Optimal strong convergence rate of a backward Euler type scheme for the Cox–Ingersoll–Ross model driven by fractional Brownian motion

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Abstract

In this paper, we investigate the optimal strong convergence rate of numerical approximations for the Cox–Ingersoll–Ross model driven by fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. To deal with the difficulties caused by the unbounded diffusion coefficient, we study an auxiliary equation based on Lamperti transformation. By means of Malliavin calculus, we prove that the backward Euler scheme applied to this auxiliary equation ensures the positivity of the numerical solution, and is of strong order one. Furthermore, a numerical approximation for the original model is obtained and converges with the same order.
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1. Introduction

The Cox–Ingersoll–Ross (CIR) model

$$dr(t) = \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dB(t), \quad t \in (0, T] \tag{1}$$

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is proposed in [4] to study the interest rate dynamics and is also used in the Heston model to describe the stochastic volatility [11]. The constants θ, σ and κ characterize the long-term mean, the volatility and the speed of adjustment, respectively. In the classical case, B is assumed to be the standard Brownian motion and there is an amount of articles devoted to numerical approximations and their strong convergence rates for (1); see [1–3,6,10,12,18,24] and references therein. According to the memory phenomena in the real market, an appropriate modification (see e.g. [8,16,20,22]) for the CIR model is to replace the standard Brownian motion by the fractional Brownian motion (fBm) with Hurst parameter H. The goal of this paper is to give the first result on strong convergence rate for the numerical approximation of (1) when $H \in (\frac{1}{2}, 1)$ in which case (1) is understood as a pathwise Riemann–Stieltjes integral equation.

Comparing with the standard Brownian setting, two main difficulties in studying equations driven by fBms are the correlation between increments and the lack of martingale property, which make the fundamental convergence theorem (see [21]) invalid in the numerical analysis. To deal with the above difficulties and to gain the strong convergence rates in multiplicative noise cases with $H \in (1/2, 1)$, a priori estimates for numerical solutions in Hölder spaces are needed (see e.g., [7,13–15]). It leads to an essential assumption that the coefficients are bounded, which is even not satisfied in general by globally Lipschitz ones.

To deal with unbounded or non-globally Lipschitz diffusion coefficients in (1), we use the Lamperti transformation $X(t) = \sqrt{r(t)}$ and study the following auxiliary equation with a singular drift and an additive noise

$$dX(t) = \frac{1}{2}\kappa \left(\frac{\theta}{X(t)} - X(t) \right) dt + \frac{1}{2}\sigma \, dB(t), \quad t \in (0, T]. \tag{2}$$

Note that the backward Euler scheme applied to (2) has an explicit expression which inherits the positivity of the exact solution, due to the special structure of the auxiliary equation. In this sense, it is more efficient to apply the backward Euler scheme to (2) than to (1). We refer to [1,2,6,24] for the study of the standard Brownian case. For the fractional Brownian case with $H \in (\frac{1}{2}, 1)$, we give the Malliavin derivative of the exact solution and the boundedness of the inverse moments of the exact solution, utilizing the techniques in [16]. Based on these a priori estimates, we prove that the backward Euler scheme applied to (2), as well as the corresponding numerical approximation to (1), converges with order one in the uniformly strong sense. Then the boundedness of the inverse moments of the numerical solution is also obtained.

Combining with the piecewise linear interpolation, we get that the continuous interpolation of the numerical solution also admits positive values on $[0, T]$. We obtain that the strong convergence rate of this continuous interpolation in $L^\infty([0, T]; L^p(\Omega; \mathbb{R}))$ is $O\left(h^H \sqrt{\log \frac{T}{h}}\right)$, where h is the time step size. This rate matches the results for equations driven by additive fBms in the cases of the globally Lipschitz coefficient [23] and the one-sided dissipative Lipschitz coefficient with polynomial growth [9]. The strong convergence rate is optimal in the sense that it achieves the Hölder regularity of fBms. In addition, we derive the Malliavin derivative of the numerical solution based on the specific form of the backward Euler scheme, which serves as a counterpart with that of the exact solution.

The paper is organized as follows. In Section 2, we introduce properties and Malliavin calculus with respect to the fBm. We give the well-posedness and the boundedness of moments of the exact solution in Section 3. The strong convergence rates of the backward Euler type scheme and the continuous interpolation are obtained in Section 4, as well as the boundedness of the inverse moments and the Malliavin derivative of the numerical solution. We present numerical experiments in Section 5 to confirm our theoretical analysis.

2. Preliminaries

In this section, we introduce some definitions and Malliavin calculus with respect to the fBm. We refer to [5,25] for more details. Throughout the paper, we use C as a generic constant and use $C(\cdot)$ if necessary to mention the parameters it depends on.

The fBm $\{B(t)\}_{t \in [0,T]}$ on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is a centered Gaussian process with continuous sample paths and covariance

$$Cov(t, s) := \mathbb{E}\Big[B(t)B(s)\Big] = \frac{1}{2}\Big(t^{2H} + s^{2H} - |t - s|^{2H}\Big), \quad \forall \, s, t \in [0, T],$$

where $H \in (0, 1)$ is the Hurst parameter. As a consequence, we have the H-Hölder regularity of the fBm in $L^p(\Omega; \mathbb{R})$, i.e.,

$$\sup_{0 \le s < t \le T} \frac{\|B(t) - B(s)\|_{L^p(\Omega;\mathbb{R})}}{|t - s|^H} \le C(p), \quad \forall \, p \ge 1.$$

Further, we also have the $(H - \varepsilon)$-Hölder regularity of continuous trajectories, i.e., for all $0 < \varepsilon < H$, there exists a nonnegative random variable $G(\varepsilon, T) \in L^p(\Omega; \mathbb{R})$, $p \ge 1$, such that

$$\sup_{0 \le s < t \le T} \frac{|B(t) - B(s)|}{|t - s|^{H - \varepsilon}} \le G(\varepsilon, T), \quad \text{a.s.}$$

In the following, we always assume $H > 1/2$. In this case,

$$Cov(t, s) = \int_0^t \int_0^s \phi(\tau, u) du d\tau$$

with $\phi(\tau, u) := \alpha_H |u - \tau|^{2H-2}$ and $\alpha_H := H(2H - 1) > 0$. Define an inner product by

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathscr{H}} := Cov(t, s).$$

Denote by $(\mathscr{H}, \langle \cdot, \cdot \rangle_{\mathscr{H}})$ the Hilbert space which is the closure of the space of all step functions on $[0, T]$ with respect to $\langle \cdot, \cdot \rangle_{\mathscr{H}}$. The map $\mathbf{1}_{[0,t]} \mapsto B(t)$ can be extended to an isometry from $\mathscr{H}$ onto the Gaussian space $\mathscr{H}_1$ associated with B ([25, Section 5]). More precisely, denote this isometry by $\varphi \mapsto B(\varphi)$, then for any $\varphi, \psi \in \mathscr{H}$,

$$\langle \varphi, \psi \rangle_{\mathscr{H}} = \mathbb{E}\Big[B(\varphi)B(\psi)\Big].$$

Introduce the kernel $K_H(t, s) := c_H s^{\frac{1}{2} - H} \int_s^t (u - s)^{H - \frac{3}{2}} u^{H - \frac{1}{2}} du \mathbf{1}_{\{s < t\}}$ with $c_H := \sqrt{\frac{\alpha_H}{\beta(2 - 2H, H - \frac{1}{2})}}$ and $\beta(\cdot, \cdot)$ being the Beta function. Then the covariance has the expression

$$Cov(t, s) = \int_0^{t \wedge s} K_H(t, u) K_H(s, u) du.$$

Define the operator K^* from $\mathscr{H}$ to $L^2([0, T]; \mathbb{R})$ by $(K^*\varphi)(s) := \int_s^T \varphi(t) \frac{\partial K_H}{\partial t}(t, s) dt$. Another isometry is then obtained between $\mathscr{H}$ and a closed subspace of $L^2([0, T]; \mathbb{R})$ ([25, Section 5]), i.e.,

$$\langle \varphi, \psi \rangle_{\mathscr{H}} = \langle K^*\varphi, K^*\psi \rangle_{L^2([0,T];\mathbb{R})}. \tag{3}$$

Define the operator K by $(K\upsilon)(t) := \int_0^t K_H(t, s)\upsilon(s) ds$, for $\upsilon \in L^2([0, T]; \mathbb{R})$. Then we have that the image space $\mathscr{H}_H := K(L^2([0, T]; \mathbb{R}))$ is the associated Cameron–Martin space [5]. The injection from $\mathscr{H}$ to $\mathscr{H}_H$ is denoted by $R_H := K \circ K^*$, i.e.,

$$(R_H\varphi)(t) := \int_0^t K_H(t, s)(K^*\varphi)(s) ds.$$

For any random variable

$$F = f(B(t_1), \ldots, B(t_N)), \tag{4}$$

where $N \in \mathbb{N}_+$, $t_1, \ldots, t_N \in [0, T]$ and $f \in \mathscr{C}_b^\infty(\mathbb{R}^N)$ is bounded smooth function with all derivatives bounded, the derivative operator D on F is defined by

$$D.F := \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(B(t_1), \ldots, B(t_N)) \mathbf{1}_{[0,t_i]}(\cdot),$$

which is an $\mathscr{H}$-valued random variable. The Sobolev space $\mathbb{D}^{1,p}$, $p \geq 1$, is the closure of the set containing all random variables in the form of (4) with respect to the norm

$$\|F\|_{\mathbb{D}^{1,p}} := \left(\mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_{\mathscr{H}}^p] \right)^{\frac{1}{p}}.$$

Consider the adjoint operator δ of the derivative operator D. If an $\mathscr{H}$-valued random variable $\varphi \in L^2(\Omega; \mathscr{H})$ satisfies

$$|\mathbb{E}[\langle \varphi, DF \rangle_{\mathscr{H}}]| \leq C(\varphi) \|F\|_{L^2(\Omega; \mathbb{R})}, \quad \forall F \in \mathbb{D}^{1,2},$$

then $\varphi \in \text{Dom}(\delta)$ and $\delta(\varphi) \in L^2(\Omega; \mathbb{R})$ is characterized by

$$\mathbb{E}[\langle \varphi, DF \rangle_{\mathscr{H}}] = \mathbb{E}[F \delta(\varphi)], \quad \forall F \in \mathbb{D}^{1,2}.$$

In fact, the definition of $\mathbb{D}^{1,p}$ can be extended to $\mathbb{D}^{1,p}(\mathscr{H})$ for $\mathscr{H}$-valued random variables and the space $\mathbb{D}^{1,2}(\mathscr{H}) \subset \text{Dom}(\delta)$ ([25, Section 1.3]).

The Skorohod integral of φ with respect to the fBm is defined by $\int_0^T \varphi(t) \delta B(t) := \delta(\varphi)$ and we have $\mathbb{E}\left[\int_0^T \varphi(t) \delta B(t) \right] = 0$. The following lemma gives a precise estimate for the Skorohod integral in $L^p(\Omega; \mathbb{R})$.

Lemma 2.1 ([25, Section 5.2.2]). *Let* $p > 1$. *Denote* $\mathbb{D}^{1,p}(|\mathscr{H}|) := \{\varphi \in \mathbb{D}^{1,p}(\mathscr{H}) : \|\mathbb{E}[\varphi]\|_{|\mathscr{H}|}^p + \mathbb{E}\left[\|D\varphi\|_{|\mathscr{H}| \otimes |\mathscr{H}|}^p \right] < \infty\}$, *where*

$$\|\varphi\|_{|\mathscr{H}|}^2 := \int_{[0,T]^2} |\varphi(\tau) \| \varphi(u)| \phi(\tau, u) d\tau du,$$

$$\|D\varphi\|_{|\mathscr{H}| \otimes |\mathscr{H}|}^2 := \int_{[0,T]^4} |D_{\tau_1} \varphi(u_1) \| D_{\tau_2} \varphi(u_2)| \phi(\tau_1, \tau_2) \phi(u_1, u_2) d\tau_1 du_1 d\tau_2 du_2.$$

If a process $\varphi \in \mathbb{D}^{1,p}(|\mathscr{H}|)$ *belongs to the domain of* δ *in* $L^p(\Omega; \mathbb{R})$, *then*

$$\mathbb{E}\left[\left| \int_0^T \varphi(t) \delta B(t) \right|^p \right] \leq C(H, p) \left(\|\mathbb{E}[\varphi]\|_{|\mathscr{H}|}^p + \mathbb{E}\left[\|D\varphi\|_{|\mathscr{H}| \otimes |\mathscr{H}|}^p \right] \right).$$

For $t \in [0, T]$, it is defined that $\int_0^t \varphi(u) \delta B(u) := \delta(\varphi \mathbf{1}_{[0,t]})$. The maximal inequality for the Skorohod integral is also established.

Lemma 2.2 ([25, Section 5.2.2]). *Let* $pH > 1$. *Denote* $\mathbb{L}_H^{1,p} := \{\varphi \in \mathbb{D}^{1,2}(|\mathscr{H}|) : \|\varphi\|_{p,1} < \infty\}$, *where*

$$\|\varphi\|_{p,1}^p := \int_0^T \mathbb{E}[|\varphi(u)|^p] du + \mathbb{E}\left[\int_0^T \left(\int_0^T |D_\tau \varphi(u)|^{\frac{1}{H}} d\tau \right)^{pH} du \right].$$

If a process $\varphi \in \mathbb{L}_H^{1,p}$, then

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \left| \int_0^t \varphi(u) \delta B(u) \right|^p \right] \leq C(T, H, p) \|\varphi\|_{p,1}^p.$$

Furthermore, the relationship between the Skorohod integral and the Riemann–Stieltjes integral can be characterized by

$$\int_0^t \varphi(u) dB(u) = \int_0^t \varphi(u) \delta B(u) + \int_0^t \int_0^T D_\tau \varphi(u) \phi(\tau, u) d\tau du. \tag{5}$$

3. The CIR model driven by FBm

Consider the CIR model

$$dr(t) = \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dB(t), \quad t \in (0, T]; \tag{6}$$
$$r(0) = r_0,$$

where $\kappa\theta > 0$, $\sigma > 0$, $r_0 > 0$ is the deterministic initial value, and $\{B(t)\}_{t\in[0,T]}$ is an fBm with Hurst parameter $H > 1/2$. The stochastic integration is treated in the Riemann–Stieltjes sense and the ordinary chain rule holds. Utilizing the Lamperti transformation $X(t) = \sqrt{r(t)}$, we have an auxiliary equation

$$dX(t) = \frac{1}{2}\kappa\left(\frac{\theta}{X(t)} - X(t)\right)dt + \frac{1}{2}\sigma dB(t), \quad t \in (0, T]; \tag{7}$$
$$X(0) = X_0 := \sqrt{r_0}.$$

The next proposition gives the well-posedness of Eq. (7), as well as the boundedness of moments of the exact solution.

Proposition 3.1. *Let $T > 0$ and $p \geq 1$. There exists a unique solution X of Eq. (7) satisfying*

$$\left(\mathbb{E}\left[\sup_{0 \leq t \leq T} |X(t)|^p\right]\right)^{\frac{1}{p}} \leq C(T, H, p, X_0, \kappa, \theta, \sigma).$$

Moreover, if the following condition holds

$$\kappa\theta \exp\left(\frac{\kappa s}{2}\right) \geq (p+1) \int_0^s \frac{\sigma^2}{2} \exp\left(\frac{\kappa\tau}{2}\right) \phi(\tau, s) d\tau, \quad \forall\, 0 \leq s \leq T, \tag{8}$$

then

$$\sup_{0 \leq t \leq T} \left(\mathbb{E}\left[|X(t)^{-1}|^p\right]\right)^{\frac{1}{p}} \leq C(T, X_0, \kappa).$$

Proof. Using the transformation $Z(t) = \exp\left(\frac{\kappa t}{2}\right) X(t)$, we have

$$dZ(t) = \frac{1}{2}\kappa\theta \exp(\kappa t) \frac{1}{Z(t)} dt + \frac{1}{2}\sigma \exp\left(\frac{\kappa t}{2}\right) dB(t), \quad t \in (0, T]; \tag{9}$$
$$Z(0) = X_0.$$

One can check that Eq. (9) satisfies the assumptions (i)–(iii) proposed in [16, Section 2]. As a result, it admits a global unique solution on $[0, T]$, which is strictly positive and never hits zero almost surely.

For moments of the exact solution, it follows from [16, Theorem 2.3 and Theorem 3.1] that

$$\left(\mathbb{E}\left[\sup_{0 \le t \le T} |Z(t)|^p \right] \right)^{\frac{1}{p}} \le C(T, H, p, X_0, \kappa, \theta, \sigma).$$

Denote $f(x) := \frac{\kappa\theta}{2x} - \frac{\kappa x}{2}$ and then $f'(x) = -\frac{\kappa\theta}{2x^2} - \frac{\kappa}{2}$. Combining the Malliavin derivative of X

$$D_s X(t) = \frac{1}{2}\sigma \exp\left(\int_s^t f'(X(\tau))d\tau \right) \mathbf{1}_{[0,t]}(s), \tag{10}$$

the chain rule of the Malliavin derivative leads to

$$\begin{aligned}
D_s Z(t) &= \exp\left(\frac{\kappa t}{2} \right) D_s X(t) \\
&= \frac{1}{2}\sigma \exp\left(\frac{\kappa s}{2} \right) \exp\left(\int_s^t \frac{-\kappa\theta \exp(\kappa\tau)}{2Z^2(\tau)}d\tau \right) \mathbf{1}_{[0,t]}(s),
\end{aligned}$$

then

$$0 \le D_s Z(t) \le \frac{1}{2}\sigma \exp\left(\frac{\kappa s}{2} \right).$$

For any $p \ge 1$ and $\varepsilon > 0$, similar to [16, Proposition 3.4], the chain rule applied to $\frac{1}{(\varepsilon+Z(t))^p}$ yields

$$\begin{aligned}
&\frac{1}{(\varepsilon + Z(t))^p} \\
&= \frac{1}{(\varepsilon + X_0)^p} - p\int_0^t \frac{\kappa\theta \exp(\kappa s)}{2Z(s)(\varepsilon + Z(s))^{p+1}}ds - p\int_0^t \frac{\sigma \exp\left(\frac{\kappa s}{2} \right)}{2(\varepsilon + Z(s))^{p+1}}dB(s) \\
&\le \frac{1}{(\varepsilon + X_0)^p} - p\int_0^t \frac{\kappa\theta \exp(\kappa s)}{2Z(s)(\varepsilon + Z(s))^{p+1}}ds - p\delta\left(\frac{\sigma \exp\left(\frac{\kappa(\cdot)}{2} \right)}{2(\varepsilon + Z(\cdot))^{p+1}}\mathbf{1}_{[0,t]}(\cdot) \right) \\
&\quad - p\int_0^t \int_0^s \left[-(p+1)\frac{\sigma \exp\left(\frac{\kappa s}{2} \right)}{2(\varepsilon + Z(s))^{p+2}}D_\tau Z(s)\phi(\tau, s) \right]d\tau ds,
\end{aligned}$$

where we use the formula (5). Therefore,

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{(\varepsilon + Z(t))^p} \right] \\
&\le \frac{1}{(\varepsilon + X_0)^p} - p\mathbb{E}\left[\int_0^t \frac{\kappa\theta \exp(\kappa s) - (p+1)\int_0^s \sigma \exp\left(\frac{\kappa s}{2} \right) D_\tau Z(s)\phi(\tau, s)d\tau}{2(\varepsilon + Z(s))^{p+2}}ds \right] \\
&\le \frac{1}{(\varepsilon + X_0)^p} - p\mathbb{E}\left[\int_0^t \frac{\kappa\theta \exp(\kappa s) - (p+1)\int_0^s \sigma \exp\left(\frac{\kappa s}{2} \right) \frac{\sigma}{2}\exp\left(\frac{\kappa\tau}{2} \right)\phi(\tau, s)d\tau}{2(\varepsilon + Z(s))^{p+2}}ds \right].
\end{aligned}$$

If constants H, T, p, κ, θ and σ satisfy condition (8), then by taking $\varepsilon \to 0$, we have

$$\sup_{0 \le t \le T} \left(\mathbb{E}\left[|Z(t)^{-1}|^p \right] \right)^{\frac{1}{p}} \le X_0^{-1}.$$

We conclude the proof according to the relationship between Z and X. $\quad\square$

Remark 3.1. *If $\kappa > 0$, we give the following condition*

$$T^{2H-1} \leq \frac{2\kappa\theta}{\sigma^2 H(p+1)}$$

as a sufficient condition for (8). *If $\kappa < 0$, a sufficient condition for* (8) *is*

$$s^{2H-1} \leq \frac{2\kappa\theta \exp\left(\frac{\kappa s}{2}\right)}{\sigma^2 H(p+1)}, \quad \forall\, 0 \leq s \leq T.$$

Note that the continuous function $y(s) := \frac{2\kappa\theta \exp\left(\frac{\kappa s}{2}\right)}{\sigma^2 H(p+1)} - s^{2H-1}$ satisfies $y(0) > 0$ since $\kappa\theta > 0$. There always exists $T > 0$ satisfying (8).

Remark 3.2. *If Eq.* (7) *is driven by the standard Brownian motion, the sufficient condition for the boundedness of inverse moments of X depends on $p, \kappa, \theta, \sigma$, but is independent of T. We refer to* [6, Section 3] *for more discussions.*

4. A backward Euler type scheme for the CIR model

In this section, we apply the backward Euler scheme to the auxiliary equation (7), which is also suggested in [1,2,6,24] for the CIR model driven by standard Brownian motion. Then the numerical solution for the original equation (6) is obtained from the inverse of the Lamperti transformation. In this sense, we call it a backward Euler type scheme for the CIR model.

We first show that the scheme ensures the positivity of the numerical solution. Given $N \in \mathbb{N}_+$, we denote by $h := T/N$ the time step size. The backward Euler scheme applied to Eq. (7) is

$$X_{n+1} = X_n + \frac{1}{2}\kappa\left(\frac{\theta}{X_{n+1}} - X_{n+1}\right)h + \frac{1}{2}\sigma \Delta B_{n+1}, \tag{11}$$

where $\Delta B_{n+1} = B(t_{n+1}) - B(t_n)$, $t_n = nh$, $n = 0, \ldots, N$. Since $\kappa\theta > 0$, Eq. (11) has a unique positive solution for any $h > 0$ satisfying $h \max\{0, -\kappa/2\} < 1$, which is

$$X_{n+1} = \frac{X_n + \frac{1}{2}\sigma \Delta B_{n+1} + \sqrt{\left(X_n + \frac{1}{2}\sigma \Delta B_{n+1}\right)^2 + \kappa h\theta\,(2 + \kappa h)}}{2 + \kappa h}. \tag{12}$$

Furthermore, we apply the piecewise linear interpolation to get the continuous interpolation X^h on $[0, T]$. Namely,

$$X^h(t) = X_n + \frac{t - t_n}{h}(X_{n+1} - X_n), \quad \forall\, t \in (t_n, t_{n+1}], \ n = 0, \ldots, N-1, \tag{13}$$

$$X^h(0) = X_0.$$

Consequently, we have X^h positive on $[0, T]$.

4.1. Optimal strong convergence rate

In the following, we give the strong convergence rate of the backward Euler scheme. Then the strong convergence rate of $r^h := (X^h)^2$, which is an approximation of the solution to the original equation (6), is obtained.

Theorem 4.1. *Let $\xi \in (0, 1)$. For $p \geq 2$, assume that*

$$\kappa\theta \exp\left(\frac{\kappa s}{2}\right) \geq (3p+1)\int_0^s \frac{\sigma^2}{2}\exp\left(\frac{\kappa\tau}{2}\right)\phi(\tau, s)d\tau, \quad \forall\, 0 \leq s \leq T. \tag{14}$$

Then for any $h \in (0, 1)$ satisfying $h\max\{0, -\kappa/2\} < 1 - \xi$, it holds that

$$\left(\mathbb{E}\left[\sup_{1 \leq n \leq N}|X(t_n) - X_n|^p\right]\right)^{\frac{1}{p}} \leq C(T, H, p, X_0, \kappa, \theta, \sigma, \xi)h,$$

where $X(\cdot)$ is the exact solution of Eq. (7) and X_n is defined by scheme (12).

Proof. Denote $e_n := X(t_n) - X_n$ and $f(x) := \frac{1}{2}\kappa\left(\frac{\theta}{x} - x\right)$. The chain rule and integration by parts of Riemann–Stieltjes integral yield

$$\begin{aligned}
e_{n+1} &= e_n + [f(X(t_{n+1})) - f(X_{n+1})]h \tag{15} \\
&\quad - \int_{t_n}^{t_{n+1}}\int_t^{t_{n+1}} f'(X(u))[f(X(u))du + \frac{1}{2}\sigma dB(u)]dt \\
&= e_n + [f(X(t_{n+1})) - f(X_{n+1})]h \\
&\quad - \int_{t_n}^{t_{n+1}}(u - t_n)f'(X(u))[f(X(u))du + \frac{1}{2}\sigma dB(u)] \\
&=: e_n + [f(X(t_{n+1})) - f(X_{n+1})]h + R_n.
\end{aligned}$$

Since $\kappa\theta > 0$, $X(t_{n+1}) > 0$ and $X_{n+1} > 0$, we have

$$f(X(t_{n+1})) - f(X_{n+1}) = \left(-\frac{\kappa\theta}{2X(t_{n+1})X_{n+1}} - \frac{\kappa}{2}\right)e_{n+1} =: \gamma_{n+1}e_{n+1},$$

where $\gamma_{n+1} \leq \tilde{\kappa}$ with $0 \leq \tilde{\kappa} = \max\{0, -\frac{\kappa}{2}\}$. It follows that

$$(1 - \gamma_{n+1}h)e_{n+1} = e_n + R_n.$$

Define $\zeta_0 := 1$, $\zeta_n := \prod_{j=1}^n(1 - \gamma_j h)$, and $\tilde{e}_n := \zeta_n e_n$. Multiplying the above formula by ζ_n, we have

$$\tilde{e}_{n+1} = \tilde{e}_n + \zeta_n R_n.$$

According to (5), we split R_n into three terms

$$\begin{aligned}
R_n &= -\int_{t_n}^{t_{n+1}}(u - t_n)f'(X(u))f(X(u))du - \int_{t_n}^{t_{n+1}}\frac{1}{2}\sigma(u - t_n)f'(X(u))\delta B(u) \\
&\quad - \int_{t_n}^{t_{n+1}}\int_0^T \frac{1}{2}\sigma(u - t_n)D_\tau[f'(X(u))]\phi(\tau, u)d\tau du.
\end{aligned} \tag{16}$$

Defining $\tilde{\zeta}_n := \zeta_n/(1 - \tilde{\kappa}h)^n$, $\lfloor\frac{u}{h}\rfloor := n$ for $u \in [t_n, t_{n+1})$, and

$$\tilde{R}(t) := -\int_0^t \frac{1}{2}\sigma(1 - \tilde{\kappa}h)^{\lfloor\frac{u}{h}\rfloor}(u - t_{\lfloor\frac{u}{h}\rfloor})f'(X(u))\delta B(u),$$

we obtain

$$\tilde{\zeta}_n(\tilde{R}(t_{n+1}) - \tilde{R}(t_n)) = -\zeta_n\int_{t_n}^{t_{n+1}}\frac{1}{2}\sigma(u - t_n)f'(X(u))\delta B(u).$$

Together with (16), we deduce

$$
\begin{aligned}
\tilde{e}_n =& \tilde{e}_{n-1} - \int_{t_{n-1}}^{t_n} \zeta_{n-1}(u - t_{n-1}) f'(X(u)) f(X(u)) du + \tilde{\zeta}_{n-1}(\tilde{R}(t_n) - \tilde{R}(t_{n-1})) \\
& - \int_{t_{n-1}}^{t_n} \int_0^T \frac{1}{2} \sigma \zeta_{n-1}(u - t_{n-1}) D_\tau[f'(X(u))] \phi(\tau, u) d\tau du \\
=& - \int_0^{t_n} \zeta_{\lfloor \frac{u}{h} \rfloor}(u - t_{\lfloor \frac{u}{h} \rfloor}) f'(X(u)) f(X(u)) du + \sum_{j=0}^{n-1} \tilde{\zeta}_j(\tilde{R}(t_{j+1}) - \tilde{R}(t_j)) \\
& - \int_0^{t_n} \int_0^T \frac{1}{2} \sigma \zeta_{\lfloor \frac{u}{h} \rfloor}(u - t_{\lfloor \frac{u}{h} \rfloor}) D_\tau[f'(X(u))] \phi(\tau, u) d\tau du.
\end{aligned}
$$

Then the error e_n satisfies

$$
\begin{aligned}
e_n =& - \int_0^{t_n} \frac{\zeta_{\lfloor \frac{u}{h} \rfloor}}{\zeta_n}(u - t_{\lfloor \frac{u}{h} \rfloor}) f'(X(u)) f(X(u)) du + \sum_{j=0}^{n-1} \frac{\tilde{\zeta}_j}{\zeta_n}(\tilde{R}(t_{j+1}) - \tilde{R}(t_j)) \\
& - \int_0^{t_n} \int_0^T \frac{1}{2} \sigma \frac{\zeta_{\lfloor \frac{u}{h} \rfloor}}{\zeta_n}(u - t_{\lfloor \frac{u}{h} \rfloor}) D_\tau[f'(X(u))] \phi(\tau, u) d\tau du.
\end{aligned}
$$

Notice that

$$
\sum_{j=0}^{n-1} \tilde{\zeta}_j(\tilde{R}(t_{j+1}) - \tilde{R}(t_j)) = \tilde{\zeta}_{n-1} \tilde{R}(t_n) + \sum_{j=1}^{n-1} (\tilde{\zeta}_{j-1} - \tilde{\zeta}_j) \tilde{R}(t_j).
$$

Since h satisfies $h\tilde{\kappa} < 1 - \xi$, we have $0 < \xi < 1 - \tilde{\kappa}h \le 1 - \gamma_j h$ and then $0 < \frac{\tilde{\zeta}_{j-1}}{\tilde{\zeta}_j} = \frac{1 - \tilde{\kappa}h}{1 - \gamma_j h} \le 1$. Hence we obtain

$$
\begin{aligned}
\left| \sum_{j=0}^{n-1} \tilde{\zeta}_j(\tilde{R}(t_{j+1}) - \tilde{R}(t_j)) \right| &\le \tilde{\zeta}_{n-1} \left| \tilde{R}(t_n) \right| + \sum_{j=1}^{n-1} (\tilde{\zeta}_j - \tilde{\zeta}_{j-1}) \left| \tilde{R}(t_j) \right| \\
&\le 2\tilde{\zeta}_n \sup_{1 \le j \le n} \left| \tilde{R}(t_j) \right|.
\end{aligned}
$$

According to $\xi < 1 - \tilde{\kappa}h \le 1$, we get

$$
\tilde{\zeta}_n / \zeta_n = (1 - \tilde{\kappa}h)^{-n} = \left(1 + \frac{\tilde{\kappa}h}{1 - \tilde{\kappa}h}\right)^n \le e^{\frac{\tilde{\kappa}T}{\xi}}, \quad n = 0, \ldots, N,
$$

which implies

$$
\sup_{1 \le n \le N} \left| \sum_{j=0}^{n-1} \frac{\tilde{\zeta}_j}{\zeta_n}(\tilde{R}(t_{j+1}) - \tilde{R}(t_j)) \right| \le 2 e^{\frac{\tilde{\kappa}T}{\xi}} \sup_{1 \le j \le N} \left| \tilde{R}(t_j) \right|.
$$

Therefore, combining with $\frac{\zeta_{\lfloor \frac{u}{h} \rfloor}}{\zeta_n} = \prod_{j=\lfloor \frac{u}{h} \rfloor + 1}^{n} (1 - \gamma_j h)^{-1} \le (1 - \tilde{\kappa}h)^{n - \lfloor \frac{u}{h} \rfloor} \le e^{\frac{\tilde{\kappa}T}{\xi}}$, we get

$$
\begin{aligned}
& \mathbb{E}\left[\sup_{1 \le n \le N} |e_n|^p \right] \\
& \le C(p) \mathbb{E}\left[\sup_{1 \le n \le N} \left| \int_0^{t_n} \frac{\zeta_{\lfloor \frac{u}{h} \rfloor}}{\zeta_n}(u - t_{\lfloor \frac{u}{h} \rfloor}) f'(X(u)) f(X(u)) du \right|^p \right]
\end{aligned}
$$

$$+ C(p)\mathbb{E}\left[\sup_{1\le n\le N}\left|\sum_{j=0}^{n-1}\frac{\tilde{\zeta}_j}{\zeta_n}(\tilde{R}(t_{j+1})-\tilde{R}(t_j))\right|^p\right]$$

$$+ C(p)\mathbb{E}\left[\sup_{1\le n\le N}\left|\int_0^{t_n}\int_0^T\frac{1}{2}\sigma\frac{\zeta_{\lfloor\frac{u}{h}\rfloor}}{\zeta_n}(u-t_{\lfloor\frac{u}{h}\rfloor})D_\tau[f'(X(u))]\phi(\tau,u)d\tau du\right|^p\right]$$

$$\le C(T,p,\kappa,\xi)\left(\mathbb{E}\left[\left(\int_0^T\left|(u-t_{\lfloor\frac{u}{h}\rfloor})f'(X(u))f(X(u))\right|du\right)^p\right]\right.$$

$$+ \mathbb{E}\left[\sup_{1\le j\le N}\left|\tilde{R}(t_j)\right|^p\right] + \left.\mathbb{E}\left[\left(\int_0^T\int_0^T\left|\frac{1}{2}\sigma(u-t_{\lfloor\frac{u}{h}\rfloor})D_\tau[f'(X(u))]\phi(\tau,u)\right|d\tau du\right)^p\right]\right)$$

$$=: C(T,p,\kappa,\xi)(I+II+III), \tag{17}$$

where the three terms I, II, III are dealt with respectively in the following.

Recall that $f(x) = \frac{\kappa\theta}{2x} - \frac{\kappa x}{2}$ and $f'(x) = -\frac{\kappa\theta}{2x^2} - \frac{\kappa}{2}$. By Proposition 3.1, the assumption (14) implies

$$\mathbb{E}[X(t)^{-3p}] \le C(T,X_0,\kappa), \quad \forall\, t\in[0,T]. \tag{18}$$

According to the Minkowski inequality, the first term satisfies

$$I = \left\|\int_0^T\left|(u-t_{\lfloor\frac{u}{h}\rfloor})f'(X(u))f(X(u))\right|du\right\|_{L^p(\Omega;\mathbb{R})}^p$$

$$\le h^p\left(\int_0^T\|f'(X(u))f(X(u))\|_{L^p(\Omega;\mathbb{R})}du\right)^p$$

$$\le C(T,p,X_0,\kappa)h^p.$$

The chain rule of the Malliavin derivative gives that for any $u\in[0,T]$,

$$D_\tau[f'(X(u))] = f''(X(u))D_\tau X(u) = f''(X(u))\frac{\sigma}{2}\exp\left(\int_\tau^u f'(X(\iota))d\iota\right)\mathbf{1}_{[0,u]}(\tau).$$

Noticing that $f'(x) = -\frac{\kappa\theta}{2x^2} - \frac{\kappa}{2} \le -\frac{\kappa}{2}$ for $x>0$ and $f''(x) = \frac{\kappa\theta}{x^3}$, we get

$$0 \le D_\tau[f'(X(u))] \le C(T,\kappa,\sigma)f''(X(u))\mathbf{1}_{[0,u]}(\tau).$$

Based on (18) and the fact

$$\int_0^T\phi(\tau,u)d\tau \le 2HT^{2H-1}, \quad \forall\, u\in[0,T],$$

we obtain for the third term that

$$III = \left\|\int_0^T\int_0^T\left|\frac{1}{2}\sigma(u-t_{\lfloor\frac{u}{h}\rfloor})D_\tau[f'(X(u))]\phi(\tau,u)\right|d\tau du\right\|_{L^p(\Omega;\mathbb{R})}^p$$

$$\le C(T,p,\kappa,\sigma)h^p\left(\int_0^T\left(\|f''(X(u))\|_{L^p(\Omega;\mathbb{R})}\int_0^T\phi(\tau,u)d\tau\right)du\right)^p$$

$$\le C(T,H,p,X_0,\kappa,\theta,\sigma)h^p.$$

Since $\frac{1}{2} < H < 1$, we know that $pH > 1$ if $p \geq 2$. For the second term, from Lemma 2.2 and Hölder's inequality, we obtain

$$
\begin{aligned}
II =& \mathbb{E}\left[\sup_{1 \leq j \leq N}\left|\tilde{R}(t_j)\right|^p\right] \\
\leq& C(T, H, p)\int_0^T \mathbb{E}\left[\left|\frac{1}{2}\sigma(1 - \tilde{\kappa}h)^{\lfloor\frac{u}{h}\rfloor}(u - t_{\lfloor\frac{u}{h}\rfloor})f'(X(u))\right|^p\right]du \\
&+ C(T, H, p)\mathbb{E}\left[\int_0^T\left(\int_0^T\left|\frac{1}{2}\sigma(1 - \tilde{\kappa}h)^{\lfloor\frac{u}{h}\rfloor}(u - t_{\lfloor\frac{u}{h}\rfloor})D_\tau[f'(X(u))]\right|^{\frac{1}{H}}d\tau\right)^{pH}du\right] \\
\leq& C(T, H, p, \sigma)h^p\int_0^T \mathbb{E}\left[\left|f'(X(u))\right|^p\right]du \\
&+ C(T, H, p, \sigma)h^p\mathbb{E}\left[\int_0^T\left(\int_0^T\left|D_\tau[f'(X(u))]\right|^{\frac{1}{H}}d\tau\right)^{pH}du\right] \\
\leq& C(T, H, p, X_0, \kappa, \theta, \sigma, \xi)h^p + C(T, H, p, \sigma)h^p\int_0^T \mathbb{E}\left[\left|f''(X(u))\right|^p\right]du \\
\leq& C(T, H, p, X_0, \kappa, \theta, \sigma, \xi)h^p.
\end{aligned}
$$

Finally, it turns out that

$$
\mathbb{E}\left[\sup_{1 \leq n \leq N}|e_n|^p\right] \leq C(T, H, p, X_0, \kappa, \theta, \sigma, \xi)h^p. \quad \square
$$

Remark 4.1. *For the CIR model driven by standard Brownian motion, we refer to [2,24] and references therein for the convergence analysis. For equations driven by globally Lipschitz drift terms and additive fBms, we refer to [19,23] for the convergence analysis.*

Together with the Hölder regularity of the fBm, we obtain the following strong convergence rate for the continuous interpolation X^h on $[0, T]$.

Theorem 4.2. *Let $\xi \in (0, 1)$. For $p \geq 2$, assume that (14) holds. Then for any $h \in (0, 1)$ satisfying $h\max\{0, -\kappa/2\} < 1 - \xi$, it holds that*

$$
\left(\mathbb{E}\left[\sup_{0 \leq t \leq T}|X(t) - X^h(t)|^p\right]\right)^{\frac{1}{p}} \leq C(T, H, p, X_0, \kappa, \theta, \sigma, \xi)h^H\sqrt{\log\frac{T}{h}},
$$

where $X(\cdot)$ is the exact solution of Eqs. (7) and $X^h(\cdot)$ is defined by scheme (12)–(13).

Proof. By definition of X^h, for any $t \in (t_n, t_{n+1}]$,

$$
\begin{aligned}
X(t) - X^h(t) =& X(t) - \frac{t - t_n}{h}X_{n+1} - \frac{t_{n+1} - t}{h}X_n \\
=& \frac{t_n - t}{h}\left[\int_t^{t_{n+1}} f(X(s))ds + \frac{1}{2}\sigma(B(t_{n+1}) - B(t))\right] \\
&+ \frac{t_{n+1} - t}{h}\left[\int_{t_n}^t f(X(s))ds + \frac{1}{2}\sigma(B(t) - B(t_n))\right] \\
&+ \frac{t - t_n}{h}[X(t_{n+1}) - X_{n+1}] + \frac{t_{n+1} - t}{h}[X(t_n) - X_n].
\end{aligned}
$$

Using the arguments in the proof of Theorem 4.1 and the result in [17, Theorem 6] which shows

$$\left(\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{t_n-t}{h}(B(t_{n+1})-B(t))+\frac{t_{n+1}-t}{h}(B(t)-B(t_n))^p\right|\right]\right)^{\frac{1}{p}}$$
$$\le C(T,H,p)h^H\sqrt{\log\frac{T}{h}},$$

we conclude the result. □

Remark 4.2. *Based on N observed values of the fBm, i.e., $\{B(t_1),\ldots,B(t_N)\}$, the strong convergence rate for the continuous interpolation of the numerical solution on $[0,T]$ in $L^p(\Omega; L^\infty([0,T];\mathbb{R}))$ is restricted by the Hölder regularity of the fBm. We refer to [23] for more discussions.*

Based on the inverse of the Lamperti transformation, we get the strong convergence rate for the backward Euler type scheme for the CIR model driven by fBm.

Theorem 4.3. *Let $\xi\in(0,1)$. For $p\ge2$, assume that (14) holds. Then for any $h\in(0,1)$ satisfying $h\max\{0,-\kappa/2\}<1-\xi$, it holds that*

$$\left(\mathbb{E}\left[\sup_{1\le n\le N}|r(t_n)-r^h(t_n)|^{\frac{p}{2}}\right]\right)^{\frac{2}{p}}\le C(T,H,p,X_0,\kappa,\theta,\sigma,\xi)h$$

and

$$\left(\mathbb{E}\left[\sup_{0\le t\le T}|r(t)-r^h(t)|^{\frac{p}{2}}\right]\right)^{\frac{2}{p}}\le C(T,H,p,X_0,\kappa,\theta,\sigma,\xi)h^H\sqrt{\log\frac{T}{h}},$$

where $r(\cdot)$ is the exact solution of Eq. (6) and $r^h(\cdot):=(X^h(\cdot))^2$.

Proof. Based on Proposition 3.1 and Theorem 4.2, we have

$$\left(\mathbb{E}\left[\sup_{1\le n\le N}|X(t_n)+X^h(t_n)|^p\right]\right)^{\frac{1}{p}}$$
$$\le\left(\mathbb{E}\left[\sup_{1\le n\le N}|2X(t_n)+(X^h(t_n)-X(t_n))|^p\right]\right)^{\frac{1}{p}}$$
$$\le C\left(\mathbb{E}\left[\sup_{1\le n\le N}|X(t_n)|^p\right]\right)^{\frac{1}{p}}+C\left(\mathbb{E}\left[\sup_{1\le n\le N}|X^h(t_n)-X(t_n)|^p\right]\right)^{\frac{1}{p}}\le C.$$

Since

$$r(t)-r^h(t)=(X(t)-X^h(t))(X(t)+X^h(t)),$$

Hölder's inequality and Theorem 4.1 yield

$$\left(\mathbb{E}\left[\sup_{1\le n\le N}|r(t_n)-r^h(t_n)|^{\frac{p}{2}}\right]\right)^{\frac{2}{p}}$$
$$\le\left(\mathbb{E}\left[\sup_{1\le n\le N}|X(t_n)-X^h(t_n)|^p\right]\right)^{\frac{1}{p}}\left(\mathbb{E}\left[\sup_{1\le n\le N}|X(t_n)+X^h(t_n)|^p\right]\right)^{\frac{1}{p}}$$
$$\le C(T,H,p,X_0,\kappa,\theta,\sigma,\xi)h.$$

Similarly, Theorem 4.2 shows

$$
\left(\mathbb{E}\left[\sup_{0 \leq t \leq T} |r(t) - r^h(t)|^{\frac{p}{2}}\right]\right)^{\frac{2}{p}} \leq C(T, H, p, X_0, \kappa, \theta, \sigma, \xi) h^H \sqrt{\log \frac{T}{h}}. \quad \square
$$

4.2. Inverse moments and the Malliavin derivative of the numerical solution

In this section, we give the boundedness of inverse moments of the numerical solution and derive the Malliavin derivative of the numerical solution, which are regarded as counterparts with properties of the exact solution.

Proposition 4.1. *Suppose that the assumptions in Theorem 4.1 hold. Then*

$$
\sup_{n=1,\dots,N} \|X_n^{-1}\|_{L^p(\Omega;\mathbb{R})} \leq C(T, H, p, X_0, \kappa, \theta, \sigma, \xi).
$$

Proof. From (11) and (15), we have

$$
X_{n+1}^{-1} = 2(\kappa\theta h)^{-1}\left(e_n - (\frac{1}{2}\kappa h + 1)e_{n+1} + \frac{1}{2}\kappa\theta h(X(t_{n+1}))^{-1} + R_n\right).
$$

According to Proposition 3.1 and Theorem 4.1, it suffices to estimate the term involving R_n. Together with the decomposition (16) of R_n, the techniques to deal with I, II, III in (17) yield

$$
h^{-1}\left(\mathbb{E}\left[\sup_{1 \leq n \leq N} |R_n|^p\right]\right) \leq C,
$$

which gives the boundedness of inverse moments of the numerical solution. $\quad \square$

Proposition 4.2. *Suppose that the assumptions in Theorem 4.1 hold. If $\kappa > 0$, then for any $t \in (t_n, t_{n+1}]$, $n = 0, \dots, N-1$,*

$$
D_s[X^h(t)] = \frac{t_{n+1} - t}{h} G_n(s)\mathbf{1}_{[0,t_n]}(s) + \frac{t - t_n}{h} G_{n+1}(s)\mathbf{1}_{[0,t_{n+1}]}(s) \tag{19}
$$

with $G_n(s) = \frac{1}{2}\sigma \prod_{j=i}^n (1 - f'(X_j)h)^{-1}$ for $s \in (t_{i-1}, t_i]$, $i = 1, \dots, n$.

Proof. For any $\varphi \in \mathcal{H}$, noticing that the operator R_H is an injection from $\mathcal{H}$ to $\mathcal{H}_H$ (see e.g. [5]), we consider the following equation

$$
dX^\varepsilon(t) = \frac{1}{2}\kappa\left(\frac{\theta}{X^\varepsilon(t)} - X^\varepsilon(t)\right)dt + \frac{1}{2}\sigma dB(t) + \frac{1}{2}\sigma\varepsilon d\tilde{\varphi}(t), \quad t \in (0, T]; \tag{20}
$$

$$
X^\varepsilon(0) = X_0
$$

with $\tilde{\varphi} = R_H\varphi$ and $\varepsilon > 0$. Based on the backward Euler scheme (12), we get numerical solution $\{X_n^\varepsilon\}_{n=0}^N$ for Eq. (20). Then the directional derivative of X_n is given by

$$
\langle DX_n, \varphi\rangle_{\mathcal{H}} = \frac{dX_n^\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0}, \quad \varphi \in \mathcal{H}.
$$

The mean value theorem shows that for any $n = 0, \ldots, N - 1$,

$$
X_{n+1}^\varepsilon - X_{n+1} = X_n^\varepsilon - X_n + (f(X_{n+1}^\varepsilon) - f(X_{n+1}))h + \frac{1}{2}\sigma\varepsilon(\tilde{\varphi}(t_{n+1}) - \tilde{\varphi}(t_n))
$$

$$
= X_n^\varepsilon - X_n + f'(\tilde{X}_{n+1}^\varepsilon)\left(X_{n+1}^\varepsilon - X_{n+1}\right)h + \frac{1}{2}\sigma\varepsilon(\tilde{\varphi}(t_{n+1}) - \tilde{\varphi}(t_n))
$$

with $\tilde{X}_{n+1}^\varepsilon = \tilde{h}_{n+1}^\varepsilon X_{n+1}^\varepsilon + (1 - \tilde{h}_{n+1}^\varepsilon)X_{n+1}$ for some $\tilde{h}_{n+1}^\varepsilon \in [0, 1]$. Denoting $a_{n+1}^\varepsilon := (1 - f'(\tilde{X}_{n+1}^\varepsilon)h)^{-1}$, we have

$$
X_{n+1}^\varepsilon - X_{n+1} = a_{n+1}^\varepsilon(X_n^\varepsilon - X_n) + \frac{1}{2}\sigma\varepsilon a_{n+1}^\varepsilon(\tilde{\varphi}(t_{n+1}) - \tilde{\varphi}(t_n)).
$$

Then

$$
X_n^\varepsilon - X_n = \frac{1}{2}\sigma\varepsilon\sum_{i=1}^n\left(\prod_{j=i}^n a_j^\varepsilon\right)(\tilde{\varphi}(t_i) - \tilde{\varphi}(t_{i-1})).
$$

Define $G_n^\varepsilon(t) := \frac{1}{2}\sigma\prod_{j=i}^n a_j^\varepsilon$, for $t \in (t_{i-1}, t_i]$ and $i = 1, \ldots, n$. Recalling the definitions of R_H and K^* introduced in Section 2, we obtain that

$$
X_n^\varepsilon - X_n = \varepsilon\int_0^{t_n} G_n^\varepsilon(\tau)d\tilde{\varphi}(\tau)
$$

$$
= \varepsilon\int_0^{t_n} G_n^\varepsilon(\tau)dR_H\varphi(\tau)
$$

$$
= \varepsilon\int_0^{t_n} G_n^\varepsilon(\tau)\left[\int_0^\tau \frac{\partial K_H(\tau, u)}{\partial\tau}(K^*\varphi)(u)du\right]d\tau
$$

$$
= \varepsilon\int_0^{t_n}\left[\int_u^{t_n} G_n^\varepsilon(\tau)\frac{\partial K_H(\tau, u)}{\partial\tau}d\tau\right](K^*\varphi)(u)du
$$

$$
= \varepsilon\int_0^{t_n}(K^*(G_n^\varepsilon\mathbf{1}_{[0,t_n]}))(u)(K^*\varphi)(u)du.
$$

The isometry (3) gives

$$
X_n^\varepsilon - X_n = \varepsilon\langle G_n^\varepsilon\mathbf{1}_{[0,t_n]}, \varphi\rangle_{\mathscr{H}}.
$$

Note that G_n^ε is bounded due to the assumption $\kappa > 0$ and the fact $f'(x) = -\frac{1}{2}\kappa\theta\frac{1}{x^2} - \frac{1}{2}\kappa < 0$. By the definition of scheme (12) and the dominated convergence theorem, we get that the following equality holds in both almost surely sense and $L^p(\Omega; \mathbb{R})$ sense for $p \geq 1$:

$$
\lim_{\varepsilon\to 0}\frac{X_n^\varepsilon - X_n}{\varepsilon} = \lim_{\varepsilon\to 0}\langle G_n^\varepsilon\mathbf{1}_{[0,t_n]}, \varphi\rangle_{\mathscr{H}} = \langle G_n\mathbf{1}_{[0,t_n]}, \varphi\rangle_{\mathscr{H}},
$$

where $G_n(t) := \frac{1}{2}\sigma\prod_{j=i}^n(1 - f'(X_j)h)^{-1}$ for any $t \in (t_{i-1}, t_i]$ and $i = 1, \ldots, n$. It follows from [26, Theorem 3.1] that $X_n \in \mathbb{D}^{1,2}$ and $DX_n = G_n\mathbf{1}_{[0,t_n]}$.
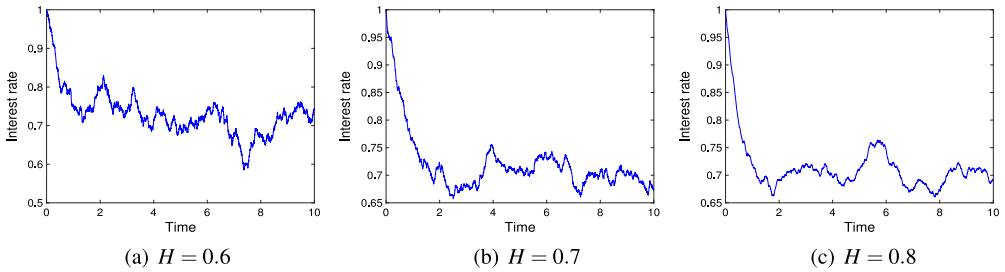
Fig. 1. Interest rate r^h vs. time.

By the definition of X^h, for any $t \in (t_n, t_{n+1}]$, we obtain the expression of the Malliavin derivative of X^h

$$D_s[X^h(t)] = D_s \left[\frac{t_{n+1} - t}{h} X_n + \frac{t - t_n}{h} X_{n+1} \right]$$

$$= \frac{t_{n+1} - t}{h} G_n(s) \mathbf{1}_{[0,t_n]}(s) + \frac{t - t_n}{h} G_{n+1}(s) \mathbf{1}_{[0,t_{n+1}]}(s). \quad \square$$

Remark 4.3. *Notice $e^x \approx 1 + x$ when x is close to 0. One can regard the expression of $D_s[X^h(t)]$ as an approximation for $D_s[X(t)]$ based on the backward Euler scheme, comparing* (10) *and* (19).

5. Numerical experiments

In the CIR model (6), we set the initial value $r_0 = 1$, the constants $\kappa = 2$, $\theta = 0.5$ and $\sigma = 0.5$. Fig. 1 presents the pathwise behaviors of the numerical solutions for three cases $H = 0.6, 0.7, 0.8$ with $T = 10$ and $h = \frac{10}{2^{12}} = 0.0024$. We observe that the solution is pushed forward along the direction from the initial value to the long-term mean value θ and the regularity of the solution is impacted by that of the fBm.

Since condition (8) is satisfied for $p = 6$ and $T = 1$, we have from Theorem 4.1 that

$$\left(\mathbb{E} \left[\sup_{1 \leq n \leq N} |X(t_n) - X_n|^2 \right] \right)^{\frac{1}{2}} \leq Ch.$$

To verify the strong convergence order of the backward Euler scheme, we simulate the 'exact' solution by using the small step size $h^* = 2^{-15}$. The five step sizes $h = 2^{-i}$, $i = 6, 7, 8, 9, 10$ are chosen for numerical solutions. In Fig. 2, we calculate the mean square error with respect to X_n by

$$\left(\mathbb{E} \left[\sup_{1 \leq n \leq T/h} |X(t_n) - X_n|^2 \right] \right)^{\frac{1}{2}}$$
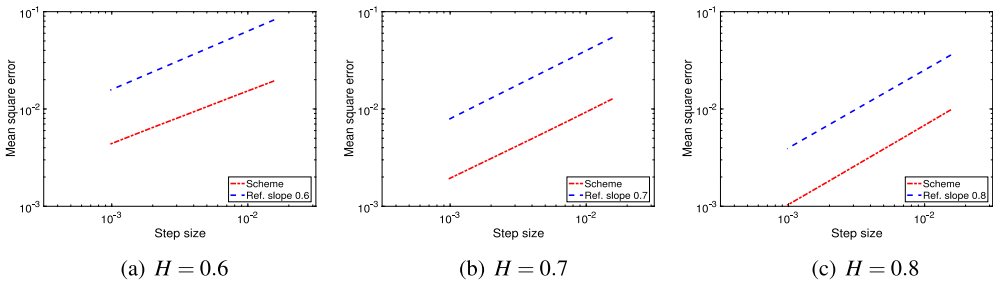
and approximate the expectation by computing averages over 500 samples. The numerical results confirm our theoretical analysis.

Under the same assumption, Theorem 4.2 leads to

$$\left(\mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t) - X^h(t)|^2 \right] \right)^{\frac{1}{2}} \leq Ch^H \sqrt{\log \frac{T}{h}}.$$
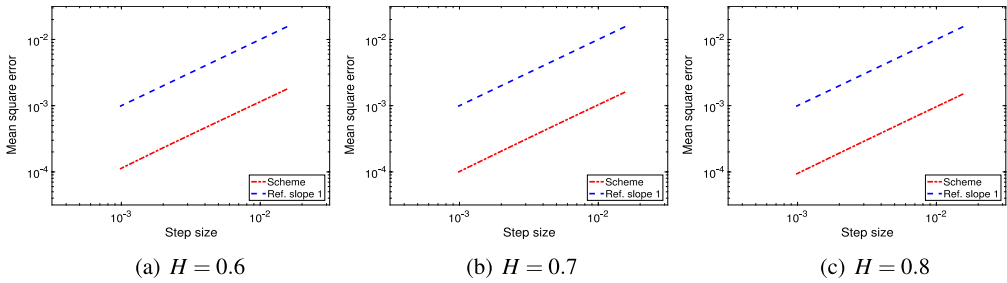
Fig. 2. Mean square error for X_n vs. step size.

Fig. 3. Mean square error for X^h vs. step size.

To demonstrate the convergence rate on $[0, T]$, we also simulate the 'exact' solution by using the small step size $h^* = 2^{-15}$. The five step sizes $h = 2^{-i}$, $i = 6, 7, 8, 9, 10$ are chosen for numerical solutions. Then we apply the piecewise linear interpolation to get the numerical solutions $X^h(nh^*)$, $n = 1, \ldots, 2^{15}$. In Fig. 3, we calculate the mean square error with respect to X^h by

$$\left(\mathbb{E} \left[\sup_{n=1,\ldots,2^{15}} |X(nh^*) - X^h(nh^*)|^2 \right] \right)^{\frac{1}{2}}$$

and approximate the expectation by computing averages over 500 samples. In consistent with our theoretical analysis, the strong convergence order for the piecewise linear interpolation is H which is mainly caused by the Hölder regularity of fBm. Additionally, the error becomes smaller when H becomes larger, due to the fact that the regularity of the fBm becomes better.

In Fig. 4, we take $T = 10$ and $h = \frac{10}{2^{15}}$. We calculate $\|(X_n)^{-1}\|_{L^2(\Omega;\mathbb{R})}$ for $n = 0, \ldots, 2^{15}$, where the expectation is approximated by computing averages over 100 samples. This coincides with our theoretical analysis for the boundedness of inverse moments of numerical solutions.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
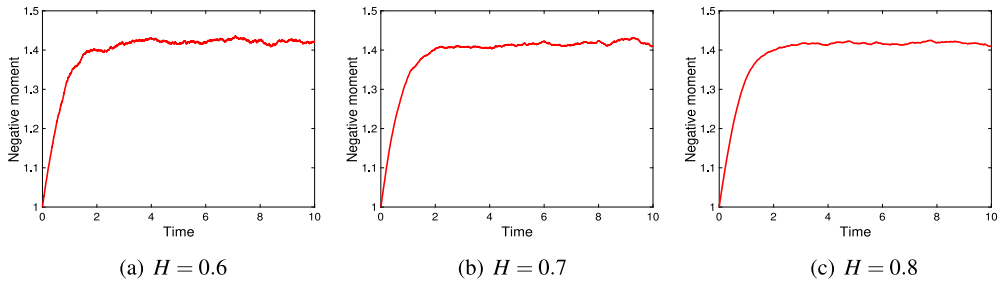
(a) $H = 0.6$ (b) $H = 0.7$ (c) $H = 0.8$

Fig. 4. Inverse moment of X_n vs. time.

Acknowledgments

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