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Regularity of distributional solutions to stochastic acoustic and elastic scattering problems ☆

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Abstract

This paper is concerned with the well-posedness and regularity of the distributional solutions for the stochastic acoustic and elastic scattering problems. We show that the regularity of the solutions depends on the regularity of both the random medium and the random source.

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Keywords: Regularity; The Helmholtz equation; The elastic wave equation; Random media; Microlocally isotropic generalized Gaussian random field

1. Introduction

The acoustic and elastic wave equations are two fundamental equations to describe wave propagation. They have significantly applications in diverse scientific areas such as remote sensing, nondestructive testing, geophysical prospecting, and medical imaging [5]. In practice, due to the unpredictability of the environments and incomplete knowledge of the systems, the radiating sources and/or the host media, and hence the radiated fields may not be deterministic but rather are modeled by random fields [7]. Their governing equations are some forms of stochastic differential equations and their solutions are random fields instead of their deterministic counterparts

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of regular functions [4,11,13]. Regularity theory of stochastic wave equations has played an important role in the study of partial differential equations and attracted a lot of attention [6,12,21]. As is known, a basic problem in classical scattering theory is the scattering of a time-harmonic wave by an inhomogeneous medium. This paper is concerned with the well-posedness and regularity of the solutions for the time-harmonic stochastic acoustic and elastic scattering problems.

For the case of acoustic waves, it is to find the induced pressure u which satisfies the Helmholtz equation

$$\Delta u + k^2(1+\rho)u = f \quad \text{in } \mathbb{R}^d, \tag{1.1}$$

where $d = 2$ or 3, $k > 0$ is the wavenumber, ρ describes the inhomogeneous medium and is assumed to be a microlocally isotropic generalized Gaussian random field (cf. Definition 2.1) defined in a bounded domain D_ρ, and f is assumed to be either a microlocally isotropic generalized Gaussian random field in a bounded domain D_f or a point source given by a delta distribution. In addition, the pressure u is required to satisfy the Sommerfeld radiation condition

$$\lim_{|x|\to\infty} |x|^{\frac{d-1}{2}} \left(\partial_{|x|} u - \mathrm{i} k u\right) = 0. \tag{1.2}$$

The elastic analogue is to find the displacement $\boldsymbol{u}$ satisfying the Navier equation

$$\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} + k^2(\boldsymbol{I}+\boldsymbol{M})\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \mathbb{R}^d, \tag{1.3}$$

where $\boldsymbol{I}$ is the identity matrix in $\mathbb{R}^d$, the Lamé parameters μ and λ satisfy $\mu > 0$ and $\lambda + 2\mu > 0$ such that the linear operator $\Delta^* := \mu\Delta + (\lambda+\mu)\nabla\nabla\cdot$ is uniformly elliptic (cf. [20, (10.4)]), $\boldsymbol{M}$ represents the anisotropic, inhomogeneous medium and is assumed to be a $\mathbb{R}^{d\times d}$-valued microlocally isotropic generalized Gaussian random field in a bounded domain $D_{\boldsymbol{M}}$, and $\boldsymbol{f}$ is either a microlocally isotropic generalized Gaussian random field in a bounded domain $D_{\boldsymbol{f}}$ or a point source given by a delta distribution. By [2], the displacement admits the Helmholtz decomposition

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{p}} + \boldsymbol{u}_{\mathrm{s}} \quad \text{in } \mathbb{R}^d \setminus (\overline{D_{\boldsymbol{M}} \cup D_{\boldsymbol{f}}}),$$

where $\boldsymbol{u}_{\mathrm{p}}$ and $\boldsymbol{u}_{\mathrm{s}}$ are the compressional and shear wave components, respectively, and are required to satisfy the Kupradze–Sommerfeld radiation condition

$$\lim_{|x|\to\infty} |x|^{\frac{d-1}{2}} \left(\partial_{|x|}\boldsymbol{u}_{\mathrm{p}} - \mathrm{i}\kappa_{\mathrm{p}}\boldsymbol{u}_{\mathrm{p}}\right) = \lim_{|x|\to\infty} |x|^{\frac{d-1}{2}} \left(\partial_{|x|}\boldsymbol{u}_{\mathrm{s}} - \mathrm{i}\kappa_{\mathrm{s}}\boldsymbol{u}_{\mathrm{s}}\right) = 0, \tag{1.4}$$

where

$$\kappa_{\mathrm{p}} = k/(\lambda+2\mu)^{1/2}, \quad \kappa_{\mathrm{s}} = k/\mu^{1/2}$$

are called the compressional and shear wavenumbers, respectively.

Recently, the microlocally isotropic generalized Gaussian random fields are adopted to characterize the random coefficients of some stochastic wave equations. The associated covariance operators can be viewed as classical pseudo-differential operators. These random fields may be too rough to be classical functions, and should be interpreted as distributions instead. Classical regularity estimates are not applicable for these stochastic equations due to the roughness of the random coefficients. The well-posedness of these equations in the distribution sense and regularity of the distributional solutions need to be investigated. We refer to [3,17] and [15,16] for the study of the well-posedness of the solutions for the acoustic and elastic wave equations with random potentials and sources, respectively. However, it remains open for the well-posedness and regularity of the solutions for the stochastic acoustic and elastic wave scattering problems in random media with random sources. The goal of this paper is to examine the well-posedness and regularity of the distributional solutions for the stochastic acoustic scattering problem (1.1)–(1.2) and the stochastic elastic scattering problem (1.3)–(1.4) by using a unified approach.

This paper is organized as follows. In Section 2, we introduce some Sobolev spaces of real order and the microlocally isotropic generalized Gaussian random fields. Sections 3 and 4 address the well-posedness and regularity of the solutions for the stochastic acoustic and elastic scattering problems, respectively.

2. Preliminaries

In this section, we briefly introduce Sobolev spaces of real order and microlocally isotropic generalized Gaussian random fields, which are used in this paper.

2.1. Sobolev spaces

Let $C_0^\infty(D)$ be the set of smooth functions compactly supported in $D \subset \mathbb{R}^d$, and $\mathcal{D}(D)$ be the space of test functions, which is $C_0^\infty(D)$ equipped with a locally convex topology (cf. [1]). The dual space $\mathcal{D}'(D)$ of $\mathcal{D}(D)$ is called the space of distributions on D equipped with a weak-star topology. Define the product

$$\langle u, v\rangle := \int_D u(x)\overline{v(x)}dx$$

for $u \in \mathcal{D}'(D)$ and $v \in \mathcal{D}(D)$. The distributional partial derivative of $u \in \mathcal{D}'(D)$ satisfies

$$\langle \partial^\zeta u, \psi\rangle = (-1)^{|\zeta|}\langle u, \partial^\zeta \psi\rangle$$

for any $\psi \in \mathcal{D}(D)$ and multi-index $\zeta = (\zeta_1, \dots, \zeta_d)$.

For any positive integer n and $1 \le p < \infty$, the Sobolev space $W^{n,p}(D)$ is defined by

$$W^{n,p}(D) = \{u \in L^p(D) : \partial^\zeta u \in L^p(D) \quad \text{for} \quad 0 \le |\zeta| \le n\},$$

which is equipped with the norm

$$\|u\|_{W^{n,p}(D)} := \left(\sum_{0\le|\zeta|\le n} \|\partial^\zeta u\|^p_{L^p(D)}\right)^{\frac{1}{p}}.$$

For any $r \in \mathbb{R}_+$, let $r = n + \mu$ with $n = [r]$ being the largest integer smaller than r and $\mu \in (0, 1)$, and define

$$W^{r,p}(D)=\{u\in W^{n,p}(D):|\partial^\zeta u|_{W^{\mu,p}(D)}<\infty \quad \text{for} \quad |\zeta|=n\}$$

equipped with the norm

$$\|u\|_{W^{r,p}(D)}:=\left(\|u\|^p_{W^{n,p}(D)}+\sum_{|\zeta|=n}|\partial^\zeta u|^p_{W^{\mu,p}(D)}\right)^{\frac{1}{p}},$$

where

$$|u|_{W^{\mu,p}(D)}:=\left(\int_D\int_D\frac{|u(x)-u(y)|^p}{|x-y|^{p\mu+d}}dxdy\right)^{\frac{1}{p}}$$

is the Slobodeckij semi-norm. Denote by $W_0^{r,p}(D)$ the closure of $C_0^\infty(D)$ in $W^{r,p}(D)$.

For any $r\in\mathbb{R}_+$, the Sobolev space $W^{-r,p}(D)$ of negative order is defined as the dual of $W_0^{r,q}(D)$ with $\frac{1}{p}+\frac{1}{q}=1$ equipped with the norm

$$\|u\|_{W^{-r,p}(D)}:=\sup_{v\in W^{r,q}(D),\|v\|_{W^{r,q}(D)}\le 1}|\langle u,v\rangle|.$$

If $D=\mathbb{R}^d$, there is another kind of Sobolev spaces defined through the Bessel potential. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing smooth functions, i.e.,

$$\mathcal{S}(\mathbb{R}^d):=\{\phi\in C^\infty(\mathbb{R}^d):\sup_{x\in\mathbb{R}^d}|x^\zeta\partial^\tau\phi(x)|<\infty \quad \text{for all multi-indices } \zeta \text{ and } \tau\},$$

and $\mathcal{S}'(\mathbb{R}^d)$ be the dual space of $\mathcal{S}(\mathbb{R}^d)$. Then $\mathcal{D}(\mathbb{R}^d)\subset\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)\subset\mathcal{D}'(\mathbb{R}^d)$. For any $s\in\mathbb{R}$, define the Bessel potential $\mathcal{J}^s:\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d)$ of order s by

$$\mathcal{J}^s u:=(I-\Delta)^{\frac{s}{2}}u=\mathcal{F}^{-1}[(1+|\cdot|^2)^{\frac{s}{2}}\hat{u}],$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. It is easy to verify that

$$(\mathcal{J}^s u,v)_{L^2(\mathbb{R}^d)}=(u,\mathcal{J}^s v)_{L^2(\mathbb{R}^d)} \quad \forall u,v\in\mathcal{S}(\mathbb{R}^d),$$

where $(\cdot,\cdot)_{L^2(\mathbb{R}^d)}$ is the inner product in $L^2(\mathbb{R}^d)$ satisfying

$$(u,v)_{L^2(\mathbb{R}^d)}=\langle u,v\rangle.$$

Based on the Bessel potential, we introduce the following Sobolev space of order $s\in\mathbb{R}$:

$$H^{s,p}(\mathbb{R}^d)=\{u\in\mathcal{S}'(\mathbb{R}^d):\mathcal{J}^s u\in L^p(\mathbb{R}^d)\}.$$

Denote $H^s(\mathbb{R}^d):=H^{s,2}(\mathbb{R}^d)$, which is a Hilbert space with the inner product

$$(u,v)_{H^s(\mathbb{R}^d)}:=(\mathcal{J}^s u,\mathcal{J}^s v)_{L^2(\mathbb{R}^d)}$$

and the induced norm

$$\|u\|_{H^s(\mathbb{R}^d)} := \|\mathcal{J}^s u\|_{L^2(\mathbb{R}^d)}.$$

For any set $D \subset \mathbb{R}^d$, define

$$H^s(D) = \{u \in \mathcal{D}'(D) : u = \tilde{u}|_D \quad \text{for some extension } \tilde{u} \in H^s(\mathbb{R}^d)\}.$$

Then it holds $H^s(D) = W^{s,2}(D)$ for any real $s \geq 0$ and $H^{-n}(D) = W^{-n,2}(D)$ for any integer $n \geq 0$ with equivalent norms (cf. [9,20]).

2.2. Microlocally isotropic generalized Gaussian random fields

Denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space, where Ω is a sample space, $\mathcal{F}$ is a σ-algebra on Ω, and $\mathbb{P}$ is a probability measure on the measurable space $(\Omega, \mathcal{F})$. Define $\mathcal{D} := \mathcal{D}(\mathbb{R}^d)$ and $\mathcal{D}' := \mathcal{D}'(\mathbb{R}^d)$. A real-valued field ρ is said to be a generalized random field if, for each $\omega \in \Omega$, the realization $\rho(\omega)$ belongs to $\mathcal{D}'$ and the mapping

$$\omega \in \Omega \longmapsto \langle \rho(\omega), \psi \rangle \in \mathbb{R} \tag{2.1}$$

is a random variable for all $\psi \in \mathcal{D}$.

In particular, a generalized random field is said to be Gaussian if (2.1) defines a Gaussian random variable for all $\psi \in \mathcal{D}$. A generalized Gaussian random field $\rho \in \mathcal{D}'$ is uniquely determined by its expectation $\mathbb{E}\rho \in \mathcal{D}'$ and covariance operator $Q_\rho : \mathcal{D} \to \mathcal{D}'$ defined by

$$\begin{aligned}
\langle \mathbb{E}\rho, \psi \rangle &:= \mathbb{E}\langle \rho, \psi \rangle \quad \forall \psi \in \mathcal{D}, \\
\langle Q_\rho \psi_1, \psi_2 \rangle &:= \mathbb{E}\left[(\langle \rho, \psi_1 \rangle - \mathbb{E}\langle \rho, \psi_1 \rangle)(\langle \rho, \psi_2 \rangle - \mathbb{E}\langle \rho, \psi_2 \rangle)\right] \quad \forall \psi_1, \psi_2 \in \mathcal{D}.
\end{aligned}$$

It follows from the continuity of Q_ρ and the Schwartz kernel theorem that there exists a unique kernel function $\mathcal{K}_\rho(x, y)$ satisfying

$$\langle Q_\rho \psi_1, \psi_2 \rangle = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathcal{K}_\rho(x, y)\psi_1(x)\overline{\psi_2(y)}dxdy \quad \forall \psi_1, \psi_2 \in \mathcal{D}.$$

The regularity of the covariance operator Q_ρ determines the regularity of the random field ρ.

Definition 2.1. A generalized Gaussian random field ρ on $\mathbb{R}^d$ is called microlocally isotropic of order $-m$ with $m \geq 0$ in D if its covariance operator Q_ρ is a classical pseudo-differential operator having an isotropic principal symbol $\phi(x)|\xi|^{-m}$ with the micro-correlation strength $\phi \in C_0^\infty(D)$ being compactly supported in D and $\phi \geq 0$.

Note that the covariance operator with a principle symbol $\phi(x)|\xi|^{-m}$ has similar regularity as the fractional Laplacian. To investigate the regularity of microlocally isotropic Gaussian random fields defined above, we introduce the centered fractional Gaussian fields (cf. [18,19]) defined by

$$h_m(x) := (-\Delta)^{-\frac{m}{4}}\dot{W}(x), \quad x \in \mathbb{R}^d, \tag{2.2}$$

where $(-\Delta)^{-\frac{m}{4}}$ is the fractional Laplacian and $\dot{W} \in \mathcal{D}'$ denotes the white noise. It is shown in [19] that h_m is a microlocally isotropic Gaussian random field of order $-m$ satisfying Definition 2.1 with $\phi \equiv 1$. Hence the fractional Gaussian field h_m defined by (2.2) has the same regularity as the microlocally isotropic Gaussian random field ρ of order $-m$ in Definition 2.1.

In particular, if $m \in (d, d+2)$, the fractional Gaussian field h_m defined above is a translation of a classical fractional Brownian motion. More precisely,

$$\tilde{h}_m(x) := \langle h_m, \delta(x-\cdot) - \delta(\cdot)\rangle, \quad x \in \mathbb{R}^d$$

has the same distribution as the classical fractional Brownian motion with Hurst parameter $H = \frac{m-d}{2} \in (0,1)$ up to a multiplicative constant, where $\delta(\cdot)$ is the Dirac function centered at the origin.

Taking advantages of the relationship between the microlocally isotropic Gaussian random fields and the fractional Gaussian fields defined in (2.2), we conclude this section by providing the regularity of microlocally isotropic Gaussian random fields, whose proof can be found in [19].

Lemma 2.2. *Let ρ be a microlocally isotropic Gaussian random field of order $-m$ in D with $m \in [0, d+2)$.*

(i) *If $m \in (d, d+2)$, then $\rho \in C^{0,\alpha}(D)$ almost surely for all $\alpha \in (0, \frac{m-d}{2})$.*
(ii) *If $m \in [0, d]$, then $\rho \in W^{\frac{m-d}{2}-\epsilon,p}(D)$ almost surely for any $\epsilon > 0$ and $p \in (1, \infty)$.*

Remark 2.3. For a microlocally isotropic Gaussian random field ρ in Definition 2.1, its kernel has the form $\mathcal{K}_\rho(x,y) = \phi(x)\mathcal{K}_{h_m}(x,y) + r(x,y)$, where $\phi\mathcal{K}_{h_m}$ is the leading term with strength ϕ and r is a smooth residual (cf. [14]).

3. The acoustic scattering problem

In this section, we consider the Helmholtz equation (1.1) and study the well-posedness for the acoustic scattering problem under the following assumptions on the medium ρ and the source f.

Assumption 1. Let the medium ρ be a real-valued centered microlocally isotropic Gaussian random field of order $-m_\rho$ with $m_\rho \in (d-1, d]$ in a bounded domain $D_\rho \subset \mathbb{R}^d$. The principal symbol of its covariance operator has the form $\phi_\rho(x)|\xi|^{-m_\rho}$ with $\phi_\rho \in C_0^\infty(D_\rho)$ and $\phi_\rho \geq 0$.

Assumption 2. Let the real-valued source f satisfy one of the following assumptions:

(i) f is a centered microlocally isotropic Gaussian random field of order $-m_f$ with $m_f \in (d-1, d]$ in a bounded domain $D_f \subset \mathbb{R}^d$. The principal symbol of its covariance operator has the form $\phi_f(x)|\xi|^{-m_f}$ with $\phi_f \in C_0^\infty(D_f)$ and $\phi_f \geq 0$.
(ii) $f = -\delta(\cdot - y)a$ is a point source with $y \in \mathbb{R}^d$ and some fixed constant $a \in \mathbb{R}$.

For such rough ρ and f, the Helmholtz equation (1.1) should be interpreted in the distribution sense. First let us consider the equivalent Lippmann–Schwinger integral equation.

3.1. The Lippmann–Schwinger equation

Based on the fundamental solution

$$\Phi_d(x,y,k) = \begin{cases} \dfrac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|), & d=2, \\ \dfrac{e^{\mathrm{i}k|x-y|}}{4\pi|x-y|}, & d=3, \end{cases} \tag{3.1}$$

of the equation $\Delta u + k^2 u = -\delta(\cdot - y)$ in $\mathbb{R}^d$, the Lippmann–Schwinger integral equation has the form

$$u(x) - k^2 \int_{\mathbb{R}^d} \Phi_d(x,z,k)\rho(z)u(z)dz = -\int_{\mathbb{R}^d} \Phi_d(x,z,k) f(z)dz. \tag{3.2}$$

Define two operators

$$(H_k v)(x) := \int_{\mathbb{R}^d} \Phi_d(x,z,k)v(z)dz,$$

$$(K_k v)(x) := \int_{\mathbb{R}^d} \Phi_d(x,z,k)\rho(z)v(z)dz,$$

which have the following properties.

Lemma 3.1. *Let ρ satisfy Assumption 1. Let $D \subset \mathbb{R}^d$ be a bounded set and $G \subset \mathbb{R}^d$ be a bounded set with a locally Lipschitz boundary.*

(i) *The operator $H_k : H_0^{-\beta}(D) \to H^\beta(G)$ is bounded for any $\beta \in (0,1]$.*
(ii) *The operator $H_k : W_0^{-\gamma,p}(D) \to W^{\gamma,q}(G)$ is compact for any $q \in (2,\infty)$, $\gamma \in (0, (\frac{1}{q} - \frac{1}{2})d + 1)$ and p satisfying $\frac{1}{p} + \frac{1}{q} = 1$.*
(iii) *The operator $K_k : W^{\gamma,q}(G) \to W^{\gamma,q}(G)$ is compact for any $q \in (2, \frac{2d}{2d-2-m_\rho})$ and $\gamma \in (\frac{d-m_\rho}{2}, (\frac{1}{q} - \frac{1}{2})d + 1)$.*

Proof. (i) It follows from [5, Theorem 8.1] that H_k is bounded from $C^{0,\alpha}(D)$ to $C^{2,\alpha}(G)$ with respect to the corresponding Hölder norms $\|\cdot\|_{C^{0,\alpha}(D)}$ and $\|\cdot\|_{C^{2,\alpha}(G)}$. Define spaces $X := C^{0,\alpha}(D)$ and $Y := C^{2,\alpha}(G)$ with scalar products

$$(f_1, f_2)_X := (\tilde{f}_1, \tilde{f}_2)_{H^{\beta-2}(\mathbb{R}^d)} \quad \forall f_1, f_2 \in X$$

and

$$(g_1, g_2)_Y := (\tilde{g}_1, \tilde{g}_2)_{H^\beta(\mathbb{R}^d)} \quad \forall g_1, g_2 \in Y,$$

respectively, where $\tilde{f}_i$ and $\tilde{g}_i$ are the zero extensions of f_i and g_i in $\mathbb{R}^d \setminus \overline{D}$ and $\mathbb{R}^d \setminus \overline{G}$, respectively. It is easy to verify that the products defined above satisfy

$$
\begin{aligned}
(f_1, f_2)_X = (J^{\beta-2}\tilde{f}_1, J^{\beta-2}\tilde{f}_2)_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} (1+|\xi|^2)^{\beta-2}\hat{\tilde{f}}_1(\xi)\overline{\hat{\tilde{f}}_2(\xi)}d\xi \\
&\lesssim \|\tilde{f}_1\|_{L^2(\mathbb{R}^d)}\|\tilde{f}_2\|_{L^2(\mathbb{R}^d)} \lesssim \|f_1\|_{C^{0,\alpha}(D)}\|f_2\|_{C^{0,\alpha}(D)}
\end{aligned}
$$

and

$$
(g_1, g_2)_Y = (J^\beta \tilde{g}_1, J^\beta \tilde{g}_2)_{L^2(\mathbb{R}^d)} \lesssim \|\tilde{g}_1\|_{H^\beta(\mathbb{R}^d)}\|\tilde{g}_2\|_{H^\beta(\mathbb{R}^d)} \lesssim \|g_1\|_{C^{2,\alpha}(G)}\|g_2\|_{C^{2,\alpha}(G)},
$$

where the notation $a \lesssim b$ denotes $a \le Cb$ for some constant $C > 0$.

We claim that there exists a bounded operator $V : Y \to X$ defined by $V = (I - \Delta)H_k(I - \Delta)$ such that

$$
(H_k f, g)_Y = (f, Vg)_X \quad \forall f \in X, g \in Y.
$$

In fact, for any $g \in Y$,

$$
\begin{aligned}
\|Vg\|_{C^{0,\alpha}(D)} = \|(I-\Delta)H_k(I-\Delta)g\|_{C^{0,\alpha}(D)} &\lesssim \|H_k(I-\Delta)g\|_{C^{2,\alpha}(D)} \\
&\lesssim \|(I-\Delta)g\|_{C^{0,\alpha}(G)} \lesssim \|g\|_{C^{2,\alpha}(G)}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
(H_k f, g)_Y &= (J^\beta H_k \tilde{f}, J^\beta \tilde{g})_{L^2(\mathbb{R}^d)} = (H_k \tilde{f}, J^{2\beta}\tilde{g})_{L^2(\mathbb{R}^d)} \\
&= (\widehat{H_k \tilde{f}}, \widehat{J^{\beta}\tilde{g}})_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \hat{\Phi}_d(\xi)\hat{\tilde{f}}(\xi)(1+|\xi|^2)^\beta \overline{\hat{\tilde{g}}(\xi)}d\xi \\
&= \int_{\mathbb{R}^d} \hat{\tilde{f}}(\xi)(1+|\xi|^2)^{\beta-2}\overline{\left[(1+|\xi|^2)\hat{\Phi}_d(\xi)(1+|\xi|^2)\hat{\tilde{g}}(\xi)\right]}d\xi \\
&= \int_{\mathbb{R}^d} \hat{\tilde{f}}(\xi)(1+|\xi|^2)^{\beta-2}\overline{\widehat{V\tilde{g}}(\xi)}d\xi = (J^{\beta-2}\tilde{f}, J^{\beta-2}V\tilde{g})_{L^2(\mathbb{R}^d)} \\
&= (f, Vg)_X,
\end{aligned}
$$

where $\hat{\Phi}_d$ is the Fourier transform of $\Phi_d(x, y, k)$ with respect to $x - y$ and satisfies $-|\xi|^2\hat{\Phi}_d(\xi) + k^2\hat{\Phi}_d(\xi) = -1$. The claim is proved.

It follows from the claim and [5, Theorem 3.5] that $H_k : X \to Y$ is bounded with respect to the norms induced by the scalar products on X and Y. More precisely, we have

$$
\|H_k f\|_Y = \|H_k f\|_{H^\beta(G)} \lesssim \|f\|_X = \|f\|_{H^{\beta-2}(D)} \le \|f\|_{H^{-\beta}(D)} \tag{3.3}
$$

for any $f \in X$ and $\beta \le 1$. It then suffices to show that (3.3) also holds for any $f \in H^{-\beta}(D)$. Noting that the subspace $C_0^\infty(D) \subset X$ is dense in $L^2(D)$ (cf. [1, Section 2.30]) and $H^{-1}(D) = \overline{L^2(D)}^{\|\cdot\|_{H^{-1}(D)}}$ (cf. [1, Section 3.13]), we get that (3.3) holds for any $f \in H^{-1}(D)$, and hence for any $f \in H^{-\beta}(D)$ since $H^{-\beta}(D) \subset H^{-1}(D)$.

(ii) For parameters p, q and γ given above, we choose $\beta = 1$ such that $\gamma < \beta$, $\frac{1}{2} - \frac{\beta-\gamma}{d} < \frac{1}{q}$, and hence the embeddings

$$W_0^{-\gamma,p}(D) \hookrightarrow H_0^{-\beta}(D), \quad H^\beta(G) \hookrightarrow W^{\gamma,q}(G)$$

are compact according to the Kondrachov compact embedding theorem (cf. [1]). Combining with the result in (i) yields that H_k is compact from $W_0^{-\gamma,p}(D)$ to $W^{\gamma,q}(G)$.

(iii) Note that $\rho \in W^{\frac{m_\rho-d}{2}-\epsilon,p'}$ for any $\epsilon > 0$ and $p' > 1$ according to Lemma 2.2. Then for any $\gamma \in (\frac{d-m_\rho}{2}, (\frac{1}{q} - \frac{1}{2})d + 1)$, there exist $\epsilon > 0$ and $p' > 1$ such that $\frac{m_\rho-d}{2} - \epsilon > -\gamma$ and $\frac{1}{p'} - \frac{\frac{m_\rho-d}{2}-\epsilon+\gamma}{d} < \frac{1}{\tilde{p}}$ with $\tilde{p} = \frac{p}{2-p}$ and p satisfying $\frac{1}{p} + \frac{1}{q} = 1$, which leads to

$$W_0^{\frac{m_\rho-d}{2}-\epsilon,p'}(D_\rho) \hookrightarrow W_0^{-\gamma,\tilde{p}}(D_\rho)$$

according to the Kondrachov compact embedding theorem, and hence $\rho \in W_0^{-\gamma,\tilde{p}}(D_\rho)$. It then follows from [14, Lemma 2] that $\rho v \in W_0^{-\gamma,p}(D_\rho)$ for any $v \in W^{\gamma,q}(G)$ with

$$\|\rho v\|_{W^{-\gamma,p}} \lesssim \|\rho\|_{W^{-\gamma,\tilde{p}}} \|v\|_{W^{\gamma,q}}. \tag{3.4}$$

Consequently, for any $v \in W^{\gamma,q}(G)$, we have $K_k v = H_k(\rho v) \in W^{\gamma,q}(G)$, which implies that $K_k : W^{\gamma,q}(G) \to W^{\gamma,q}(G)$ is compact according to (ii). □

Before showing the well-posedness of the Lippmann–Schwinger equation (3.2), we present the unique continuation principle which ensures the uniqueness of the solution of (3.2).

Theorem 3.2. *Let ρ satisfy Assumption 1. If $u \in W^{\gamma,q}_{comp}(\mathbb{R}^d)$ with $\gamma \in (0, (\frac{1}{q} - \frac{1}{2})\frac{d}{2} + \frac{1}{2})$ and $q \in (2, \frac{2d}{d-2})$ is a solution of the homogeneous equation*

$$\Delta u + k^2(1+\rho)u = 0$$

in the distribution sense, then $u \equiv 0$.

Proof. For any fixed $k > 0$, define an auxiliary function $v(x) := e^{-\mathrm{i}\eta\cdot x}u(x)$ with

$$\eta := (kt, 0, \cdots, 0, \mathrm{i}k\sqrt{t^2+1}) \in \mathbb{C}^d, \quad t > 1$$

such that $\eta \cdot \eta = -k^2$ and $\lim_{t\to\infty} |\eta| = \infty$, which satisfies

$$(\Delta + 2\mathrm{i}\eta \cdot \nabla)v = -k^2 \rho v.$$

The equation above is equivalent to

$$v = G_\eta(\rho v),$$

where $v \in W^{\gamma,q}_{comp}(\mathbb{R}^d)$ and the operator G_η is defined by

$$G_\eta(f)(x) := \mathcal{F}^{-1}\left[\frac{k^2}{|\xi|^2 + 2\eta\cdot\xi}\hat{f}\right](x)$$

with $\xi = (\xi_1, \cdots, \xi_d)^\top \in \mathbb{R}^d$.

We first give the estimate of the operator G_η. Let $D \subset \mathbb{R}^d$ be a bounded domain containing the supports of both u and ρ. For any $f, g \in C_0^\infty(D)$, we still denote the zero extensions of f and g in $\mathbb{R}^d$ by f and g, respectively. For any $s \in [0, \frac{1}{2}]$, by denoting $\xi^- := (\xi_1, \cdots, \xi_{d-1})^\top \in \mathbb{R}^{d-1}$ and $\xi^{--} := (\xi_2, \cdots, \xi_{d-1})^\top \in \mathbb{R}^{d-2}$ with $\xi^{--} = 0$ if $d = 2$, we get

$$\begin{aligned}
\langle G_\eta f, g\rangle =& \langle \widehat{G_\eta f}, \hat{g}\rangle = \int_{\mathbb{R}^d} \frac{k^2}{|\xi|^2 + 2\eta\cdot\xi}\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi \\
=& \int_{\mathbb{R}^d} \frac{k^2}{|\xi|^2 + 2kt\xi_1 + 2\mathrm{i}k\sqrt{t^2+1}\xi_d}\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi \\
=& \int_{\mathbb{R}^d} \frac{k^2}{(\xi_1 + kt)^2 - k^2t^2 + |\xi^{--}|^2 + \xi_d^2 + 2\mathrm{i}k\sqrt{t^2+1}\xi_d}\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi \\
=& \int_{\mathbb{R}^d} \frac{k^2}{|\xi^-|^2 - k^2t^2 + \xi_d^2 + 2\mathrm{i}k\sqrt{t^2+1}\xi_d}\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi \\
=& \int_{\Omega_{\mathrm{I}}} \frac{k^2(1+|\xi|^2)^s}{|\xi^-|^2 - k^2t^2 + \xi_d^2 + 2\mathrm{i}k\sqrt{t^2+1}\xi_d}\widehat{\mathcal{J}^{-s}f}(\xi)\overline{\widehat{\mathcal{J}^{-s}g}(\xi)}d\xi \\
&+ \int_{\Omega_{\mathrm{II}}} \frac{k^2(1+|\xi|^2)^s}{|\xi^-|^2 - k^2t^2 + \xi_d^2 + 2\mathrm{i}k\sqrt{t^2+1}\xi_d}\widehat{\mathcal{J}^{-s}f}(\xi)\overline{\widehat{\mathcal{J}^{-s}g}(\xi)}d\xi \\
=&: \mathrm{I} + \mathrm{II}
\end{aligned}$$

with

$$\Omega_{\mathrm{I}} := \left\{\xi : ||\xi^-| - kt| > \frac{kt}{2}\right\} = \left\{\xi : |\xi^-| > \frac{3kt}{2} \text{ or } |\xi^-| < \frac{kt}{2}\right\}$$

and

$$\Omega_{\mathrm{II}} := \left\{\xi : ||\xi^-| - kt| < \frac{kt}{2}\right\} = \left\{\xi : \frac{kt}{2} < |\xi^-| < \frac{3kt}{2}\right\},$$

where the transformation of variables $(\xi_1 + kt, \xi_2, \cdots, \xi_d)^\top \mapsto (\xi_1, \xi_2, \cdots, \xi_d)^\top$ and the fact $\hat{f}(\xi_1 - kt, \xi_2, \cdots, \xi_d) = e^{-\mathrm{i}kt\xi_1}\hat{f}(\xi_1, \xi_2, \cdots, \xi_d)$ are used.

The first term I satisfies

$$
\begin{aligned}
|\mathrm{I}| \le & \int\limits_{\Omega_\mathrm{I}} \frac{k^2(1+|\xi|^2)^s}{\left[(|\xi^-|^2 - k^2t^2 + \xi_d^2)^2 + 4k^2(t^2+1)\xi_d^2\right]^{\frac12}} |\widehat{\mathcal{J}^{-s}f}||\widehat{\mathcal{J}^{-s}g}|d\xi \\
= & \int\limits_{\Omega_\mathrm{I}} \frac{k^2(1+|\xi|^2)^s}{\left[((|\xi^-| - kt)^2 + \xi_d^2)((|\xi^-| + kt)^2 + \xi_d^2) + 4k^2\xi_d^2\right]^{\frac12}} |\widehat{\mathcal{J}^{-s}f}||\widehat{\mathcal{J}^{-s}g}|d\xi \\
\le & \int\limits_{\Omega_\mathrm{I}} \frac{k^2(1+|\xi|^2)^s}{||\xi^-| - kt|((|\xi^-| + kt)^2 + \xi_d^2)^{\frac12}} |\widehat{\mathcal{J}^{-s}f}||\widehat{\mathcal{J}^{-s}g}|d\xi \\
\le & \frac{2k}{t}\Bigg[\int\limits_{\{\xi : |\xi^-| > \frac{3kt}{2}\}} \frac{(1+|\xi|^2)^s}{((|\xi^-| + kt)^2 + \xi_d^2)^{\frac12}} |\widehat{\mathcal{J}^{-s}f}||\widehat{\mathcal{J}^{-s}g}|d\xi \\
& + \int\limits_{\{\xi : |\xi^-| < \frac{kt}{2}, |\xi_d| < \frac{kt}{2}\}} \frac{(1+|\xi|^2)^s}{((|\xi^-| + kt)^2 + \xi_d^2)^{\frac12}} |\widehat{\mathcal{J}^{-s}f}||\widehat{\mathcal{J}^{-s}g}|d\xi \\
& + \int\limits_{\{\xi : |\xi^-| < \frac{kt}{2}, |\xi_d| > \frac{kt}{2}\}} \frac{(1+|\xi|^2)^s}{((|\xi^-| + kt)^2 + \xi_d^2)^{\frac12}} |\widehat{\mathcal{J}^{-s}f}||\widehat{\mathcal{J}^{-s}g}|d\xi \Bigg] \\
=: & \frac{2k}{t}\left[\mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3\right],
\end{aligned}
\tag{3.5}
$$

where in the third step we use the fact

$$
\begin{aligned}
& (|\xi^-|^2 - k^2t^2 + \xi_d^2)^2 + 4k^2(t^2+1)\xi_d^2 \\
= & \left(|\xi^-|^2 + \xi_d^2 + k^2t^2\right)^2 - 4k^2t^2|\xi^-|^2 + 4k^2\xi_d^2 \\
= & \left[(|\xi^-|^2 - kt)^2 + \xi_d^2\right]\left[(|\xi^-|^2 + kt)^2 + \xi_d^2\right] + 4k^2\xi_d^2.
\end{aligned}
$$

For sufficiently large $t > 0$, the following estimates hold:

$$
\mathrm{I}_1 \lesssim \int\limits_{\{\xi : |\xi| > \frac{3kt}{2}\}} \frac{1}{|\xi|^{1-2s}} |\widehat{\mathcal{J}^{-s}f}||\widehat{\mathcal{J}^{-s}g}|d\xi \lesssim \frac{1}{(kt)^{1-2s}} \|f\|_{H^{-s}(D)}\|g\|_{H^{-s}(D)},
$$

$$
\mathrm{I}_2 \lesssim \int\limits_{\{\xi : |\xi| < kt\}} \frac{(1+|\xi|^2)^s}{kt} |\widehat{\mathcal{J}^{-s}f}||\widehat{\mathcal{J}^{-s}g}|d\xi \lesssim \frac{1}{(kt)^{1-2s}} \|f\|_{H^{-s}(D)}\|g\|_{H^{-s}(D)}
$$

and

$$\mathrm{I}_3 \lesssim \int_{\{\xi:|\xi^-|<\frac{kt}{2},|\xi_d|>\frac{kt}{2}\}} \frac{(1+|\xi^-|^2+\xi_d^2)^s}{|\xi_d|} |\widehat{\mathcal{J}^{-s}f}||\widehat{\mathcal{J}^{-s}g}| d\xi$$
$$\lesssim \int_{\{\xi:|\xi^-|<\frac{kt}{2},|\xi_d|>\frac{kt}{2}\}} \left(\frac{|\xi^-|^{2s}}{|\xi_d|} + \frac{1}{|\xi_d|^{1-2s}}\right) |\widehat{\mathcal{J}^{-s}f}||\widehat{\mathcal{J}^{-s}g}| d\xi$$
$$\lesssim \frac{1}{(kt)^{1-2s}} \|f\|_{H^{-s}(D)} \|g\|_{H^{-s}(D)},$$

which, together with (3.5), lead to

$$|\mathrm{I}| \lesssim \frac{k^{2s}}{t^{2-2s}} \|f\|_{H^{-s}(D)} \|g\|_{H^{-s}(D)}. \tag{3.6}$$

For term II, a simple calculation yields

$$\mathrm{II} = \int_{\{\xi:\frac{kt}{2}<|\xi^-|<\frac{3kt}{2},|\xi_d|>\frac{kt}{2}\}} \frac{k^2(1+|\xi|^2)^s \widehat{\mathcal{J}^{-s}f}(\xi)\overline{\widehat{\mathcal{J}^{-s}g}(\xi)}}{|\xi^-|^2 - k^2t^2 + \xi_d^2 + 2\mathrm{i}k\sqrt{t^2+1}\xi_d} d\xi$$
$$+ \int_{\{\xi:\frac{kt}{2}<|\xi^-|<\frac{3kt}{2},|\xi_d|<\frac{kt}{2}\}} \frac{k^2(1+|\xi|^2)^s \widehat{\mathcal{J}^{-s}f}(\xi)\overline{\widehat{\mathcal{J}^{-s}g}(\xi)}}{|\xi^-|^2 - k^2t^2 + \xi_d^2 + 2\mathrm{i}k\sqrt{t^2+1}\xi_d} d\xi$$
$$=: \mathrm{II}_1 + \mathrm{II}_2, \tag{3.7}$$

where II_1 satisfies

$$|\mathrm{II}_1| \le \int_{\{\xi:\frac{kt}{2}<|\xi^-|<\frac{3kt}{2},|\xi_d|>\frac{kt}{2}\}} \frac{k^2(1+|\xi|^2)^s \left|\widehat{\mathcal{J}^{-s}f}\right| \left|\widehat{\mathcal{J}^{-s}g}\right|}{\left[(|\xi^-|^2 - k^2t^2 + \xi_d^2)^2 + 4k^2(t^2+1)\xi_d^2\right]^{\frac{1}{2}}} d\xi$$
$$\lesssim \int_{\{\xi:\frac{kt}{2}<|\xi^-|<\frac{3kt}{2},|\xi_d|>\frac{kt}{2}\}} \left(\frac{k^2|\xi^-|^{2s}}{kt|\xi_d|} + \frac{k^2}{kt|\xi_d|^{1-2s}}\right) \left|\widehat{\mathcal{J}^{-s}f}\right| \left|\widehat{\mathcal{J}^{-s}g}\right| d\xi$$
$$\lesssim \frac{k^{2s}}{t^{2-2s}} \|f\|_{H^{-s}(D)} \|g\|_{H^{-s}(D)}. \tag{3.8}$$

It then suffices to estimate II_2. Define a function

$$m_t(\xi) := \frac{k^2}{|\xi^-|^2 - k^2t^2 + \xi_d^2 + 2\mathrm{i}k\sqrt{t^2+1}\xi_d}$$

and the transformation of variables $\xi \mapsto \xi^* = (\xi', -\xi_d)$ with

$$\xi' = \left(\frac{2kt}{|\xi^-|} - 1\right)\xi^-$$

and the Jacobian

$$J_t(\xi) = \left|\det \frac{\partial \xi^*}{\partial \xi}\right| = \left(\frac{2kt}{|\xi^-|} - 1\right)^{d-2}$$

such that $|\xi'| = 2kt - |\xi^-|$. Clearly, the transformation maps the subdomain

$$\Omega_1 := \{\xi : \frac{kt}{2} < |\xi^-| < kt, |\xi_d| < \frac{kt}{2}\}$$

to the subdomain

$$\Omega_2 := \{\xi : kt < |\xi^-| < \frac{3kt}{2}, |\xi_d| < \frac{kt}{2}\}.$$

Hence, II_2 satisfies

$$\begin{aligned}
\mathrm{II}_2 &= \int\limits_{\Omega_1 \cup \Omega_2} m_t(\xi)(1+|\xi|^2)^s \widehat{\mathcal{J}^{-s} f}(\xi)\overline{\widehat{\mathcal{J}^{-s} g}(\xi)}d\xi \\
&= \int\limits_{\Omega_2} \Big[m_t(\xi)(1+|\xi|^2)^s \widehat{\mathcal{J}^{-s} f}(\xi)\overline{\widehat{\mathcal{J}^{-s} g}(\xi)} \\
&\quad + m_t(\xi^*)(1+|\xi^*|^2)^s \widehat{\mathcal{J}^{-s} f}(\xi^*)\overline{\widehat{\mathcal{J}^{-s} g}(\xi^*)} J_t(\xi)\Big]d\xi \\
&= \int\limits_{\Omega_2} \big[m_t(\xi) + m_t(\xi^*)J_t(\xi)\big](1+|\xi|^2)^s \widehat{\mathcal{J}^{-s} f}(\xi)\overline{\widehat{\mathcal{J}^{-s} g}(\xi)}d\xi \\
&\quad + \int\limits_{\Omega_2} m_t(\xi^*)J_t(\xi)\Big[(1+|\xi^*|^2)^s - (1+|\xi|^2)^s\Big]\widehat{\mathcal{J}^{-s} f}(\xi)\overline{\widehat{\mathcal{J}^{-s} g}(\xi)}d\xi \\
&\quad + \int\limits_{\Omega_2} m_t(\xi^*)J_t(\xi)(1+|\xi^*|^2)^s\big[\widehat{\mathcal{J}^{-s} f}(\xi^*) - \widehat{\mathcal{J}^{-s} f}(\xi)\big]\overline{\widehat{\mathcal{J}^{-s} g}(\xi)}d\xi \\
&\quad + \int\limits_{\Omega_2} m_t(\xi^*)J_t(\xi)(1+|\xi^*|^2)^s \widehat{\mathcal{J}^{-s} f}(\xi^*)\big[\overline{\widehat{\mathcal{J}^{-s} g}(\xi^*)} - \overline{\widehat{\mathcal{J}^{-s} g}(\xi)}\big]d\xi \\
&=: \mathrm{II}_{21} + \mathrm{II}_{22} + \mathrm{II}_{23} + \mathrm{II}_{24}.
\end{aligned}$$

For any $\xi \in \Omega_2$, we define the function

$$h(\xi_d^2) := |m_t(\xi) + m_t(\xi^*)J_t(\xi)|.$$

If $d = 2$, it can be easily shown that

$$
\begin{aligned}
h(\xi_d^2) :=& \left| \frac{k^2}{|\xi^-|^2 - k^2t^2 + \xi_d^2 + 2\mathrm{i}k\sqrt{t^2+1}\xi_d} + \frac{k^2}{|\xi'|^2 - k^2t^2 + \xi_d^2 - 2\mathrm{i}k\sqrt{t^2+1}\xi_d} \right| \\
=& \frac{k^2(|\xi^-|^2 + |\xi'|^2 - 2k^2t^2 + 2\xi_d^2)}{\left[(|\xi^-|^2 - k^2t^2 + \xi_d^2)^2 + 4k^2(t^2+1)\xi_d^2\right]^{\frac{1}{2}} \left[(|\xi'|^2 - k^2t^2 + \xi_d^2)^2 + 4k^2(t^2+1)\xi_d^2\right]^{\frac{1}{2}}}
\end{aligned}
$$

is decreasing with respect to $\xi_d^2 \in [0, \frac{k^2t^2}{4})$ and hence

$$
h(\xi_d^2) \le h(0) = \frac{2k^2}{(|\xi^-| + kt)(3kt - |\xi^-|)} \lesssim \frac{1}{t^2}.
$$

If $d = 3$, similarly, we have

$$
\begin{aligned}
h(\xi_d^2) \le& h(0) = k^2 \left| \frac{1}{|\xi^-|^2 - k^2t^2} + \frac{(\frac{2kt}{|\xi^-|} - 1)}{|\xi'|^2 - k^2t^2} \right| \\
=& \frac{k^2}{|\xi^-|} \frac{2kt}{(|\xi^-| + kt)(3kt - |\xi^-|)} \lesssim \frac{1}{t^2}.
\end{aligned}
$$

As a result, we obtain

$$
|\mathrm{II}_{21}| \lesssim \frac{1}{t^2} \int\limits_{\Omega_2} (1 + |\xi|^2)^s \left| \widehat{\mathcal{J}^{-s} f} \right| \left| \widehat{\mathcal{J}^{-s} g} \right| d\xi \lesssim \frac{k^{2s}}{t^{2-2s}} \|f\|_{H^{-s}(D)} \|g\|_{H^{-s}(D)}. \tag{3.9}
$$

By the mean value theorem, similar to the estimate of $h(\xi_d^2)$ above, we get for some $\theta \in (0, 1)$ that

$$
\begin{aligned}
& \left| m_t(\xi^*) J_t(\xi) \left[(1 + |\xi^*|^2)^s - (1 + |\xi|^2)^s \right] \right| \\
=& \left| m_t(\xi^*) J_t(\xi) s \left(1 + \theta|\xi^*|^2 + (1 - \theta)|\xi|^2 \right)^{s-1} (|\xi^*|^2 - |\xi|^2) \right| \\
=& \left| \frac{k^2(\frac{2kt}{|\xi^-|} - 1)^{d-2}(|\xi'|^2 - |\xi^-|^2)}{|\xi'|^2 - k^2t^2 + \xi_d^2 - 2\mathrm{i}k\sqrt{t^2+1}\xi_d} \right| s \left(1 + \theta|\xi^*|^2 + (1 - \theta)|\xi|^2 \right)^{s-1} \lesssim \frac{k^{2s}}{t^{2-2s}},
\end{aligned}
$$

which leads to

$$
|\mathrm{II}_{22}| \lesssim \frac{k^{2s}}{t^{2-2s}} \|f\|_{H^{-s}(D)} \|g\|_{H^{-s}(D)}. \tag{3.10}
$$

To estimate II_{23} and II_{24}, we employ the following characterization of $W^{1,p}(\mathbb{R}^d)$ introduced in [10].

Lemma 3.3. *For $1 < p \le \infty$, the function $u \in W^{1,p}(\mathbb{R}^d)$ if and only if there exist $g \in L^p(\mathbb{R}^d)$ and $C > 0$ such that*

$$|u(x) - u(y)| \le C|x - y|(g(x) + g(y)).$$

Moreover, we can choose $g = M(|\nabla u|)$, *where* M *is defined by*

$$M(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)| dy$$

and is called the Hardy–Littlewood maximal function of f.

For $f, g \in C_0^\infty(D)$, we have $\widehat{\mathcal{J}^{-s} f}, \widehat{\mathcal{J}^{-s} g} \in \mathcal{S}(\mathbb{R}^d) \subset H^1(\mathbb{R}^d)$. An application of Lemma 3.3 gives

$$\left|\widehat{\mathcal{J}^{-s} f}(\xi^*) - \widehat{\mathcal{J}^{-s} f}(\xi)\right| \lesssim \big||\xi^*| - |\xi|\big| \big[M(|\nabla \widehat{\mathcal{J}^{-s} f}|)(\xi^*) + M(|\nabla \widehat{\mathcal{J}^{-s} f}|)(\xi)\big],$$

where $M(|\nabla \widehat{\mathcal{J}^{-s} f}|)$ satisfies

$$\begin{aligned}
\|M(|\nabla \widehat{\mathcal{J}^{-s} f}|)\|_{L^2(\mathbb{R}^d)} &\lesssim \|\nabla \widehat{\mathcal{J}^{-s} f}\|_{L^2(\mathbb{R}^d)} \lesssim \|(I - \Delta)^{\frac{1}{2}} \widehat{\mathcal{J}^{-s} f}\|_{L^2(\mathbb{R}^d)} \\
&= \|(I - \Delta)^{\frac{1}{2}} (1 + |\cdot|^2)^{-\frac{s}{2}} \hat{f}(\cdot)\|_{L^2(\mathbb{R}^d)} \\
&= \|(I + |\cdot|)^{\frac{1}{2}} (1 - \Delta)^{-\frac{s}{2}} f(\cdot)\|_{L^2(\mathbb{R}^d)} \\
&\lesssim \|f\|_{H^{-s}(D)}
\end{aligned}$$

according to [22, Theorem 2.1], and the same for g. The above estimates then lead to

$$\begin{aligned}
|\mathrm{II}_{23}| &\lesssim \frac{k^{1+2s}}{t^{1-2s}} \int\limits_{\Omega_2} \big[M(|\nabla \widehat{\mathcal{J}^{-s} f}|)(\xi^*) + M(|\nabla \widehat{\mathcal{J}^{-s} f}|)(\xi)\big] |\widehat{\mathcal{J}^{-s} g}(\xi)| d\xi \\
&\lesssim \frac{k^{1+2s}}{t^{1-2s}} \|f\|_{H^{-s}(D)} \|g\|_{H^{-s}(D)}
\end{aligned}$$

and

$$|\mathrm{II}_{24}| \lesssim \frac{k^{1+2s}}{t^{1-2s}} \|f\|_{H^{-s}(D)} \|g\|_{H^{-s}(D)},$$

which, together with (3.9) and (3.10), yield

$$|\mathrm{II}_2| \lesssim \frac{(1+k)k^{2s}}{t^{1-2s}} \|f\|_{H^{-s}(D)} \|g\|_{H^{-s}(D)}. \tag{3.11}$$

We conclude from (3.6)–(3.8) and (3.11) that

$$|\langle G_\eta f, g\rangle| \lesssim \frac{(1+k)k^{2s}}{t^{1-2s}} \|f\|_{H^{-s}(D)} \|g\|_{H^{-s}(D)}$$

for any $f, g \in C_0^\infty(D)$, which can be easily extended to $f, g \in H^{-s}(D)$ by taking the procedure used in the estimate of (3.3). Hence, we get

$$\|G_\eta\|_{\mathcal{L}(H^{-s}(D),H^s(D))} \lesssim \frac{(1+k)k^{2s}}{t^{1-2s}},$$

which, together with [14, Proposition 2], leads to

$$\|G_\eta\|_{\mathcal{L}(W^{-\gamma,p}(D),W^{\gamma,q}(D))} \lesssim \frac{(1+k)^\theta k^{2s\theta}}{t^{(1-2s)\theta}},$$

where p satisfies $\frac{1}{p}+\frac{1}{q}=1$, $\theta=(\frac{1}{q}-\frac{1}{2})d+1$ and $\gamma=\theta s \in (0,(\frac{1}{q}-\frac{1}{2})\frac{d}{2}+\frac{1}{2})$. Utilizing (3.4) gives

$$\|v\|_{W^{\gamma,q}(D)} = \|G_\eta(\rho v)\|_{W^{\gamma,q}(D)} \lesssim \frac{(1+k)^\theta k^{2s\theta}}{t^{(1-2s)\theta}}\|\rho\|_{W^{-\gamma,\tilde{p}}(D)}\|v\|_{W^{\gamma,q}(D)},$$

where $\tilde{p}=\frac{p}{2-p}$. The proof is completed by letting $t\to\infty$. $\square$

Theorem 3.4. *Let ρ satisfy Assumption 1. Then the Lippmann–Schwinger equation (3.2) admits a unique solution $u \in W^{\gamma,q}_{loc}(\mathbb{R}^d)$ almost surely with $\gamma \in (\frac{d-m}{2},(\frac{1}{q}-\frac{1}{2})\frac{d}{2}+\frac{1}{2})$ and $q \in (2,\frac{2d}{3d-2-2m})$, where*

(i) *$m=m_\rho \wedge m_f$ if the condition (i) in Assumption 2 holds*

or

(ii) *$m=m_\rho$ if the condition (ii) in Assumption 2 holds.*

Proof. Let $G\subset\mathbb{R}^d$ be any bounded set with a locally Lipschitz boundary. Based on the definition of the operator K_k, the Lippmann–Schwinger equation can be written in the form

$$(I-k^2K_k)u=-H_kf, \tag{3.12}$$

where the operator $I-k^2K_k : W^{\gamma,q}(G)\to W^{\gamma,q}(G)$ is Fredholm according to Lemma 3.1. It follows from the Fredholm alternative theorem that (3.12) has a unique solution $u\in W^{\gamma,q}(G)$ if

$$(I-k^2K_k)u=0 \tag{3.13}$$

has only the trivial solution $u\equiv 0$, which has been proved in Theorem 3.2.

Next is to show $H_kf\in W^{\gamma,q}(G)$. We consider the following two cases:

If the condition (i) in Assumption 2 holds, for any q,γ given above and p satisfying $\frac{1}{p}+\frac{1}{q}=1$, there exist $\epsilon>0$ and $p'>1$ such that

$$f\in W_0^{\frac{m_f-d}{2}-\epsilon,p'}(D_f)\hookrightarrow W_0^{-\gamma,p}(D_f)$$

and hence $H_k f \in W^{\gamma,q}(G)$ according to Lemma 3.1.

If the condition (ii) in Assumption 2 holds, it holds $H_k f = \Phi_d(\cdot, y, k) \in W^{1,p'}(G)$ for any $p' \in (1, 3 - \frac{d}{2})$ according to [19, Lemma 3.1]. Then there exists $p' = 3 - \frac{d}{2} - \epsilon$ for a sufficiently small $\epsilon > 0$ satisfying $\frac{1}{p'} - \frac{1-\gamma}{d} < \frac{1}{q}$, such that

$$W^{1,p'}(G) \hookrightarrow W^{\gamma,q}(G),$$

and hence $H_k f = \Phi_d(\cdot, y, k) \in W^{\gamma,q}(G)$. □

3.2. Well-posedness

Now we present the well-posedness on the solution of (1.1)–(1.2) in the distribution sense by showing the equivalence to the Lippmann–Schwinger equation.

Theorem 3.5. *Let ρ satisfy Assumption 1. The acoustic scattering problem (1.1)–(1.2) is well-defined in the distribution sense, and admits a unique solution $u \in W^{\gamma,q}_{loc}(\mathbb{R}^d)$ almost surely with $\gamma \in (\frac{d-m}{2}, (\frac{1}{q} - \frac{1}{2})\frac{d}{2} + \frac{1}{2})$ and $q \in (2, \frac{2d}{3d-2-2m})$, where*

(i) *$m = m_\rho \wedge m_f$ if the condition (i) in Assumption 2 holds*

or

(ii) *$m = m_\rho$ if the condition (ii) in Assumption 2 holds.*

Proof. First we show the existence of the solution of (1.1)–(1.2). Specifically, we show that the solution of the Lippmann–Schwinger equation (3.2) is also a solution of (1.1)–(1.2) in the distribution sense. Suppose that $u^* \in W^{\gamma,q}_{loc}(\mathbb{R}^d)$ is the solution of (3.2) and satisfies

$$u^*(x) - k^2 \int_{\mathbb{R}^d} \Phi_d(x, z, k)\rho(z)u^*(z)dz = -\int_{\mathbb{R}^d} \Phi_d(x, z, k) f(z)dz, \quad x \in \mathbb{R}^d.$$

Note that the Green tensor Φ_d is the fundamental solution for the operator $\Delta + k^2 I$:

$$(\Delta + k^2)\Phi_d(\cdot, z, k) = -\delta(\cdot - z),$$

where the Dirac delta function δ is a distribution, i.e., $\delta \in \mathcal{D}'$. It indicates that, for any $\psi \in \mathcal{D}$,

$$\langle (\Delta + k^2)\Phi_d(\cdot, z, k), \psi\rangle = -\langle \delta(\cdot - z), \psi\rangle = -\overline{\psi(y)}.$$

We obtain for any $\psi \in \mathcal{D}$ that

$$\begin{aligned} &\langle \Delta u^* + k^2 u^* + k^2\rho u^*, \psi\rangle \\ &= k^2 \left\langle \int_{\mathbb{R}^d} \left(\Delta + k^2\right) \Phi_d(\cdot, z, k)\rho(z)u^*(z)dz, \psi \right\rangle \end{aligned}$$

$$
\begin{aligned}
&-\left\langle \int\limits_{\mathbb{R}^d} \left(\Delta + k^2\right) \Phi_d(x,z,k) f(z) dz, \psi \right\rangle + k^2 \langle \rho u^*, \psi\rangle \\
&= k^2 \int\limits_{\mathbb{R}^d} \rho(z) u^*(z) \Big\langle \left(\Delta + k^2\right) \Phi_d(\cdot, z, k), \psi \Big\rangle dz \\
&\quad - \int\limits_{\mathbb{R}^d} f(z) \Big\langle \left(\Delta + k^2\right) \Phi_d(\cdot, y, k), \psi \Big\rangle dz + k^2 \langle \rho u^*, \psi\rangle \\
&= -k^2 \int\limits_{\mathbb{R}^d} \rho(z) u^*(z) \overline{\psi(z)} dz + \int\limits_{\mathbb{R}^d} f(z) \overline{\psi(z)} dz + k^2 \langle \rho u^*, \psi \rangle \\
&= \langle f, \psi \rangle.
\end{aligned}
$$

Hence, $u^* \in W^{\gamma,q}_{loc}(\mathbb{R}^d)$ is also a solution of (1.1)–(1.2) in the distribution sense, which shows the existence of the solution of (1.1)–(1.2) according to Theorem 3.4.

The uniqueness of the solution of (1.1)–(1.2) can be proved by using the same procedure as that of the Lippmann–Schwinger equation. Let u_0 be any solution of (1.1) with $f = 0$ in the distribution sense. It then suffices to show that u_0 is also a solution of (3.13) with $f = 0$, i.e., $u_0 \equiv 0$. In fact, u_0 satisfies

$$
\Delta u_0 + k^2 u_0 = -k^2 \rho u_0
$$

in the distribution sense, where $\rho \in W^{-\gamma,\tilde{p}}(D_\rho)$, $u_0 \in W^{\gamma,q}_{loc}(\mathbb{R}^d)$ and hence $\rho u_0 \in W_0^{-\gamma,p}(D_\rho)$ with $\tilde{p} = \frac{p}{2-p}$ according to the proof of Lemma 3.1. Let B_r be an open ball with radius r large enough such that $D_\rho \subset B_r$.

Moreover, it has been shown in Theorem 3.4 that $\Phi_d(\cdot, y, k) \in W^{1,p'}(B_r) \hookrightarrow W^{\gamma,q}(B_r)$ with $p' = 3 - \frac{d}{2} - \epsilon$ for a sufficiently small $\epsilon > 0$. It then indicates that

$$
\begin{aligned}
&\int\limits_{B_r} \Phi_d(x,z,k) \left[\Delta u_0(z) + k^2 u_0(z)\right] dz \\
&\quad = -k^2 \int\limits_{B_r} \Phi_d(x,z,k) \rho(z) u_0(z) dz. \qquad (3.14)
\end{aligned}
$$

Define the operator T by

$$
(T\psi)(x) := \int\limits_{B_r} \Phi_d(x,z,k) \left[\Delta \psi(z) + k^2 \psi(z)\right] dz
$$

for $\psi \in \mathcal{D}$. By the similar arguments as those in the proof of [15, Lemma 4.3], we obtain

$$
(T\psi)(x) = -\psi(x) + \int\limits_{\partial B_r} \left[\Phi_d(x,z,k) \partial_\nu \psi(z) - \partial_\nu \Phi_d(x,z,k) \psi(z)\right] ds(z),
$$

where ν is the unit outward normal vector on the boundary ∂B_r. Then (3.14) turns to be

$$u_0(x) - \int\limits_{\partial B_r} \Big[\Phi_d(x,z,k)\partial_\nu u_0(z) - \partial_\nu \Phi_d(x,z,k)u_0(z)\Big]ds(z)$$
$$= k^2 \int\limits_{B_r} \Phi_d(x,z,k)\rho(z)u_0(z)dz.$$

Let $r \to \infty$ and applying the radiation condition, we get

$$u_0(x) = k^2 \int\limits_{\mathbb{R}^d} \Phi_d(x,z,k)\rho(z)u_0(z)dz,$$

which implies that u_0 is also a solution of the Lippmann–Schwinger equation (3.2) with $f = 0$, and hence $u_0 \equiv 0$ according to Theorem 3.4. $\square$

4. The elastic scattering problem

In this section, we discuss the well-posedness of the elastic wave equation (1.3) in the distribution sense, where the medium $\boldsymbol{M}$ and the source $\boldsymbol{f}$ satisfy the following assumptions.

Assumption 3. Let the medium $\boldsymbol{M} = (M_{ij})_{d\times d}$ be a $\mathbb{R}^{d\times d}$-valued and centered microlocally isotropic Gaussian random field of order $-m_{\boldsymbol{M}}$ with $m_{\boldsymbol{M}} \in (d-1, d]$ in a bounded domain $D_{\boldsymbol{M}} \subset \mathbb{R}^d$. The principal symbol of the covariance operator of each component M_{ij} has the form $\phi_{ij}(x)|\xi|^{-m_{\boldsymbol{M}}}$ with $\phi_{ij} \in C_0^\infty(D_{\boldsymbol{M}})$, $\phi_{ij} \ge 0$ and $i, j = 1, \cdots, d$.

Remark 4.1. For a random medium $\boldsymbol{M} = (M_{ij})_{d\times d}$, if the components are centered microlocally isotropic Gaussian random fields of different orders, denoted by $-m_{ij}$, then $\boldsymbol{M}$ satisfies Assumption 3 with $m_{\boldsymbol{M}} := \min_{i,j\in\{1,\cdots,d\}} m_{ij}$. Moreover, for any component M_{ij} with $m_{ij} > m_{\boldsymbol{M}}$, it holds $\phi_{ij} \equiv 0$.

Assumption 4. Let the $\mathbb{R}^d$-valued source $\boldsymbol{f}$ satisfy one of the following assumptions:

(i) $\boldsymbol{f}$ is a centered microlocally isotropic Gaussian random vector field of order $-m_{\boldsymbol{f}}$ with $m_{\boldsymbol{f}} \in (d-1, d]$ in a bounded domain $D_{\boldsymbol{f}} \subset \mathbb{R}^d$. The principal symbol of its covariance operator has the form $A_{\boldsymbol{f}}(x)|\xi|^{-m_{\boldsymbol{f}}}$ with $A_{\boldsymbol{f}} \in C_0^\infty(D_{\boldsymbol{f}}; \mathbb{R}^{d\times d})$.
(ii) $\boldsymbol{f} = -\delta(\cdot - y)\boldsymbol{a}$ is a point source with $y \in \mathbb{R}^d$ and some fixed vector $\boldsymbol{a} \in \mathbb{R}^d$.

In the sequel, we denote by

$$\boldsymbol{X} := X^d = \{\boldsymbol{g} = (g_1, \cdots, g_d)^\top : g_j \in X \ \forall\, j = 1, \cdots, d\}$$

the Cartesian product vector space, and use notations $\boldsymbol{W}^{r,p} := (W^{r,p}(\mathbb{R}^d))^d$ and $\boldsymbol{H}^r := \boldsymbol{W}^{r,2}$ for simplicity.

4.1. The Lippmann–Schwinger equation

Similarly, we consider the equivalent Lippmann–Schwinger equation for the elastic wave scattering problem. Denote by $\boldsymbol{\Phi}_d(x,y,k) \in \mathbb{C}^{d\times d}$ the Green tensor for the Navier equation which has the following form:

$$\boldsymbol{\Phi}_d(x,y,k) = \frac{1}{\mu}\Phi_d(x,y,\kappa_{\rm s})\boldsymbol{I} + \frac{1}{\omega^2}\nabla_x\nabla_x^\top\Big[\Phi_d(x,y,\kappa_{\rm s}) - \Phi_d(x,y,\kappa_{\rm p})\Big], \tag{4.1}$$

where $\boldsymbol{I}$ is the $d\times d$ identity matrix and Φ_d is the fundamental solution for the Helmholtz equation and is defined in (3.1).

Based on the Green tensor $\boldsymbol{\Phi}_d$ given in (4.1), the Lippmann–Schwinger equation has the form

$$\boldsymbol{u}(x) - k^2\int_{\mathbb{R}^d}\boldsymbol{\Phi}_d(x,z,k)\boldsymbol{M}(z)\boldsymbol{u}(z)dz = -\int_{\mathbb{R}^d}\boldsymbol{\Phi}_d(x,z,k)\boldsymbol{f}(z)dz. \tag{4.2}$$

Define two operators $\boldsymbol{H}_k$ and $\boldsymbol{K}_k$ by

$$(\boldsymbol{H}_k\boldsymbol{v})(x) = \int_{\mathbb{R}^d}\boldsymbol{\Phi}_d(x,z,k)\boldsymbol{v}(z)dz,$$
$$(\boldsymbol{K}_k\boldsymbol{v})(x) = \int_{\mathbb{R}^d}\boldsymbol{\Phi}_d(x,z,k)\boldsymbol{M}(z)\boldsymbol{v}(z)dz,$$

which have the following properties.

Lemma 4.2. *Let $\boldsymbol{M}$ satisfy Assumption 3. Let $D\subset\mathbb{R}^d$ be a bounded set and $G\subset\mathbb{R}^d$ be a bounded set with a locally Lipschitz boundary.*

(i) *The operator $\boldsymbol{H}_k : \boldsymbol{H}_0^{-\beta}(D) \to \boldsymbol{H}^\beta(G)$ is bounded for any $\beta\in(0,1]$.*
(ii) *The operator $\boldsymbol{H}_k : \boldsymbol{W}_0^{-\gamma,p}(D) \to \boldsymbol{W}^{\gamma,q}(G)$ is compact for any $q\in(0,\infty)$, $\gamma\in(0,(\frac{1}{q}-\frac{1}{2})d+1)$ and p satisfying $\frac{1}{p}+\frac{1}{q}=1$.*
(iii) *The operator $\boldsymbol{K}_k : \boldsymbol{W}^{\gamma,q}(G) \to \boldsymbol{W}^{\gamma,q}(G)$ is compact almost surely for any $q\in(2,\frac{2d}{2d-2-m_{\boldsymbol{M}}})$ and $\gamma\in(\frac{d-m_{\boldsymbol{M}}}{2},(\frac{1}{q}-\frac{1}{2})d+1)$.*

Proof. Noting that the linear operator in (1.3) is uniformly elliptic and $\boldsymbol{H}_k$ is bounded from $\boldsymbol{C}^{0,\alpha}(D)$ to $\boldsymbol{C}^{2,\alpha}(G)$ (cf. [8, Theorem 6.8]), we may obtain the result in (i) by following essentially the same procedure as that for Lemma 3.1. The details are omitted for brevity.

The proof of (ii) can also be obtained by using the same procedure as the proof of Lemma 3.1 and noting that the embeddings

$$\boldsymbol{W}_0^{-\gamma,p}(D) \hookrightarrow \boldsymbol{H}_0^{-\beta}(D), \quad \boldsymbol{H}^\beta(G) \hookrightarrow \boldsymbol{W}^{\gamma,q}(G)$$

hold by choosing $\beta=1$ such that $\gamma<\beta$ and $\frac{1}{2}-\frac{\beta-\gamma}{d}<\frac{1}{q}$.

It then suffices to show (iii). Note that $\boldsymbol{M} \in (W^{\frac{m_{\boldsymbol{M}}-d}{2}-\epsilon,p'})^{d\times d}$ almost surely for any $\epsilon > 0$ and $p' > 1$ according to Lemma 2.2, and there must exist $\epsilon > 0$ and $p' > 1$ such that the embedding

$$W_0^{\frac{m_{\boldsymbol{M}}-d}{2}-\epsilon,p'}(D_{\boldsymbol{M}}) \hookrightarrow W_0^{-\gamma,\tilde{p}}(D_{\boldsymbol{M}})$$

holds according to the Kondrachov compact embedding theorem with $\tilde{p} = \frac{p}{2-p}$. Hence, $\boldsymbol{M} \in (W^{-\gamma,\tilde{p}}(\mathbb{R}^d))^{d\times d}$ and $\boldsymbol{M}\boldsymbol{v} \in \boldsymbol{W}^{-\gamma,p}$ almost surely for any $\boldsymbol{v} \in \boldsymbol{W}^{\gamma,q}(G)$ with

$$\|\boldsymbol{M}\boldsymbol{v}\|_{\boldsymbol{W}^{-\gamma,p}} \lesssim \|\boldsymbol{M}\|_{(W^{-\gamma,\tilde{p}}(\mathbb{R}^d))^{d\times d}}\|\boldsymbol{v}\|_{\boldsymbol{W}^{\gamma,q}}$$

according to [14, Lemma 2], where

$$\|\boldsymbol{M}\|_{(W^{-\gamma,\tilde{p}})^{d\times d}} := \left(\sum_{i,j=1}^{d}\|M_{ij}\|^2_{W^{-\gamma,\tilde{p}}(\mathbb{R}^d)}\right)^{\frac{1}{2}}, \quad \|\boldsymbol{v}\|_{\boldsymbol{W}^{\gamma,q}} := \left(\sum_{j=1}^{d}\|v_j\|^2_{W^{\gamma,q}(\mathbb{R}^d)}\right)^{\frac{1}{2}}$$

for any $\boldsymbol{v} = (v_1, \cdots, v_d)^\top$. As a result, for any $\boldsymbol{v} \in \boldsymbol{W}^{\gamma,q}(G)$, we get $\boldsymbol{K}_k v = \boldsymbol{H}_k(\boldsymbol{M}\boldsymbol{v}) \in \boldsymbol{W}^{\gamma,q}(G)$ almost surely, which completes the proof. □

Theorem 4.3. *Let $\boldsymbol{M}$ satisfy Assumption 3. Then the Lippmann–Schwinger equation (4.2) admits a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{\gamma,q}_{loc}$ almost surely with $\gamma \in (\frac{d-m}{2}, (\frac{1}{q}-\frac{1}{2})\frac{d}{2}+\frac{1}{2})$ and $q \in (2, \frac{2d}{3d-2-2m})$, where*

(i) *$m = m_{\boldsymbol{M}} \wedge m_{\boldsymbol{f}}$ if the condition (i) in Assumption 4 holds*

or

(ii) *$m = m_{\boldsymbol{M}}$ if the condition (ii) in Assumption 4 holds.*

Note that, according to the Helmholtz decomposition, the solution $\boldsymbol{u}$ of the homogeneous elastic wave equation with $\boldsymbol{f} \equiv 0$ in (1.3) can be decomposed into the compressional part $\boldsymbol{u}_{\rm p}$ and the shear part $\boldsymbol{u}_{\rm s}$ such that $\boldsymbol{u} = \boldsymbol{u}_{\rm p} + \boldsymbol{u}_{\rm s}$. Both $\boldsymbol{u}_{\rm p}$ and $\boldsymbol{u}_{\rm s}$ satisfy the Helmholtz equation with Sommerfeld radiation condition (cf. [16]). Hence, the proof of Theorem 4.3 can be obtained similarly by following the same procedure as the proof of Theorem 3.4 utilizing the fact $\boldsymbol{\Phi}_d(\cdot, y, k) \in (W^{1,p'}(G))^{d\times d}$ with $p' \in (1, 3-\frac{d}{2})$ shown in [15, Lemma 4.1]. The details are omitted for brevity.

4.2. Well-posedness

Now we present the existence and uniqueness of the solution of (1.3)–(1.4) in the distribution sense by utilizing the Lippmann–Schwinger equation for the elastic scattering problem.

Theorem 4.4. *Let $\boldsymbol{M}$ satisfy Assumption 3. The elastic scattering problem (1.3)–(1.4) is well-defined in the distribution sense, and admits a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{\gamma,q}_{loc}$ almost surely with $\gamma \in (\frac{d-m}{2}, (\frac{1}{q}-\frac{1}{2})\frac{d}{2}+\frac{1}{2})$ and $q \in (2, \frac{2d}{3d-2-2m})$, where*

(i) *$m = m_{\boldsymbol{M}} \wedge m_{\boldsymbol{f}}$ if the condition (i) in Assumption 4 holds*

or

(ii) *$m = m_{\boldsymbol{M}}$ if the condition (ii) in Assumption 4 holds.*

Proof. To show the existence of the solution, we first show that the solution of the Lippmann–Schwinger equation (4.2) is also a solution of (1.3)–(1.4) in the distribution sense. Suppose that $\boldsymbol{u}^* \in \boldsymbol{W}_{loc}^{\gamma,q}$ is the solution of (4.2) and satisfies

$$\boldsymbol{u}^*(x) - k^2 \int_{\mathbb{R}^d} \boldsymbol{\Phi}_d(x,z,k)\boldsymbol{M}(z)\boldsymbol{u}^*(z)dz = -\int_{\mathbb{R}^d} \boldsymbol{\Phi}_d(x,z,k)\boldsymbol{f}(z)dz, \quad x \in \mathbb{R}^d.$$

The Green tensor $\boldsymbol{\Phi}_d$ is the fundamental solution of the equation

$$(\Delta^* + k^2)\boldsymbol{\Phi}_d(\cdot, y, k) = -\delta(\cdot - y)\boldsymbol{I},$$

where $\Delta^* := \mu\Delta + (\mu+\lambda)\nabla\nabla\cdot$. For any $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}$, it is easy to note that

$$\langle (\Delta^* + k^2)\boldsymbol{\Phi}_d(\cdot, y, k), \boldsymbol{\psi}\rangle = -\langle \delta(\cdot - y)\boldsymbol{I}, \boldsymbol{\psi}\rangle = -\overline{\boldsymbol{\psi}(y)}.$$

Hence, for any $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}$, we get

$$\begin{aligned}
&\langle \Delta^*\boldsymbol{u}^* + k^2(\boldsymbol{I}+\boldsymbol{M})\boldsymbol{u}^*, \boldsymbol{\psi}\rangle \\
&= k^2\left\langle \int_{\mathbb{R}^d} \left(\Delta^* + k^2\right)\boldsymbol{\Phi}_d(\cdot, z, k)\boldsymbol{M}(z)\boldsymbol{u}^*(z)dz, \boldsymbol{\psi}\right\rangle \\
&\quad - \left\langle \int_{\mathbb{R}^d} \left(\Delta^* + k^2\right)\boldsymbol{\Phi}_d(\cdot, z, k)\boldsymbol{f}(z)dz, \boldsymbol{\psi}\right\rangle + k^2\langle \boldsymbol{M}\boldsymbol{u}^*, \boldsymbol{\psi}\rangle \\
&= k^2 \int_{\mathbb{R}^d} \left(\boldsymbol{M}(z)\boldsymbol{u}^*(z)\right)^\top \left\langle \left(\Delta^* + k^2\right)\boldsymbol{\Phi}_d(\cdot, z, k), \boldsymbol{\psi}\right\rangle dz \\
&\quad - \int_{\mathbb{R}^d} \boldsymbol{f}(z)^\top \left\langle \left(\Delta^* + k^2\right)\boldsymbol{\Phi}_d(\cdot, z, k), \boldsymbol{\psi}\right\rangle dz + k^2\langle \boldsymbol{M}\boldsymbol{u}^*, \boldsymbol{\psi}\rangle \\
&= -k^2 \int_{\mathbb{R}^d} \left(\boldsymbol{M}(z)\boldsymbol{u}^*(z)\right)^\top \overline{\boldsymbol{\psi}(z)}dz + \int_{\mathbb{R}^d} \boldsymbol{f}(z)^\top \overline{\boldsymbol{\psi}(z)}dz + k^2\langle \boldsymbol{M}\boldsymbol{u}^*, \boldsymbol{\psi}\rangle \\
&= \langle \boldsymbol{f}, \boldsymbol{\psi}\rangle.
\end{aligned}$$

Hence, $\boldsymbol{u}^* \in \boldsymbol{W}_{loc}^{\gamma,q}$ is also a solution of (1.3)–(1.4) in the distribution sense, which shows the existence of the solution of (1.3)–(1.4) according to Theorem 4.3.

The uniqueness of the solution of (1.3)–(1.4) is obtained based on the same procedure as the proof of Theorem 3.5. □

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