partial k_{si}}{\partial z_j} = 0, \quad \forall\, i, j, s = 1, \cdots, 2d. \tag{2.7}
$$

Then a $2d$*-dimensional stochastic differential equation in the form of (2.12) possesses the following symplectic conservation law almost surely:*

$$
\mathrm{d}Z(t) \wedge K(Z(t))\mathrm{d}Z(t) = \mathrm{d}Z_0 \wedge K(Z_0)\mathrm{d}Z_0, \quad \forall\, t \geq 0,\ Z_0 \in \mathbb{R}^{2d}. \tag{2.8}
$$

Proof Throughout the proof, we omit the dependence of Z on t and the dependence of K on Z unless it is necessary to avoid confusions.

Rewriting Z and K by their components, i.e., $Z = (z_1, \cdots, z_{2d})^\top$ and $K = [k_{ij}]_{2d\times 2d}$, we have

$$
\begin{aligned}
d\Big(\mathrm{d}Z \wedge K\mathrm{d}Z\Big) &= \sum_{i,j=1}^{2d} \mathrm{d}(dz_i) \wedge k_{ij}\mathrm{d}z_j + \sum_{i,j=1}^{2d} \mathrm{d}z_i \wedge (dk_{ij})\mathrm{d}z_j + \sum_{i,j=1}^{2d} \mathrm{d}z_i \wedge k_{ij}\mathrm{d}(dz_j)\\
&=: I + II + III.
\end{aligned}
$$

Denote $K^{-1} = \left[\tilde{k}_{ij}\right]_{2d\times 2d}$, which satisfies

$$
\sum_{r=1}^{2d} k_{ir}\tilde{k}_{rj} = -\sum_{r=1}^{2d} k_{ri}\tilde{k}_{rj} = \delta_{ij} := \begin{cases} 1, & i = j,\\ 0, & i \neq j. \end{cases}
$$

Then for terms I and III, we get

$$
\begin{aligned}
I = & \sum_{i,j,r,s=1}^{2d} k_{ij}\Bigg[\left(\frac{\partial \tilde{k}_{ir}}{\partial z_s}\frac{\partial H_0}{\partial z_r} + \tilde{k}_{ir}\frac{\partial^2 H_0}{\partial z_s \partial z_r}\right)dt + \left(\frac{\partial \tilde{k}_{ir}}{\partial z_s}\frac{\partial H_1}{\partial z_r} + \tilde{k}_{ir}\frac{\partial^2 H_1}{\partial z_s \partial z_r}\right)\circ dW\Bigg]\mathrm{d}z_s \wedge \mathrm{d}z_j \\
= & \sum_{i,j,r,s=1}^{2d}\left(k_{sj}\frac{\partial \tilde{k}_{sr}}{\partial z_i}\frac{\partial H_0}{\partial z_r}dt + k_{sj}\frac{\partial \tilde{k}_{sr}}{\partial z_i}\frac{\partial H_1}{\partial z_r}\circ dW\right)\mathrm{d}z_i \wedge \mathrm{d}z_j \\
& - \sum_{j,r,s=1}^{2d}\delta_{jr}\left(\frac{\partial^2 H_0}{\partial z_s \partial z_r}dt + \frac{\partial^2 H_1}{\partial z_s \partial z_r}\circ dW\right)\mathrm{d}z_s \wedge \mathrm{d}z_j
\end{aligned}
$$

and

$$
\begin{aligned}
III = & \sum_{i,j,r,s=1}^{2d} k_{ij}\Bigg[\left(\frac{\partial \tilde{k}_{jr}}{\partial z_s}\frac{\partial H_0}{\partial z_r} + \tilde{k}_{jr}\frac{\partial^2 H_0}{\partial z_s \partial z_r}\right)dt + \left(\frac{\partial \tilde{k}_{jr}}{\partial z_s}\frac{\partial H_1}{\partial z_r} + \tilde{k}_{jr}\frac{\partial^2 H_1}{\partial z_s \partial z_r}\right)\circ dW\Bigg]\mathrm{d}z_i \wedge \mathrm{d}z_s \\
= & \sum_{i,j,r,s=1}^{2d}\left(k_{is}\frac{\partial \tilde{k}_{sr}}{\partial z_j}\frac{\partial H_0}{\partial z_r}dt + k_{is}\frac{\partial \tilde{k}_{sr}}{\partial z_j}\frac{\partial H_1}{\partial z_r}\circ dW\right)\mathrm{d}z_i \wedge \mathrm{d}z_j \\
& + \sum_{i,r,s=1}^{2d}\delta_{ir}\left(\frac{\partial^2 H_0}{\partial z_s \partial z_r}dt + \frac{\partial^2 H_1}{\partial z_s \partial z_r}\circ dW\right)\mathrm{d}z_i \wedge \mathrm{d}z_s.
\end{aligned}
$$

For term II, it holds

$$
\begin{aligned}
II = & \sum_{i,j,r,s=1}^{2d}\left(\tilde{k}_{rs}\frac{\partial k_{ij}}{\partial z_r}\frac{\partial H_0}{\partial z_s}dt + \tilde{k}_{rs}\frac{\partial k_{ij}}{\partial z_r}\frac{\partial H_1}{\partial z_s}\circ dW\right)\mathrm{d}z_i \wedge \mathrm{d}z_j \\
= & \sum_{i,j,r,s=1}^{2d}\left(\tilde{k}_{sr}\frac{\partial k_{ij}}{\partial z_s}\frac{\partial H_0}{\partial z_r}dt + \tilde{k}_{sr}\frac{\partial k_{ij}}{\partial z_s}\frac{\partial H_1}{\partial z_r}\circ dW\right)\mathrm{d}z_i \wedge \mathrm{d}z_j.
\end{aligned}
$$

Adding these three equalities and utilizing the fact that

$$
\sum_{s=1}^{2d}\left(k_{is}\frac{\partial \tilde{k}_{sr}}{\partial z_j} + \frac{\partial k_{is}}{\partial z_j}\tilde{k}_{sr}\right) = 0 \quad \text{and} \quad k_{is} = -k_{si},
$$

we get

$$
\begin{aligned}
d\Big(\mathrm{d}Z\wedge K\mathrm{d}Z\Big) &= \sum_{i,j,r,s=1}^{2d}\left(k_{sj}\frac{\partial\tilde{k}_{sr}}{\partial z_i}+\tilde{k}_{sr}\frac{\partial k_{ij}}{\partial z_s}+k_{is}\frac{\partial\tilde{k}_{sr}}{\partial z_j}\right)\left(\frac{\partial H_0}{\partial z_r}dt+\frac{\partial H_1}{\partial z_r}\circ dW\right)\mathrm{d}z_i\wedge\mathrm{d}z_j\\
&\quad+2\sum_{r,s=1}^{2d}\left(\frac{\partial^2 H_0}{\partial z_s\partial z_r}\mathrm{d}t+\frac{\partial^2 H_1}{\partial z_s\partial z_r}\circ dW\right)\mathrm{d}z_r\wedge\mathrm{d}z_s\\
&=\sum_{i,j,r,s=1}^{2d}\left(\frac{\partial k_{js}}{\partial z_i}+\frac{\partial k_{ij}}{\partial z_s}+\frac{\partial k_{si}}{\partial z_j}\right)\tilde{k}_{sr}\left(\frac{\partial H_0}{\partial z_r}dt+\frac{\partial H_1}{\partial z_r}\circ dW\right)\mathrm{d}z_i\wedge\mathrm{d}z_j\\
&\quad+2\sum_{r,s=1}^{2d}\left(\frac{\partial^2 H_0}{\partial z_s\partial z_r}\mathrm{d}t+\frac{\partial^2 H_1}{\partial z_s\partial z_r}\circ dW\right)\mathrm{d}z_r\wedge\mathrm{d}z_s\\
&=0
\end{aligned}
$$

almost surely due to the skew-symmetry of the 2-form $\mathrm{d}z_r\wedge\mathrm{d}z_s=-\mathrm{d}z_s\wedge\mathrm{d}z_r$, which completes the proof. □

Remark 2.1 To avoid confusion, we would like to mention that the operator 'd' in equation (2.12) denotes the differential with respect to time, while the operator 'd' in the differential 2-forms in (2.8) denotes the exterior derivative on the manifold.

Theorem 2.3 *Assume that the coefficient matrices $\Gamma^{(l)}$ with $l=1,2$ are diagonal. Then the phase flow of (2.5) is a stochastic symplectic transformation and possesses the stochastic symplectic conservation law*

$$
\mathrm{d}Z(t)\wedge K(Z(t))\mathrm{d}Z(t)=\mathrm{d}Z_0\wedge K(Z_0)\mathrm{d}Z_0,\quad\forall\, t\geq 0 \tag{2.9}
$$

almost surely, where $Z=(X,Y)=(x_1,\cdots,x_d,y_1,\cdots,y_d)^\top$ with $Z_0=(X_0,Y_0)\in\mathbb{R}_+^{2d}$ and K is a skew-symmetric matrix given by

$$
K(Z)=\begin{bmatrix}0 & -K^*(Z)\\ K^*(Z) & 0\end{bmatrix},\quad K^*(Z)=\mathrm{diag}\left\{(x_1y_1)^{-1},\cdots,(x_dy_d)^{-1}\right\}. \tag{2.10}
$$

Equivalently, the phase flow $\varphi_t:Z_0\mapsto Z(t)$ satisfies

$$
\left[\frac{\partial\varphi_t(Z_0)}{\partial Z_0}\right]^\top K(\varphi_t(Z_0))\left[\frac{\partial\varphi_t(Z_0)}{\partial Z_0}\right]=K(Z_0). \tag{2.11}
$$

Proof We first show that (2.5) possesses the stochastic symplectic conservation law. Denote

$$\begin{aligned} H_0(Z) := & \sum_{i=1}^{d}\left[-\gamma_{ii}^{(1)}x_i + \left(\eta_i^{(1)} + \frac{1}{2}\sum_{j=1}^{d}\left(\sigma_{ij}^{(1)}\right)^2 \ln x_i\right)\right] \\ & + \sum_{i=1}^{d}\left[-\gamma_{ii}^{(2)}y_i + \left(\eta_i^{(2)} - \frac{1}{2}\sum_{j=1}^{d}\left(\sigma_{ij}^{(2)}\right)^2 \ln y_i\right)\right], \\ H_1(Z) := & \left[\sum_{i=1}^{d}\left(-\sigma_{i1}^{(1)}\ln x_i + \sigma_{i1}^{(2)}\ln y_i\right), \cdots, \sum_{i=1}^{d}\left(-\sigma_{im}^{(1)}\ln x_i + \sigma_{im}^{(2)}\ln y_i\right)\right]_{1\times m}. \end{aligned}$$

Due to Theorem 2.1, functions $H_0(Z(t))$ and $H_1(Z(t))$ are well-defined, and (2.5) can be rewritten as

$$dZ(t) = K^{-1}(Z(t))\nabla_Z H_0(Z(t))dt + K^{-1}(Z(t))\nabla_Z H_1(Z(t)) \circ dW(t) \quad (2.12)$$

with K given in (2.10). According to Theorem 2.2, it suffices to show that condition (2.7) is satisfied.

In fact, the matrix $K = [k_{ij}]_{2d\times 2d}$ given in (2.10) satisfies

$$k_{ij} = \begin{cases} -(x_i y_i)^{-1}, & |i-j| = d,\ i<j \\ (x_j y_j)^{-1}, & |i-j| = d,\ i>j \\ 0, & \text{otherwise.} \end{cases}$$

By noting that $z_{i+d} = y_i$ for $i = 1, \cdots, d$, it can be verified that $\frac{\partial k_{js}}{\partial z_i} \neq 0$ if and only if $|j-s| = d$ and $i \in \{j, s\}$. For the case $i = j$, we get (2.7) holds with

$$\frac{\partial k_{js}}{\partial z_i} + \frac{\partial k_{ij}}{\partial z_s} + \frac{\partial k_{si}}{\partial z_j} = \frac{\partial k_{js}}{\partial z_j} + \frac{\partial k_{sj}}{\partial z_j} = 0,$$

and the case $i = s$ can be proved similarly.

Next, we show the equivalence between (2.9) and (2.11). Note that

$$\begin{aligned} \mathrm{d}Z(t) \wedge K(Z(t))\mathrm{d}Z(t) &= \frac{\partial\varphi_t(Z_0)}{\partial Z_0}\mathrm{d}Z_0 \wedge K(Z(t))\frac{\partial\varphi_t(Z_0)}{\partial Z_0}\mathrm{d}Z_0 \\ &= \mathrm{d}Z_0 \wedge \left(\left[\frac{\partial\varphi_t(Z_0)}{\partial Z_0}\right]^\top K(\varphi_t(Z_0))\left[\frac{\partial\varphi_t(Z_0)}{\partial Z_0}\right]\right)\mathrm{d}Z_0. \end{aligned}$$

If (2.11) holds, then (2.9) holds apparently. If (2.9) holds, then we get

$$\mathrm{d}Z_0 \wedge \left(\left[\frac{\partial\varphi_t(Z_0)}{\partial Z_0}\right]^\top K(\varphi_t(Z_0))\left[\frac{\partial\varphi_t(Z_0)}{\partial Z_0}\right] - K(Z_0)\right)\mathrm{d}Z_0 = 0.$$

On one hand, since K is a skew-symmetric matrix, the matrix

$$M := \left[\frac{\partial \varphi_t(Z_0)}{\partial Z_0}\right]^\top K(\varphi_t(Z_0)) \left[\frac{\partial \varphi_t(Z_0)}{\partial Z_0}\right] - K(Z_0)$$

is also skew-symmetric. On the other hand, the equation $\mathrm{d}Z_0 \wedge M\mathrm{d}Z_0 = 0$, together with the property for wedge product, implies that M is symmetric. We then conclude that $M = 0$, that is, (2.11) holds. □

3 Positivity-preserving symplectic methods

In this section, we design a Runge–Kutta type method for the stochastic LV model (1.1), which is obtained by applying the stochastic Runge–Kutta method to an auxiliary equation. We show the positivity of the numerical solution, and present the conditions for the proposed method to be symplectic.

Based on the positivity of the solution (X, Y) to (2.5), we introduce the auxiliary stochastic processes

$$U = \ln X := (\ln x_1, \cdots, \ln x_d)^\top,$$
$$V = \ln Y := (\ln y_1, \cdots, \ln y_d)^\top,$$

and define notations $e^V := (e^{v_1}, \cdots, e^{v_d})^\top$ and $V^p := (v_1^p, \cdots, v_d^p)^\top$ for any vector $V = (v_1, \ldots, v_d)^\top$ and $p \in \mathbb{R}_+$. Then (2.5) can be rewritten into a system related to (U, V):

$$\begin{aligned} dU(t) &= \left(-\Gamma^{(2)} e^{V(t)} + \eta^{(2)} - \frac{1}{2}\Lambda^{(2)}\right) dt + \Sigma^{(2)} dW(t), \\ dV(t) &= \left(\Gamma^{(1)} e^{U(t)} - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)}\right) dt + \Sigma^{(1)} dW(t). \end{aligned} \tag{3.1}$$

For the time interval $[0, T]$, we introduce the uniform partition $0 = t_0 < t_1 < \cdots < t_N = T$ with step-size $h = T/N$. For any fixed $s \in \mathbb{N}$, applying the s-stage stochastic Runge–Kutta method to (3.1), we obtain

$$
\begin{aligned}
U_{n,i} &= U_n + h\sum_{j=1}^{s} a_{ij}\left(-\Gamma^{(2)}e^{V_{n,j}} + \eta^{(2)} - \frac{1}{2}\Lambda^{(2)}\right) + \sum_{j=1}^{s} b_{ij}\,\Sigma^{(2)} J_{n+1},\\
V_{n,i} &= V_n + h\sum_{j=1}^{s} a_{ij}\left(\Gamma^{(1)}e^{U_{n,j}} - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)}\right) + \sum_{j=1}^{s} b_{ij}\,\Sigma^{(1)} J_{n+1},\\
U_{n+1} &= U_n + h\sum_{i=1}^{s} \alpha_{i}\left(-\Gamma^{(2)}e^{V_{n,i}} + \eta^{(2)} - \frac{1}{2}\Lambda^{(2)}\right) + \sum_{i=1}^{s} \beta_{i}\,\Sigma^{(2)} J_{n+1},\\
V_{n+1} &= V_n + h\sum_{i=1}^{s} \alpha_{i}\left(\Gamma^{(1)}e^{U_{n,i}} - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)}\right) + \sum_{i=1}^{s} \beta_{i}\,\Sigma^{(1)} J_{n+1}
\end{aligned}
\tag{3.2}
$$

starting from $U_0 = \ln X_0$ and $V_0 = \ln X_0$ for $i = 1, \cdots, s$ and $n = 0, \cdots, N-1$. Here, $J_{n+1} := W(t_{n+1}) - W(t_n)$ denotes the increment of the Wiener process. By defining $X_{n,i} = e^{U_{n,i}}$, $Y_{n,i} = e^{V_{n,i}}$, $X_n = e^{U_n}$ and $Y_n = e^{V_n}$, we finally get the following s-stage method starting from (X_0, Y_0) for the original system (1.1):

$$
\begin{aligned}
X_{n,i} &= X_n .* \exp\left(h\sum_{j=1}^{s} a_{ij}\left(-\Gamma^{(2)}Y_{n,j} + \eta^{(2)} - \frac{1}{2}\Lambda^{(2)}\right)\right) .* \exp\left(\sum_{j=1}^{s} b_{ij}\,\Sigma^{(2)} J_{n+1}\right),\\
Y_{n,i} &= Y_n .* \exp\left(h\sum_{j=1}^{s} a_{ij}\left(\Gamma^{(1)}X_{n,j} - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)}\right)\right) .* \exp\left(\sum_{j=1}^{s} b_{ij}\,\Sigma^{(1)} J_{n+1}\right),\\
X_{n+1} &= X_n .* \exp\left(h\sum_{i=1}^{s} \alpha_{i}\left(-\Gamma^{(2)}Y_{n,i} + \eta^{(2)} - \frac{1}{2}\Lambda^{(2)}\right)\right) .* \exp\left(\sum_{i=1}^{s} \beta_{i}\,\Sigma^{(2)} J_{n+1}\right),\\
Y_{n+1} &= Y_n .* \exp\left(h\sum_{i=1}^{s} \alpha_{i}\left(\Gamma^{(1)}X_{n,i} - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)}\right)\right) .* \exp\left(\sum_{i=1}^{s} \beta_{i}\,\Sigma^{(1)} J_{n+1}\right),
\end{aligned}
\tag{3.3}
$$

where the product '$.*$' is defined by

$$
U .* V = V .* U := (u_1 v_1, \cdots, u_d v_d)^\top \in \mathbb{R}^d, \quad c .* V = V .* c := cV
$$

for any vectors $U = (u_1, \cdots, u_d)^\top$, $V = (v_1, \cdots, v_d)^\top \in \mathbb{R}^d$ and scalar $c \in \mathbb{R}$.

The parameters of method (3.3) can be characterized by the Butcher tableau

$$
\begin{array}{c|cc}
 & A & B \\
\hline
 & \alpha^\top & \beta^\top
\end{array}
$$

with $A = [a_{ij}]_{s\times s}$, $B = [b_{ij}]_{s\times s}$, $\alpha = (\alpha_1, \cdots, \alpha_s)^\top$ and $\beta = (\beta_1, \cdots, \beta_s)^\top$.

For the stochastic Runge–Kutta type method (3.3), the positivity of its solution can be obtained directly.

Theorem 3.1 *For any deterministic initial value* $(X_0, Y_0) \in \mathbb{R}_+^{2d}$, *the solution of* (3.3) *satisfies* $(X_n, Y_n) \in \mathbb{R}_+^{2d}$ *for all* $n = 0, 1, \cdots, N$.

3.1 Symplectic condition of stochastic Runge–Kutta type method

In this subsection, we proceed to analyze the condition of stochastic symplecticity for the proposed method (3.3).

Theorem 3.2 *Assume that assumptions in Theorem* 2.3 *hold and coefficients* a_{ij}, α_i *in* (3.3) *satisfy*

$$\alpha_i a_{ij} + \alpha_j a_{ji} = \alpha_i \alpha_j, \quad \forall\, i, j = 1, 2, \ldots, s. \tag{3.4}$$

Then the stochastic Runge–Kutta type method (3.3) *preserves the discrete stochastic symplectic conservation law*

$$\mathrm{d}Z_{n+1} \wedge K(Z_{n+1})\mathrm{d}Z_{n+1} = \mathrm{d}Z_n \wedge K(Z_n)\mathrm{d}Z_n, \quad \forall\, n \in \mathbb{N}$$

almost surely, where $Z_n = (X_n, Y_n)$ *and* K *is defined in* (2.10).

Proof Based on (3.2), we denote

$$F_j = -\Gamma^{(2)} e^{V_{n,j}}, \quad G_j = \Gamma^{(1)} e^{U_{n,j}}.$$

Then the exterior differential applied to (3.2) leads to

$$\begin{aligned} \mathrm{d}U_{n,i} &= \mathrm{d}U_n + h\sum_{j=1}^{s} a_{ij}\mathrm{d}F_j, \quad \mathrm{d}V_{n,i} = \mathrm{d}V_n + h\sum_{j=1}^{s} a_{ij}\mathrm{d}G_j, \\ \mathrm{d}U_{n+1} &= \mathrm{d}U_n + h\sum_{i=1}^{s} \alpha_i\mathrm{d}F_i, \quad \mathrm{d}V_{n+1} = \mathrm{d}V_n + h\sum_{i=1}^{s} \alpha_i\mathrm{d}G_i. \end{aligned} \tag{3.5}$$

Therefore, we obtain

$$\begin{aligned} \mathrm{d}U_{n+1} \wedge \mathrm{d}V_{n+1} &= \mathrm{d}U_n \wedge \mathrm{d}V_n + h\sum_{i=1}^{s} \alpha_i \mathrm{d}U_n \wedge \mathrm{d}G_i \\ &\quad + h\sum_{i=1}^{s} \alpha_i \mathrm{d}F_i \wedge \mathrm{d}V_n + h^2 \sum_{i,j=1}^{s} \alpha_i\alpha_j \mathrm{d}F_i \wedge \mathrm{d}G_j \\ &= \mathrm{d}U_n \wedge \mathrm{d}V_n + h\sum_{i=1}^{s} \alpha_i \left(\mathrm{d}U_{n,i} \wedge \mathrm{d}G_i + \mathrm{d}F_i \wedge \mathrm{d}V_{n,i}\right) \\ &\quad + h^2 \sum_{i,j=1}^{s} \left(\alpha_i\alpha_j - \alpha_i a_{ij} - \alpha_j a_{ji}\right) \mathrm{d}F_i \wedge \mathrm{d}G_j, \end{aligned} \tag{3.6}$$

where we used the facts

$$\mathrm{d}U_n \wedge \mathrm{d}G_i = \mathrm{d}U_{n,i} \wedge \mathrm{d}G_i - h\sum_{j=1}^{s} a_{ij}\mathrm{d}F_j \wedge \mathrm{d}G_i,$$

$$\mathrm{d}F_i \wedge \mathrm{d}V_n = \mathrm{d}F_i \wedge \mathrm{d}V_{n,i} - h\sum_{j=1}^{s} a_{ij}\mathrm{d}F_i \wedge \mathrm{d}G_j$$

according to (3.5). Note that

$$\mathrm{d}U_{n,i} \wedge \mathrm{d}G_i = \mathrm{d}U_{n,i} \wedge \left(\Gamma^{(1)}\overline{e^{U_{n,i}}}\right)\mathrm{d}U_{n,i} = 0,$$

where the notation $\overline{e^{U_{n,i}}}$ is defined in the introduction, and the last step holds due to the fact that $\Gamma^{(1)}\overline{e^{U_{n,i}}}$ is a symmetric matrix. Similarly, we can also get $\mathrm{d}F_i \wedge \mathrm{d}V_{n,i} = 0$. Combined with condition (3.4), formula (3.6) turns to be

$$\mathrm{d}U_{n+1} \wedge \mathrm{d}V_{n+1} = \mathrm{d}U_n \wedge \mathrm{d}V_n.$$

We then complete the proof by noting the facts

$$\mathrm{d}U_n \wedge \mathrm{d}V_n = K^*(Z_n)\mathrm{d}X_n \wedge \mathrm{d}Y_n,$$

and

$$\mathrm{d}Z_n \wedge K(Z_n)\mathrm{d}Z_n = -2K^*(Z_n)\mathrm{d}X_n \wedge \mathrm{d}Y_n = -2\mathrm{d}U_n \wedge \mathrm{d}V_n,$$

where K^* is the symmetric matrix defined in (2.10). □

3.2 Uniform boundedness of the numerical solution

In this subsection, we consider a special case that $\Sigma^{(2)} = 0$, that is, only the death rate of the predator is perturbed by a random noise which may be caused by poaching. The uniform boundedness of the numerical solution given by the stochastic Runge–Kutta type method (3.3) is studied.

Theorem 3.3 *Assume that $\Sigma^{(2)} = 0$ and $\alpha_i \geq 0$ for $i = 1, \cdots, s$. For any deterministic initial value $(X_0, Y_0) \in \mathbb{R}_+^{2d}$, the stochastic Runge–Kutta type method (3.3) admits an $\mathscr{F}_{t_n}$-adapted solution $(X_n, Y_n) \in \mathbb{R}_+^{2d}$. Furthermore, for any integer $p \geq 1$, the pth moment of the numerical solution is uniformly bounded*

$$\sup_{n=1,\cdots,N} \mathbb{E}\left[|X_n|^p + |Y_n|^p\right] \leq C,$$

where $C = C(X_0, Y_0, \Gamma^{(l)}, \eta^{(l)}, \Sigma^{(l)}, p, T, A, \alpha, \beta)$ with $l = 1, 2$ is a positive constant, and $|\cdot|$ denotes the Euclidean norm of a vector.

Proof The existence and adaptness of the solution to (3.3) can be proved by means of the procedure given in [5]. Noting that $(X_{n,i}, Y_{n,i}) \in \mathbb{R}_+^{2d}$ with

$$X_{n,i} \leq X_n . * \exp\left(h \sum_{j=1}^{s} a_{ij} \eta^{(2)}\right),$$

and denoting constants $C_\alpha = \sum_{j=1}^{s} \alpha_j$ and $C_\beta = \sum_{j=1}^{s} \beta_j$, we have for all $n = 0, 1, \cdots, N-1$ that

$$\begin{aligned}
X_{n+1} \leq & X_n . * \exp\left(hC_\alpha \eta^{(2)}\right) \leq X_0 . * \exp\left(TC_\alpha \eta^{(2)}\right), \\
Y_{n+1} \leq & Y_n . * \exp\left(h \sum_{i=1}^{s} \alpha_i \Gamma^{(1)} X_{n,i}\right) . * \exp\left(C_\beta \Sigma^{(1)} J_{n+1}\right) \\
\leq & Y_n . * \exp\left[h \sum_{i=1}^{s} \alpha_i \Gamma^{(1)} X_n . * \exp\left(h \sum_{j=1}^{s} a_{ij} \eta^{(2)}\right)\right] . * \exp\left(C_\beta \Sigma^{(1)} J_{n+1}\right) \\
\leq & Y_n . * \exp\left[h \sum_{i=1}^{s} \alpha_i \Gamma^{(1)} X_0 . * \exp\left(TC_\alpha \eta^{(2)} + h \sum_{j=1}^{s} a_{ij} \eta^{(2)}\right)\right] . * \exp\left(C_\beta \Sigma^{(1)} J_{n+1}\right) \\
\leq & Y_0 . * \exp(\tilde{C}) . * \exp\left(C_\beta \Sigma^{(1)} W(t_{n+1})\right)
\end{aligned}$$

with

$$\tilde{C} := T \sum_{i=1}^{s} \alpha_i \Gamma^{(1)} X_0 . * \exp\left(TC_\alpha \eta^{(2)} + h \sum_{j=1}^{s} a_{ij} \eta^{(2)}\right)$$

being a deterministic vector.

Note also that $|U .* V| \leq |U||V|$ for any vectors $U, V \in \mathbb{R}^d$. Then the pth moment of the numerical solution satisfies

$$\begin{aligned}
\mathbb{E}\left[|X_{n+1}|^p + |Y_{n+1}|^p\right] \leq & |X_0|^p \left|\exp\left(TC_\alpha \eta^{(2)}\right)\right|^p \\
& + |Y_0|^p \left|\exp\left(\tilde{C}\right)\right|^p \mathbb{E}\left|\exp\left(C_\beta \Sigma^{(1)} W(t_{n+1})\right)\right|^p,
\end{aligned}$$

which is uniformly bounded due to the fact

$$\begin{aligned}
\mathbb{E}\left|\exp\left(C_\beta \Sigma^{(1)} W(t_{n+1})\right)\right|^p = & \mathbb{E}\left[\sum_{i=1}^{d} \left(\exp\left(C_\beta \sum_{r=1}^{m} \sigma_{i,r}^{(1)} W_r(t_{n+1})\right)\right)^2\right]^{\frac{p}{2}} \\
\lesssim & \sum_{i=1}^{d} \mathbb{E}\left[\exp\left(pC_\beta \sum_{r=1}^{m} \sigma_{i,r}^{(1)} W_r(t_{n+1})\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{d} \prod_{r=1}^{m} \mathbb{E}\left[\exp\left(pC_\beta \sigma_{i,r}^{(1)} W_r(t_{n+1})\right)\right] \\
&= \sum_{i=1}^{d} \prod_{r=1}^{m} \exp\left(\frac{1}{2} p^2 C_\beta^2 \left(\sigma_{i,r}^{(1)}\right)^2 t_{n+1}\right) \\
&\leq \sum_{i=1}^{d} \exp\left(\frac{1}{2} p^2 C_\beta^2 \sum_{r=1}^{m} \left(\sigma_{i,r}^{(1)}\right)^2 T\right)
\end{aligned}
$$

according to the independence of W_r and Itô's formula. □

Remark 3.1 The assumption $\Sigma^{(2)} = 0$ leads to the uniform boundedness of X_n in the almost surely sense, which results in the boundedness of Y_n. For the case $\Sigma^{(2)} \neq 0$, it can be shown that the pth moment of X_n is still uniformly bounded: based on the fact $(X_{n,i}, Y_{n,i}) \in \mathbb{R}_+^{2d}$, we get from (3.3) that

$$
\begin{aligned}
X_{n+1} &\leq X_n.*\exp\left(hC_\alpha \eta^{(2)} + C_\beta \Sigma^{(2)} J_{n+1}\right) \\
&\leq X_0.*\exp\left(TC_\alpha \eta^{(2)} + C_\beta \Sigma^{(2)} W(t_{n+1})\right),
\end{aligned}
$$

and hence $\mathbb{E}|X_{n+1}|^p \leq C$ based on the procedure used in the proof of Theorem 3.3. However, in this case, the boundedness of Y_n can not be obtained based on the same procedure, and some other techniques need to be explored to investigate the boundedness of the numerical solution.

3.3 Convergence order conditions

Based on the uniform boundedness of both the exact solution and the numerical one, the convergence order conditions are stated in the following theorem.

Theorem 3.4 *Assume that assumptions in Theorem 3.3 and the condition*

$$
\sum_{i=1}^{s} \alpha_i = \sum_{i=1}^{s} \beta_i = 1 \tag{3.7}
$$

hold, then the stochastic Runge–Kutta type method (3.3) converges with global order one in the $\mathbb{L}^1(\Omega)$-norm.

Proof It follows from Theorem 2.1 that the auxiliary processes $U(t) = \ln X(t)$ and $V(t) = \ln Y(t)$ are well-defined and satisfy (3.1). Considering the local error between (3.1) and (3.2) with $\Sigma^{(2)} = 0$, and utilizing (3.7), we get

$$
U(h) - U_1 = \Gamma^{(2)} \int_0^h \sum_{i=1}^{s} \alpha_i e^{V_{0,i}} - e^{V(t)} dt,
$$

where

$$
\begin{aligned}
e^{V(t)} &= e^{V_0} + \int_0^t e^{V(s)} .* dV(s) \\
&= e^{V_0} + \int_0^t e^{V(s)} .* \left[\left(\Gamma^{(1)} e^{U(s)} - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)}\right) ds + \Sigma^{(1)} dW(s)\right].
\end{aligned}
$$

Denote $C_i^B := \sum_{j=1}^s b_{ij}$. According to the Taylor expansion, there exists some $\theta \in (0, 1)$, such that

$$
\begin{aligned}
&\sum_{i=1}^s \alpha_i e^{V_{0,i}} \\
&= \sum_{i=1}^s \alpha_i \Bigg[e^{V_0} + e^{V_0} .* \left(h \sum_{j=1}^s a_{ij} \left(\Gamma^{(1)} e^{U_{0,j}} - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)}\right) + C_i^B \Sigma^{(1)} J_1 \right) \\
&+ \frac{1}{2} e^{\theta V_0 + (1-\theta) V_{0,i}} .* \left(h \sum_{j=1}^s a_{ij} \left(\Gamma^{(1)} e^{U_{0,j}} - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)}\right) + C_i^B \Sigma^{(1)} J_1 \right)^2 \Bigg] \\
&= e^{V_0} + h \sum_{i,j=1}^s \alpha_i a_{ij} e^{V_0} .* \left(\Gamma^{(1)} e^{U_{0,j}} - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)}\right) + e^{V_0} .* \Sigma^{(1)} J_1 \sum_{i,j=1}^s \alpha_i b_{ij} \\
&+ \sum_{i=1}^s \frac{\alpha_i}{2} e^{\theta V_0 + (1-\theta) V_{0,i}} .* \left(h \sum_{j=1}^s a_{ij} \left(\Gamma^{(1)} e^{U_{0,j}} - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)}\right) + C_i^B \Sigma^{(1)} J_1 \right)^2.
\end{aligned}
$$

We then conclude that the weak local error between $U(h)$ and U_1 is of order two:

$$
\begin{aligned}
&|\mathbb{E}[U(h) - U_1]| \\
&= \Bigg| \Gamma^{(2)} \int_0^h \mathbb{E}\Bigg[h \sum_{i,j=1}^s \alpha_i a_{ij} e^{V_0} .* \left(\Gamma^{(1)} e^{U_{0,j}} - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)}\right) \\
&- \int_0^t e^{V(s)} .* \left(\Gamma^{(1)} e^{U(s)} - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)}\right) ds \\
&+ \sum_{i=1}^s \frac{\alpha_i}{2} e^{\theta V_0 + (1-\theta) V_{0,i}} .* \left(h \sum_{j=1}^s a_{ij} \left(\Gamma^{(1)} e^{U_{0,j}} - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)}\right) + C_i^B \Sigma^{(1)} J_1 \right)^2 \Bigg] dt \Bigg| \\
&\leq \left|\Gamma^{(2)}\right|_F \Bigg\{ h^2 \sum_{i,j=1}^s |\alpha_i a_{ij}| \left|e^{V_0}\right| \left| \Gamma^{(1)} \mathbb{E}\left[e^{U_{0,j}}\right] - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)} \right| \\
&+ \int_0^h \int_0^t \mathbb{E}\left[\left|e^{V(s)}\right| \left| \Gamma^{(1)} e^{U(s)} - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)} \right| \right] ds dt \\
&+ h \sum_{i=1}^s \frac{\alpha_i}{2} \mathbb{E}\Bigg[\left| e^{\theta V_0 + (1-\theta) V_{0,i}} \right| \left| h \sum_{j=1}^s a_{ij} \left(\Gamma^{(1)} e^{U_{0,j}} - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)}\right) + C_i^B \Sigma^{(1)} J_1 \right|^2 \Bigg] \Bigg\} \\
&\lesssim h^2,
\end{aligned}
$$

where we used the uniform boundedness of the solution $X(t) = e^{U(t)}$ and $Y(t) = e^{V(t)}$, as well as the facts $|U .* V| \leq |U||V|$ for any vectors $U, V \in \mathbb{R}^d$ and $|AU| \leq |A|_F|U|$ for a matrix A and a vector U with $|\cdot|_F$ being the Frobenius norm of a matrix.

Similarly, according to the Taylor expansion, we obtain

$$\sum_{i=1}^{s} \alpha_i e^{V_{0,i}} = e^{V_0} + \sum_{i=1}^{s} \alpha_i e^{\tilde{\theta} V_0 + (1-\tilde{\theta}) V_{0,i}} .* \left[h \sum_{j=1}^{s} a_{ij} \left(\Gamma^{(1)} e^{U_{0,j}} - \eta^{(1)} - \frac{1}{2} \Lambda^{(1)} \right) \right.$$
$$\left. + C_i^B \Sigma^{(1)} J_1 \right]$$

for some $\tilde{\theta} \in (0, 1)$. Then the strong local error satisfies

$$\begin{aligned}
&\mathbb{E}\,|U(h) - U_1|^2 \\
&= \mathbb{E} \left| \int_0^h \sum_{i=1}^{s} \alpha_i e^{\tilde{\theta} V_0 + (1-\tilde{\theta}) V_{0,i}} .* \left(h \sum_{j=1}^{s} a_{ij} \left(\Gamma^{(1)} e^{U_{0,j}} - \eta^{(1)} - \frac{1}{2} \Lambda^{(1)} \right) + C_i^B \Sigma^{(1)} J_1 \right) dt \right. \\
&\left. - \int_0^h \int_0^t e^{V(s)} .* \left[\left(\Gamma^{(1)} e^{U(s)} - \eta^{(1)} - \frac{1}{2} \Lambda^{(1)} \right) ds + \Sigma^{(1)} dW(s) \right] dt \right|^2 \\
&\lesssim h^2 \sum_{i=1}^{s} \alpha_i^2 \mathbb{E} \left[\left| e^{\tilde{\theta} V_0 + (1-\tilde{\theta}) V_{0,i}} \right|^2 \left| h \sum_{j=1}^{s} a_{ij} \left(\Gamma^{(1)} e^{U_{0,j}} - \eta^{(1)} - \frac{1}{2} \Lambda^{(1)} \right) + C_i^B \Sigma^{(1)} J_1 \right|^2 \right] \\
&+ h \int_0^h \mathbb{E} \left| \int_0^t e^{V(s)} .* \left[\left(\Gamma^{(1)} e^{U(s)} - \eta^{(1)} - \frac{1}{2} \Lambda^{(1)} \right) ds + \Sigma^{(1)} dW(s) \right] \right|^2 dt \\
&\leq C h^3.
\end{aligned}$$

The estimates for $V(h) - V_1$ can be obtained similarly. Based on the fundamental theorem on the mean-square order of convergence (cf. [13]), one has

$$\left(\mathbb{E} \left[|U(t_n) - U_n|^2 + |V(t_n) - V_n|^2 \right] \right)^{\frac{1}{2}} \leq C h$$

for any $t_n = nh \in [0, T]$, which ensures the final result

$$\begin{aligned}
&\mathbb{E}\left(|X(t_n) - X_n| + |Y(t_n) - Y_n| \right) \\
&= \mathbb{E} \left(\left| e^{U(t_n)} - e^{U_n} \right| + \left| e^{V(t_n)} - e^{V_n} \right| \right) \\
&\leq \mathbb{E} \left(\left| e^{\theta_1 U_n + (1-\theta_1) U(t_n)} (U(t_n) - U_n) \right| + \left| e^{\theta_2 V_n + (1-\theta_2) V(t_n)} (V(t_n) - V_n) \right| \right) \\
&\leq \max \left\{ \left| X_n^{\theta_1} X(t_n)^{1-\theta_1} \right|_{L^2(\Omega)}, \left| Y_n^{\theta_2} Y(t_n)^{1-\theta_2} \right|_{L^2(\Omega)} \right\} \left(\mathbb{E} \left[|U(t_n) - U_n|^2 + |V(t_n) - V_n|^2 \right] \right)^{\frac{1}{2}} \\
&\leq C h
\end{aligned}$$

according to the uniform boundedness of the solution $(X(t), Y(t))$ and (X_n, Y_n) shown in Proposition 2.1 and Theorem 3.3 with $\theta_1, \theta_2 \in (0, 1)$. □

Remark 3.2 Based on the procedure above, we can also construct positivity-preserving symplectic methods for (1.1) based on the following stochastic partitioned Runge–Kutta method

$$\begin{aligned} X_{n,i} &= X_n.*\exp\left(h\sum_{j=1}^{s}a_{ij}\left(-\Gamma^{(2)}Y_{n,j}+\eta^{(2)}-\frac{1}{2}\Lambda^{(2)}\right)\right).*\exp\left(\sum_{j=1}^{s}b_{ij}\Sigma^{(2)}J_{n+1}\right),\\ Y_{n,i} &= Y_n.*\exp\left(h\sum_{j=1}^{s}\tilde{a}_{ij}\left(\Gamma^{(1)}X_{n,j}-\eta^{(1)}-\frac{1}{2}\Lambda^{(1)}\right)\right).*\exp\left(\sum_{j=1}^{s}\tilde{b}_{ij}\Sigma^{(1)}J_{n+1}\right),\\ X_{n+1} &= X_n.*\exp\left(h\sum_{j=1}^{s}\alpha_j\left(-\Gamma^{(2)}Y_{n,j}+\eta^{(2)}-\frac{1}{2}\Lambda^{(2)}\right)\right).*\exp\left(\sum_{j=1}^{s}\beta_j\Sigma^{(2)}J_{n+1}\right),\\ Y_{n+1} &= Y_n.*\exp\left(h\sum_{j=1}^{s}\tilde{\alpha}_j\left(\Gamma^{(1)}X_{n,j}-\eta^{(1)}-\frac{1}{2}\Lambda^{(1)}\right)\right).*\exp\left(\sum_{j=1}^{s}\tilde{\beta}_j\Sigma^{(1)}J_{n+1}\right). \end{aligned} \tag{3.8}$$

The parameters of method (3.8) can be characterized by the Butcher tableau

$$\begin{array}{c|cccc} & A & \tilde{A} & B & \tilde{B} \\ \hline & \alpha^\top & \tilde{\alpha}^\top & \beta^\top & \tilde{\beta}^\top \end{array}$$

with $A = [a_{ij}]_{s\times s}$, $\tilde{A} = [\tilde{a}_{ij}]_{s\times s}$, $B = [b_{ij}]_{s\times s}$, $\tilde{B} = [\tilde{b}_{ij}]_{s\times s}$, $\alpha = (\alpha_1,\cdots,\alpha_s)^\top$, $\tilde{\alpha} = (\tilde{\alpha}_1,\cdots,\tilde{\alpha}_s)^\top$, $\beta = (\beta_1,\cdots,\beta_s)^\top$ and $\tilde{\beta} = (\tilde{\beta}_1,\cdots,\tilde{\beta}_s)^\top$. In this case, the stochastic symplectic conditions (3.4) is modified as

$$\alpha_i = \tilde{\alpha}_i,\ \ \alpha_i\tilde{a}_{ij}+\tilde{\alpha}_j a_{ji} = \alpha_i\tilde{\alpha}_j,$$

and order conditions (3.7) turns to be

$$\sum_{j=1}^{s}\alpha_j = \sum_{j=1}^{s}\tilde{\alpha}_j = \sum_{j=1}^{s}\tilde{\beta}_j = 1$$

for the case $\Sigma^{(2)} = 0$.

4 Numerical examples

In Sect. 4.1, we present several low-stage positive-preserving symplectic methods satisfying both the order conditions and the stochastic symplectic conditions for (1.1). In Sect. 4.2, numerical experiments are given for two specific models, where convergence errors and the evolution of the phase area are tested for a one-stage stochastic Runge–Kutta method (Scheme 1) and a two-stage stochastic partitioned Runge–Kutta method (Scheme 4) compared with Euler–Maruyama and Milstein schemes.

4.1 Some low-stage positive-preserving symplectic methods

Four specific s-stage Runge–Kutta methods are presented below whose coefficients A, B, α and β satisfy the assumptions in Theorems 3.2 and 3.4, and thus are all positivity-preserving symplectic methods.

Scheme 1 *The one-stage stochastic Runge–Kutta method with the Butcher tableau*

$$\begin{array}{c|cc} & 1/2 & 1/2 \\ \hline & 1 & 1 \end{array}.$$

More precisely, the scheme has the following form

$$\begin{aligned} X_{n+1} &= X_n .* \exp\left(h\left(-\Gamma^{(2)}(Y_{n+1} .* Y_n)^{\frac{1}{2}} + \eta^{(2)} - \frac{1}{2}\Lambda^{(2)}\right) + \Sigma^{(2)} J_{n+1}\right), \\ Y_{n+1} &= Y_n .* \exp\left(h\left(\Gamma^{(1)}(X_{n+1} .* X_n)^{\frac{1}{2}} - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)}\right) + \Sigma^{(1)} J_{n+1}\right). \end{aligned}$$

Scheme 2 *The two-stage stochastic Runge–Kutta method with the Butcher tableau*

$$\begin{array}{c|cccc} & 1/8 & 0 & 1/4 & 0 \\ & 1/4 & 3/8 & 1/2 & 1/4 \\ & 1/4 & 3/4 & 1/2 & 1/2 \\ \hline \end{array}.$$

Scheme 3 *The one-stage stochastic partitioned Runge–Kutta method with the Butcher tableau*

$$\begin{array}{c|cccc} & 1/2 & 1/2 & 1/2 & 1/2 \\ \hline & 1 & 1 & 1 & 1 \end{array}.$$

Scheme 4 *The two-stage stochastic partitioned Runge–Kutta method with the Butcher tableau*

$$\begin{array}{c|cccccccc} & 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ & 1/2 & 1/2 & 1/2 & 0 & 1/2 & 1/2 & 1/2 & 0 \\ \hline & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \end{array}.$$

More precisely, the scheme has the following explicit form

$$\begin{aligned} Y_{n,1} &= Y_n .* \exp\left(\frac{h}{2}\left(\Gamma^{(1)} X_n - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)}\right) + \frac{1}{2}\Sigma^{(1)} J_{n+1}\right), \\ X_{n+1} &= X_n .* \exp\left(h\left(-\Gamma^{(2)} Y_{n,1} + \eta^{(2)} - \frac{1}{2}\Lambda^{(2)}\right) + \Sigma^{(2)} J_{n+1}\right), \\ Y_{n+1} &= Y_n .* \exp\left(h\left(\frac{1}{2}\Gamma^{(1)}(X_{n+1} + X_n) - \eta^{(1)} - \frac{1}{2}\Lambda^{(1)}\right) + \Sigma^{(1)} J_{n+1}\right). \end{aligned}$$

4.2 Numerical experiments

In this section, we consider the following two-dimensional stochastic LV model perturbed by an m-dimensional Wiener process $W = (W_1, \cdots, W_m)^\top$:

$$\begin{aligned} dx(t) &= x(t)\left[(-\gamma_2 y(t) + \eta_2)\, dt + \sigma_2 dW(t)\right], \quad x(0) = x_0, \\ dy(t) &= y(t)\left[(\gamma_1 x(t) - \eta_1)\, dt + \sigma_1 dW(t)\right], \quad \ \ y(0) = y_0, \end{aligned} \tag{4.1}$$

where $\gamma_l, \eta_l > 0$ and $\sigma_l = (\sigma_{l1}, \cdots, \sigma_{lm}) \in \mathbb{R}^{1\times m}$. We will use Schemes 1 and 4 to present some numerical experiments to show the favorable performance of the proposed numerical methods in comparison with the following two widely used numerical methods applied to the stochastic LV model (4.1):

- The Euler–Maruyama (EM) scheme

$$\begin{aligned} X_{n+1} &= X_n + h(-\gamma_2 X_n Y_n + \eta_2 X_n) + X_n \sigma_2 J_{n+1}, \\ Y_{n+1} &= Y_n + h\,(\gamma_1 X_n Y_n - \eta_1 Y_n) + Y_n \sigma_1 J_{n+1}. \end{aligned}$$

- The Milstein scheme

$$X_{n+1} = X_n + h(-\gamma_2 X_n Y_n + \eta_2 X_n) + X_n \sigma_2 J_{n+1} + X_n \sum_{k,l=1}^{r} \sigma_{2,k}\sigma_{2,l} I_{kl}^{(n+1)},$$

$$Y_{n+1} = Y_n + h\,(\gamma_1 X_n Y_n - \eta_1 Y_n) + Y_n \sigma_1 J_{n+1} + Y_n \sum_{k,l=1}^{r} \sigma_{1,k}\sigma_{1,l} I_{kl}^{(n+1)},$$

  where

$$I_{kl}^{(n+1)} = \int_{nh}^{(n+1)h} (W_k(t) - W_k(nh))\, dW_l(t).$$

Throughout these experiments, the expectation is approximated by taking averaged value over 1000 realizations. We use the solution of Scheme 4 with a fine step size 2^{-12} as the reference value of the exact solution.

4.2.1 The single noise case

We first consider the single noise case $m = 1$ and choose parameters $\gamma_1 = \gamma_2 = \eta_1 = \eta_2 = \sigma_1 = 1$ and $\sigma_2 = 0$.

The convergence order in the $\mathbb{L}^1(\Omega)$-norm as well as the convergence error is investigated for the EM scheme, the Milstein scheme, Schemes 1 and 4. It can be observed from Fig. 1 that the EM scheme is of order 0.5 while the other three schemes are all of order 1 in the $\mathbb{L}^1(\Omega)$-norm compared with the reference lines, which coincides with the theoretical analysis in Theorem 3.4. The convergence errors for the above four schemes are given in Table 1 over different time intervals $T = 0.5, 1, 5, 10, 20$.

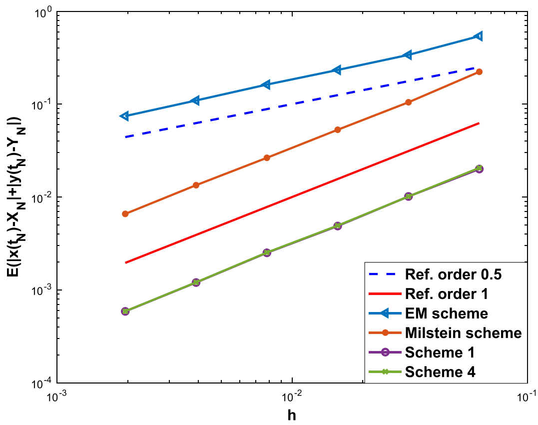


Fig. 1 $\mathbb{L}^1(\Omega)$-convergence order for numerical solution (X_N, Y_N) at $T = 1$ in log-log scale with step size $h \in \{2^{-i}, 4 \le i \le 9\}$

Table 1 $\mathbb{L}^1(\Omega)$ convergence error for schemes with $h = 2^{-6}$

T	0.5	1	5	10	20
EM scheme	5.18e−01	3.93e−01	2.35e−01	1.89	3.39
Milstein scheme	5.20e−02	4.99e−02	1.52e−01	1.26	2.91
Scheme 1	6.80e−03	5.00e−03	1.74e−02	8.08e−02	4.82e−01
Scheme 4	7.00e−03	5.20e−03	1.67e−02	1.08e−01	7.67e−01

It shows that the errors for Schemes 1 and 4 are smaller than that of the EM scheme and the Milstein scheme.

The performance of two positivity-preserving symplectic methods—Schemes 1 and 4—are investigated, compared with EM and Milstein schemes which are not stochastic symplectic or positivity-preserving. We present the evolution of the phase area and the error of the phase area for the EM scheme, the Milstein scheme, Schemes 1 and 4 in Fig. 2. The triangle determined by three points $w_0^1 = (1, 7)$, $w_0^2 = (7, 1)$ and $w_0^3 = (4, 4)$ is chosen as the initial area. We can get three series of points $\{w_n^i\}_{n \ge 0}, i = 1, 2, 3$, under the propagation of a specific scheme. At each step n in the time interval $[0, 20]$, the phase area is the area of the triangle determined by points $\{w_n^1, w_n^2, w_n^3\}$.

Figure 2 shows the evolution of the phase area and the error of the phase area. As shown in (a), the evolution of the phase area of either Schemes 1 or 4 is almost the same as the exact one, while the phase areas of both EM and Milstein schemes turn to deviate from the exact one. This phenomenon appears more evident in (b), where we simulate the error of the phase area for the EM scheme, the Milstein scheme, Schemes 1 and 4. The good performance of Schemes 1 and 4 benefits from the preservation of the geometric structure, which shows the superiority of our proposed numerical methods in this work.

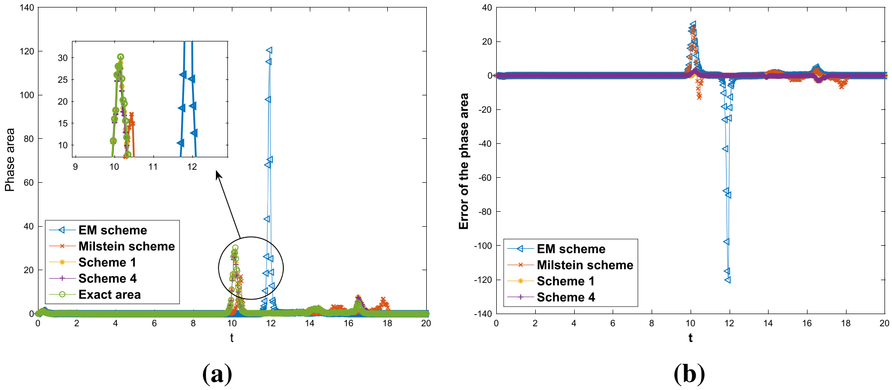


Fig. 2 Evolution of **a** the phase area and **b** the error of the phase area for the EM scheme, the Milstein scheme, Scheme 1 and Scheme 4 ($h = 2^{-5}$, $T = 20$)

4.2.2 A multiple noise case

In this subsection, we consider a multiple noise case with $m = 3$, and choose parameters $(\gamma_1, \gamma_2, \eta_1, \eta_2) = (0.5, 0.6, 0.7, 0.8)$, $\sigma_1 = (0, -0.3, 0.4)$ and $\sigma_2 = (0.2, 0, 0)$ in (4.1) such that the species X is perturbed by W_1, while Y is perturbed by independent random processes W_2 and W_3. Under this setting, (4.1) turns to be

$$dx(t) = x(t)\left[(-0.6y(t) + 0.8)\,dt + 0.2dW_1(t)\right], \quad x(0) = x_0,$$
$$dy(t) = y(t)\left[(0.5x(t) - 0.7)\,dt - 0.3dW_2(t) + 0.4dW_3(t)\right], \qquad y(0) = y_0.$$

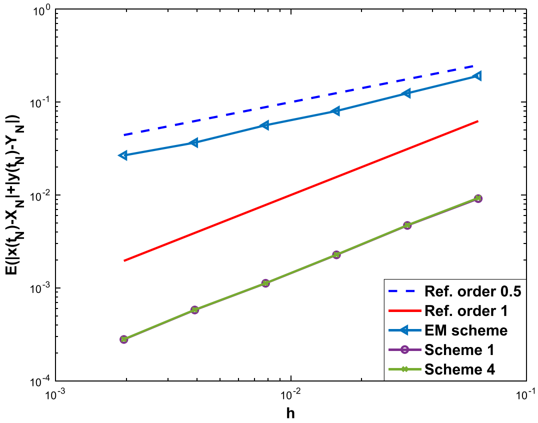
For the multiple noise case, the Milstein scheme is not easy to implement due to the presence of the double Wiener integral $I_{kl}^{(n)}$ with $k \neq l$. More precisely, $I_{kl}^{(n)}$ can not be expressed in terms of simpler stochastic integrals when $k \neq l$, and the Fourier series expansion is usually used to approximate this kind of integral. Details of the implementation of the Milstein scheme can be found in [9], and won't be discussed here. In the following, Schemes 1 and 4 are tested in comparison with the EM scheme.

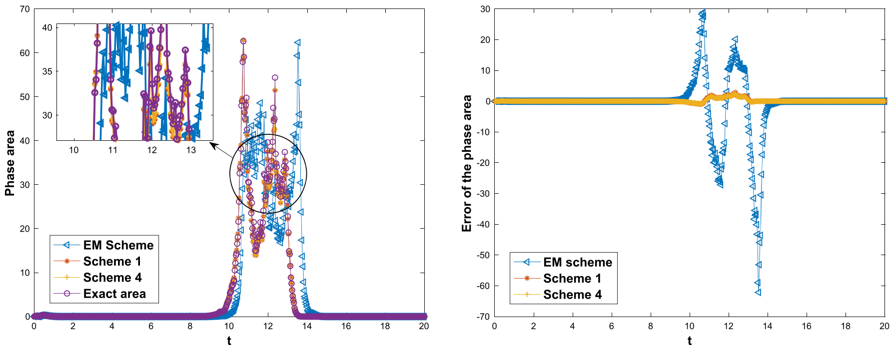
Similar to the single noise case, the convergence order in the $\mathbb{L}^1(\Omega)$-norm, and the evolution of the area as well as its error are given in Figs. 3 and 4, respectively. One can figure it out from Fig. 3 that Schemes 1 and 4 are both of order 1 while the EM scheme is of order 0.5. The errors of the phase area for Schemes 1 and 4 are much smaller than that for the EM scheme as shown in Fig. 4.

5 Conclusion

We have studied the behavior for the stochastic Lotka–Volterra predator-prey model, including the positivity and the stochastic symplecticity of the solution. A class of Runge–Kutta methods is presented to preserve the positivity. Both symplectic conditions and order conditions are given to ensure the stochastic symplecticity and the convergence order one in the $\mathbb{L}^1(\Omega)$-norm of the proposed methods.

Fig. 3 $\mathbb{L}^1(\Omega)$-convergence order for numerical solution (X_N, Y_N) at $T = 1$ in log-log scale with step size $h \in \{2^{-i}, 4 \leq i \leq 9\}$





(a) **(b)**

Fig. 4 Evolution of **a** the phase area and **b** the error of the phase area for the EM scheme, Schemes 1 and 4 ($h = 2^{-5}$, $T = 20$)

For more complex biological systems, such as models with competition or cooperation, it is still not clear if the solution is positive mathematically when perturbed by random noises or if it possesses some kinds of stochastic conservation laws. Moreover, constructions and analysis of numerical methods are more challenging due to the interacting among different species and non-global Lipschitz nonlinear term involved in the equation. Some other tools or techniques need to be explored to study the other biological systems. We hope to be able to report the progress on these problems elsewhere in the future.

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