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AN INVERSE RANDOM SOURCE PROBLEM FOR MAXWELL'S EQUATIONS*

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Abstract. This paper is concerned with an inverse random source problem for the three-dimensional time-harmonic Maxwell equations. The source is assumed to be a centered complex-valued Gaussian vector field with correlated components, and its covariance operator is a pseudodifferential operator. The well-posedness of the direct source scattering problem is established, and the regularity of the electromagnetic field is given. For the inverse source scattering problem, the microcorrelation strength matrix of the covariance operator is shown to be uniquely determined by the high frequency limit of the expectation of the electric field measured in an open bounded domain disjoint with the support of the source. In particular, we show that the diagonal entries of the strength matrix can be uniquely determined by only using the amplitude of the electric field. Moreover, this result is extended to the almost surely sense by deducing an ergodic relation for the electric field over the frequencies.

Key words. inverse source problem, Maxwell's equations, complex random source, fractional Gaussian field, pseudodifferential operator, principal symbol

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1. Introduction.

Inverse source scattering problems are to infer the information of the radiating sources by using the measured wave fields generated by the unknown sources. These problems arise naturally and have significant applications in many scientific areas, such as biomedical engineering, medical imaging, and optical tomography [3, 10, 12, 22]. They have attracted much attention by many researchers in both the engineering and the mathematical communities. Consequently, a great number of computational and mathematical results are available [2, 6, 7, 13]. In particular, modeled by Maxwell's equations, the inverse source scattering problem for electromagnetic waves is an important research subject not only from the viewpoint of engineering and industrial applications but also from the mathematical aspect. For instance, the model can be used to determine the source currents in the brain based on the electric or magnetic measurements on the surface of the human head [2]. As for the mathematical studies, we refer the reader to [2] for the unique recovery of surface current density, to [23, 24, 25] for the unique recovery of volume current density, and to [6] for the stability analysis on the inverse source problems for elastic and electromagnetic waves.

So far, all the sources have been considered to be deterministic functions in the existing mathematical models for the inverse electromagnetic source scattering problem. However, in many practical situations, the source of the system should be described by a random field instead of a deterministic function due to the unpredictability of the surrounding environment or uncertainties associated with the source itself [9].

Compared with the deterministic counterparts, if the source is a random field whose covariance operator is not regular enough, then the source would be too rough

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to exist pointwisely. In this case, the source should be understood as a distribution, and the corresponding problem and its solution should be studied in the distribution sense. For instance, for a d-dimensional problem, if the random source is microlocally isotropic with order s, i.e., its covariance operator is a pseudodifferential operator with the principal symbol $\phi(x)|\xi|^{-2s}$, where $s \in (0, \frac{d}{2}]$ is a real number and $\phi \in C_0^\infty(\mathbb{R}^d)$ is a positive function representing the microcorrelation strength of the random source, then the source is a distribution in the Sobolev space $W^{s-\frac{d}{2}-\epsilon}(\mathbb{R}^d)$ for any $\epsilon > 0$ (cf. [20]). There are already some mathematical studies on the inverse random source problems for acoustic and elastic waves, which are to recover the strength ϕ by using measured wave fields in a domain which has a positive distance to the support of the source. If $s = 0$, then the source is as rough as a random field of the form $\sqrt{\phi}\dot{W}$ with $\dot{W}$ being the white noise (cf. [20]). When the source is modeled by a white noise type random field, the Itô isometry can be resorted to recover the variance of the random source. We refer the reader to [4, 19] and [5] on the study of the inverse acoustic and elastic source scattering problems, respectively. If $s \in (0, \frac{d}{2}+1)$, the Itô isometry is not valid anymore since the increments of the random source may be correlated. It turns out that the microlocal analysis is effective to handle such a random source. For the inverse acoustic random source scattering problems, we refer the reader to [15] for the case $s \in [\frac{d}{2}, \frac{d}{2}+1)$ and to [20] for the case $s \in (0, \frac{d}{2}+1)$. The results can be found in [15, 16] on the inverse elastic random source scattering problems with $s \in [\frac{d}{2}, \frac{d}{2}+1)$. We refer the reader to [17, 18] for related inverse problems on the stochastic Schrödinger equation. To the best of our knowledge, the inverse random source problem for Maxwell's equations is completely open! This work initializes the mathematical study on the direct and inverse source scattering problems for the stochastic Maxwell equations driven by a random electric current density.

In this paper, we consider the three-dimensional time-harmonic stochastic Maxwell equations

$$\nabla \times \boldsymbol{E} = \mathrm{i}k\boldsymbol{H}, \quad \nabla \times \boldsymbol{H} = -\mathrm{i}k\boldsymbol{E} + \boldsymbol{J}, \tag{1.1}$$

where $k > 0$ is the wavenumber, $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric field and the magnetic field, respectively, and $\boldsymbol{J}$ is the electric current density, which is assumed to be a complex-valued random vector field defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a compact support $\mathcal{O} \subset \mathbb{R}^3$. Moreover, the source $\boldsymbol{J}$ is assumed to be microlocally isotropic such that its covariance operator is a pseudodifferential operator with the principal symbol given by $A(x)|\xi|^{-2s}$, $s \in (0, \frac{5}{2})$, where the complex-valued matrix $A \in C_0^\infty(\mathbb{R}^3; \mathbb{C}^{3\times 3})$ describes the microcorrelation strength of the random source and the entries are assumed to be smooth functions with compact supports contained in $\mathcal{O}$. The parameter s indicates how many irregular realizations such a random process has. This large class of random fields includes stochastic processes like the fractional Brownian motion and Markov field [14]. Hence, we consider a more general principal symbol than that studied in [15, 16, 20], where the principal symbol is characterized by a scalar real-valued smooth function. Given the electric current density $\boldsymbol{J}$, the direct scattering problem is to study the well-posedness of (1.1); the inverse scattering problem is to determine $\boldsymbol{J}$ from a knowledge of the electric field $\boldsymbol{E}$. Due to such a random $\boldsymbol{J}$, both the direct and the inverse scattering problems are challenging.

The work contains three contributions. First, by considering an equivalent problem, the well-posedness is established for (1.1) in the distribution sense. The regularity is given for both the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$. A key ingredient is to find an appropriate function space for the electric current density $\boldsymbol{J}$, which is required to satisfy a divergence-free condition in the weak sense. Second, we show that the microcorrelation strength matrix A is uniquely determined by the high frequency limit of the expectation of the electric field measured in a bounded open domain disjoint with the support of $\boldsymbol{J}$. The result also implies that the diagonal entries of the strength matrix A can be uniquely determined by the high frequency limit of the amplitude of the electric field, which is known as the phaseless data. Third, if only the amplitude of the electric field is available, then we show that the diagonal entries of the strength matrix A can be uniquely recovered by the energy of the electric field averaged over the frequency band at a single realization of the random source, which indicates that it is statistically stable to recover the strength matrix. The idea is to deduce an ergodic relation for the electric field over the frequencies in order to obtain such a strong result.

The paper is organized as follows. In section 2, we address the direct source scattering problem. The properties and assumptions are introduced for the random source, and the well-posedness of (1.1) and the regularity of the electromagnetic field are examined. Sections 3 and 4 are devoted to the inverse source scattering problem. In section 3, we discuss the uniqueness to recover the microcorrelation strength matrix by using the expectation of the electric field, while in section 4, we present the uniqueness result by using the amplitude of the electric field at a single path. The paper concludes with some general remarks and directions for future work in section 5.

2. Direct scattering problem.

In this section, we introduce some basic notation for complex isotropic Gaussian random fields and establish the well-posedness for the direct scattering problem if the current density is a complex-valued isotropic Gaussian random field.

2.1. Complex isotropic Gaussian random fields.

Let $\boldsymbol{J}(x)$ be a complex-valued Gaussian random vector field. It can be determined by the mean $\boldsymbol{m}(x) = \mathbb{E}[\boldsymbol{J}(x)]$, the covariance

$$C_{\boldsymbol{J}}(x,y) = \mathbb{E}\left[(\boldsymbol{J}(x) - \mathbb{E}[\boldsymbol{J}(x)])(\overline{\boldsymbol{J}(y) - \mathbb{E}[\boldsymbol{J}(y)]})^\top\right],$$

and the relation

$$R_{\boldsymbol{J}}(x,y) = \mathbb{E}\left[(\boldsymbol{J}(x) - \mathbb{E}[\boldsymbol{J}(x)])(\boldsymbol{J}(y) - \mathbb{E}[\boldsymbol{J}(y)])^\top\right]$$

if they exist. It is easy to verify the following properties for the complex-valued covariance and relation matrix functions: for any $x, y \in \mathbb{R}^3$,

(i) $C^*_{\boldsymbol{J}}(y,x) = \overline{C_{\boldsymbol{J}}(y,x)}^\top = C_{\boldsymbol{J}}(x,y)$;
(ii) $\overline{R_{\boldsymbol{J}}(x,y)} = R_{\overline{\boldsymbol{J}}}(x,y)$ and $R_{\boldsymbol{J}}(y,x)^\top = R_{\boldsymbol{J}}(x,y)$;
(iii) $C_{\boldsymbol{J}}(x,y) = R_{\boldsymbol{J}}(x,y)$ if $\boldsymbol{J}(x)$ is real-valued;
(iv) $R_{\boldsymbol{J}}(x,y) = \boldsymbol{0}$ if the real and imaginary parts of $\boldsymbol{J}$ are independent and identically distributed.

For a complex-valued Gaussian random vector $\boldsymbol{Z} = \boldsymbol{X} + \mathrm{i}\boldsymbol{Y}$, the variance matrices of $\boldsymbol{X}$ and $\boldsymbol{Y}$ and the covariance matrices between $\boldsymbol{X}$ and $\boldsymbol{Y}$ are uniquely determined by the covariance and relation of $\boldsymbol{Z}$, and vice versa. More precisely, let $V_{\boldsymbol{XX}}$ and $V_{\boldsymbol{YY}}$ be the variance matrices of $\boldsymbol{X}$ and $\boldsymbol{Y}$, and let $V_{\boldsymbol{XY}}$ and $V_{\boldsymbol{YX}}$ be the covariance matrices between $\boldsymbol{X}$ and $\boldsymbol{Y}$. Denote by C and R the covariance and relation matrices of $\boldsymbol{Z}$. Then it is easy to note that

$$V_{\boldsymbol{XX}}=\frac{1}{2}\Re[C+R],\quad V_{\boldsymbol{YY}}=\frac{1}{2}\Re[C-R],$$
$$V_{\boldsymbol{XY}}=\frac{1}{2}\Im[R-C],\quad V_{\boldsymbol{YX}}=\frac{1}{2}\Im[R+C],$$

where $\Re[\cdot]$ and $\Im[\cdot]$ stand for the real and imaginary parts of a complex number or matrix, respectively. Conversely, we have from a simple calculation that

$$C=V_{\boldsymbol{XX}}+V_{\boldsymbol{YY}}+\mathrm{i}(V_{\boldsymbol{YX}}-V_{\boldsymbol{XY}}),\quad R=V_{\boldsymbol{XX}}-V_{\boldsymbol{YY}}+\mathrm{i}(V_{\boldsymbol{YX}}+V_{\boldsymbol{XY}}).$$

If $\boldsymbol{J}$ is not regular enough, the covariance and relation matrix functions may not exist pointwisely. Hence, it is necessary to give rigorous definitions of the covariance and the relation of $\boldsymbol{J}$. Let $\mathcal{D}:=\mathcal{D}(\mathbb{R}^3)$ be the space of test functions on $\mathbb{R}^3$, which is $C_0^\infty(\mathbb{R}^3)$ equipped with a locally convex topology. Denote by $\mathcal{D}':=\mathcal{D}'(\mathbb{R}^3)$ the space of distributions on $\mathbb{R}^3$, which is the dual space of $\mathcal{D}$ equipped with the weak-star topology. Denote by $\langle\cdot,\cdot\rangle$ the dual product between $(\mathcal{D}')^3$ and $\mathcal{D}^3$. Then the derivative of a distribution $\boldsymbol{\psi}\in(\mathcal{D}')^3$ is defined by

$$\langle\partial_{x_j}\boldsymbol{\psi},\boldsymbol{\varphi}\rangle=-\langle\boldsymbol{\psi},\partial_{x_j}\boldsymbol{\varphi}\rangle\quad\forall\,\boldsymbol{\varphi}\in\mathcal{D}^3$$

for $j=1,2,3$. We refer the reader to [1] and references cited therein for more details about distributions. Define the covariance operator $Q_{\boldsymbol{J}}^C$ and the relation operator $Q_{\boldsymbol{J}}^R$ by

$$\begin{aligned}\langle Q_{\boldsymbol{J}}^C\boldsymbol{\varphi},\boldsymbol{\psi}\rangle&:=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\boldsymbol{\psi}^*(y)\mathbb{E}\left[\boldsymbol{J}(x)\boldsymbol{J}^*(y)\right]\boldsymbol{\varphi}(x)dxdy\\&=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\boldsymbol{\psi}^*(y)C_{\boldsymbol{J}}(x,y)\boldsymbol{\varphi}(x)dxdy\end{aligned}$$

and

$$\begin{aligned}\langle Q_{\boldsymbol{J}}^R\boldsymbol{\varphi},\boldsymbol{\psi}\rangle&:=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\boldsymbol{\psi}^*(y)\mathbb{E}[\boldsymbol{J}(x)\boldsymbol{J}(y)^\top]\boldsymbol{\varphi}(x)dxdy\\&=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\boldsymbol{\psi}^*(y)R_{\boldsymbol{J}}(x,y)\boldsymbol{\varphi}(x)dxdy\end{aligned}$$

for any $\boldsymbol{\varphi},\boldsymbol{\psi}\in\mathcal{D}^3$, where the star denotes the complex conjugate.

Hereafter, we use the notation $W^{r,p}:=W^{r,p}(\mathbb{R}^3)$ and $C^{r,\alpha}:=C^{r,\alpha}(\mathbb{R}^3)$ for simplicity. For any space X, we denote by $\boldsymbol{X}$ the Cartesian product vector space X^3 for convenience. Without loss of generality, we may assume that the current density $\boldsymbol{J}$ is a centered Gaussian random field. If not, it is essentially a deterministic inverse source problem to determine the nonzero mean, which has been well studied in [6]. In addition, the current density $\boldsymbol{J}$ is required to satisfy the following conditions.

Assumption 2.1. Let $\boldsymbol{J}\in\boldsymbol{\mathcal{D}}'$ be a complex-valued isotropic centered Gaussian random vector field compactly supported in $\mathcal{O}\subset\mathbb{R}^3$ with the covariance kernel $C_{\boldsymbol{J}}(x,y)$ and the relation kernel $R_{\boldsymbol{J}}(x,y)$ depending only on $|x-y|$. Assume the following:

(i) the real and imaginary parts of $\boldsymbol{J}$ are independent and identically distributed with the relation operator $Q_{\boldsymbol{J}}^R=0$;

(ii) the covariance operator $Q_{\boldsymbol{J}}^C$ defined through the kernel $C_{\boldsymbol{J}}$ is a pseudodifferential operator of order $s \in (0, \frac{5}{2})$, which implies that $Q_{\boldsymbol{J}}^C$ has a principal symbol $A(x)|\xi|^{-2s}$, where $A(x) \in C_0^\infty(\mathbb{R}^3; \mathbb{C}^{3\times 3})$ is a smooth matrix function with a compact support contained in $\mathcal{O}$.

Given the current density $\boldsymbol{J}$ satisfying Assumption 2.1, the direct scattering problem is to study the well-posedness of Maxwell's equations (1.1). We intend to answer the following questions: What are the conditions of $\boldsymbol{J}$ such that Maxwell's equations (1.1) admit a unique solution $(\boldsymbol{E}, \boldsymbol{H})$? What are the regularities for $\boldsymbol{E}$ and $\boldsymbol{H}$ if there is a unique solution? For the inverse scattering problem, the goal is not to determine the random current density $\boldsymbol{J}$ but to determine the matrix A, which represents the microcorrelation strength of the current density $\boldsymbol{J}$, from a knowledge of the measured electric field $\boldsymbol{E}$. We are concerned with the uniqueness for the inverse scattering problem: Can A or what part of A be uniquely determined by the available data? To give a detailed explanation of A, we rewrite $\boldsymbol{J} = (J_1, J_2, J_3)^\top$ by its components. Then a simple calculation yields that

$$\boldsymbol{J}(x)\boldsymbol{J}^*(y) \overset{d}{=} \begin{bmatrix} J_1(x)\overline{J_1(y)} & \cdots & J_1(x)\overline{J_3(y)} \\ \vdots & \ddots & \vdots \\ J_3(x)\overline{J_1(y)} & \cdots & J_3(x)\overline{J_3(y)} \end{bmatrix},$$

where $\overset{d}{=}$ means "equals in distribution." As a result, each entry in $A(x)$ is determined by the strength of covariance operator between J_j and J_l with $j, l = 1, 2, 3$.

2.2. Well-posedness. If the current density $\boldsymbol{J} \in \boldsymbol{\mathcal{D}}'$ is a distribution, then Maxwell's equations (1.1) no longer hold pointwisely. To establish the well-posedness of (1.1) in some proper sense, we impose the weak Silver–Müller radiation condition

$$\lim_{r\to\infty} \int_{|x|=r} \left(\boldsymbol{H} \times \frac{x}{|x|} - \boldsymbol{E} \right) \cdot \boldsymbol{\phi} ds = 0 \quad \forall \boldsymbol{\phi} \in \boldsymbol{\mathcal{D}},$$

which characterizes the behavior of solutions to (1.1) at infinity.

Eliminating the magnetic field $\boldsymbol{H}$ from (1.1), multiplying a test function $\boldsymbol{\phi} \in \boldsymbol{\mathcal{D}}$, and integrating over $\mathbb{R}^3$, we get

$$\int_{\mathbb{R}^3} (\nabla \times (\nabla \times \boldsymbol{E})) \cdot \boldsymbol{\phi} dx - k^2 \int_{\mathbb{R}^3} \boldsymbol{E} \cdot \boldsymbol{\phi} dx = \mathrm{i}k \int_{\mathbb{R}^3} \boldsymbol{J} \cdot \boldsymbol{\phi} dx \quad \forall \boldsymbol{\phi} \in \boldsymbol{\mathcal{D}},$$

which, by derivatives of distributions, leads to

$$\int_{\mathbb{R}^3} (-\Delta - k^2)\boldsymbol{E} \cdot \boldsymbol{\phi} dx - \int_{\mathbb{R}^3} (\nabla \cdot \boldsymbol{E})(\nabla \cdot \boldsymbol{\phi}) dx = \mathrm{i}k \int_{\mathbb{R}^3} \boldsymbol{J} \cdot \boldsymbol{\phi} dx \quad \forall \boldsymbol{\phi} \in \boldsymbol{\mathcal{D}}. \tag{2.1}$$

Moreover, for any $\boldsymbol{\phi} \in \boldsymbol{\mathcal{D}}$, it follows from the second equation in (1.1) that we get $\nabla(\nabla \cdot \boldsymbol{\phi}) \in \boldsymbol{\mathcal{D}}$ and hence

$$\int_{\mathbb{R}^3} (\nabla \times \boldsymbol{H}) \cdot (\nabla(\nabla \cdot \boldsymbol{\phi})) dx = -\mathrm{i}k \int_{\mathbb{R}^3} \boldsymbol{E} \cdot (\nabla(\nabla \cdot \boldsymbol{\phi})) dx + \int_{\mathbb{R}^3} \boldsymbol{J} \cdot (\nabla(\nabla \cdot \boldsymbol{\phi})) dx,$$

which implies

$$\int_{\mathbb{R}^3} (\nabla \cdot \boldsymbol{E})(\nabla \cdot \boldsymbol{\phi}) dx = \frac{\mathrm{i}}{k} \int_{\mathbb{R}^3} \boldsymbol{J} \cdot (\nabla(\nabla \cdot \boldsymbol{\phi})) dx \quad \forall \boldsymbol{\phi} \in \boldsymbol{\mathcal{D}}. \tag{2.2}$$

Define the space

$$\mathbb{X} := \left\{ \boldsymbol{U} \in \boldsymbol{\mathcal{D}}' : \int_{\mathbb{R}^3} \boldsymbol{U} \cdot (\nabla(\nabla \cdot \boldsymbol{\phi}))\, dx = 0 \quad \forall \boldsymbol{\phi} \in \boldsymbol{\mathcal{D}} \right\}.$$

Apparently, $\mathbb{X}$ is nonempty since all divergence-free vector fields are included. If $\boldsymbol{J} \in \mathbb{X}$, we obtain from (2.1)–(2.2) that

$$\int_{\mathbb{R}^3} \left[(\Delta + k^2)\boldsymbol{E} + \mathrm{i}k\boldsymbol{J} \right] \cdot \boldsymbol{\phi} dx = 0 \quad \forall \boldsymbol{\phi} \in \boldsymbol{\mathcal{D}},$$

which indicates that the following Helmholtz equation holds in the distribution sense:

$$(\Delta + k^2)\boldsymbol{E} = -\mathrm{i}k\boldsymbol{J}. \tag{2.3}$$

THEOREM 2.2. *Let* $p \in (\frac{3}{2}, 2]$, $s \in (\frac{3}{p} - \frac{1}{2}, \frac{3}{2}]$, *and* $H = s - \frac{3}{2} \in (\frac{3}{p} - 2, 0]$. *Assume that* $\boldsymbol{J} \in \mathbb{X} \cap \boldsymbol{W}^{H-\epsilon,p}_{comp}$ *for any* $\epsilon > 0$ *with a compact support contained in* $\mathcal{O}$. *Then* (2.3) *admits a unique solution*

$$\boldsymbol{E}(x) = \mathrm{i}k \int_{\mathbb{R}^3} \Phi_k(x,y)\boldsymbol{J}(y)dy \quad a.s.$$

in $\mathbb{X} \cap \boldsymbol{W}^{-H+\epsilon,q}_{loc}$ *with* q *satisfying* $\frac{1}{p} + \frac{1}{q} = 1$ *and*

$$\Phi_k(x,y) = \frac{e^{\mathrm{i}k|x-y|}}{4\pi|x-y|}$$

being the fundamental solution for the three-dimensional Helmholtz equation.

Proof. It has been shown in [20] that the scalar Helmholtz equation in $\mathbb{R}^3$ has a unique solution in $W^{-H+\epsilon,q}$, which implies the well-posedness of (2.3) in $\boldsymbol{W}^{-H+\epsilon,q}$. It then suffices to show that $\boldsymbol{E} \in \mathbb{X}$. In fact, noting that $\nabla_x \Phi_k(x,y) = -\nabla_y \Phi_k(x,y)$, we have for any $\boldsymbol{\phi} \in \boldsymbol{\mathcal{D}}$ that

$$\begin{aligned}
\int_{\mathbb{R}^3} \boldsymbol{E}(x) \cdot \nabla_x (\nabla_x \cdot \boldsymbol{\phi})\, dx &= \mathrm{i}k \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}^3} \Phi_k(x,y) \nabla_x (\nabla_x \cdot \boldsymbol{\phi})\, dx \right] \cdot \boldsymbol{J}(y) dy \\
&= \mathrm{i}k \int_{\mathbb{R}^3} \nabla_y \left[\int_{\mathbb{R}^3} \Phi_k(x,y) (\nabla_x \cdot \boldsymbol{\phi})\, dx \right] \cdot \boldsymbol{J}(y) dy \\
&= -\mathrm{i}k \int_{\mathbb{R}^3} \nabla_y \left[\int_{\mathbb{R}^3} (\nabla_x \Phi_k(x,y)) \cdot \boldsymbol{\phi}(x) dx \right] \cdot \boldsymbol{J}(y) dy \\
&= \mathrm{i}k \int_{\mathbb{R}^3} \nabla_y \left[\int_{\mathbb{R}^3} (\nabla_y \Phi_k(x,y)) \cdot \boldsymbol{\phi}(x) dx \right] \cdot \boldsymbol{J}(y) dy \\
&= \mathrm{i}k \int_{\mathcal{O}} \left(\nabla_y \left(\nabla_y \cdot \left[\int_{\mathbb{R}^3} \Phi_k(x,y) \boldsymbol{\phi}(x) dx \right] \right) \right) \cdot \boldsymbol{J}(y) dy. \qquad (2.4)
\end{aligned}$$

Let

$$\boldsymbol{f}(y) = \int_{\mathbb{R}^3} \Phi_k(x,y)\boldsymbol{\phi}(x)dx, \quad y \in \mathcal{O},$$

and choose a sufficiently large ball B such that $\mathcal{O} \subsetneq B$. Define a smooth extension $\widetilde{\boldsymbol{f}}$ on $\mathbb{R}^3$ such that

$$\widetilde{\boldsymbol{f}}(y) = \begin{cases} \boldsymbol{f}(y), & y \in \mathcal{O}, \\ \boldsymbol{0}, & y \in \mathbb{R}^3 \setminus \overline{B}. \end{cases}$$

It is easy to note that $\widetilde{\boldsymbol{f}} \in \boldsymbol{\mathcal{D}}$. Since $\boldsymbol{J} \in \mathbb{X}$, we have from (2.4) that

$$\begin{aligned}\int_{\mathbb{R}^3} \boldsymbol{E}(x) \cdot \nabla_x (\nabla_x \cdot \boldsymbol{\phi})\, dx &= \mathrm{i}k \int_{\mathcal{O}} (\nabla_y (\nabla_y \cdot \boldsymbol{f}(y))) \cdot \boldsymbol{J}(y) dy \\ &= \mathrm{i}k \int_{\mathbb{R}^3} \left(\nabla_y \left(\nabla_y \cdot \widetilde{\boldsymbol{f}}(y)\right)\right) \cdot \boldsymbol{J}(y) dy \\ &= 0,\end{aligned}$$

which completes the proof. ∎

COROLLARY 2.3. *Under the assumptions in Theorem* 2.2*, the Helmholtz equation* (2.3) *together with*

$$\nabla \times \boldsymbol{E} = \mathrm{i}k\boldsymbol{H} \tag{2.5}$$

is equivalent to Maxwell's equations (1.1) *in the distribution sense. Moreover, it holds that* $\boldsymbol{H} \in \left(\boldsymbol{W}^{H-\epsilon,p}(curl)\right)'$*, which is the dual space of* $\boldsymbol{W}^{H-\epsilon,p}(curl)$ *equipped with norm*

$$\|\boldsymbol{h}\|_{\boldsymbol{W}^{H-\epsilon,p}(curl)} = \left(\|\boldsymbol{h}\|^2_{\boldsymbol{W}^{H-\epsilon,p}} + \|\nabla \times \boldsymbol{h}\|^2_{\boldsymbol{W}^{H-\epsilon,p}}\right)^{\frac{1}{2}}.$$

Proof. Based on the above discussions, it has been shown that any solution of (1.1) is also a solution of (2.3)–(2.5). Next, we show that if $\boldsymbol{J} \in \mathbb{X} \cap \boldsymbol{W}^{H-\epsilon,p}_{comp}$ and $\boldsymbol{E} \in \mathbb{X} \cap \boldsymbol{W}^{-H+\epsilon,q}$ is a solution of (2.3)–(2.5) as stated in Theorem 2.2, then $\boldsymbol{E}$ also solves (1.1).

Noting that $\boldsymbol{E} \in \mathbb{X}$ and using (2.3) and (2.5), we get for any $\boldsymbol{\phi} \in \boldsymbol{\mathcal{D}}$ that

$$\begin{aligned}-\mathrm{i}k \int_{\mathbb{R}^3} \boldsymbol{J} \cdot \boldsymbol{\phi} dx &= \int_{\mathbb{R}^3} (\Delta + k^2)\boldsymbol{E} \cdot \boldsymbol{\phi} dx \\ &= \int_{\mathbb{R}^3} \left[-\nabla \times (\nabla \times \boldsymbol{E}) + \nabla(\nabla \cdot \boldsymbol{E}) + k^2\boldsymbol{E}\right] \cdot \boldsymbol{\phi} dx \\ &= -\int_{\mathbb{R}^3} \nabla \times (\mathrm{i}k\boldsymbol{H}) \cdot \boldsymbol{\phi} dx + \int_{\mathbb{R}^3} \boldsymbol{E} \cdot (\nabla(\nabla \cdot \boldsymbol{\phi}))\, dx + k^2 \int_{\mathbb{R}^3} \boldsymbol{E} \cdot \boldsymbol{\phi} dx \\ &= -\mathrm{i}k \int_{\mathbb{R}^3} (\nabla \times \boldsymbol{H} + \mathrm{i}k\boldsymbol{E}) \cdot \boldsymbol{\phi} dx,\end{aligned}$$

which implies that

$$\nabla \times \boldsymbol{H} = -\mathrm{i}k\boldsymbol{E} + \boldsymbol{J}.$$

Moreover, since $\boldsymbol{E} \in \mathbb{X} \cap \boldsymbol{W}^{-H+\epsilon,q}_{loc}$, we have for any $\boldsymbol{\phi} \in \boldsymbol{\mathcal{D}}$ that

$$\begin{aligned}\left|\int_{\mathbb{R}^3} \boldsymbol{H} \cdot \boldsymbol{\phi} dx\right| = \left|\frac{1}{\mathrm{i}k}\int_{\mathbb{R}^3} (\nabla \times \boldsymbol{E}) \cdot \boldsymbol{\phi} dx\right| &= \frac{1}{k}\left|\int_{\mathbb{R}^3} \boldsymbol{E} \cdot (\nabla \times \boldsymbol{\phi}) dx\right| \\ &\le \frac{1}{k}\|\boldsymbol{E}\|_{\boldsymbol{W}^{-H+\epsilon,q}} \|\nabla \times \boldsymbol{\phi}\|_{\boldsymbol{W}^{H-\epsilon,p}} \\ &\le \frac{1}{k}\|\boldsymbol{E}\|_{\boldsymbol{W}^{-H+\epsilon,q}} \|\boldsymbol{\phi}\|_{\boldsymbol{W}^{H-\epsilon,p}(curl)},\end{aligned}$$

which completes the proof. ∎

It should be pointed out that if $\boldsymbol{J} \in \mathbb{X}$ satisfies Assumption 2.1 with $s \in (\frac{3}{p} - \frac{1}{2}, \frac{3}{2}]$, then it also holds that $\boldsymbol{J} \in \boldsymbol{W}^{H-\epsilon,p}$ with $H = s - \frac{3}{2}$ and $p > 1$ according to Lemma 2.6 in [20], i.e., the assumptions in Theorem 2.2 are satisfied. If $\boldsymbol{J} \in \mathbb{X}$ satisfies

Assumption 2.1 with $s \in (\frac{3}{2}, \frac{5}{2})$, the current density $\boldsymbol{J}$ turns to be smooth such that $\boldsymbol{J} \in \boldsymbol{C}^{0,\alpha}$ for all $\alpha \in (0, s - \frac{3}{2})$ according to Lemma 2.6 in [20]. The well-posedness of the problem in this case has been investigated in [21]. Therefore, we only need to consider the current density $\boldsymbol{J} \in \mathbb{X}$ which satisfies Assumption 2.1 with $s \in (\frac{3}{p} - \frac{1}{2}, \frac{5}{2})$ and $p \in (\frac{3}{2}, 2]$ in the following sections.

3. Inverse scattering problem. This section addresses the inverse scattering problem. According to Assumption 2.1, the centered Gaussian random field $\boldsymbol{J}$ is determined by its covariance operator $Q_{\boldsymbol{J}}^C$. To recover the strength matrix $A(x)$ of the operator $Q_{\boldsymbol{J}}^C$, it is required to recover the strength of the covariance operator between J_j and J_l, $j, l = 1, 2, 3$, where $\boldsymbol{J} = (J_1, J_2, J_3)^\top$. For convenience, we denote by $a_{jl}(x) = a_{jl}^{\rm r}(x) + {\rm i} a_{jl}^{\rm i}(x)$ the (j, l)-entry of the strength matrix $A(x)$. We discuss the covariance for each component of $\boldsymbol{J}$ and the covariance between different components of $\boldsymbol{J}$ separately.

3.1. Covariance for each component of $\boldsymbol{J}$. First, we consider the covariance operator for each component of $\boldsymbol{J}$. By Theorem 2.2, the energy of each of the components of $\boldsymbol{E}$ is

$$\begin{aligned}\mathbb{E}|E_j(x)|^2 &= k^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi_k(x, y) \overline{\Phi_k(x, z)} \mathbb{E}[J_j(y) \overline{J_j(z)}] dy dz \\ &= \frac{k^2}{(4\pi)^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{e^{{\rm i}k|x-y| - {\rm i}k|x-z|}}{|x-y||x-z|} C_{jj}(y, z) dy dz,\end{aligned}$$

where C_{jj}, $j = 1, 2, 3$, is the (j, j)-entry of the kernel $C_{\boldsymbol{J}}$.

Let $C_{jj} = C_{jj}^{\rm r} + {\rm i} C_{jj}^{\rm i}$, where $C_{jj}^{\rm r}$ and $C_{jj}^{\rm i}$ are the real and imaginary parts of C_{jj}, respectively. It follows from Assumption 2.1 that the principal symbols of $C_{jj}^{\rm r}$ and $C_{jj}^{\rm i}$ are $a_{jj}^{\rm r} |\xi|^{-2s}$ and $a_{jj}^{\rm i} |\xi|^{-2s}$, respectively.

THEOREM 3.1. *Let Assumption* 2.1 *hold, and let* $\mathcal{U} \subset \mathbb{R}^3$ *be a bounded open set which has a positive distance to* $\mathcal{O}$. *For* $j = 1, 2, 3$, *the strength* $a_{jj}^{\rm r}$ *is uniquely determined by*

$$\lim_{k \to \infty} k^{2s-2} \mathbb{E}|E_j(x)|^2 = \frac{1}{(4\pi)^2} \int_{\mathbb{R}^3} \frac{1}{|x-y|^2} a_{jj}^{\rm r}(y) dy, \quad x \in \mathcal{U},$$

and $a_{jj}^{\rm i} \equiv 0$.

Remark 3.2. The diagonal entry a_{jj} of the strength matrix A is a real-valued function, and it can be uniquely determined by the high frequency limit of the phaseless data $\mathbb{E}|E_j|^2$ on an open set $\mathcal{U}$, $j = 1, 2, 3$.

Proof. Rewriting $\mathbb{E}|E_j(x)|^2$ through $C_{jj}^{\rm r}$ and $C_{jj}^{\rm i}$, one gets

$$\begin{aligned}\mathbb{E}|E_j(x)|^2 = &\frac{k^2}{(4\pi)^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \\ &\frac{\cos(k|x-y| - k|x-z|) C_{jj}^{\rm r}(y, z) - \sin(k|x-y| - k|x-z|) C_{jj}^{\rm i}(y, z)}{|x-y||x-z|} dy dz \\ &+ \frac{{\rm i}k^2}{(4\pi)^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \\ &\frac{\sin(k|x-y| - k|x-z|) C_{jj}^{\rm r}(y, z) + \cos(k|x-y| - k|x-z|) C_{jj}^{\rm i}(y, z)}{|x-y||x-z|} dy dz,\end{aligned}$$

which apparently leads to

$$\mathbb{E}|E_j(x)|^2 = \frac{k^2}{(4\pi)^2}\Re\left[\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}k(|x-y|-|x-z|)}C^{\mathrm{r}}_{jj}(y,z)}{|x-y||x-z|}dydz\right] - \frac{k^2}{(4\pi)^2}\Im\left[\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}k(|x-y|-|x-z|)}C^{\mathrm{i}}_{jj}(y,z)}{|x-y||x-z|}dydz\right] \tag{3.1}$$

and

$$\Im\left[\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}k(|x-y|-|x-z|)}C^{\mathrm{r}}_{jj}(y,z)}{|x-y||x-z|}dydz\right] + \Re\left[\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}k(|x-y|-|x-z|)}C^{\mathrm{i}}_{jj}(y,z)}{|x-y||x-z|}dydz\right] = 0. \tag{3.2}$$

It then suffices to consider the integrals

$$\mathrm{I}_1(x) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}k(|x-y|-|x-z|)}C^{\mathrm{r}}_{jj}(y,z)}{|x-y||x-z|}dydz$$

and

$$\mathrm{I}_2(x) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}k(|x-y|-|x-z|)}C^{\mathrm{i}}_{jj}(y,z)}{|x-y||x-z|}dydz.$$

Note that the kernel $C_{\boldsymbol{J}}(y,z)$, which is also called a conormal distribution, is singular only at the diagonal: an application of a first order differential operator does not make it more singular. Hence, the covariance operator $Q^C_{\boldsymbol{J}}$ defined through the kernel $C_{\boldsymbol{J}}$ is invariant under coordinate transformations due to a characterization of the kernel as a conormal distribution (cf. [11]). The proof is to utilize the invariance of coordinate transformations and is divided into the following four steps.

Step 1. For any $x \in \mathcal{U}$, by introducing a smooth function $\theta \in C_0^\infty(\mathbb{R}^3)$ such that $\theta|_{\mathcal{U}} \equiv 1$ and $\mathrm{supp}(\theta) \subset \mathbb{R}^3\backslash\overline{\mathcal{O}}$, we get

$$\mathrm{I}_1(x) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}k(|x-y|-|x-z|)}}{|x-y||x-z|}C^{\mathrm{r}}_{jj}(y,z)\theta(x)dydz.$$

Denote

$$S_1(y,z,x) := C^{\mathrm{r}}_{jj}(y,z)\theta(x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{\mathrm{i}(y-z)\cdot\xi}s_1(y,x,\xi)d\xi.$$

It can be easily verified that the symbol $s_1(y,x,\xi) = c^{\mathrm{r}}_{jj}(y,\xi)\theta(x)$, where $c^{\mathrm{r}}_{jj}(y,\xi)$ is the symbol of the kernel C^{r}_{jj} (cf. [20]). By Assumption 2.1, the leading term of s_1, which is the principal symbol of S_1, has the form

$$s_1^p(y,\xi) = a^{\mathrm{r}}_{jj}(y)\theta(x)|\xi|^{-2s}.$$

Following [20], we define an invertible transformation $\tau : \mathbb{R}^9 \to \mathbb{R}^9$ given by

$\tau(y,z,x)=(g,h,x)$, where $g=(g_1,g_2,g_3)$ and $h=(h_1,h_2,h_3)$ with

$$g_1=\frac{1}{2}\left(|x-y|-|x-z|\right),\quad h_1=\frac{1}{2}\left(|x-y|+|x-z|\right),$$
$$g_2=\frac{1}{2}\left[|x-y|\arccos\left(\frac{y_3-x_3}{|x-y|}\right)-|x-z|\arccos\left(\frac{z_3-x_3}{|x-z|}\right)\right],$$
$$h_2=\frac{1}{2}\left[|x-y|\arccos\left(\frac{y_3-x_3}{|x-y|}\right)+|x-z|\arccos\left(\frac{z_3-x_3}{|x-z|}\right)\right],$$
$$g_3=\frac{1}{2}\left[|x-y|\arctan\left(\frac{y_2-x_2}{y_1-x_1}\right)-|x-z|\arctan\left(\frac{z_2-x_2}{z_1-x_1}\right)\right],$$
$$h_3=\frac{1}{2}\left[|x-y|\arctan\left(\frac{y_2-x_2}{y_1-x_1}\right)+|x-z|\arctan\left(\frac{z_2-x_2}{z_1-x_1}\right)\right].$$

Then

$$\mathrm{I}_1(x)=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{2ik(e_1\cdot g)}S_2(g,h,x)dgdh,$$

where $e_1=(1,0,0)$ and

$$\begin{aligned}S_2(g,h,x)&=S_1(\tau^{-1}(g,h,x))\frac{\det\left((\tau^{-1})'(g,h,x)\right)}{((g+h)\cdot e_1)((h-g)\cdot e_1)}\\ &=: S_1(\tau^{-1}(g,h,x))L^\tau(g,h,x).\end{aligned}\tag{3.3}$$

Step 2. To get an explicit expression of S_2 with respect to (g,h,x), we define another invertible transformation $\eta:\mathbb{R}^9\to\mathbb{R}^9$ given by $\eta(y,z,x)=(v,w,x)$ with $v=y-z$ and $w=y+z$. Let the kernel

$$\begin{aligned}S_3(v,w,x)&:=S_1\circ\eta^{-1}(v,w,x)=S_1\Big(\frac{v+w}{2},\frac{w-v}{2},x\Big)\\ &=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}e^{\mathrm{i}v\cdot\xi}s_1\Big(\frac{v+w}{2},x,\xi\Big)d\xi\\ &=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}e^{\mathrm{i}v\cdot\xi}s_3\left(w,x,\xi\right)d\xi,\end{aligned}\tag{3.4}$$

where we have used the properties of symbols in the last step (cf. [11, Lemma 18.2.1]). More precisely, the symbol s_3 is defined by

$$s_3(w,x,\xi)=e^{\langle-\mathrm{i}D_v,D_\xi\rangle}s_1\left(\frac{v+w}{2},x,\xi\right)\Big|_{v=0},$$

which has an asymptotic expansion

$$s_3(w,x,\xi)\sim\sum_{n=1}^{\infty}\frac{\langle-\mathrm{i}D_v,D_\xi\rangle^n}{n!}s_1\left(\frac{v+w}{2},x,\xi\right)\Big|_{v=0}.$$

Hence, the principal symbol of s_3 is

$$s_3^p(w,x,\xi)=s_1^p\left(\frac{v+w}{2},x,\xi\right)\Big|_{v=0}=a_{jj}^r\left(\frac{w}{2}\right)|\xi|^{-2s}\theta(x).$$

Next, define the diffeomorphism $\gamma:=\eta\circ\tau^{-1}:(g,h,x)\mapsto(v,w,x)$. It preserves the plane $\{(g,h,x)\in\mathbb{R}^9:g=0\}$, i.e., $\gamma(0,h,x)=(0,w,x)$. Now we are able to consider the kernel $S_1\circ\tau^{-1}$ in (3.3):

$$S_1\circ\tau^{-1}(g,h,x)=S_1\circ\eta^{-1}\circ\eta\circ\tau^{-1}(g,h,x)=S_3\circ\gamma(g,h,x),$$

where the kernel $S_3 \circ \gamma$ admits a symbol $\tilde{s}_3(h, x, \xi)$ under the diffeomorphism γ satisfying

$$S_3 \circ \gamma(g, h, x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{\mathrm{i}g\cdot\xi} \tilde{s}_3(h, x, \xi) d\xi.$$

Comparing the above kernel $S_3 \circ \gamma(g, h, x)$ with $S_3(v, w, x)$ defined in (3.4), we may check that their symbols have the following relationship (cf. [11, Theorem 18.2.9] or [14]):

$$\begin{aligned}
\tilde{s}_3(h, x, \xi) &= s_3 \left(w(0, h, x), x, \left(\frac{\partial v}{\partial g}(0, h, x) \right)^{-\top} \xi \right) \left| \det \left(\frac{\partial v}{\partial g}(0, h, x) \right) \right|^{-1} + r_1(h, x, \xi) \\
&= s_3^p \left(w(0, h, x), x, \left(\frac{\partial v}{\partial g}(0, h, x) \right)^{-\top} \xi \right) \left| \det \left(\frac{\partial v}{\partial g}(0, h, x) \right) \right|^{-1} + r_2(h, x, \xi) \\
&= a_{jj}^{\mathrm{r}} \left(\frac{w(0, h, x)}{2} \right) \left| \left(\frac{\partial v}{\partial g}(0, h, x) \right)^{-\top} \xi \right|^{-2s} \left| \det \left(\frac{\partial v}{\partial g}(0, h, x) \right) \right|^{-1} \theta(x) \\
&\quad + r_2(h, x, \xi) \\
&=: \tilde{s}_3^p(h, x, \xi) + r_2(h, x, \xi),
\end{aligned}$$

where the residuals $r_1, r_2 \in \mathcal{S}^{-2s-1}$. Here $\mathcal{S}^m$ denotes the space of symbols of order m (cf. [11]).

We conclude from the above discussions that

$$\begin{aligned}
S_2(g, h, x) &= S_1(\tau^{-1}(g, h, x)) L^\tau(g, h, x) \\
&= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{\mathrm{i}g\cdot\xi} \tilde{s}_3(h, x, \xi) L^\tau(g, h, x) d\xi \\
&= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{\mathrm{i}g\cdot\xi} s_2(h, x, \xi) d\xi,
\end{aligned}$$

where in the last step we have used the same property as that used in (3.4). Here the symbol s_2 satisfies

$$s_2(h, x, \xi) = s_2^p(h, x, \xi) + r_3(h, x, \xi),$$

where the residual $r_3 \in \mathcal{S}^{-2s-1}$ and the principal symbol

$$s_2^p(h, x, \xi) = \tilde{s}_3^p(h, x, \xi) L^\tau(0, h, x).$$

Step 3. Based on the expression of S_2, the integral $I_1(x)$ has the form

$$\begin{aligned}
\mathrm{I}_1(x) &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{2\mathrm{i}k(e_1\cdot g)} S_2(g, h, x) dg dh \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{2\mathrm{i}k(e_1\cdot g)} \left[\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{\mathrm{i}g\cdot\xi} [\tilde{s}_3^p(h, x, \xi) L^\tau(0, h, x) + r_3(h, x, \xi)] d\xi \right] dg dh \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left[\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{\mathrm{i}g\cdot(\xi + 2ke_1)} dg \right] [\tilde{s}_3^p(h, x, \xi) L^\tau(0, h, x) + r_3(h, x, \xi)] d\xi dh \\
&= \int_{\mathbb{R}^3} [\tilde{s}_3^p(h, x, -2ke_1) L^\tau(0, h, x) + r_3(h, x, -2ke_1)] dh.
\end{aligned}$$

It then suffices to calculate

$$\tilde{s}_3^p(h,x,-2ke_1) = a^{\mathrm{r}}_{jj}\left(\frac{w(0,h,x)}{2}\right)\left|\left(\frac{\partial v}{\partial g}(0,h,x)\right)^{-\top}(-2ke_1)\right|^{-2s} \times\left|\det\left(\frac{\partial v}{\partial g}(0,h,x)\right)\right|^{-1}\theta(x)$$

and

$$L^\tau(0,h,x) = \frac{\left|\det\left((\tau^{-1})'(0,h,x)\right)\right|}{(h\cdot e_1)^2}.$$

Noting that

$$\begin{aligned}
&h_1+g_1=|x-y|, \quad h_1-g_1=|x-z|,\\
&\frac{h_2+g_2}{h_1+g_1}=\arccos\Big(\frac{y_3-x_3}{|x-y|}\Big), \quad \frac{h_2-g_2}{h_1-g_1}=\arccos\Big(\frac{z_3-x_3}{|x-z|}\Big),\\
&\frac{h_3+g_3}{h_1+g_1}=\arctan\Big(\frac{y_2-x_2}{y_1-x_1}\Big), \quad \frac{h_3-g_3}{h_1-g_1}=\arctan\Big(\frac{z_2-x_2}{z_1-x_1}\Big),
\end{aligned}$$

we get

$$\begin{aligned}
y_1&=x_1+(h_1+g_1)\sin\left(\frac{h_2+g_2}{h_1+g_1}\right)\cos\left(\frac{h_3+g_3}{h_1+g_1}\right),\\
y_2&=x_2+(h_1+g_1)\sin\left(\frac{h_2+g_2}{h_1+g_1}\right)\sin\left(\frac{h_3+g_3}{h_1+g_1}\right),\\
y_3&=x_3+(h_1+g_1)\cos\left(\frac{h_2+g_2}{h_1+g_1}\right),\\
z_1&=x_1+(h_1-g_1)\sin\left(\frac{h_2-g_2}{h_1-g_1}\right)\cos\left(\frac{h_3-g_3}{h_1-g_1}\right),\\
z_2&=x_2+(h_1-g_1)\sin\left(\frac{h_2-g_2}{h_1-g_1}\right)\sin\left(\frac{h_3-g_3}{h_1-g_1}\right),\\
z_3&=x_3+(h_1-g_1)\cos\left(\frac{h_2-g_2}{h_1-g_1}\right).
\end{aligned}$$

A simple calculation yields that

$$\frac{\partial v}{\partial g}(0,h,x) = 2 \times\begin{bmatrix} \sin\alpha\cos\beta-\alpha\cos\alpha\cos\beta+\beta\sin\alpha\sin\beta & \cos\alpha\cos\beta & -\sin\alpha\sin\beta\\ \sin\alpha\sin\beta-\alpha\cos\alpha\sin\beta-\beta\sin\alpha\cos\beta & \cos\alpha\sin\beta & \sin\alpha\cos\beta\\ \cos\alpha+\alpha\sin\alpha & -\sin\alpha & 0\end{bmatrix},$$

where $\alpha:=\frac{h_2}{h_1}$, $\beta:=\frac{h_3}{h_1}$, and

$$(\tau^{-1})'(0,h,x) = \begin{bmatrix} \frac12\frac{\partial v}{\partial g} & \frac12\frac{\partial v}{\partial g} & I\\ -\frac12\frac{\partial v}{\partial g} & \frac12\frac{\partial v}{\partial g} & I\\ 0 & 0 & I\end{bmatrix}.$$

Here I is the 3×3 identity matrix. It then leads to

$$\det\left(\frac{\partial v}{\partial g}(0,h,x)\right) = 8\sin\alpha,$$

$$\left(\frac{\partial v}{\partial g}(0,h,x)\right)^{-\top} e_1 = \frac{1}{2}\begin{bmatrix} \sin\alpha\cos\beta \\ \sin\alpha\sin\beta \\ \cos\alpha \end{bmatrix},$$

and thus

$$\tilde{s}_3^p(h,x,-2ke_1) = a_{jj}^{\mathrm{r}}\left(\frac{w(0,h,x)}{2}\right)\frac{k^{-2s}}{8|\sin\alpha|}\theta(x).$$

To get $L^\tau(0,h,x)$, we next consider the matrix

$$(\tau^{-1})'(0,h,x) = \frac{\partial(y,z,x)}{\partial(g,h,x)}\bigg|_{g=0} = \begin{bmatrix} \frac{1}{2}\frac{\partial v}{\partial g} & \frac{1}{2}\frac{\partial v}{\partial g} & I \\ -\frac{1}{2}\frac{\partial v}{\partial g} & \frac{1}{2}\frac{\partial v}{\partial g} & I \\ 0 & 0 & I \end{bmatrix},$$

which gives $\det\left((\tau^{-1})'(0,h,x)\right) = 8\sin^2\alpha$ and

$$L^\tau(0,h,x) = \frac{8\sin^2\alpha}{(h\cdot e_1)^2}.$$

Step 4. Based on the a priori estimates above, we obtain

$$\mathrm{I}_1(x) = \int_{\mathbb{R}^3}\left[a_{jj}^{\mathrm{r}}\left(\frac{w(0,h,x)}{2}\right)\frac{|\sin\alpha|}{(h\cdot e_1)^2}k^{-2s}\theta(x) + r_3(h,x,-2ke_1)\right]dh,$$

where $\frac{w(0,h,x)}{2} = (h_1\sin\alpha\cos\beta, h_1\sin\alpha\sin\beta, h_1\cos\alpha) + x$.

Define another coordinate transform $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\rho(h) = \zeta := (h_1\sin\alpha\cos\beta, h_1\sin\alpha\sin\beta, h_1\cos\alpha) + x.$$

Noting that $|\zeta - x| = h_1 = h\cdot e_1$ and $\det((\rho^{-1})') = \frac{1}{\det(\rho')}$ with

$$\rho' = \begin{bmatrix} \sin\alpha\cos\beta - \alpha\cos\alpha\cos\beta + \beta\sin\alpha\sin\beta & \cos\alpha\cos\beta & -\sin\alpha\sin\beta \\ \sin\alpha\sin\beta - \alpha\cos\alpha\sin\beta - \beta\sin\alpha\cos\beta & \cos\alpha\sin\beta & \sin\alpha\cos\beta \\ \cos\alpha + \alpha\sin\alpha & -\sin\alpha & 0 \end{bmatrix},$$

we have that

$$\mathrm{I}_1(x) = \left[\int_{\mathbb{R}^3}\frac{1}{|\zeta - x|^2}a_{jj}^{\mathrm{r}}(\zeta)d\zeta\right]k^{-2s} + O(k^{-2s-1}), \quad x \in \mathcal{U}.$$

Following the same procedure as above, we may show that

$$\mathrm{I}_2(x) = \left[\int_{\mathbb{R}^3}\frac{1}{|\zeta - x|^2}a_{jj}^{\mathrm{i}}(\zeta)d\zeta\right]k^{-2s} + O(k^{-2s-1}), \quad x \in \mathcal{U}.$$

It follows from (3.1)–(3.2) that

$$\lim_{k\to\infty} k^{2s-2}\mathbb{E}|E_j(x)|^2 = \frac{1}{(4\pi)^2}\int_{\mathbb{R}^3}\frac{1}{|\zeta - x|^2}a_{jj}^{\mathrm{r}}(\zeta)d\zeta$$

and

$$\int_{\mathbb{R}^3}\frac{1}{|\zeta - x|^2}a_{jj}^{\mathrm{i}}(\zeta)d\zeta = 0,$$

which imply that a_{jj}^{r} and a_{jj}^{i} can be uniquely determined (cf. [20, Theorem 4.6]), in particular $a_{jj}^{\mathrm{i}} = 0$. ∎

3.2. Covariance between different components of $\boldsymbol{J}$. To recover the nondiagonal entries of the strength matrix $A(x)$, we now consider the covariance between different components of $\boldsymbol{J}$. By Theorem 2.2, we have

$$\begin{aligned}\mathbb{E}[E_j(x)\overline{E_l(x)}] &= k^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Phi_k(x,y)\overline{\Phi_k(x,z)}\mathbb{E}[J_j(y)\overline{J_l(z)}]dydz\\ &= \frac{k^2}{(4\pi)^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}k|x-y|-\mathrm{i}k|x-z|}}{|x-y||x-z|}C_{jl}(y,z)dydz\end{aligned}$$

for $j\neq l$ and $j,l=1,2,3$. Denote by C_{jl}^{r} and C_{jl}^{i} the real and imaginary parts of C_{jl}, respectively. The recovery of strengths a_{jl}^{r} and a_{jl}^{i} of C_{jl}^{r} and C_{jl}^{i} are stated in the following theorem.

THEOREM 3.3. *Let Assumption* 2.1 *hold, and let $\mathcal{U}\subset\mathbb{R}^3$ be a bounded open set which has a positive distance to $\mathcal{O}$. For $j,l=1,2,3$ and $j\neq l$, the strengths a_{jl}^{r} and a_{jl}^{i} are uniquely determined by*

$$\lim_{k\to\infty}k^{2s-2}\Re\mathbb{E}[E_j(x)\overline{E_l(x)}] = \frac{1}{(4\pi)^2}\int_{\mathbb{R}^3}\frac{1}{|x-y|^2}a_{jl}^{\mathrm{r}}(y)dy,\quad x\in\mathcal{U},$$

and

$$\lim_{k\to\infty}k^{2s-2}\Im\mathbb{E}[E_j(x)\overline{E_l(x)}] = \frac{1}{(4\pi)^2}\int_{\mathbb{R}^3}\frac{1}{|x-y|^2}a_{jl}^{\mathrm{i}}(y)dy,\quad x\in\mathcal{U}.$$

Remark 3.4. The nondiagonal entry a_{jl} of the strength matrix A is a complex-valued function, and it can be uniquely determined by the high frequency limit of the phased data $\mathbb{E}[E_j\overline{E_l}]$ on an open set $\mathcal{U}$ with $j,l=1,2,3$ and $j\neq l$.

Proof. Using C_{jl}^{r} and C_{jl}^{i}, we may split $\mathbb{E}[E_j(x)\overline{E_l(x)}]$ into the real and imaginary parts

$$\begin{aligned}&\mathbb{E}[E_j(x)\overline{E_l(x)}]\\ &= \frac{k^2}{(4\pi)^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\\ &\quad\frac{\cos(k|x-y|-k|x-z|)C_{jl}^{\mathrm{r}}(y,z)-\sin(k|x-y|-k|x-z|)C_{jl}^{\mathrm{i}}(y,z)}{|x-y||x-z|}dydz\\ &\quad+\frac{\mathrm{i}k^2}{(4\pi)^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\\ &\quad\frac{\sin(k|x-y|-k|x-z|)C_{jl}^{\mathrm{r}}(y,z)+\cos(k|x-y|-k|x-z|)C_{jl}^{\mathrm{i}}(y,z)}{|x-y||x-z|}dydz\\ &= \frac{k^2}{(4\pi)^2}\left(\Re[\mathrm{I}_3(x)]-\Im[\mathrm{I}_4(x)]\right)+\frac{\mathrm{i}k^2}{(4\pi)^2}\left(\Im[\mathrm{I}_3(x)]+\Re[\mathrm{I}_4(x)]\right),\end{aligned}$$

where

$$\mathrm{I}_3(x) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}k(|x-y|-|x-z|)}C_{jl}^{\mathrm{r}}(y,z)}{|x-y||x-z|}dydz$$

and

$$\mathrm{I}_4(x) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}k(|x-y|-|x-z|)}C_{jl}^{\mathrm{i}}(y,z)}{|x-y||x-z|}dydz.$$

Following the same procedure as that in the proof of Theorem 3.1, we may show for any $x \in \mathcal{U}$ that

$$\mathrm{I}_3(x) = \left[\int_{\mathbb{R}^3} \frac{1}{|\zeta - x|^2} a_{jl}^{\mathrm{r}}(\zeta)\theta(x)d\zeta\right] k^{-2s} + O(k^{-2s-1})$$

and

$$\mathrm{I}_4(x) = \left[\int_{\mathbb{R}^3} \frac{1}{|\zeta - x|^2} a_{jl}^{\mathrm{i}}(\zeta)\theta(x)d\zeta\right] k^{-2s} + O(k^{-2s-1}).$$

Consequently, we have for any $x \in \mathcal{U}$ that

$$\lim_{k\to\infty} k^{2s}\Im[\mathrm{I}_3(x)] = \lim_{k\to\infty} k^{2s}\Im[\mathrm{I}_4(x)] = 0$$

and

$$\lim_{k\to\infty} k^{2s}\Re[\mathrm{I}_3(x)] = \int_{\mathbb{R}^3} \frac{1}{|\zeta - x|^2} a_{jl}^{\mathrm{r}}(\zeta)\theta(x)d\zeta,$$
$$\lim_{k\to\infty} k^{2s}\Re[\mathrm{I}_4(x)] = \int_{\mathbb{R}^3} \frac{1}{|\zeta - x|^2} a_{jl}^{\mathrm{i}}(\zeta)\theta(x)d\zeta,$$

which completes the proof. ∎

Remark 3.5. The above results can be combined into

$$\lim_{k\to\infty} k^{2s-2}\mathbb{E}[E_j(x)\overline{E_l(x)}] = \frac{1}{(4\pi)^2}\int_{\mathbb{R}^3} \frac{1}{|x-y|^2} a_{jl}(y)dy, \quad j,l = 1,2,3,\ x \in \mathcal{U}. \tag{3.5}$$

Equivalently, we have the matrix form

$$\lim_{k\to\infty} k^{2s-2}\mathbb{E}\left[\boldsymbol{E}(x)\boldsymbol{E}^*(x)\right] = \frac{1}{(4\pi)^2}\int_{\mathbb{R}^3} \frac{1}{|x-y|^2} A(y)dy, \quad x \in \mathcal{U}, \tag{3.6}$$

which shows that the microcorrelation strength matrix function $A(x)$ can be uniquely determined by the high frequency limit of the data $\mathbb{E}\left[\boldsymbol{E}\boldsymbol{E}^*\right]$ on an open set $\mathcal{U}$.

Remark 3.6. If the covariance operators between components J_j and J_l are pseudodifferential operators of the same order with the principal symbols $a_{jl}(x)|\xi|^{-2s}$, then all the strength $\{a_{jl}\}_{j,l=1,2,3}$ can be recovered at the same time by (3.6).

However, if the covariance operators between J_j and J_l are of different orders with the principal symbols $a_{jl}(x)|\xi|^{-2s_{jl}}$ where $s_{jl} \in [0, \frac{5}{2})$, then only the strength of the roughest term can be recovered by (3.6). For example, if $s_{11} < s_{jl}$ for any $(j,l) \neq (1,1)$ and $j,l = 1,2,3$, then the principal symbol of the covariance operator of $\boldsymbol{J}$ is $A(x)|\xi|^{-2s_{11}}$ with $A(x) = \mathrm{diag}\{a_{11}(x), 0, 0\}$. In this case, the other strength $a_{jl}(x)$ can be recovered by modifying (3.5) as follows:

$$\lim_{k\to\infty} k^{2s_{jl}-2}\mathbb{E}[E_j(x)\overline{E_l(x)}] = \frac{1}{(4\pi)^2}\int_{\mathbb{R}^3} \frac{1}{|x-y|^2} a_{jl}(y)dy, \quad j,l = 1,2,3,\ x \in \mathcal{U}.$$

By Theorems 3.1 and 3.3, we conclude that the strength matrix $A(x)$ of the covariance operator $Q_{\boldsymbol{J}}^C$ can be uniquely determined by the high frequency limit of the expectation of the electric field $\boldsymbol{E}$ measured on an open set $\mathcal{U}$. Moreover, if only the energy of the electric field $|E_j(x)|^2$, $j = 1,2,3$, can be observed on an open bounded domain $\mathcal{U}$, then the strength of J_j can be uniquely determined by a single realization of the phaseless data almost surely, which is discussed in the following section.

4. Recovery by a single path. In this section, we present some ergodicity results to avoid using all the sample paths in the recovery of the strength. We show that the diagonal entries of the microcorrelation strength matrix can be uniquely determined almost surely by the amplitude of the electric field averaged over the frequency band at a single path.

To indicate the dependence on the wavenumber k of the electric field, we use the notation $E_j(x;k)$ instead of $E_j(x)$ from now on. The following theorem is the main result of this section.

THEOREM 4.1. *Let Assumption* 2.1 *hold, and let $\mathcal{U}\subset\mathbb{R}^3$ be a bounded open set which has a positive distance to $\mathcal{O}$. The strength a_{jj} is uniquely determined almost surely by*

$$\lim_{K\to\infty}\frac{1}{K-1}\int_1^K k^{2s-2}|E_j(x;k)|^2dk=\frac{1}{(4\pi)^2}\int_{\mathbb{R}^3}\frac{1}{|x-y|^2}a_{jj}(y)dy,\quad x\in\mathcal{U}.$$

The above theorem indicates that it is statistically stable to recover the diagonal entries of the microcorrelation strength matrix since only a single realization is needed for the random source. We present some preliminaries on ergodicity before showing the proof of Theorem 4.1.

4.1. Ergodic relation. For $j=1,2,3$, define

$$T_j(x):=\frac{1}{(4\pi)^2}\int_{\mathbb{R}^3}\frac{1}{|x-y|^2}a_{jj}(y)dy.$$

According to Theorem 3.1, it holds that $a_{jj}=a_{jj}^{\rm r}+{\rm i}a_{jj}^{\rm i}=a_{jj}^{\rm r}$ and

$$\lim_{k\to\infty}k^{2s-2}\mathbb{E}|E_j(x;k)|^2=T_j(x),\quad x\in\mathcal{U},\tag{4.1}$$

which implies that

$$\lim_{K\to\infty}\frac{1}{K-1}\int_1^K k^{2s-2}\mathbb{E}|E_j(x;k)|^2dk=T_j(x).\tag{4.2}$$

In fact, for any $\epsilon>0$, it follows from (4.1) that there exists some $k^*=k^*(\epsilon)>0$ such that

$$\left|k^{2s-2}\mathbb{E}|E_j(x;k)|^2-T_j(x)\right|<\frac{\epsilon}{2}\quad\forall\ k>k^*.$$

On the other hand, there exists $K^*=K^*(\epsilon)>0$ such that for any $K>K^*$

$$\begin{aligned}&\left|\frac{1}{K-1}\int_1^K\left(k^{2s-2}\mathbb{E}|E_j(x;k)|^2-T_j(x)\right)dk\right|\\&\le\frac{1}{K-1}\int_1^{k^*}\left|k^{2s-2}\mathbb{E}|E_j(x;k)|^2-T_j(x)\right|dk\\&\quad+\frac{1}{K-1}\int_{k^*}^K\left|k^{2s-2}\mathbb{E}|E_j(x;k)|^2-T_j(x)\right|dk\\&\le\frac{C}{K-1}+\frac{K-k^*}{K-1}\frac{\epsilon}{2}<\epsilon\end{aligned}$$

for some constant $C>0$, and hence (4.2) holds.

To prove the result given in Theorem 4.1, due to (4.2), it then suffices to show that

$$\lim_{K\to\infty}\frac{1}{K-1}\int_1^K k^{2s-2}\left(|E_j(x;k)|^2-\mathbb{E}|E_j(x;k)|^2\right)dk=0. \tag{4.3}$$

The following propositions are required in order to get the ergodic relation (4.3). The proofs can be found in [8, 14, 15].

PROPOSITION 4.2. *Let $Y(t)$ be a centered random field with $\mathbb{E}[Y(t)]=0$. If the covariance function $R(\cdot,\cdot)$ is continuous and satisfies*

$$|R(t,u)|=|\mathbb{E}[Y(t)Y(u)]|\lesssim\frac{t^\alpha+u^\alpha}{1+|t-u|^\beta},$$

where the constants α,β satisfy $0\le 2\alpha<\beta<1$, then

$$\lim_{K\to\infty}\frac{1}{K-1}\int_1^K Y(k)dk=0$$

holds almost surely.

PROPOSITION 4.3. *Let X and Y be centered Gaussian random variables with $\mathbb{E}[X]=\mathbb{E}[Y]=0$. Then the following identity holds:*

$$\mathbb{E}\left[(X^2-\mathbb{E}[X^2])(Y^2-\mathbb{E}[Y^2])\right]=2(\mathbb{E}[XY])^2.$$

4.2. Proof of Theorem 4.1. Define

$$Y_j(x;k):=k^{2s-2}\left(|E_j(x;k)|^2-\mathbb{E}|E_j(x;k)|^2\right),\quad x\in\mathcal{U},$$

for $j=1,2,3$, which apparently satisfies $\mathbb{E}[Y(x;k)]=0$. Next, we estimate

$$\mathbb{E}[Y_j(x;k_1)Y_j(x;k_2)]\quad\forall\, k_1,k_2\ge 1.$$

Let $E_j=E_j^{\rm r}+{\rm i}E_j^{\rm i}$, $j=1,2,3$, where $E_j^{\rm r}$ and $E_j^{\rm i}$ are the real and imaginary parts of E_j. A simple calculation yields

$$Y_j(x;k)=k^{2s-2}\left((E_j^{\rm r}(x;k))^2-\mathbb{E}(E_j^{\rm r}(x;k))^2+(E_j^{\rm i}(x;k))^2-\mathbb{E}(E_j^{\rm i}(x;k))^2\right)$$

and

$$\begin{aligned}
&\frac{|\mathbb{E}[Y_j(x;k_1)Y_j(x;k_2)]|}{k_1^{2s-2}k_2^{2s-2}}\\
&=\mathbb{E}\Big[\left((E_j^{\rm r}(x;k_1))^2-\mathbb{E}(E_j^{\rm r}(x;k_1))^2\right)\left((E_j^{\rm r}(x;k_2))^2-\mathbb{E}(E_j^{\rm r}(x;k_2))^2\right)\Big]\\
&\quad+\mathbb{E}\Big[\left((E_j^{\rm r}(x;k_1))^2-\mathbb{E}(E_j^{\rm r}(x;k_1))^2\right)\left((E_j^{\rm i}(x;k_2))^2-\mathbb{E}(E_j^{\rm i}(x;k_2))^2\right)\Big]\\
&\quad+\mathbb{E}\Big[\left((E_j^{\rm i}(x;k_1))^2-\mathbb{E}(E_j^{\rm i}(x;k_1))^2\right)\left((E_j^{\rm r}(x;k_2))^2-\mathbb{E}(E_j^{\rm r}(x;k_2))^2\right)\Big]\\
&\quad+\mathbb{E}\Big[\left((E_j^{\rm i}(x;k_1))^2-\mathbb{E}(E_j^{\rm i}(x;k_1))^2\right)\left((E_j^{\rm i}(x;k_2))^2-\mathbb{E}(E_j^{\rm i}(x;k_2))^2\right)\Big]\\
&=2\Big(\mathbb{E}[E_j^{\rm r}(x;k_1)E_j^{\rm r}(x;k_2)]\Big)^2+2\Big(\mathbb{E}[E_j^{\rm r}(x;k_1)E_j^{\rm i}(x;k_2)]\Big)^2\\
&\quad+2\Big(\mathbb{E}[E_j^{\rm i}(x;k_1)E_j^{\rm r}(x;k_2)]\Big)^2+2\Big(\mathbb{E}[E_j^{\rm i}(x;k_1)E_j^{\rm i}(x;k_2)]\Big)^2\\
&=:{\rm I}_{j,1}+{\rm I}_{j,2}+{\rm I}_{j,3}+{\rm I}_{j,4},
\end{aligned}\tag{4.4}$$

where we have used Proposition 4.3.

Using the fact that

$$\Re[g]\Re[h] = \frac{1}{2}\Re[gh + g\overline{h}],$$

$$\Re[g]\Im[h] = -\Re[g]\Re[\mathrm{i}h] = -\frac{1}{2}\Re[\mathrm{i}gh - \mathrm{i}g\overline{h}] = \frac{1}{2}\Im[gh - g\overline{h}],$$

$$\Im[g]\Im[h] = \Re[\mathrm{i}g]\Re[\mathrm{i}h] = \frac{1}{2}\Re[g\overline{h} - gh]$$

for any $g, h \in \mathbb{C}$, we get

$$\begin{aligned}\mathrm{I}_{j,1} &= \frac{1}{2}\Re\left[\mathbb{E}[E_j(x;k_1)E_j(x;k_2)] + \mathbb{E}[E_j(x;k_1)\overline{E_j(x;k_2)}]\right]^2\\ &\le |\mathbb{E}[E_j(x;k_1)E_j(x;k_2)]|^2 + \left|\mathbb{E}[E_j(x;k_1)\overline{E_j(x;k_2)}]\right|^2,\end{aligned}$$

$$\begin{aligned}\mathrm{I}_{j,2} &= \frac{1}{2}\Im\left[\mathbb{E}[E_j(x;k_1)E_j(x;k_2)] - \mathbb{E}[E_j(x;k_1)\overline{E_j(x;k_2)}]\right]^2\\ &\le |\mathbb{E}[E_j(x;k_1)E_j(x;k_2)]|^2 + \left|\mathbb{E}[E_j(x;k_1)\overline{E_j(x;k_2)}]\right|^2,\end{aligned}$$

$$\begin{aligned}\mathrm{I}_{j,3} &= \frac{1}{2}\Im\left[\mathbb{E}[E_j(x;k_1)E_j(x;k_2)] - \mathbb{E}[\overline{E_j(x;k_1)}E_j(x;k_2)]\right]^2\\ &\le |\mathbb{E}[E_j(x;k_1)E_j(x;k_2)]|^2 + \left|\mathbb{E}[E_j(x;k_1)\overline{E_j(x;k_2)}]\right|^2,\end{aligned}$$

$$\begin{aligned}\mathrm{I}_{j,4} &= \frac{1}{2}\Re\left[\mathbb{E}[E_j(x;k_1)\overline{E_j(x;k_2)}] - \mathbb{E}[E_j(x;k_1)E_j(x;k_2)]\right]^2\\ &\le |\mathbb{E}[E_j(x;k_1)E_j(x;k_2)]|^2 + \left|\mathbb{E}[E_j(x;k_1)\overline{E_j(x;k_2)}]\right|^2.\end{aligned}$$

For $k_1, k_2 \ge 1$, let

$$\begin{aligned}\mathscr{A}_j(k_1,k_2) &= |\mathbb{E}[E_j(x;k_1)E_j(x;k_2)]|^2,\\ \mathscr{B}_j(k_1,k_2) &= \left|\mathbb{E}[E_j(x;k_1)\overline{E_j(x;k_2)}]\right|^2.\end{aligned}$$

It suffices to estimate $\mathscr{A}_j(k_1,k_2)$ and $\mathscr{B}_j(k_1,k_2)$.

By Assumption 2.1, we may easily verify that

$$\begin{aligned}&\mathbb{E}\left[E_j(x;k_1)E_j(x;k_2)\right]\\ &= k_1k_2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Phi_{k_1}(x,y)\Phi_{k_2}(x,z)\mathbb{E}\left[J_j(y)J_j(z)\right]dydz\\ &= k_1k_2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Phi_{k_1}(x,y)\Phi_{k_2}(x,z)R_{jj}(y,z)dydz\\ &= 0,\end{aligned}$$

where R_{jj} is the (j,j)-entry of the relation kernel $R_{\boldsymbol{J}}$ of the relation operator $Q_{\boldsymbol{J}}^R$. Consequently,

$$\mathscr{A}_j(k_1,k_2) = 0 \quad \forall\ k_1,k_2 \ge 1.$$

For the term $\mathscr{B}_j(k_1,k_2)$, we have

$$\mathbb{E}\left[E_j(x;k_1)\overline{E_j(x;k_2)}\right] = k_1k_2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Phi_{k_1}(x,y)\overline{\Phi_{k_2}(x,z)}\mathbb{E}\left[J_j(y)\overline{J_j(z)}\right]dydz$$
$$= \frac{k_1k_2}{(4\pi)^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}(k_1|x-y|-k_2|x-z|)}}{|x-y||x-z|}C_{jj}(y,z)dydz.$$

Noting that

$$k_1|x-y|-k_2|x-z| = (k_1+k_2)\frac{|x-y|-|x-z|}{2}+(k_1-k_2)\frac{|x-y|+|x-z|}{2}$$

and using the coordinate transform and the symbols defined in the proof of Theorem 3.1, we get

$$\mathscr{B}_j(k_1,k_2) = \left|\frac{k_1k_2}{(4\pi)^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{\mathrm{i}((k_1+k_2)e_1\cdot g+(k_1-k_2)e_1\cdot h)}S_2(g,h,x)dgdh\right|^2$$
$$= \left|\frac{k_1k_2}{(4\pi)^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{\mathrm{i}((k_1+k_2)e_1\cdot g+(k_1-k_2)e_1\cdot h)}\right.$$
$$\left.\times\left[\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}e^{\mathrm{i}g\cdot\xi}s_2(h,x,\xi)d\xi\right]dgdh\right|^2$$
$$= \left|\frac{k_1k_2}{(4\pi)^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{\mathrm{i}(k_1-k_2)e_1\cdot h}\right.$$
$$\left.\times\left[\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}e^{\mathrm{i}g\cdot(\xi+(k_1+k_2)e_1)}dg\right]s_2(h,x,\xi)d\xi dh\right|^2$$
$$= \left|\frac{k_1k_2}{(4\pi)^2}\int_{\mathbb{R}^3}e^{\mathrm{i}(k_1-k_2)e_1\cdot h}s_2(h,x,-(k_1+k_2)e_1)dh\right|^2.$$

If $|k_1-k_2|<1$, due to the fact that $A(x)$ is compactly supported, then we obtain

$$\mathscr{B}_j(k_1,k_2) = \left|\frac{k_1k_2}{(4\pi)^2}\int_{\mathbb{R}^3}e^{\mathrm{i}(k_1-k_2)e_1\cdot h}\left[a_{jj}\left(\frac{w(0,h,x)}{2}\right)\frac{|\sin(h_2/h_1)|}{(h\cdot e_1)^2}\right.\right.$$
$$\left.\left.\times\left(\frac{k_1+k_2}{2}\right)^{-2s}\theta(x)+r_3(h,x,-(k_1+k_2)e_1)\right]dh\right|^2$$
$$\lesssim\left(\frac{k_1k_2}{(k_1+k_2)^{2s}}\right)^2.$$

If $|k_1-k_2|\ge 1$, then for arbitrary $\beta\in(0,1)$, we deduce that

$$\mathscr{B}_j(k_1,k_2) = \left|\frac{k_1k_2}{(4\pi)^2}\frac{1}{\mathrm{i}(k_1-k_2)}\int_{\mathbb{R}^3}s_2(h,x,-(k_1+k_2)e_1)de^{\mathrm{i}(k_1-k_2)h_1}dh_2dh_3\right|^2$$
$$= \left|\frac{k_1k_2}{(4\pi)^2}\frac{1}{\mathrm{i}(k_1-k_2)}\int_{\mathbb{R}^3}e^{\mathrm{i}(k_1-k_2)h_1}\partial_{h_1}s_2(h,x,-(k_1+k_2)e_1)dh_1dh_2dh_3\right|^2$$
$$\lesssim\left(\frac{k_1k_2}{(k_1+k_2)^{2s}|k_1-k_2|}\right)^2$$
$$\le\left(\frac{k_1k_2}{(k_1+k_2)^{2s}}\right)^2\frac{1}{|k_1-k_2|^{\beta}}$$

since the symbol s_2 is also compactly supported and $|\partial_{h_1} s_2(h,x,\xi)| \lesssim |\xi|^{-2s}$ for $x \in \mathcal{U}$. We conclude from the above estimates that

$$\mathscr{A}_j(k_1,k_2) + \mathscr{B}_j(k_1,k_2) \lesssim \frac{k_1^2 k_2^2}{(k_1+k_2)^{4s}} \frac{1}{1+|k_1-k_2|^\beta}.$$

Finally, it follows from (4.4) that

$$\begin{aligned}|\mathbb{E}[Y_j(x;k_1)Y_j(x;k_2)]| &\le 4k_1^{2s-2}k_2^{2s-2}\left(\mathscr{A}_j(k_1,k_2)+\mathscr{B}_j(k_1,k_2)\right)\\ &\lesssim \left(\frac{k_1k_2}{(k_1+k_2)^2}\right)^{2s}\frac{1}{1+|k_1-k_2|^\beta}\\ &\lesssim \frac{1}{1+|k_1-k_2|^\beta}.\end{aligned}$$

Using Proposition 4.2 with $\alpha = 0$, we get

$$\lim_{K\to\infty}\frac{1}{K-1}\int_1^K Y_j(x;k)dk = 0 \quad \forall\, x\in\mathcal{U},$$

and hence (4.3) holds. This completes the proof of Theorem 4.1.

5. Conclusion. In this paper, we have studied the three-dimensional Maxwell's equations driven by a rough complex-valued Gaussian vector field, where the covariance operator of the random source is a pseudodifferential operator with a complex-valued strength matrix. Under an appropriate assumption of the random source, the well-posedness of the direct scattering problem is established in the distribution sense. The regularity of the electromagnetic field is also given. The microcorrelation strength matrix of the random source is shown to be uniquely determined by the high frequency limit of the expectation of the electric field. Moreover, the diagonal entries of the strength matrix are shown to be uniquely determined by the amplitude of the electric field averaged over the frequency band at a single path due to the ergodicity.

In this work, we assume that the real and imaginary parts of the random source are independent and identically distributed, i.e., they are uncorrelated. A possible future work is to remove the assumption and consider more general complex-valued Gaussian vector fields where the real and imaginary parts are correlated. In this case, the centered random source would be determined by not only its covariance operator but also its relation operator. The recovery of the strength matrix of the relation operator is open since the microlocal analysis no longer seems to work. The same issue appears in the inverse elastic wave scattering problem. If the random source is a real-valued Gaussian vector field and the components are independent and identically distributed, the recovery of the scalar strength function for the elastic scattering problem has been investigated in [15, 16]. However, there are no results on the problem if the random source is complex and correlated. It is also unclear how the nondiagonal entries of the strength matrix can be uniquely determined by only the amplitude of the electric field averaged over the frequency band at a single path. We hope to be able to report the progress on these problems elsewhere in the future.

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