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INVERSE RANDOM SOURCE SCATTERING FOR THE HELMHOLTZ EQUATION WITH ATTENUATION*

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Abstract. In this paper, a new model is proposed for the inverse random source scattering problem of the Helmholtz equation with attenuation. The source is assumed to be driven by a fractional Gaussian field whose covariance is represented by a classical pseudodifferential operator. The work contains three contributions. First, the connection is established between fractional Gaussian fields and rough sources characterized by their principal symbols. Second, the direct source scattering problem is shown to be well-posed in the distribution sense. Third, we demonstrate that the microcorrelation strength of the random source can be uniquely determined by the passive measurements of the wave field in a set which is disjoint with the support of the strength function. The analysis relies on careful studies on the Green function and Fourier integrals for the Helmholtz equation.

Key words. inverse scattering problem, the Helmholtz equation, random source, fractional Gaussian field, pseudodifferential operator, principal symbol

AMS subject classifications. 78A46, 65C30

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1. Introduction.

The inverse source scattering in waves is an important and active research subject in inverse scattering theory. It is an important mathematical tool for the solution of many medical imaging modalities [3, 13]. The inverse source scattering problems are to determine the unknown sources that generate prescribed wave patterns. These problems have attracted much research. The mathematical and numerical results can be found in [7, 8, 17] and the references cited therein.

Stochastic modeling is widely introduced to mathematical systems due to unpredictability of the environments, incomplete knowledge of the systems and measurements, and fine-scale fluctuations in simulation. In many situations, the source, hence the wave field, may not be deterministic but are rather modeled by random processes [12]. Due to the extra challenge of randomness and uncertainties, little is known for the inverse random source scattering problems.

In this paper, we consider the Helmholtz equation with a random source

$$\Delta u + (k^2 + \mathrm{i}k\sigma)u = f, \quad x \in \mathbb{R}^d, \tag{1.1}$$

where $d = 2$ or 3, $k > 0$ is the wavenumber, the attenuation coefficient $\sigma \geq 0$ describes the electrical conductivity of the medium, u denotes the wave field, and f representing the electric current density is a random function compactly supported in a bounded domain $\mathcal{O}$.

In [4], the white noise model was studied for the inverse random source problem of the stochastic Helmholtz equation without attenuation

$$\Delta u + k^2 u = g + h\dot{W}, \quad x \in \mathbb{R}^d,$$

where g and h are deterministic and compactly supported functions, and $\dot{W}$ is the spatial white noise. It was shown that g and h can be determined by statistics of the

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wave fields at multiple frequencies. The white noise model can also be found in [6] and [5] for the one-dimensional problem and the stochastic elastic wave equation, respectively. Recently, the model of a generalized Gaussian field was developed to handle random processes [9, 14]. The random function is said to be microlocally isotropic of order $2s$ if the covariance operator is a pseudodifferential operator with principal symbol given by $\mu(x)|\xi|^{-2s}$, where $\mu \geq 0$ is a smooth and compactly support function and is called the microcorrelation strength of the random function. It was shown that μ can be uniquely determined by the wave field averaged over the frequency band at a single realization of the random function. This model was also investigated in [20, 21] for the inverse random source problems of the elastic wave equation and the Helmholtz equation without attenuation. In these works, the parameter $s \in [\frac{d}{2}, \frac{d}{2}+1)$ and the random functions are smoother than the white noise (cf. Lemma 2.6): They can be interpreted as distributions in $W^{-\epsilon,p}(\mathcal{O})$ for any small $\epsilon > 0$ and $p \in (1,\infty)$ if $s = \frac{d}{2}$; they are functions in $C^{0,\alpha}(\mathbb{R}^d)$ for any $\alpha \in (0, s-\frac{d}{2})$ if $s \in (\frac{d}{2}, \frac{d}{2}+1)$. To the best of our knowledge, it is still open on the problem of the stochastic Helmholtz equation driven by rougher random sources with $s < \frac{d}{2}$, where the distributional solution should be studied since it is less regular than the one considered in the previous work.

In this paper, we propose a new model for the stochastic Helmholtz equation (1.1), where the random source f is driven by a fractional Gaussian field (FGF) $(-\Delta)^{-\frac{s}{2}}\dot{W}$ with $s \in [0, \frac{d}{2}+1)$ (cf. Definition 2.2). The FGFs include various processes such as Brownian motion ($d = 1, s = 1$), fractional Brownian motion ($d = 1, \frac{1}{2} < s < \frac{3}{2}$), white noise ($s = 0$), Gaussian free field ($s = 1$), bi-Laplacian Gaussian field ($s = 2$), the log-correlated Gaussian field ($s = \frac{d}{2}$), Lévy's Brownian motion ($s = \frac{d}{2} + \frac{1}{2}$), and multidimensional fractional Brownian motion ($\frac{d}{2} < s < \frac{d}{2} + 1$). A survey can be found in [25] on some basic results of FGFs. These fields have significant applications in finance, statistical physics, quantum field theory, early-universe cosmology, image processing, and many other disciplines. In particular, the model problem considered in this paper has an important application in medical imaging of lossy media such as the human body.

The work contains three contributions in addition to the new model. First, we demonstrate that the FGFs include the classical fractional Brownian fields. Moreover, we establish the connection between the FGFs and rough sources characterized by their principal symbols (cf. Proposition 2.5). Second, we examine the regularity of the random source and show that the direct scattering problem is well-posed in the distribution sense (cf. Theorem 3.2). Third, for the inverse problem, we prove that the strength μ of the random source can be uniquely determined by the high frequency limit of the second moment of the wave field, which is stated as follows (cf. Theorems 4.2, 4.4, and 4.5).

THEOREM 1.1. *Let $d = 2, 3$. Assume that μ is compactly supported in $\mathcal{O}$ and $\mathcal{U} \subset \mathbb{R}^d$ is a bounded open set such that dist$(\mathcal{U}, \mathcal{O}) =: r_0 > 0$. For any $x \in \mathcal{U}$, it holds*

$$\lim_{k\to\infty} k^{2s+3-d}\mathbb{E}|u(x;k)|^2 = C_d \int_{\mathcal{O}} \frac{e^{-\sigma|x-y|}}{|x-y|^{d-1}}\mu(y)dy =: T^{(d)}(x),$$

where $C_d = \frac{1}{2^2(2\pi)^{d-1}}$. Moreover, the strength μ can be uniquely determined by the integral $T^{(d)}(x)$.

In particular, if $\sigma = 0$, the strength μ can also be determined uniquely by the amplitude of the wave field averaged over the frequency band at a single realization of the random source. It is worthy to be pointed out that (1) if $s \in [0, \frac{d}{2}]$, the random function is a distribution in $f \in W^{s-\frac{d}{2}-\epsilon,p}(\mathcal{O})$ for any $\epsilon > 0$ and $p \in (1,\infty)$ (cf. Lemma 2.6), which is rougher than those considered in [9, 14, 20, 21]; (2) if $\sigma = 0$ and $s \in [\frac{d}{2}, \frac{d}{2}+1)$, the results obtained in this paper coincide with the ones given in [20].

The paper is organized as follows. In section 2, the random source model is introduced. The relationship is established between the FGF and the classical fractional Brownian motion; the regularity is studied for the random source. Section 3 addresses the well-posedness and regularity of the solution for the direct problem. The inverse problem is discussed in section 4, where the two- and three-dimensional problems are considered separately. The paper is concluded with some general remarks and directions for future work in section 5.

2. Random source. In this section, we give a general description of the random source on $\mathbb{R}^d$. Let f be a real-valued centered random field defined on a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Introduce the following Sobolev spaces. The details can be found in [2].

- $W^{s,p} := W^{s,p}(\mathbb{R}^d)$ for $s \in \mathbb{R}$ and $p \in (1,\infty)$. In particular, if $p = 2$, denote $H^s := W^{s,2}$.
- Denote by $W^{s,p}_{\rm loc}$ the space of functions which are locally in $W^{s,p}$. More precisely, for any precompact subset $D \subset \mathbb{R}^d$, $u|_D \in W^{s,p}(D)$.
- Denote by $W^{s,p}_0(D)$ the closure of $C_0^\infty(D)$ in $W^{s,p}(D)$ with $D \subset \mathbb{R}^d$. In particular, if $D = \mathbb{R}^d$, $W^{s,p}_0 = W^{s,p}$.

Let $f : \Omega \to \mathcal{D}'$ be measurable such that the mapping $\omega \mapsto \langle f(\omega), \phi\rangle$ defines a Gaussian random variable for any $\phi \in \mathcal{D}$. Here, $\mathcal{D}'$ is the space of distributions on $\mathbb{R}^d$, which is the dual space of the test function space $\mathcal{D}$. The covariance operator $Q_f : \mathcal{D} \to \mathcal{D}'$ is given by

$$\langle \varphi, Q_f\psi\rangle = \mathbb{E}[\langle f,\varphi\rangle\langle f,\psi\rangle] \quad \forall\, \varphi,\psi \in \mathcal{D},$$

where $\langle\cdot,\cdot\rangle$ denotes the dual product. The derivative of a distribution $g \in \mathcal{D}'$ is defined by

$$\langle \partial_{x_j} g, \varphi\rangle = -\langle g, \partial_{x_j}\varphi\rangle \quad \forall\, \varphi \in \mathcal{D}.$$

Denote by $K_f(x,y)$ the Schwartz kernel of Q_f, which satisfies

$$\langle \varphi, Q_f\psi\rangle = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_f(x,y)\varphi(x)\psi(y)dxdy.$$

Hence we have the following formal expression of the Schwartz kernel:

$$K_f(x,y) = \mathbb{E}[f(x)f(y)].$$

Assumption 2.1. The covariance operator Q_f of the source f is a classical pseudodifferential operator with the principal symbol $\mu(x)|\xi|^{-2s}$, where $s \in \big[0, \frac{d}{2}+1\big)$ and $0 \le \mu \in C_0^\infty(\mathcal{O})$.

The positive function μ stands for the microcorrelation strength of the random field f. The assumption implies that the covariance operator Q_f satisfies

$$(Q_f\psi)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{{\rm i}x\cdot\xi} c(x,\xi)\hat{\psi}(\xi)d\xi \quad \forall\, \psi \in \mathcal{D},$$

where the symbol $c(x,\xi)$ has the leading term $\mu(x)|\xi|^{-2s}$ and

$$\hat{\psi}(\xi) = \mathcal{F}[\psi](\xi) = \int_{\mathbb{R}^d} e^{-{\rm i}x\cdot\xi}\psi(x)dx$$

is the Fourier transform of ψ [15, 16]. By the expression of $Q_f\psi$, we can deduce the relationship between the kernel K_f and the symbol $c(x,\xi)$. In fact, noting that

$$\begin{aligned}\langle\varphi, Q_f\psi\rangle &= \int_{\mathbb{R}^d}\varphi(x)\left[\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{\mathrm{i}x\cdot\xi}c(x,\xi)\hat{\psi}(\xi)d\xi\right]dx\\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\varphi(x)\int_{\mathbb{R}^d}e^{\mathrm{i}x\cdot\xi}c(x,\xi)\left[\int_{\mathbb{R}^d}e^{-\mathrm{i}y\cdot\xi}\psi(y)dy\right]d\xi dx\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left[\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{\mathrm{i}(x-y)\cdot\xi}c(x,\xi)d\xi\right]\varphi(x)\psi(y)dxdy,\end{aligned}$$

we get that the kernel K_f is an oscillatory integral of the form

$$K_f(x,y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{\mathrm{i}(x-y)\cdot\xi}c(x,\xi)d\xi. \tag{2.1}$$

2.1. FGFs. We introduce the FGFs, which can be used to generate random fields satisfying Assumption 2.1.

Definition 2.2. *The FGF h^s on $\mathbb{R}^d$ with parameter $s\in\mathbb{R}$ is given by*

$$h^s := (-\Delta)^{-\frac{s}{2}}\dot{W},$$

where $(-\Delta)^{-\frac{s}{2}}$ is the fractional Laplacian on $\mathbb{R}^d$ defined by

$$(-\Delta)^{\alpha}u = \mathcal{F}^{-1}\left[|\xi|^{2\alpha}\mathcal{F}[u](\xi)\right], \quad \alpha\in\mathbb{R}, \tag{2.2}$$

and $\dot{W}\in\mathcal{D}'$ is the white noise on $\mathbb{R}^d$ determined by the covariance operator $Q_{\dot{W}}$: $L^2\to L^2$ as follows:

$$\langle\varphi, Q_{\dot{W}}\psi\rangle := \mathbb{E}[\langle\dot{W},\varphi\rangle\langle\dot{W},\psi\rangle] = (\varphi,\psi)_{L^2} \quad \forall\,\varphi,\psi\in L^2.$$

We denote by $\mathbb{G}_s(\mathbb{R}^d)$ the space of FGFs with parameter s. Let $h^s\sim\mathbb{G}_s(\mathbb{R}^d)$ if h^s is a FGF on $\mathbb{R}^d$ with parameter s. If $d=1$ and $s=1$, h^1 turns to be the classical one-dimensional Brownian motion. If $s=0$, $h^0=\dot{W}$ is the white noise on $\mathbb{R}^d$. If $s<0$, h^s is even rougher than the white noise. We refer to [25] and references therein for more details about the FGFs and the fractional Laplacian.

To make sense of the expression $h^s=(-\Delta)^{-\frac{s}{2}}\dot{W}$, we define

$$\mathcal{S}_r := \begin{cases}\left\{\varphi\in\mathcal{S} : \int_{\mathbb{R}^d}x^{\alpha}\varphi(x)dx = 0 \quad \forall\,|\alpha|\le r\right\} & \text{if } r\ge 0,\\ \mathcal{S} & \text{if } r<0,\end{cases}$$

where $\mathcal{S}$ denotes the Schwartz space. Denote by T_s the closure of $\mathcal{S}_{s-\frac{d}{2}}$ in H^{-s}. Then the expression $h^s=(-\Delta)^{-\frac{s}{2}}\dot{W}$ in Definition 2.2 is interpreted by

$$\langle h^s,\varphi\rangle := \langle\dot{W},(-\Delta)^{-\frac{s}{2}}\varphi\rangle = \int_{\mathbb{R}^d}(-\Delta)^{-\frac{s}{2}}\varphi(x)dW(x) \quad \forall\,\varphi\in T_s.$$

The kernel K_{h^s} for the covariance operator Q_{h^s} of h^s satisfies

$$\langle\varphi, Q_{h^s}\psi\rangle = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K_{h^s}(x,y)\varphi(x)\psi(y)dxdy \quad \forall\,\varphi,\psi\in\mathcal{D}\cap T_s.$$

Moreover, the kernel has the following expression. The proof can be found in [25].

LEMMA 2.3. *Let $h^s \sim \mathbb{G}_s(\mathbb{R}^d)$ with parameter $s \in [0,\infty)$. Denote $H := s - \frac{d}{2}$.*

(i) *If $s \in (0,\infty)$ and H is not a nonnegative integer, then*

$$K_{h^s}(x,y) = C_1(s,d)|x-y|^{2H},$$

where $C_1(s,d) = 2^{-2s}\pi^{-\frac{d}{2}}\Gamma(\frac{d}{2}-s)/\Gamma(s)$ with $\Gamma(\cdot)$ being the Gamma function.

(ii) *If $s \in (0,\infty)$ and H is a nonnegative integer, then*

$$K_{h^s}(x,y) = C_2(s,d)|x-y|^{2H}\ln|x-y|,$$

where $C_2(s,d) = (-1)^{H+1}2^{-2s+1}\pi^{-\frac{d}{2}}/(H!\Gamma(s))$.

(iii) *If $s = 0$, then*

$$K_{h^s}(x,y) = \delta(x-y),$$

where $\delta(\cdot)$ is the Dirac delta function centered at 0.

2.2. Relationship with classical fractional Brownian fields. For any $h^s \sim \mathbb{G}_s(\mathbb{R}^d)$, we define its generalized Hurst parameter $H = s - \frac{d}{2}$. If $s \in (\frac{d}{2}, \frac{d}{2}+1)$, h^s coincides with the classical fractional Brownian fields B^H determined by the covariance operator Q_{B^H}:

$$\langle \varphi, Q_{B^H}\psi\rangle = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{2}\left[|x|^{2H}+|y|^{2H}-|x-y|^{2H}\right]\varphi(x)\psi(y)dxdy, \tag{2.3}$$

where the Hurst parameter $H \in (0,1)$.

LEMMA 2.4. *Let $s \in (\frac{d}{2}, \frac{d}{2}+1)$ and $h^s \sim \mathbb{G}_s(\mathbb{R}^d)$. Then the stochastic process defined by*

$$\tilde{h}^s(x) = \langle h^s, \delta_x - \delta_0\rangle$$

has the same distribution as the fractional Brownian field B^H with $H = s - \frac{d}{2} \in (0,1)$ up to a multiplicative constant, where $\delta_x(\cdot) \in H^{-s}$ is the Dirac measure centered at $x \in \mathbb{R}^d$.

Proof. By Lemma 2.3, the kernel of the covariance operator reads

$$\begin{aligned}\mathbb{E}[\tilde{h}^s(x)\tilde{h}^s(y)] &= \mathbb{E}[\langle h^s, \delta_x-\delta_0\rangle\langle h^s,\delta_y-\delta_0\rangle]\\ &= C_1(s,d)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|r_1-r_2|^{2H}(\delta_x-\delta_0)(r_1)(\delta_y-\delta_0)(r_2)dr_1dr_2\\ &= C_1(s,d)\left(|x-y|^{2H}-|x|^{2H}-|y|^{2H}\right),\end{aligned}$$

which is a scalar multiple of the kernel of the covariance operator Q_{B^H} defined in (2.3). The result then follows from the fact that the distribution of a centered Gaussian random field is unique determined by its covariance operator. ∎

Note that $\langle h^s, \delta_x - \delta_0\rangle$ is actually a translation of h^s. It indicates that we can identify $h^s \sim \mathbb{G}_s(\mathbb{R}^d)$ as the fractional Brownian field B^H with $H = s - \frac{d}{2} \in (0,1)$ by fixing its value to be zero at the origin. Define a random function

$$f(x,\omega) := a(x)h^s(x,\omega), \quad x \in \mathbb{R}^d,\ \omega \in \Omega, \tag{2.4}$$

where $s \in (\frac{d}{2}, \frac{d}{2}+1)$ and $a \in C_0^\infty$ with $\text{supp}(a) \subset \mathcal{O}$. We claim that such an f defined above satisfies Assumption 2.1. More precisely, the covariance operator Q_f of f has the principal symbol $a^2(x)|\xi|^{-2s}$ up to a multiplicative constant.

Proposition 2.5. *The random field f defined in* (2.4) *with $s\in[0,\frac d2+1)$ satisfies Assumption* 2.1 *with $\mu=a^2$.*

Proof. According to Definition 2.2, the covariance operator Q_f of f satisfies

$$\begin{aligned}\langle\varphi,Q_f\psi\rangle =&\ \mathbb{E}\left[\langle ah^s,\varphi\rangle\langle ah^s,\psi\rangle\right]=\mathbb{E}\left[\int_{\mathbb{R}^d}(-\Delta)^{-\frac s2}(a\varphi)dW\int_{\mathbb{R}^d}(-\Delta)^{-\frac s2}(a\psi)dW\right]\\ =&\int_{\mathbb{R}^d}(-\Delta)^{-\frac s2}(a\varphi)(-\Delta)^{-\frac s2}(a\psi)dx\\ =&\ \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\overline{\mathcal{F}\left[(-\Delta)^{-\frac s2}(a\varphi)\right](\xi)}\mathcal{F}\left[(-\Delta)^{-\frac s2}(a\psi)\right](\xi)d\xi,\end{aligned}$$

where the Plancherel theorem is used in the last step. It follows from the definition of the fractional Laplacian given in (2.2) that we get

$$\begin{aligned}\langle\varphi,Q_f\psi\rangle =&\ \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi|^{-2s}\overline{\widehat{(a\varphi)}(\xi)}\widehat{(a\psi)}(\xi)d\xi\\ =&\ \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi|^{-2s}\left[\int_{\mathbb{R}^d}a(x)\varphi(x)e^{\mathrm{i}x\cdot\xi}dx\right]\left[\int_{\mathbb{R}^d}a(y)\psi(y)e^{-\mathrm{i}y\cdot\xi}dy\right]d\xi\\ =&\ \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(x)\psi(y)e^{\mathrm{i}(x-y)\cdot\xi}a^2(x)|\xi|^{-2s}d\xi dxdy\\ &-\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(x)\psi(y)e^{\mathrm{i}(x-y)\cdot\xi}a(x)(a(x)-a(y))|\xi|^{-2s}d\xi dxdy\\ =:&\ I_1+I_2.\end{aligned}$$

Noting $a(x)-a(y)=\nabla a(\theta x+(1-\theta)y)\cdot(x-y)$ for some $\theta\in(0,1)$ and

$$\begin{aligned}&\int_{\mathbb{R}^d}e^{\mathrm{i}(x-y)\cdot\xi}\nabla a(\theta x+(1-\theta)y)\cdot(x-y)|\xi|^{-2s}d\xi\\ &\quad=-\mathrm{i}\int_{\mathbb{R}^d}\nabla a(\theta x+(1-\theta)y)\cdot(x-y)[(x-y)^\top(x-y)]^{-1}(x-y)^\top|\xi|^{-2s}de^{\mathrm{i}(x-y)\cdot\xi}\\ &\quad=-2\mathrm{i}s\int_{\mathbb{R}^d}e^{\mathrm{i}(x-y)\cdot\xi}\nabla a(\theta x+(1-\theta)y)\cdot(x-y)\\ &\qquad[(x-y)^\top(x-y)]^{-1}(x-y)^\top|\xi|^{-2s-2}\xi^\top d\xi,\end{aligned}$$

we conclude that the term I_2 contains only higher order terms of the symbol and the term I_1 contains the principal symbol $a^2(x)|\xi|^{-2s}$, which completes the proof. ∎

2.3. Regularity of random sources. By Proposition 2.5, for any function f satisfying Assumption 2.1 with parameter $s\in[0,\frac d2+1)$, its principal symbol has the same order as the principal symbol of the random field ah^s. Without loss of generality, we only need to investigate the regularity of random fields given by $f=ah^s$, where $a\in C_0^\infty$ and $\mathrm{supp}(a)\subset\mathcal{O}$. Moreover, we assume that f is a centered random field to avoid using the modification $\langle h^s,\delta_x-\delta_0\rangle$.

Lemma 2.6. *Let $s\in[0,\frac d2+1)$ and $h^s\sim\mathbb{G}_s(\mathbb{R}^d)$. Define the random field $f:=ah^s$ with $a\in C_0^\infty$ and supp$(a)\subset\mathcal{O}$.*

(i) *If $s\in(\frac d2,\frac d2+1)$, then $f\in C^{0,\alpha}$ almost surely for all $\alpha\in(0,s-\frac d2)$.*

(ii) *If $s\in[0,\frac d2]$, then $f\in W^{s-\frac d2-\epsilon,p}(\mathcal{O})$ almost surely for any $\epsilon>0$ and $p\in(1,\infty)$.*

Proof. (i) If $s \in (\frac{d}{2}, \frac{d}{2}+1)$, it follows from Lemma 2.4 that f has the same distribution as aB^H, where B^H is a classical fractional Brownian field on $\mathbb{R}^d$ with the Hurst parameter $H = s - \frac{d}{2} \in (0,1)$. Note that B^H is $(H-\epsilon)$-Hölder continuous for any $\epsilon \in (0,H)$. Hence, $f \in C^{0,\alpha}$ with $\alpha \in (0,H) = (0, s-\frac{d}{2})$.

(ii) We first consider the case $s=0$. Note that $f = ah^0 = a\dot{W}$ has the same regularity as the white noise. Hence, $f \in W^{-\frac{d}{2}-\epsilon,p}(\mathcal{O})$ for any $\epsilon > 0$ and $p \in (1,\infty)$ (cf. [26]).

If $s \in (0, \frac{d}{2}]$, as a smoothing of the white noise, it holds $f = ah^s = a(-\Delta)^{-\frac{s}{2}}\dot{W} \in W^{s-\frac{d}{2}-\epsilon,p}(\mathcal{O})$ for any $\epsilon > 0$ and $p \in (1,\infty)$. ∎

The readers are also referred to [9, Proposition 2.4] for an alternative proof of the regularity of random fields satisfying Assumption 2.1.

3. Direct scattering problem. This section is to investigate the well-posedness and study the regularity of the solution for the direct scattering problem.

3.1. Fundamental solution. Let $\kappa^2 = k^2 + \mathrm{i}k\sigma$ with $\Re[\kappa] > 0$. A simple calculation yields that

$$\Re[\kappa] = \kappa_{\rm r} = \left(\frac{\sqrt{k^4+k^2\sigma^2}+k^2}{2}\right)^{\frac{1}{2}}, \quad \Im[\kappa] = \kappa_{\rm i} = \left(\frac{\sqrt{k^4+k^2\sigma^2}-k^2}{2}\right)^{\frac{1}{2}},$$

and

$$\lim_{k\to\infty}\frac{\kappa_{\rm r}}{k} = 1, \quad \lim_{k\to\infty}\kappa_{\rm i} = \frac{\sigma}{2}. \tag{3.1}$$

Then the Helmholtz equation (1.1) can be written as

$$\Delta u + \kappa^2 u = f \quad \text{in } \mathbb{R}^d. \tag{3.2}$$

The Helmholtz equation (3.2) with a complex-valued wavenumber has the fundamental solution

$$\Phi_\kappa(x,y) = \begin{cases} \frac{\mathrm{i}}{4}H_0^{(1)}(\kappa|x-y|), & d=2,\\ \frac{1}{4\pi}\frac{e^{\mathrm{i}\kappa|x-y|}}{|x-y|}, & d=3,\end{cases}$$

where $H_0^{(1)}$ is the Hankel function of the first kind with order 0.

3.2. Well-posedness and regularity. Using the fundamental solution Φ_κ, we define a volume potential

$$(V_\kappa f)(x) := -\int_{\mathbb{R}^d}\Phi_\kappa(x,y)f(y)dy.$$

The mollifier V_κ has the following property, which will be used to show the well-posedness of the direct scattering problem.

LEMMA 3.1. *Let D and U be two bounded domains in $\mathbb{R}^d$. The operator V_κ : $H_0^{-\beta}(D) \to H^\beta(U)$ is bounded for $\beta \in (0,1]$.*

The proof of the above lemma can be found in [24] and is omitted here. Now we are at the position to show the well-posedness of (1.1) in the distribution sense.

THEOREM 3.2. *Let f satisfy Assumption* 2.1 *with $s \in (\frac{d}{p}-1, \frac{d}{2}]$ and $p \in (\frac{2d}{d+2}, 2]$. Denote $H = s - \frac{d}{2} \in ((\frac{1}{p}-\frac{1}{2})d-1, 0]$. Then the scattering problem* (1.1) *admits a* *unique solution* $u \in W_{\rm loc}^{-H+\epsilon,q}$ *almost surely in the distribution sense with* q *satisfying* $\frac{1}{p}+\frac{1}{q}=1$. *Moreover, the solution is given by*

$$u(x;k) = -\int_{\mathbb{R}^d} \Phi_\kappa(x,y) f(y) dy.$$

Proof. We only need to show the existence of the solution since the uniqueness follows directly from the deterministic case. Since f satisfies Assumption 2.1, according to Proposition 2.5, f has the same regularity as the random field defined in (2.4). Hence, it follows from Lemma 2.6 that $f \in W^{H-\epsilon,p}(\mathcal{O})$ for any $\epsilon > 0$. For any $x \in \mathbb{R}^d$, define the volume potential

$$u_*(x;k) := -\int_{\mathcal{O}} \Phi_\kappa(x,y) f(y) dy = -\int_{\mathbb{R}^d} \Phi_\kappa(x,y) f(y) dy.$$

First we show that u_* is a solution of (1.1) in the distribution sense. In fact, we have for any $v \in \mathcal{D}$ that

$$\begin{aligned}
&\langle \Delta u_* + \kappa^2 u_*, v\rangle = -\langle \nabla u_*, \nabla v\rangle + \kappa^2 \langle u_*, v\rangle \\
&= \int_{\mathbb{R}^d} \nabla_x \Big[\int_{\mathbb{R}^d} \Phi_\kappa(x,y) f(y) dy\Big] \nabla v(x) dx - \kappa^2 \int_{\mathbb{R}^d} \Big[\int_{\mathbb{R}^d} \Phi_\kappa(x,y) f(y) dy\Big] v(x) dx \\
&= -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_x \Phi_\kappa(x,y) v(x) f(y) dx dy - \kappa^2 \int_{\mathbb{R}^d} \Big[\int_{\mathbb{R}^d} \Phi_\kappa(x,y) f(y) dy\Big] v(x) dx \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big(\kappa^2 \Phi_\kappa(x,y) + \delta(x-y)\big) v(x) f(y) dx dy - \kappa^2 \int_{\mathbb{R}^d} \Big[\int_{\mathbb{R}^d} \Phi_\kappa(x,y) f(y) dy\Big] v(x) dx \\
&= \langle f, v\rangle.
\end{aligned}$$

It then suffices to show that $u_* \in W_{\rm loc}^{-H+\epsilon,q}$, which is equivalent to show that $\phi u_* \in W^{-H+\epsilon,q}$ for any $\phi \in C_0^\infty$ compactly supported in $\mathcal{U} \subset \mathbb{R}^d$ with a C^1-boundary. Define a weighted potential

$$(\tilde{V}_\kappa f)(x) := -\phi(x) \int_{\mathbb{R}^d} \Phi_\kappa(x,y) f(y) dy, \quad x \in \mathcal{U}.$$

By Lemma 3.1, the operator $\tilde{V}_\kappa : H_0^{-\beta}(\mathcal{O}) \to H^\beta(\mathcal{U})$ is bounded for any $\beta \in (0,1]$. For parameters p, q, and H satisfying assumptions in the theorem, by choosing $\beta = 1$ such that $\frac{1}{q} > \frac{1}{2} - \frac{\beta-(-H+\epsilon)}{d}$, we get from the Kondrachov embedding theorem that the embeddings

$$W_0^{H-\epsilon,p}(\mathcal{O}) \hookrightarrow H_0^{-\beta}(\mathcal{O}), \quad H^\beta(\mathcal{U}) \hookrightarrow W^{-H+\epsilon,q}(\mathcal{U})$$

are continuous. Consequently, $\tilde{V}_\kappa : W_0^{H-\epsilon,p}(\mathcal{O}) \to W^{-H+\epsilon,q}(\mathcal{U})$ is bounded, which shows that $\phi u_* = \tilde{V}_\kappa f \in W^{-H+\epsilon,q}$ and completes the proof. ∎

Remark 3.3. It follows from Lemma 2.6 that the random source is a continuous function for $s \in (\frac{d}{2}, \frac{d}{2}+1)$. The well-posedness of the scattering problem (1.1) is well known since the source f is compactly supported and regular enough [10].

4. Inverse scattering problem. This section addresses the inverse scattering problem. The goal is to determine the strength μ of the random source f. We discuss the two- and three-dimensional cases separately.

4.1. Two-dimensional case. First we consider $d=2$ in which $s \in [0, \frac{d}{2}+1) = [0,2)$. Recall that the Hankel function has the following asymptotic expansion [1]:

$$H_0^{(1)}(z) \sim \sum_{j=0}^{\infty} a_j z^{-\left(j+\frac{1}{2}\right)} e^{\mathrm{i}z}, \quad z \in \mathbb{C},\ |z| \to \infty, \tag{4.1}$$

where $a_0 = \sqrt{\frac{2}{\pi}} e^{-\frac{\mathrm{i}\pi}{4}}$ and $a_j = \sqrt{\frac{2}{\pi}} (\frac{\mathrm{i}}{8})^j (\prod_{l=1}^{j} (2l-1)^2/j!) e^{-\frac{\mathrm{i}\pi}{4}}, j \geq 1$. Denoting

$$H_{0,N}^{(1)}(z) := \sum_{j=0}^{N} a_j z^{-\left(j+\frac{1}{2}\right)} e^{\mathrm{i}z}, \quad \Phi_\kappa^N(x,y) := \frac{\mathrm{i}}{4} H_{0,N}^{(1)}(\kappa|x-y|),$$

we have

$$\Phi_\kappa(x,y) = \Phi_\kappa^N(x,y) + O\big(|\kappa|x-y||^{-(N+\frac{3}{2})}\big), \quad N \in \mathbb{N},$$

as $|\kappa|x-y|| \to \infty$ due to $\kappa_{\mathrm{i}} > 0$. Based on the truncated fundamental solution $\Phi_\kappa^2(x,y)$ by choosing $N=2$, we first consider the approximate solution

$$\begin{aligned} u^2(x;k) = &-\int_{\mathbb{R}^2} \Phi_\kappa^2(x,y) f(y) dy = -\frac{\mathrm{i}a_0}{4} \int_{\mathbb{R}^2} (\kappa|x-y|)^{-\frac{1}{2}} e^{\mathrm{i}\kappa|x-y|} f(y) dy \\ &-\frac{\mathrm{i}a_1}{4} \int_{\mathbb{R}^2} (\kappa|x-y|)^{-\frac{3}{2}} e^{\mathrm{i}\kappa|x-y|} f(y) dy - \frac{\mathrm{i}a_2}{4} \int_{\mathbb{R}^2} (\kappa|x-y|)^{-\frac{5}{2}} e^{\mathrm{i}\kappa|x-y|} f(y) dy. \end{aligned} \tag{4.2}$$

Let $\mathcal{U} \subset \mathbb{R}^2$ be a bounded domain satisfying $\operatorname{dist}(\mathcal{U}, \mathcal{O}) = r_0 > 0$. First we show that the strength μ of the source f given in Assumption 2.1 can be reconstructed uniquely by the variance of the solution u on $\mathcal{U}$.

PROPOSITION 4.1. *Let $d=2$, $k \geq 1$ and the assumptions in Theorem 3.2 hold. Then the following estimate holds:*

$$\mathbb{E}|u^2(x;k)|^2 = T_\kappa(x)|\kappa|^{-1}\kappa_{\mathrm{r}}^{-2s} + O\left(\kappa_{\mathrm{r}}^{-2s-2}\right), \quad x \in \mathcal{U},$$

where

$$T_\kappa(x) := \frac{1}{2^3\pi} \int_{\mathbb{R}^2} \frac{e^{-2\kappa_{\mathrm{i}}|x-y|}}{|x-y|} \mu(y)\, dy$$

and the residual term $O(\kappa_{\mathrm{r}}^{-2s-2})$ is an infinitesimal function equivalent to $\kappa_{\mathrm{r}}^{-2s-2}$ as $\kappa_{\mathrm{r}} \to \infty$.

Proof. For any $x \in \mathcal{U}$, we have from the expression of u^2 given in (4.2) that

$$
\begin{aligned}
\mathbb{E}|u^2(x;k)|^2 = &\frac{|a_0|^2}{16|\kappa|}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{e^{\mathrm{i}\kappa|x-y|-\mathrm{i}\bar{\kappa}|x-z|}}{|x-y|^{\frac12}|x-z|^{\frac12}}\mathbb{E}[f(y)f(z)]dydz\\
&+\Re\left[\frac{a_0\bar{a}_1}{8|\kappa|\bar{\kappa}}\right]\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{e^{\mathrm{i}\kappa|x-y|-\mathrm{i}\bar{\kappa}|x-z|}}{|x-y|^{\frac12}|x-z|^{\frac32}}\mathbb{E}[f(y)f(z)]dydz\\
&+\frac{|a_1|^2}{16|\kappa|^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{e^{\mathrm{i}\kappa|x-y|-\mathrm{i}\bar{\kappa}|x-z|}}{|x-y|^{\frac32}|x-z|^{\frac32}}\mathbb{E}[f(y)f(z)]dydz\\
&+\Re\left[\frac{a_0\bar{a}_2}{8|\kappa|\bar{\kappa}^2}\right]\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{e^{\mathrm{i}\kappa|x-y|-\mathrm{i}\bar{\kappa}|x-z|}}{|x-y|^{\frac12}|x-z|^{\frac52}}\mathbb{E}[f(y)f(z)]dydz\\
&+\Re\left[\frac{a_1\bar{a}_2}{8|\kappa|^3\bar{\kappa}}\right]\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{e^{\mathrm{i}\kappa|x-y|-\mathrm{i}\bar{\kappa}|x-z|}}{|x-y|^{\frac32}|x-z|^{\frac52}}\mathbb{E}[f(y)f(z)]dydz\\
&+\frac{|a_2|^2}{16|\kappa|^5}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{e^{\mathrm{i}\kappa|x-y|-\mathrm{i}\bar{\kappa}|x-z|}}{|x-y|^{\frac52}|x-z|^{\frac52}}\mathbb{E}[f(y)f(z)]dydz. \qquad (4.3)
\end{aligned}
$$

To estimate all the above terms, it suffices to consider the following integral with $l_1, l_2 \in \{0,1,2\}$:

$$
\begin{aligned}
I_{l_1,l_2}(x;k) :=& \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{e^{\mathrm{i}\kappa|x-y|-\mathrm{i}\overline{\kappa}|x-z|}}{|x-y|^{\frac12+l_1}|x-z|^{\frac12+l_2}}K_f(y,z)\theta(x)dydz\\
=& \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{e^{\mathrm{i}\kappa|x-y|-\mathrm{i}\overline{\kappa}|x-z|}}{|x-y|^{\frac12+l_1}|x-z|^{\frac12+l_2}}C_1(y,z,x)dydz, \qquad (4.4)
\end{aligned}
$$

where $C_1(y,z,x) := K_f(y,z)\theta(x)$ and $\theta \in C_0^\infty$ such that $\theta|_{\mathcal{U}} \equiv 1$ and $\mathrm{supp}(\theta) \subset \mathbb{R}^2\backslash\overline{\mathcal{O}}$. According to (2.1), the kernel C_1 in (4.4) is also an oscillatory integral

$$
C_1(y,z,x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{\mathrm{i}(y-z)\cdot\xi}c_1(y,x,\xi)d\xi
$$

and is compactly supported in $\mathcal{O}^\theta := \mathcal{O}\times\mathcal{O}\times\mathrm{supp}(\theta)$, where $c_1(y,x,\xi) := c(y,\xi)\theta(x)$ and $c(y,\xi)$ is the symbol of the covariance operator Q_f of the random field f. Since f satisfies Assumption 2.1, we get $c_1 \in S^{-2s}$ with the principal symbol

$$
c_1^p(y,x,\xi) = \mu(y)\theta(x)|\xi|^{-2s},
$$

where S^m denotes the space of symbols of order m. Moreover, C_1 is a conormal distribution in $\mathbb{R}^6$ of the Hörmander type having conormal singularity on the surface $S := \{(y,z,x) \in \mathbb{R}^6 : y - z = 0\}$ and is invariant under the change of coordinates [16].

To calculate the integral in (4.4), it is necessary to consider different coordinate systems. Define an invertible transformation $\tau : \mathbb{R}^6 \to \mathbb{R}^6$ by

$$
\tau(y,z,x) = (g,h,x),
$$

where $g = (g_1, g_2)$ and $h = (h_1, h_2)$ with

$$
\begin{aligned}
g_1 &= \frac12(|x-y| - |x-z|),\ g_2 = \frac12\left[|x-y|\arcsin\left(\frac{y_1-x_1}{|x-y|}\right) - |x-z|\arcsin\left(\frac{z_1-x_1}{|x-z|}\right)\right],\\
h_1 &= \frac12(|x-y| + |x-z|),\ h_2 = \frac12\left[|x-y|\arcsin\left(\frac{y_1-x_1}{|x-y|}\right) + |x-z|\arcsin\left(\frac{z_1-x_1}{|x-z|}\right)\right].
\end{aligned}
$$

Under the new coordinates system, (4.4) can be written as

$$I_{l_1,l_2}(x;k) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\mathrm{i}\kappa_{\mathrm{r}}(|x-y|-|x-z|)-\kappa_{\mathrm{i}}(|x-y|+|x-z|)} \frac{C_1(y,z,x)}{|x-y|^{\frac{1}{2}+l_1}|x-z|^{\frac{1}{2}+l_2}}dydz$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\mathrm{i}2\kappa_{\mathrm{r}}(e_1\cdot g)-2\kappa_{\mathrm{i}}(e_1\cdot h)}C_2(g,h,x)dgdh, \tag{4.5}$$

where $e_1 = (1,0)$ and

$$C_2(g,h,x) = C_1(\tau^{-1}(g,h,x))\frac{|\det((\tau^{-1})'(g,h,x))|}{((g+h)\cdot e_1)^{\frac{1}{2}+l_1}\,((h-g)\cdot e_1)^{\frac{1}{2}+l_2}}$$
$$=: C_1(\tau^{-1}(g,h,x))L^{\tau}(g,h,x). \tag{4.6}$$

To get a detailed expression of C_2 as well as its principal symbol, we define another invertible transformation $\eta : \mathbb{R}^6 \to \mathbb{R}^6$ by

$$\eta(y,z,x) = (v,w,x),$$

where $v = y - z$ and $w = y + z$. Consider the pull-back $C_3 := C_1 \circ \eta^{-1}$ satisfying

$$C_3(v,w,x) = C_1(\eta^{-1}(v,w,x)) = C_1\Big(\frac{v+w}{2}, \frac{w-v}{2}, x\Big)$$
$$= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{\mathrm{i}v\cdot\xi}c_1\Big(\frac{v+w}{2},x,\xi\Big)d\xi = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{\mathrm{i}v\cdot\xi}c_3\left(w,x,\xi\right)d\xi,$$

where we have used the properties of symbols (cf. [16, Lemma 18.2.1]) and that c_3 has the following asymptotic expansion:

$$c_3(w,x,\xi) = e^{-\mathrm{i}\langle D_v, D_\xi\rangle}c_1\Big(\frac{v+w}{2},x,\xi\Big)\Big|_{v=0}$$
$$\sim \sum_{j=0}^{\infty}\frac{\langle -\mathrm{i}D_v, D_\xi\rangle^j}{j!}c_1\Big(\frac{v+w}{2},x,\xi\Big)\Big|_{v=0}.$$

Moreover, the principal symbol of c_3 is

$$c_3^p(w,x,\xi) = c_1^p\Big(\frac{w}{2},x,\xi\Big) = \mu\left(\frac{w}{2}\right)\theta(x)|\xi|^{-2s}.$$

Finally, we define a diffeomorphism $\gamma := \eta\circ\tau^{-1} : (g,h,x) \mapsto (v,w,x)$, which preserves the plane $\{(g,h,x)\in\mathbb{R}^6 : g = 0\}$; i.e., if $g = 0$, then $v = 0$. By Theorem 18.2.9 in [16], the pull-back $C_4 := C_3\circ\gamma$ can be calculated by

$$C_4(g,h,x) = C_3(\gamma(g,h,x)) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{\mathrm{i}g\cdot\xi}c_4\left(h,x,\xi\right)d\xi,$$

where

$$c_4(h,x,\xi) = c_3\big(\gamma_2(0,h,x),(\gamma'_{11}(0,h,x))^{-\top}\xi\big)\left|\det\left(\gamma'_{11}(0,h,x)\right)\right|^{-1} + r_3(h,x,\xi)$$
$$= c_3^p\big(\gamma_2(0,h,x),(\gamma'_{11}(0,h,x))^{-\top}\xi\big)\left|\det\left(\gamma'_{11}(0,h,x)\right)\right|^{-1} + r_4(h,x,\xi).$$

Here the residuals $r_3, r_4 \in S^{-2s-1}$, $\gamma = (\gamma_1,\gamma_2)$ with $\gamma_1(g,h,x) = v$ and $\gamma_2(g,h,x) = (w,x)$, and γ'_{11} is determined by the Jacobian matrix

$$\gamma' = \begin{bmatrix} \gamma'_{11} & \gamma'_{12} \\ \gamma'_{21} & \gamma'_{22} \end{bmatrix}.$$

Hence, $c_4 \in S^{-2s}$ is still C^∞-smooth and compactly supported in the variables (h, x) with the principal symbol

$$c_4^p(h,x,\xi) = \mu\Big(\frac{w(0,h,x)}{2}\Big)\theta(x)\left|(\gamma_{11}'(0,h,x))^{-\top}\xi\right|^{-2s}\left|\det(\gamma_{11}'(0,h,x))\right|^{-1}. \tag{4.7}$$

Noting that $C_4 = C_3 \circ \gamma = C_1 \circ \eta^{-1} \circ \eta \circ \tau^{-1} = C_1 \circ \tau^{-1}$ and combining with (4.6), we obtain

$$\begin{aligned} C_2(g,h,x) &= C_4(g,h,x)L^\tau(g,h,x) \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{\mathrm{i}g\cdot\xi}c_4(h,x,\xi)\,L^\tau(g,h,x)d\xi = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{\mathrm{i}g\cdot\xi}c_5(h,x,\xi)d\xi, \end{aligned} \tag{4.8}$$

where we have used Lemma 18.2.1 in [16] again and the fact that the function $L^\tau(g,h,x)$ is smooth in the domain $\tau(\mathcal{O}^\theta)$. Similar to the asymptotic expansion of c_3, we have

$$c_5(h,x,\xi) \sim \sum_{j=0}^{\infty}\frac{\langle -\mathrm{i}D_g, D_\xi\rangle^j}{j!}\left(c_4(h,x,\xi)\,L^\tau(g,h,x)\right)\Big|_{g=0}.$$

Using (4.7) and the expression of L^τ defined in (4.6), we obtain the principal symbol

$$\begin{aligned} c_5^p(h,x,\xi) &= c_4^p(h,x,\xi)\,L^\tau(0,h,x) \\ &= \mu\Big(\frac{w(0,h,x)}{2}\Big)\theta(x)\left|(\gamma_{11}'(0,h,x))^{-\top}\xi\right|^{-2s}\frac{|\det((\tau^{-1})'(0,h,x))|}{|\det(\gamma_{11}'(0,h,x))|\,(h\cdot e_1)^{1+l_1+l_2}}, \end{aligned} \tag{4.9}$$

and residual $r_5 := c_5 - c_5^p \in S^{-2s-1}$.

Let $\alpha = \frac{h_2}{h_1}$. Simple calculations show that

$$\begin{aligned} \gamma_{11}'(0,h,x) = \frac{\partial v}{\partial g}(0,h,x) &= \begin{bmatrix} \frac{\partial v_1}{\partial g_1} & \frac{\partial v_1}{\partial g_2} \\ \frac{\partial v_2}{\partial g_1} & \frac{\partial v_2}{\partial g_2} \end{bmatrix}(0,h,x) \\ &= 2\begin{bmatrix} \sin\alpha - \alpha\cos\alpha & \cos\alpha \\ \cos\alpha + \alpha\sin\alpha & -\sin\alpha \end{bmatrix} \end{aligned}$$

is invertible since $\det(\gamma_{11}'(0,h,x)) = -4$ and $\gamma_2(0,h,x) = (w(0,h,x), x)$ with

$$w(0,h,x) = \Big(2h_1\sin\Big(\frac{h_2}{h_1}\Big) + 2x_1, 2h_1\cos\Big(\frac{h_2}{h_1}\Big) + 2x_2\Big). \tag{4.10}$$

Moreover, a straightforward calculation gives

$$\begin{aligned} \left(\tau^{-1}\right)'(0,h,x) &= \frac{\partial(y,z,x)}{\partial(g,h,x)}\bigg|_{g=0} \\ &= \begin{bmatrix} \sin\alpha - \alpha\cos\alpha & \cos\alpha & \sin\alpha - \alpha\cos\alpha & \cos\alpha & 1 & 0 \\ \cos\alpha + \alpha\sin\alpha & -\sin\alpha & \cos\alpha + \alpha\sin\alpha & -\sin\alpha & 0 & 1 \\ -\sin\alpha + \alpha\cos\alpha & -\cos\alpha & \sin\alpha - \alpha\cos\alpha & \cos\alpha & 1 & 0 \\ -\cos\alpha - \alpha\sin\alpha & \sin\alpha & \cos\alpha + \alpha\sin\alpha & -\sin\alpha & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

and $\det((\tau^{-1})'(0,h,x)) = 4$.

Combining (4.5) and (4.8)–(4.9), we obtain

$$
\begin{aligned}
I_{l_1,l_2}(x;k) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\mathrm{i}2\kappa_{\mathrm{r}}(e_1\cdot g)-2\kappa_{\mathrm{i}}(e_1\cdot h)} \\
&\qquad \times \left[\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{\mathrm{i}g\cdot\xi}\Big(c_4^p(h,x,\xi)L^\tau(0,h,x)+r_5(h,x,\xi)\Big)d\xi\right]dgdh \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-2\kappa_{\mathrm{i}}(e_1\cdot h)}\Big[c_4^p(h,x,\xi)L^\tau(0,h,x)+r_5(h,x,\xi)\Big]\delta(2\kappa_{\mathrm{r}}e_1+\xi)d\xi dh \\
&= \int_{\mathbb{R}^2} e^{-2\kappa_{\mathrm{i}}(e_1\cdot h)}\Big[c_4^p\left(h,x,-2\kappa_{\mathrm{r}}e_1\right)L^\tau(0,h,x)+r_5(h,x,-2\kappa_{\mathrm{r}}e_1)\Big]dh \\
&= \Big[\int_{\mathbb{R}^2} e^{-2\kappa_{\mathrm{i}}(e_1\cdot h)}\mu\Big(\frac{w(0,h,x)}{2}\Big)\theta(x)\left|(\gamma_{11}'(0,h,x))^{-\top}(-2\kappa_{\mathrm{r}}e_1)\right|^{-2s} \\
&\qquad \times\frac{1}{(e_1\cdot h)^{1+l_1+l_2}}dh+O(\kappa_{\mathrm{r}}^{-2s-1})\Big] \\
&= \Big[\int_{\mathbb{R}^2}\frac{e^{-2\kappa_{\mathrm{i}}(e_1\cdot h)}}{(e_1\cdot h)^{1+l_1+l_2}}\mu\Big(\frac{w(0,h,x)}{2}\Big)\theta(x)dh\Big]\kappa_{\mathrm{r}}^{-2s}+O(\kappa_{\mathrm{r}}^{-2s-1}) \\
&=: M_{l_1,l_2}^\kappa(x)\kappa_{\mathrm{r}}^{-2s}+O(\kappa_{\mathrm{r}}^{-2s-1}),
\end{aligned}
\tag{4.11}
$$

where we have used the fact that $\delta(\xi)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{\mathrm{i}x\cdot\xi}dx$ in the second step and

$$
M_{l_1,l_2}^\kappa(x)=\int_{\mathbb{R}^2}\frac{e^{-2\kappa_{\mathrm{i}}(e_1\cdot h)}}{(e_1\cdot h)^{1+l_1+l_2}}\mu\Big(\frac{w(0,h,x)}{2}\Big)\theta(x)dh.
$$

To simplify the expression of $M_{l_1,l_2}^\kappa(x)$, we consider the following coordinate transformation from h to ζ:

$$
\zeta=\left(h_1\sin\left(\frac{h_2}{h_1}\right),h_1\cos\left(\frac{h_2}{h_1}\right)\right)+x,
$$

which has the Jacobian

$$
\det\left(\frac{\partial\zeta}{\partial h}\right)=\begin{vmatrix}\sin\left(\frac{h_2}{h_1}\right)-\frac{h_2}{h_1}\cos\left(\frac{h_2}{h_1}\right) & \cos\left(\frac{h_2}{h_1}\right)\\ \cos\left(\frac{h_2}{h_1}\right)+\frac{h_2}{h_1}\sin\left(\frac{h_2}{h_1}\right) & -\sin\left(\frac{h_2}{h_1}\right)\end{vmatrix}=-1.
$$

Noting also that $|x-\zeta|=e_1\cdot h$ and $w(0,h,x)=2\zeta$ according to (4.10), we obtain for $x\in\mathcal{U}$ that

$$
\begin{aligned}
M_{l_1,l_2}^\kappa(x) &= \int_{\mathbb{R}^2}\frac{e^{-2\kappa_{\mathrm{i}}(e_1\cdot h)}}{(e_1\cdot h)^{1+l_1+l_2}}\mu\Big(\frac{w(0,h,x)}{2}\Big)\theta(x)dh \\
&= \int_{\mathbb{R}^2}\frac{e^{-2\kappa_{\mathrm{i}}|x-\zeta|}}{|x-\zeta|^{1+l_1+l_2}}\mu\left(\zeta\right)\theta(x)\left|\det\left(\frac{\partial\zeta}{\partial h}\right)^{-1}\right|d\zeta \\
&= \int_{\mathbb{R}^2}\frac{e^{-2\kappa_{\mathrm{i}}|x-\zeta|}}{|x-\zeta|^{1+l_1+l_2}}\mu\left(\zeta\right)d\zeta.
\end{aligned}
$$

By the definition of I_{l_1,l_2} defined in (4.4) and its estimate given in (4.11), the energy $\mathbb{E}|u^2(x;k)|^2$ given in (4.3) turns to be

$$
\begin{aligned}
\mathbb{E}|u^2(x;k)|^2 &= \frac{|a_0|^2}{16|\kappa|} I_{0,0}(x;k) + \Re\left[\frac{a_0\bar{a}_1}{8|\kappa|\kappa} I_{0,1}(x;k)\right] + \frac{|a_1|^2}{16|\kappa|^3} I_{1,1}(x;k) \\
&\quad + \Re\left[\frac{a_0\bar{a}_2}{8|\kappa|\kappa^2} I_{0,2}(x;k)\right] + \Re\left[\frac{a_1\bar{a}_2}{8|\kappa|^3\kappa} I_{1,2}(x;k)\right] + \frac{|a_2|^2}{16|\kappa|^5} I_{2,2}(x;k) \\
&= \frac{|a_0|^2}{16|\kappa|}\left[M_{0,0}^{\kappa}(x)\kappa_{\rm r}^{-2s} + O(\kappa_{\rm r}^{-2s-1})\right] \\
&\quad + \Re\left[\frac{a_0\bar{a}_1}{8|\kappa|\kappa}\left(M_{0,1}^{\kappa}(x)\kappa_{\rm r}^{-2s} + O(\kappa_{\rm r}^{-2s-1})\right)\right] \\
&\quad + \frac{|a_1|^2}{16|\kappa|^3}\left[M_{1,1}^{\kappa}(x)\kappa_{\rm r}^{-2s} + O(\kappa_{\rm r}^{-2s-1})\right] \\
&\quad + \Re\left[\frac{a_0\bar{a}_2}{8|\kappa|\kappa^2}\left(M_{0,2}^{\kappa}(x)\kappa_{\rm r}^{-2s} + O(\kappa_{\rm r}^{-2s-1})\right)\right] \\
&\quad + \Re\left[\frac{a_1\bar{a}_2}{8|\kappa|^3\kappa}\left(M_{1,2}^{\kappa}(x)\kappa_{\rm r}^{-2s} + O(\kappa_{\rm r}^{-2s-1})\right)\right] \\
&\quad + \frac{|a_2|^2}{16|\kappa|^5}\left[M_{2,2}^{\kappa}(x)\kappa_{\rm r}^{-2s} + O(\kappa_{\rm r}^{-2s-1})\right] \\
&= \frac{|a_0|^2}{16} M_{0,0}^{\kappa}(x)|\kappa|^{-1}\kappa_{\rm r}^{-2s} + O(\kappa_{\rm r}^{-2s-2}),
\end{aligned}
$$

which completes the proof. ∎

THEOREM 4.2. *Let $d = 2$ and assumptions in Theorem 3.2 hold. For any $x \in \mathcal{U}$, it holds*

$$
\lim_{k\to\infty} k^{2s+1}\mathbb{E}|u(x;k)|^2 = \frac{1}{2^3\pi}\int_{\mathbb{R}^2}\frac{e^{-\sigma|x-y|}}{|x-y|}\mu(y)\,dy =: T^{(2)}(x).
$$

Proof. Note that

$$
\begin{aligned}
k^{2s+1}\mathbb{E}|u(x;k)|^2 &= k^{2s+1}\mathbb{E}|u^2(x;k)|^2 + 2k^{2s+1}\mathbb{E}\Re\left[\overline{u^2(x;k)}(u(x;k) - u^2(x;k))\right] \\
&\quad + k^{2s+1}\mathbb{E}\left|u(x;k) - u^2(x;k)\right|^2 \\
&=: V_1(k) + V_2(k) + V_3(k).
\end{aligned}
$$

Next we calculate the limits of V_1, V_2, and V_3, respectively.

Using the asymptotic expansions of the Hankel function in (4.1), we get

$$
\left|H_n^{(1)}(\kappa|x-y|) - H_{n,N}^{(1)}(\kappa|x-y|)\right| = O\big(|\kappa|x-y||^{-(N+\frac{3}{2})}\big), \quad k\to\infty.
$$

Noting $H_0^{(1)'}(z) = -H_1^{(1)}(z)$, we have

$$
\left|\partial_{y_i}H_0^{(1)}(\kappa|x-y|) - \partial_{y_i}H_{0,N}^{(1)}(\kappa|x-y|)\right| = O\big(|\kappa|^{-(N+\frac{1}{2})}|x-y|^{-(N+\frac{3}{2})}\big), \quad k\to\infty.
$$

Hence

$$
\begin{aligned}
\mathbb{E}|u(x;k) - u^2(x;k)|^2 &= \mathbb{E}\left|\int_{\mathcal{O}}\left(\Phi_\kappa(x,y) - \Phi_\kappa^2(x,y)\right)f(y)dy\right|^2 \\
&\lesssim \|\Phi_\kappa(x,\cdot) - \Phi_\kappa^2(x,\cdot)\|_{W^{1,q}(\mathcal{O})}^2\mathbb{E}\|f\|_{W^{-1,p}(\mathcal{O})}^2 \\
&\lesssim \|\Phi_\kappa(x,\cdot) - \Phi_\kappa^2(x,\cdot)\|_{W^{1,q}(\mathcal{O})}^2\mathbb{E}\|f\|_{W^{H-\epsilon,p}(\mathcal{O})}^2 \lesssim |\kappa|^{-5},
\end{aligned}
$$

where $f \in L^2(\Omega, W^{H-\epsilon,p}(\mathcal{O})) \subset L^2(\Omega, W^{-1,p}(\mathcal{O}))$ for $H \in (\frac{2}{p} - 2, 0]$ and $p \in (1, 2]$ and $\frac{1}{q} + \frac{1}{p} = 1$ according to Theorem 3.2 with $d = 2$. It then indicates that

$$V_3(k) \lesssim k^{2s+1}|\kappa|^{-5} = k^{2s+1}(k^4 + k^2\sigma^2)^{-\frac{5}{4}} \to 0$$

as $k \to \infty$ since $s < 2$ for $d = 2$.

For $V_2(k)$, we have

$$V_2(k) \leq 2\left(k^{2s+1}\mathbb{E}|u^2(x;k)|^2\right)^{\frac{1}{2}}\left(k^{2s+1}\mathbb{E}|u(x;k) - u^2(x;k)|^2\right)^{\frac{1}{2}} = 2V_1(k)^{\frac{1}{2}}V_3(k)^{\frac{1}{2}},$$

which converges to 0 if the limit of $V_1(k)$ exists.

For $V_1(k)$, by Proposition 4.1,

$$V_1(k) = T_\kappa(x)k^{2s+1}|\kappa|^{-1}\kappa_{\rm r}^{-2s} + O(k^{2s+1}\kappa_{\rm r}^{-2s-2}).$$

We have from (3.1) that

$$\lim_{k\to\infty} V_1(k) = \lim_{k\to\infty} T_\kappa(x) = \frac{|a_0|^2}{16}\int_{\mathbb{R}^2}\frac{e^{-\sigma|x-y|}}{|x-y|}\mu(y)\,dy,$$

which completes the proof. ∎

Remark 4.3. It can be seen from the above proof that only two terms are needed in the truncation of (4.1) if the source is extremely rough with $s \in [0, \frac{d}{2})$. More precisely, it suffices to consider the approximate solution

$$u^1(x;k) := -\int_{\mathbb{R}^d}\Phi_\kappa^1(x,y)f(y)dy$$

instead of u^2, where $V_3(k) \lesssim k^{2s+1}|\kappa|^{-3} \to 0$ as $k \to \infty$ since $s < \frac{d}{2} = 1$.

Next, we show that the strength μ is uniquely determined by the measurement $T^{(2)}(x)$ in the bounded open set $\mathcal{U}$ given in Theorem 4.2. To have the three-dimensional case included, the following uniqueness result is given for the d-dimensional case with $d = 2, 3$.

THEOREM 4.4. *The strength μ is uniquely determined by*

$$T^{(d)}(x) = C_d\int_{\mathbb{R}^d}\frac{e^{-\sigma|x-y|}}{|x-y|^{d-1}}\mu(y)\,dy, \quad x \in \mathcal{U} \subset \mathbb{R}^d,$$

where $C_d = \frac{1}{2^2(2\pi)^{d-1}}$ and $d = 2, 3$.

Proof. Denote $g(x) := e^{-\sigma|x|}/|x|^{d-1}$ such that $T^{(d)}(x) = C_d(g * \mu)(x)$. We claim that $g \in L^1_{\rm loc}(\mathbb{R}^d)$, and hence $T^{(d)} = C_d(g*\mu) \in C^\infty(\mathbb{R}^d)$ for $\mu \in C_0^\infty(\mathbb{R}^d)$. In fact, for any compact set $K \subset \mathbb{R}^d$, there exists a positive constant R such that $K \subset B(0, R)$, where $B(0, R)$ denotes the open ball centered at 0 with radius R, and

$$\int_K |g(x)|dx \lesssim \int_0^R \frac{e^{-\sigma r}}{r^{d-1}}r^{d-1}dr \leq R,$$

which completes the proof of the claim.

To ensure that the Fourier transform can be applied to $T^{(d)}$, we next show that $T^{(d)}$ is real analytic in the open set $\mathcal{U}$ and hence can be uniquely extended to $\mathbb{R}^d$.

As an alternative characterization of real analytic functions, we only need to show equivalently that $T^{(d)} \in C^\infty(\mathcal{U})$ and for any compact set $K \subset \mathcal{U}$, there exist constants C_1 and C_2 such that

$$\|\Delta^m T^{(d)}\|_{L^2(K)} \le C_1(C_2 m)^{2m}$$

for any $m \in \mathbb{N}$ (cf. [18]). Note that

$$\Delta\left(\frac{e^{-\sigma|x|}}{|x|^j}\right) = \left[\frac{j^2}{|x|^{j+2}} + \frac{(2j-1)\sigma}{|x|^{j+1}} + \frac{\sigma^2}{|x|^j}\right] e^{-\sigma|x|} \quad \forall\, j \in \mathbb{N}$$

and that $|x-y| > r_0 = \mathrm{dist}(\mathcal{U},\mathcal{O}) > 0$ for any $x \in K \subset \mathcal{U}$ and $y \in \mathcal{O}$. Without loss of generality, we assume that $r_0 < 1$. If $r_0 > 1$, we can always find a positive constant $\tilde r_0 < 1$ such that $|x-y| > \tilde r_0$ for any $x \in K \subset \mathcal{U}$ and $y \in \mathcal{O}$. For simplicity, we denote $l := d-1$, and derive for any $x \in K$ that

$$\begin{aligned}
\left|\frac{\Delta^m T^{(d)}(x)}{C_d}\right| &= \Delta^{m-1}\int_{\mathbb{R}^d}\left(\frac{l^2}{|x-y|^{l+2}} + \frac{(2l-1)\sigma}{|x-y|^{l+1}} + \frac{\sigma^2}{|x-y|^l}\right) e^{-\sigma|x-y|}\mu(y)\,dy \\
&= \Delta^{m-2}\int_{\mathbb{R}^d}\Bigg[l^2\left(\frac{(l+2)^2}{|x-y|^{l+2+2}} + \frac{(2(l+2)-1)\sigma}{|x-y|^{l+2+1}} + \frac{\sigma^2}{|x-y|^{l+2}}\right) \\
&\quad +(2l-1)\sigma\left(\frac{(l+1)^2}{|x-y|^{l+1+2}} + \frac{(2(l+1)-1)\sigma}{|x-y|^{l+1+1}} + \frac{\sigma^2}{|x-y|^{l+1}}\right) \\
&\quad +\sigma^2\left(\frac{l^2}{|x-y|^{l+2}} + \frac{(2l-1)\sigma}{|x-y|^{l+1}} + \frac{\sigma^2}{|x-y|^l}\right)\Bigg] e^{-\sigma|x-y|}\mu(y)\,dy \\
&= \int_{\mathbb{R}^d}\Bigg[l^2(l+2)^2\cdots(l+2(m-2))^2\Bigg(\frac{(l+2(m-1))^2}{|x-y|^{l+2m}} \\
&\quad +\frac{(2(l+2(m-1))-1)\sigma}{|x-y|^{l+2m-1}} + \frac{\sigma^2}{|x-y|^{l+2m-2}}\Bigg) + \cdots \\
&\quad +\sigma^{2(m-1)}\left(\frac{l^2}{|x-y|^{l+2}} + \frac{(2l-1)\sigma}{|x-y|^{l+1}} + \frac{\sigma^2}{|x-y|^l}\right)\Bigg] e^{-\sigma|x-y|}\mu(y)\,dy \\
&\le \Bigg[l^2(l+2)^2\cdots(l+2(m-2))^2\Bigg(\frac{(l+2(m-1))^2}{r_0^{l+2m}} \\
&\quad +\frac{(2(l+2(m-1))-1)\sigma}{r_0^{l+2m-1}} + \frac{\sigma^2}{r_0^{l+2m-2}}\Bigg) + \cdots \\
&\quad +\sigma^{2(m-1)}\left(\frac{l^2}{r_0^{l+2}} + \frac{(2l-1)\sigma}{r_0^{l+1}} + \frac{\sigma^2}{r_0^l}\right)\Bigg]\int_{\mathcal{O}}\mu(y)dy.
\end{aligned} \tag{4.12}$$

Note also that

$$\frac{j^2}{r_0^{j+2}} + \frac{(2j-1)\sigma}{r_0^{j+1}} + \frac{\sigma^2}{r_0^j} \le \frac{1}{r_0^j}\left(\frac{j}{r_0} + \sigma\right)^2 \quad \forall\, j \in \mathbb{N}$$

and

$$(2j-1)\sigma \le j^2 + \sigma^2 \quad \forall\, j \in \mathbb{N}.$$

Hence, (4.12) leads to

$$
\begin{aligned}
&\|\Delta^m T^{(d)}\|_{L^2(K)} \\
&\lesssim \frac{l^2\cdots(l+2(m-2))^2}{r_0^{l+2m-2}}\left(\frac{l+2m-2}{r_0}+\sigma\right)^2+\cdots+\frac{\sigma^{2(m-1)}}{r_0^l}\left(\frac{l}{r_0}+\sigma\right)^2 \\
&\lesssim 3^m\frac{\max\{l^2\cdots(l+2(m-2))^2,\cdots,\sigma^{2(m-1)}\}}{r_0^{l+2m-2}}\left(\frac{l+2m-2}{r_0}+\sigma\right)^2 \\
&\lesssim \left(\frac{3}{r_0^2}\right)^m l^2\cdots(l+2(m-2))^2\left(1\vee\sigma^{2(m-1)}\right)\left[\left(\frac{l+2m-2}{r_0}\right)^2+\sigma^2\right] \\
&\lesssim \left(\frac{3}{r_0^2}\right)^m ((2m-2)!!)^2\left(1\vee\sigma^{2m}\right) \\
&\lesssim (Cm)^{2m}.
\end{aligned}
$$

Finally, we conclude that the Fourier transform of μ can be uniquely determined by

$$\hat{\mu}(\xi)=\frac{\mathcal{F}[T^{(d)}](\xi)}{\mathcal{F}[g](\xi)},$$

provided that $\mathcal{F}[g]$ is a well-defined nonzero function. It is clear from the Fourier transform of g that $\mathcal{F}[g]$ is positive for any $\xi\in\mathbb{R}^d$. Next is to show that $\mathcal{F}[g]$ is well defined. In fact, for any constant $R>0$, we may verify from simple calculations that

$$
\begin{aligned}
|\mathcal{F}[g](\xi)| &= \left|\left(\mathcal{F}[e^{-\sigma|x|}]*\mathcal{F}[|x|^{-(d-1)}]\right)(\xi)\right| \\
&\lesssim \int_{\mathbb{R}^d}\frac{1}{\sigma^2+|\tau|^2}|\xi-\tau|^{-1}d\tau \\
&\lesssim \int_{\{|\xi-\tau|<R\}}\frac{1}{\sigma^2}|\xi-\tau|^{-1}d\tau+\int_{\{|\xi-\tau|>R,|\tau|>R\}}\frac{1}{|\tau|^2}R^{-1}d\tau \\
&\quad+\int_{\{|\xi-\tau|>R,|\tau|<R\}}\frac{1}{\sigma^2 R}d\tau \\
&\lesssim \int_0^R r^{-1}r^{d-1}dr+\int_R^\infty r^{-2}r^{d-1}dr+\int_{B(0,R)}1d\tau<\infty,
\end{aligned}
$$

which completes the proof. ∎

4.2. Three-dimensional case. Now we consider $d=3$. By Theorem 3.2, the solution of the direct problem is

$$u(x;k)=-\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}\kappa|x-y|}}{|x-y|}f(y)dy. \tag{4.13}$$

Following the same procedure as that for the two-dimensional case, we first show that the strength μ is uniquely determined by the variance of the solution u.

THEOREM 4.5. *Let $d=3$ and assumptions in Theorem 3.2 hold. For any $x\in\mathcal{U}$, it holds*

$$\lim_{k\to\infty}k^{2s}\mathbb{E}|u(x;k)|^2=\frac{1}{2^4\pi^2}\int_{\mathbb{R}^3}\frac{e^{-\sigma|x-y|}}{|x-y|^2}\mu(y)\,dy=:T^{(3)}(x).$$

Moreover, the strength μ is uniquely determined by $T^{(3)}$ in $\mathcal{U}$.

Proof. Using the formula given in (4.13), we have for any $x\in\mathcal{U}$ that

$$
\begin{aligned}
\mathbb{E}|u(x;k)|^2 &= \frac{1}{16\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}\kappa|x-y|-\mathrm{i}\bar{\kappa}|x-z|}}{|x-y||x-z|}\mathbb{E}[f(y)f(z)]dydz\\
&= \frac{1}{16\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}\kappa|x-y|-\mathrm{i}\bar{\kappa}|x-z|}}{|x-y||x-z|}K_f(y,z)\theta(x)dydz\\
&= \frac{1}{16\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{\mathrm{i}\kappa_{\mathrm{r}}(|x-y|-|x-z|)-\kappa_{\mathrm{i}}(|x-y|+|x-z|)}\frac{C_1(y,z,x)}{|x-y||x-z|}dydz, \qquad (4.14)
\end{aligned}
$$

where $\theta \in C_0^\infty$ such that $\theta|_{\mathcal{U}} \equiv 1$ and $\mathrm{supp}(\theta) \subset \mathbb{R}^3\backslash\overline{\mathcal{O}}$,

$$
C_1(y,z,x) := K_f(y,z)\theta(x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}e^{\mathrm{i}(y-z)\cdot\xi}c_1(y,x,\xi)d\xi.
$$

Here $c_1(y,x,\xi) := c(y,\xi)\theta(x)$ with the symbol $c(y,\xi)$ satisfying (2.1). Then the principal symbol of c_1 has the form

$$
c_1^p(y,x,\xi) = \mu(y)\theta(x)|\xi|^{-2s}.
$$

We first define an invertible transformation $\tau : \mathbb{R}^9 \to \mathbb{R}^9$ by $\tau(y,z,x) = (g,h,x)$, where $g = (g_1,g_2,g_3)$ and $h = (h_1,h_2,h_3)$ with

$$
\begin{aligned}
g_1 &= \frac{1}{2}\left(|x-y|-|x-z|\right), \quad h_1 = \frac{1}{2}\left(|x-y|+|x-z|\right),\\
g_2 &= \frac{1}{2}\left[|x-y|\arccos\left(\frac{y_3-x_3}{|x-y|}\right)-|x-z|\arccos\left(\frac{z_3-x_3}{|x-z|}\right)\right],\\
h_2 &= \frac{1}{2}\left[|x-y|\arccos\left(\frac{y_3-x_3}{|x-y|}\right)+|x-z|\arccos\left(\frac{z_3-x_3}{|x-z|}\right)\right],\\
g_3 &= \frac{1}{2}\left[|x-y|\arctan\left(\frac{y_2-x_2}{y_1-x_1}\right)-|x-z|\arctan\left(\frac{z_2-x_2}{z_1-x_1}\right)\right],\\
h_3 &= \frac{1}{2}\left[|x-y|\arctan\left(\frac{y_2-x_2}{y_1-x_1}\right)+|x-z|\arctan\left(\frac{z_2-x_2}{z_1-x_1}\right)\right].
\end{aligned}
$$

Under the transformation defined above, (4.14) turns to be

$$
\mathbb{E}|u(x;k)|^2 = \frac{1}{16\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{2\mathrm{i}\kappa_{\mathrm{r}}(e_1\cdot g)-2\kappa_{\mathrm{i}}(e_1\cdot h)}C_2(g,h,x)dgdh, \qquad (4.15)
$$

where $e_1 = (1,0,0)$ and

$$
\begin{aligned}
C_2(g,h,x) &= C_1(\tau^{-1}(g,h,x))\frac{|\det\left((\tau^{-1})'(g,h,x)\right)|}{((g+h)\cdot e_1)((h-g)\cdot e_1)}\\
&=: C_1(\tau^{-1}(g,h,x))L^\tau(g,h,x). \qquad (4.16)
\end{aligned}
$$

Next is to get an explicit expression of C_2 with respect to (g,h,x). We define another invertible transformation $\eta : \mathbb{R}^9 \to \mathbb{R}^9$ by $\eta(y,z,x) = (v,w,x)$ with $v = y-z$ and $w = y+z$, and define the diffeomorphism $\gamma := \eta\circ\tau^{-1} : (g,h,x) \mapsto (v,w,x)$. Following the same procedure as that used in Proposition 4.1, by defining $C_3 := C_1\circ\eta^{-1}$, we obtain

$$C_3(v,w,x) = C_1(\eta^{-1}(v,w,x)) = C_1\Big(\frac{v+w}{2}, \frac{w-v}{2}, x\Big)$$
$$= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{\mathrm{i}v\cdot\xi} c_1\Big(\frac{v+w}{2}, x, \xi\Big) d\xi = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{\mathrm{i}v\cdot\xi} c_3\,(w,x,\xi)\, d\xi,$$

where c_3 has the principal symbol $c_3^p(w,x,\xi) = c_1^p\left(\frac{v+w}{2}, x, \xi\right)|_{v=0} = \mu(\frac{w}{2})|\xi|^{-2s}\theta(x)$. By Theorem 18.2.9 in [16],

$$C_4(g,h,x) := C_3\circ\gamma(g,h,x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{\mathrm{i}g\cdot\xi} c_4(h,x,\xi) d\xi, \tag{4.17}$$

where c_4 has the principal symbol

$$c_4^p(h,x,\xi) = c_3^p\left(\gamma_2(0,h,x), (\gamma_{11}'(0,h,x))^{-\top}\xi\right) |\det(\gamma_{11}'(0,h,x))|^{-1},$$

and $\gamma_2(0,h,x) = (w(0,h,x),x), \gamma_{11}'(0,h,x) = \frac{\partial v}{\partial g}(0,h,x)$. Noting that $C_4 = C_3\circ\gamma = (C_1\circ\eta^{-1})\circ(\eta\circ\tau^{-1}) = C_1\circ\tau^{-1}$, we are able to give the expression of C_2 defined in (4.16) based on the expression of C_4 in (4.17):

$$C_2(g,h,x) = C_1\circ\tau^{-1}(g,h,x)L^\tau(g,h,x)$$
$$= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{\mathrm{i}g\cdot\xi} c_4(h,x,\xi)L^\tau(g,h,x) d\xi = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{\mathrm{i}g\cdot\xi} c_5(h,x,\xi) d\xi, \tag{4.18}$$

where the principal symbol of c_5, according to the asymptotic expansion of c_4, is

$$c_5^p(h,x,\xi) = c_4^p(h,x,\xi)L^\tau(0,h,x)$$
$$= \mu\Big(\frac{w(0,h,x)}{2}\Big)\theta(x)\Big|\Big(\frac{\partial v}{\partial g}(0,h,x)\Big)^{-\top}\xi\Big|^{-2s}\Big|\det\Big(\frac{\partial v}{\partial g}(0,h,x)\Big)\Big|^{-1} L^\tau(0,h,x) \tag{4.19}$$

and the residual $r_5 := c_5 - c_5^p \in S^{-2s-1}$.

It then suffices to calculate c_5^p. Noting that

$$h_1 + g_1 = |x-y|, \quad h_1 - g_1 = |x-z|,$$
$$\frac{h_2+g_2}{h_1+g_1} = \arccos\Big(\frac{y_3-x_3}{|x-y|}\Big), \quad \frac{h_2-g_2}{h_1-g_1} = \arccos\Big(\frac{z_3-x_3}{|x-z|}\Big),$$
$$\frac{h_3+g_3}{h_1+g_1} = \arctan\Big(\frac{y_2-x_2}{y_1-x_1}\Big), \quad \frac{h_3-g_3}{h_1-g_1} = \arctan\Big(\frac{z_2-x_2}{z_1-x_1}\Big),$$

we get

$$y_1 = x_1 + (h_1+g_1)\sin\left(\frac{h_2+g_2}{h_1+g_1}\right)\cos\left(\frac{h_3+g_3}{h_1+g_1}\right),$$
$$y_2 = x_2 + (h_1+g_1)\sin\left(\frac{h_2+g_2}{h_1+g_1}\right)\sin\left(\frac{h_3+g_3}{h_1+g_1}\right),$$
$$y_3 = x_3 + (h_1+g_1)\cos\left(\frac{h_2+g_2}{h_1+g_1}\right),$$
$$z_1 = x_1 + (h_1-g_1)\sin\left(\frac{h_2-g_2}{h_1-g_1}\right)\cos\left(\frac{h_3-g_3}{h_1-g_1}\right),$$
$$z_2 = x_2 + (h_1-g_1)\sin\left(\frac{h_2-g_2}{h_1-g_1}\right)\sin\left(\frac{h_3-g_3}{h_1-g_1}\right),$$
$$z_3 = x_3 + (h_1-g_1)\cos\left(\frac{h_2-g_2}{h_1-g_1}\right).$$

A simple calculation yields that

$$\frac{\partial v}{\partial g}(0,h,x)=2\begin{bmatrix}\sin\alpha\cos\beta-\alpha\cos\alpha\cos\beta+\beta\sin\alpha\sin\beta & \cos\alpha\cos\beta & -\sin\alpha\sin\beta\\ \sin\alpha\sin\beta-\alpha\cos\alpha\sin\beta-\beta\sin\alpha\cos\beta & \cos\alpha\sin\beta & \sin\alpha\cos\beta\\ \cos\alpha+\alpha\sin\alpha & -\sin\alpha & 0\end{bmatrix},$$

where $\alpha:=\frac{h_2}{h_1},\beta:=\frac{h_3}{h_1}$, and

$$(\tau^{-1})'(0,h,x)=\begin{bmatrix}\frac{1}{2}\frac{\partial v}{\partial g} & \frac{1}{2}\frac{\partial v}{\partial g} & I\\ -\frac{1}{2}\frac{\partial v}{\partial g} & \frac{1}{2}\frac{\partial v}{\partial g} & I\\ 0 & 0 & I\end{bmatrix}.$$

Here I is the 3×3 identity matrix. It can be verified that

$$\det\left(\frac{\partial v}{\partial g}(0,h,x)\right)=8\sin\alpha,\quad L^\tau(0,h,x)=\frac{|\det\left((\tau^{-1})'(0,h,x)\right)|}{(h\cdot e_1)^2}=\frac{8\sin^2\alpha}{(h\cdot e_1)^2},$$

and

$$\left(\frac{\partial v}{\partial g}(0,h,x)\right)^{-\top}=\frac{1}{2}\begin{bmatrix}\sin\alpha\cos\beta & \cos\alpha\cos\beta+\alpha\sin\alpha\cos\beta & -\frac{\sin\beta}{\sin\alpha}+\beta\sin\alpha\cos\beta\\ \sin\alpha\sin\beta & \cos\alpha\sin\beta+\alpha\sin\alpha\sin\beta & \frac{\cos\beta}{\sin\alpha}+\beta\sin\alpha\sin\beta\\ \cos\alpha & -\frac{\cos\beta}{\sin\alpha}-\beta\sin\alpha\sin\beta & \beta\cos\alpha\end{bmatrix}.$$

By (4.18)–(4.19) and the above estimates on $\frac{\partial v}{\partial g}(0,h,x)$ and $L^\tau(0,h,x)$, the energy $\mathbb{E}|u(x;k)|^2$ in (4.15) can be written as

(4.20)

$$\begin{aligned}\mathbb{E}|u(x;k)|^2&\\ &=\frac{1}{16\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{2i\kappa_r(e_1\cdot g)-2\kappa_i(e_1\cdot h)}C_2(g,h,x)dgdh\\ &=\frac{1}{16\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{2i\kappa_r(e_1\cdot g)-2\kappa_i(e_1\cdot h)}\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}e^{ig\cdot\xi}c_5(h,x,\xi)d\xi dgdh\\ &=\frac{1}{16\pi^2}\int_{\mathbb{R}^3}e^{-2\kappa_i(e_1\cdot h)}c_5(h,x,-2\kappa_r e_1)dh\\ &=\frac{1}{16\pi^2}\int_{\mathbb{R}^3}e^{-2\kappa_i(e_1\cdot h)}\left[\mu\Big(\frac{w(0,h,x)}{2}\Big)\theta(x)\kappa_r^{-2s}\frac{|\sin\alpha|}{(h\cdot e_1)^2}+r_5(h,x,-2\kappa_r e_1)\right]dh,\end{aligned}$$

where

$$\frac{w(0,h,x)}{2}=(h_1\sin\alpha\cos\beta,h_1\sin\alpha\sin\beta,h_1\cos\alpha)+x.$$

Define another coordinate transform $\rho:\mathbb{R}^3\to\mathbb{R}^3$ by

$$\rho(h)=\zeta:=(h_1\sin\alpha\cos\beta,h_1\sin\alpha\sin\beta,h_1\cos\alpha)+x.$$

By noting that $|\zeta-x|=h_1=h\cdot e_1$ and $\det((\rho^{-1})')=\frac{1}{\det(\rho')}$ with

$$\rho'=\begin{bmatrix}\sin\alpha\cos\beta-\alpha\cos\alpha\cos\beta+\beta\sin\alpha\sin\beta & \cos\alpha\cos\beta & -\sin\alpha\sin\beta\\ \sin\alpha\sin\beta-\alpha\cos\alpha\sin\beta-\beta\sin\alpha\cos\beta & \cos\alpha\sin\beta & \sin\alpha\cos\beta\\ \cos\alpha+\alpha\sin\alpha & -\sin\alpha & 0\end{bmatrix},$$

we obtain from (4.20) that

$$\mathbb{E}|u(x;k)|^2=\left[\frac{1}{2^4\pi^2}\int_{\mathbb{R}^3}\frac{e^{-2\kappa_i|\zeta-x|}}{|\zeta-x|^2}\mu(\zeta)\theta(x)d\zeta\right]\kappa_r^{-2s}+O(\kappa_r^{-2s-1}).$$

Finally, for any $x \in \mathcal{U}$, we have from (3.1) that

$$\lim_{k\to\infty} k^{2s}\mathbb{E}|u(x;k)|^2 = \lim_{k\to\infty} \frac{1}{2^4\pi^2}\int_{\mathbb{R}^3} \frac{e^{-2\kappa_{\rm i}|\zeta-x|}}{|\zeta-x|^2}\mu(\zeta)d\zeta\left(\frac{k}{\kappa_{\rm r}}\right)^{2s} = T^{(3)}(x).$$

Moreover, the strength μ is uniquely determined by $T^{(3)}(x)$ for $x \in \mathcal{U}$ according to Theorem 4.4, which completes the proof. ∎

4.3. The case $\boldsymbol{\sigma = 0}$ and ergodicity. If $\sigma = 0$, the model (1.1) reduces to the one considered in [20]. In this case, the ergodicity of the solution can be obtained by following the same way which was investigated in [19, 20]. This result makes it possible to uniquely recover the strength μ by a single realization of the measurements.

PROPOSITION 4.6. *Assume that $f \in L^2(\Omega, W^{H-\epsilon,p}(\mathcal{O}))$ with H, ϵ, and p satisfying the conditions given in Theorem* 3.2. *Let $s = H + \frac{d}{2}$. Then*

(i) *if $d = 2$,*

$$\lim_{K\to\infty} \frac{1}{K-1}\int_1^K k^{2s+1}|u(x;k)|^2 dk = T^{(2)}(x) \quad \textit{almost surely,}$$

(ii) *if $d = 3$,*

$$\lim_{K\to\infty} \frac{1}{K-1}\int_1^K k^{2s}|u(x;k)|^2 dk = T^{(3)}(x) \quad \textit{almost surely,}$$

where $T^{(2)}$ and $T^{(3)}$ are defined in Theorems 4.2 *and* 4.5, *respectively.*

Proof. If $\sigma = 0$, following the same procedure as that of Lemma 3.4 in [20] or Proposition 4.1, we may obtain for any $k_1, k_2 \geq 1$ that

$$\left|\mathbb{E}\left[u^2(x;k_1)\overline{u^2(x;k_2)}\right]\right| \leq C(1+|k_1-k_2|)^{-2s},$$
$$\left|\mathbb{E}\left[u^2(x;k_1)u^2(x;k_2)\right]\right| \leq C(1+|k_1-k_2|)^{-2s},$$

which, together with the fact that

$$\lim_{K\to\infty} \frac{1}{K-1}\int_1^K X(t)dt = 0, \quad \text{almost surely,}$$

if $|\mathbb{E}X(t_1)X(t_2)| \leq C(1+|t_1-t_2|)^{-\varepsilon}$ for a centered real-valued stochastic process X with continuous paths and some $\varepsilon > 0$ (cf. [11, 19, 20]), one can get the desired results by following the proof in Theorem 3.10 in [20]. The details are omitted for brevity. ∎

5. Conclusion. We have studied the inverse random source scattering problem for the Helmholtz equations with attenuation. The source is assumed to be a fractional Gaussian random field. The relationship is established between the FGFs and the generalized Gaussian random fields. The well-posedness of the direct problem is examined. For the inverse problem, we show that the microcorrelation strength of the random source can be uniquely determined by the passive measurement of the wave fields.

There are some future works which can be considered. For instance, if the medium is inhomogeneous, the solution cannot be expressed explicitly through the fundamental solution. The present method is not applicable, a new approach is needed. Another interesting problem is to consider that both the medium and the source are random functions. Similar problems for the Schrödinger equation were investigated in [22, 23]. The Helmholtz equation is more difficult because of the coupling of the medium with the wavenumber. It is an open problem for the Maxwell equations with a random source. The singularity of Green's tensor may limit the roughness of the source. We hope to be able to report the progress on these problems elsewhere in the future.

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