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An inverse random source problem for the time fractional diffusion equation driven by a fractional Brownian motion

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Abstract

This paper is concerned with the mathematical analysis of an inverse random source problem for the time fractional diffusion equation, where the source is driven by a fractional Brownian motion. Given the random source, the direct problem is to study the stochastic time fractional diffusion equation. The inverse problem is to determine the statistical properties of the source from the expectation and variance of the final time data. For the direct problem, we show that it is well-posed and has a unique mild solution under a certain condition. For the inverse problem, the uniqueness is proved and the instability is characterized. The major ingredients of the analysis are based on the properties of the Mittag–Leffler function and the stochastic integrals associated with the fractional Brownian motion.

Keywords: fractional diffusion equation, inverse source problem, fractional Brownian motion, uniqueness, ill-posedness

(Some figures may appear in colour only in the online journal)

1. Introduction

In the last two decades, the fractional differential equations (FDEs) have received ever-increasing attention by many researchers due to their potential applications in modeling real physical phenomena. For examples, the FDE can be used to describe the anomalous diffusion in a

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highly heterogeneous aquifer [1], the relaxation phenomena in complex viscoelastic materials [10], the anomalous diffusion in an underground environmental problem [13], and a non-Markovian diffusion process with memory [26]. We refer to [11] for some recent advances in theory and simulation of the fractional diffusion processes.

Motivated by significant scientific and industrial applications, the field of inverse problems has undergone a tremendous growth in the last several decades since Calderón proposed an inverse conductivity problem. Recently, the inverse problems on FDEs have also progressed into an area of intense research activity. In particular, for the time or time-space fractional diffusion equations, the inverse source problems have been widely investigated mathematically and numerically. Compared with the semilinear problem [25], many more results are available for the linear problems. The linear inverse source problems for fractional diffusion equations can be broadly classified into the following six cases: (1) determining a space-dependent source term from the space-dependent data [3, 9, 18, 19, 36–38, 40–42, 44, 47, 48]; (2) determining a time-dependent source term from the time-dependent data [14, 23, 24, 35, 45]; (3) determining a time-dependent source term from the space-dependent data [2, 15]; (4) determining a space-dependent source term from the time-dependent data [50]; (5) determining a space-dependent source term from the boundary data [43]; (6) determining a general source from the time-dependent data [28]. Despite a considerable amount of work done so far, the rigorous mathematical theory is still lacking [16], especially for the inverse problems where the sources contain uncertainties, which are known as the inverse random source problems.

The inverse random source problems belong to a category of stochastic inverse problems, which refer to inverse problems that involve uncertainties. Compared to deterministic inverse problems, stochastic inverse problems have substantially more difficulties on top of the existing obstacles due to the randomness and uncertainties. There are some work done for the inverse random source scattering problems, where the wave propagation is governed by the stochastic Helmholtz equation driven by the white noise. In [8], it was shown that the correlation of the random source could be determined uniquely by the correlation of the random wave field. Recently, an effective computational model was developed in [4–7, 20–22], the goal was to reconstruct the statistical properties of the random source such as the mean and variance from the boundary measurement of the radiated random wave field at multiple frequencies.

The work is very rare for the inverse random source problems of the fractional diffusion equations. In [30], the authors presented a study on the random source problem for the fractional diffusion equation. Specifically, they considered the initial-boundary value problem

$$\begin{cases} \partial_t^\alpha u(x,t) - \Delta u(x,t) = f(x)h(t) + g(x)\dot{W}(t), & (x,t) \in D \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial D \times [0,T], \\ u(x,0) = 0, & x \in \overline{D}, \end{cases} \tag{1.1}$$

where D is a bounded domain with the Lipschitz boundary ∂D, f and g are deterministic functions with compact supports contained in D, h is also a deterministic function, W and $\dot{W}$ are the Brownian motion and the white noise, respectively, and $\partial_t^\alpha u(x,t), 0 < \alpha \leqslant 1$ is the Caputo fractional derivative given by

$$\partial_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{\mathrm{d}s}{(t-s)^\alpha}, & \alpha \in (0,1), \\ \partial_t u(t,x), & \alpha = 1. \end{cases}$$

Here $\Gamma(\alpha) = \int_0^\infty \mathrm{e}^{-s} s^{\alpha-1} \mathrm{d}s$ is the Gamma function. For the model problem (1.1), the authors studied the inverse problem of reconstructing $f(x)$ and $g(x)$ up to sign, i.e. $\pm g(x)$ or equivalently $|g(x)|$, from the statistics of the final time data $u(x,T)$ with $\frac{1}{2} < \alpha < 1$. It was shown that f and $|g|$ can be uniquely determined by the expectation and covariance of the final data, respectively. Besides, they also showed that the inverse problem is not stable in the sense that a small variance of the data may lead to a huge error of the reconstruction. Naturally, one may ask the following two questions:

Q1. Can the results be extended to $0 < \alpha < 1$ for the Brownian motion?
Q2. Can the results be extended to the fractional Brownian motion?

Motivated by above reasons, the main purpose of this paper is to study the inverse source problem for the time fractional diffusion equation, where the source is assumed to be driven by a more general stochastic process: the fractional Brownian motion $B^H(t)$ with $\alpha \in (0, 1]$ and $H \in (0, 1)$, where H is called the Hurst index of the fractional Brownian motion (FBM). The FBM is a widely used stochastic process that is particularly suited to model short- and long-range dependent phenomena, and anomalous diffusion in a variety of fields including physics (e.g. motion of ultra-cold atoms), hydrology (e.g. ground water flow and solute transport), biology (e.g. motion of tracer particles in living biological cells), network research (e.g. traffic in communication networks), financial mathematics (e.g. derivatives of the stock), etc. Clearly, the model equation (1.1) is reduced to the classical heat conduction equation with the Brownian motion for $\alpha = 1$. In this work, we give affirmative answers to Q1 and Q2. For Q1, due to the singular integral (see lemma 3.4 in [30] or the proof later in this paper), the results can not be extended; for Q2, the results can be extended as long as $\alpha + H > 1$. For the restriction $\alpha + H > 1$, it is not difficult to understand since both H and α imply some smoothness requirement of the solution for the model equation.

The rest of this paper is organized as follows. In section 2, we introduce some preliminaries for the time-fractional diffusion equation and the Mittag–Leffler function. Section 3 is concerned with the well-posedness of the direct problem. Section 4 is devoted to the inverse problem. The two cases $H \in (0, \frac{1}{2})$ and $H \in (\frac{1}{2}, 1)$ are discussed separately for both of the direct and inverse problems. The paper is concluded with some general remarks and directions for future research in section 5. To make the paper easily accessible, some necessary notation and useful results are provided in appendix on the fractional Brownian motion.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which the fractional Brownian motion B^H is defined with $H \in (0, 1)$. Here Ω is a sample space, $\mathcal{F}$ is a σ-algebra on Ω, and $\mathbb{P}$ is a probability measure on the measurable space $(\Omega, \mathcal{F})$. For a random variable X, we denote by $\mathbb{E}(X)$ and $\mathbb{V}(X) = \mathbb{E}(X - \mathbb{E}(X))^2 = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$ the expectation and variance of X, respectively. For two random variables X and Y, $\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))]$ denotes the covariance of X and Y. In the sequel, the dependence of random variables on the sample $\omega \in \Omega$ will be omitted unless it is necessary to avoid confusions.

Consider initial-boundary value problem of the fractional diffusion equation with a random source driven by the fractional Brownian motion

$$\begin{cases} \partial_t^\alpha u(x,t) - \Delta u(x,t) = f(x)h(t) + g(x)\dot{B}^H(t), & (x,t) \in D \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial D \times [0,T], \\ u(x,0) = 0, & x \in \overline{D}. \end{cases} \tag{2.1}$$

Let $\{\lambda_k, \varphi_k\}_{k=1}^\infty$ be an eigen-system of the operator $-\Delta$ with the homogeneous Dirichlet boundary condition. The eigenvalues satisfy $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots$ with $\lambda_k \to \infty$ as

$k \to \infty$, and the eigen-functions $\{\varphi_k\}_{k=1}^{\infty}$ form a complete and orthonormal basis for $L^2(D)$. Hence, any function v in $L^2(D)$ can be written as

$$v = \sum_{k=1}^{\infty} v_k \varphi_k$$

with coefficients $\{v_k\}_{k=1}^{\infty}$. Noting that $\dot{B}^H$ is a distribution instead of a classical function, the equation (2.1) does not hold pointwisely. Instead, it should be interpreted as an integral equation, and its mild solution is defined as follows based on the Mittag–Leffler function. We refer to [12] and references therein for more details on the Mittag–Leffler function.

Definition 1. A stochastic process u taking values in $L^2(D)$ is called a mild solution of (2.1) if

$$u(x,t) = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big((t-\tau)^{\alpha}\Delta\big) f(x) h(\tau) \mathrm{d}\tau + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big((t-\tau)^{\alpha}\Delta\big) g(x) \mathrm{d}B^H(\tau)$$

is well-defined almost surely, where $E_{\alpha,\beta}$ is the two-parametric Mittag–Leffler function defined as

$$E_{\alpha,\beta}(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(l\alpha+\beta)}, \quad \alpha, \beta \in \mathbb{R}.$$

If $u \in L^2(D)$ is a mild solution of (2.1), then we have equivalently

$$u(\cdot,t) = \sum_{k=1}^{\infty} u_k(t) \varphi_k, \tag{2.2}$$

where

$$\begin{aligned} u_k(t) &= (u(\cdot,t), \varphi_k)_{L^2(D)} \\ &= f_k \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (t-\tau)^{\alpha}) h(\tau) \mathrm{d}\tau + g_k \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (t-\tau)^{\alpha}) \mathrm{d}B^H(\tau) \\ &:= I_{k,1}(t) + I_{k,2}(t), \end{aligned} \tag{2.3}$$

where $f_k = (f, \varphi_k)_{L^2(D)}$ and $g_k = (g, \varphi_k)_{L^2(D)}$, and $u_k(t)$ satisfies the stochastic fractional differential equation

$$\begin{cases} D_t^{\alpha} u_k(t) + \lambda_k u_k(t) = f_k h(t) + g_k \dot{B}^H(t), & t \in (0,T), \\ u_k(0) = 0, \end{cases} \tag{2.4}$$

where D_t^{α} denotes the fractional derivative for ordinary differential equations.

In particular, if $g \equiv 0$, the stochastic fractional differential equation (2.4) reduces to a deterministic fractional differential equation

$$\begin{cases} D_t^{\alpha} u_k(t) + \lambda_k u_k(t) = f_k h(t), & t \in (0,T), \\ u_k(0) = 0, \end{cases}$$

whose solution can be obtained directly by applying the following lemma.

Lemma 2.1. *Consider the Cauchy problem for the fractional differential equation:*

$$\begin{cases} D_t^{\alpha} v(t) - \lambda v(t) = f(t), & t \in (0,T), \\ \frac{\mathrm{d}^n v}{\mathrm{d}t^n}(0) = v_n, & n = 0, \ldots, \lfloor \alpha \rfloor. \end{cases} \tag{2.5}$$

If $f(t) \in C^{0,\gamma}$ *with* $\gamma \in [0, \alpha]$*, then the Cauchy problem (2.5) has a unique solution given by*

$$v(t) = \sum_{n=0}^{\lfloor\alpha\rfloor} v_n t^n E_{\alpha,n+1}(\lambda t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha) f(\tau) \mathrm{d}\tau.$$

The proof of the lemma above can be found in [17, page 230] or [32, example 4.3] by utilizing some *a priori* estimates of the Mittag–Leffler function. The Mittag–Leffler function is important when proving the well-posedness of the stochastic problem (2.1). For convenience, some of its properties are given as follows.

Lemma 2.2 ([32, theorem 1.6]). *If* $\alpha \in (0, 2)$*,* β *is an arbitrary real number and* μ *satisfies* $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$*, then there exists a positive constant* $C = C(\alpha, \beta)$ *such that*

$$|E_{\alpha,\beta}(z)| \leqslant \frac{C}{1+|z|}, \quad \mu \leqslant |\arg(z)| \leqslant \pi, \quad |z| \geqslant 0.$$

Lemma 2.3 ([35, lemma 3.2]). *For* $\lambda > 0, \alpha > 0$*, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), \quad t > 0.$$

Lemma 2.4. *For* $\lambda, z \in \mathbb{C}$*, we have*

$$\frac{\mathrm{d}}{\mathrm{d}z}(z^{\alpha-1} E_{\alpha,\alpha}(-\lambda z^\alpha)) = z^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda z^\alpha).$$

Proof. By [12, 4.3.1]

$$\frac{\mathrm{d}}{\mathrm{d}z}(z^{\alpha-1} E_{\alpha,\alpha}(z^\alpha)) = z^{\alpha-2} E_{\alpha,\alpha-1}(z^\alpha),$$

which completes the proof after using the chain rule. □

Lemma 2.5. *For any* $s \in (0, t)$ *and* $\lambda_k > 0$*, there exists some constant* $C = C(\alpha)$ *such that*

$$|t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k t^\alpha) - s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k s^\alpha)| \leqslant C \int_s^t \frac{r^{\alpha-2}}{1+\lambda_k r^\alpha} \mathrm{d}r.$$

Proof. By lemmas 2.4 and 2.2, we have

$$\frac{\mathrm{d}}{\mathrm{d}r}[r^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k r^\alpha)] = r^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_k r^\alpha)$$

and

$$|E_{\alpha,\alpha-1}(-\lambda_k r^\alpha)| \leqslant \frac{C}{1+\lambda_k r^\alpha}.$$

A simple calculation yields that

$$\begin{aligned}|t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k t^{\alpha}) - s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k s^{\alpha})| &= \int_s^t r^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_k r^{\alpha})\mathrm{d}r\\ &\leqslant C\int_s^t \frac{r^{\alpha-2}}{1+\lambda_k r^{\alpha}}\mathrm{d}r,\end{aligned}$$

which completes the proof. □

Lemma 2.6 ([34]). *For $\alpha \in [0,1]$, the function $E_{\alpha,1}$ is completely monotonic, i.e.*

$$(-1)^n \frac{\mathrm{d}^n E_{\alpha,1}(-x)}{\mathrm{d}x^n} \geqslant 0, \quad x \geqslant 0,\ n \in \mathbb{N}.$$

By lemmas 2.3 and 2.6, we have the following property of $E_{\alpha,\alpha}$.

Lemma 2.7. *For $\alpha \in (0,1]$ and $x \geqslant 0$, there holds $E_{\alpha,\alpha}(-x) \geqslant 0$ and $x^{\alpha-1}E_{\alpha,\alpha}(-\lambda x^{\alpha})$ is monotonically decreasing.*

3. The direct problem

In this section, the well-posedness of the direct problem is studied. We show that the mild solution (2.2) of the initial-boundary value problem (2.1) is well-defined under the following assumptions.

Assumption 1. *Let $H \in (0,1)$ and $\alpha \in (0,1]$ such that $\alpha + H > 1$. Let $f, g \in L^2(D)$ with $\|g\|_{L^2(D)} \neq 0$. Assume in addition that $h \in L^{\infty}(0,T)$ is a nonnegative function, i.e. $h \geqslant 0$, whose support has a positive measure.*

It is easy to note that the mild solution (2.2) satisfies

$$\begin{aligned}\|u(\cdot,t)\|^2_{L^2(D)} &= \left\|\sum_{k=1}^{\infty}(I_{k,1}(t)+I_{k,2}(t))\varphi_k(\cdot)\right\|^2_{L^2(D)}\\ &= \sum_{k=1}^{\infty}(I_{k,1}(t)+I_{k,2}(t))^2 \leqslant 2\left(\sum_{k=1}^{\infty}I^2_{k,1}(t)+\sum_{k=1}^{\infty}I^2_{k,2}(t)\right),\end{aligned}$$

where $I_{k,1}$ and $I_{k,2}$ are defined by (2.3) with $I_{k,2}$ being a stochastic integral. Hence,

$$\begin{aligned}\mathbb{E}\left[\|u\|^2_{L^2(D\times[0,T])}\right] &= \mathbb{E}\left[\int_0^T \|u(\cdot,t)\|^2_{L^2(D)}\mathrm{d}t\right]\\ &\lesssim \mathbb{E}\left[\int_0^T\left(\sum_{k=1}^{\infty}I^2_{k,1}(t)+\sum_{k=1}^{\infty}I^2_{k,2}(t)\right)\mathrm{d}t\right]\\ &= \int_0^T\left(\sum_{k=1}^{\infty}I^2_{k,1}(t)\right)\mathrm{d}t + \mathbb{E}\left[\int_0^T\left(\sum_{k=1}^{\infty}I^2_{k,2}(t)\right)\mathrm{d}t\right]\\ &= \sum_{k=1}^{\infty}\|I_{k,1}\|^2_{L^2(0,T)} + \int_0^T\left(\sum_{k=1}^{\infty}\mathbb{E}\left[I^2_{k,2}(t)\right]\right)\mathrm{d}t\\ &:= S_1 + S_2.\end{aligned} \tag{3.1}$$

Hereinafter $a \lesssim b$ stands for $a \leqslant Cb$, where $C > 0$ is a constant. It then suffices to show the boundedness of S_1 and S_2, respectively.

For S_1, by denoting $G_{\alpha,k}(t) = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k t^\alpha)$, it is easy to see that $I_{k,1}(t) = f_k\,(G_{\alpha,k} * h)\,(t)$. Applying the Young convolution inequality yields

$$\|I_{k,1}\|_{L^2(0,T)} \leqslant T^{\frac{1}{2}}|f_k|\|G_{\alpha,k}\|_{L^1(0,T)}\|h\|_{L^\infty(0,T)}. \tag{3.2}$$

It follows from lemma 2.2 that

$$\|G_{\alpha,k}\|_{L^1(0,T)} = \int_0^T |t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k t^\alpha)|\mathrm{d}t \lesssim \int_0^T t^{\alpha-1}\mathrm{d}t = \frac{T^\alpha}{\alpha}. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain

$$S_1 = \sum_{k=1}^\infty \|I_{k,1}\|^2_{L^2(0,T)} \lesssim \frac{T^{2\alpha+1}}{\alpha^2}\sum_{k=1}^\infty |f_k|^2\|h\|^2_{L^\infty(0,T)} = \frac{T^{2\alpha+1}}{\alpha^2}\|h\|^2_{L^\infty(0,T)}\|f\|^2_{L^2(D)}. \tag{3.4}$$

For S_2, it is easy to note that

$$\begin{aligned}\mathbb{E}\left[I^2_{k,2}(t)\right] &= \mathbb{E}\left[g_k^2\left(\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\tau)^\alpha)\mathrm{d}B^H(\tau)\right)^2\right]\\ &= g_k^2\mathbb{E}\left[\left(\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\tau)^\alpha)\mathrm{d}B^H(\tau)\right)^2\right].\end{aligned} \tag{3.5}$$

The case $H = \frac{1}{2}$ and $\alpha \in (\frac{1}{2}, 1)$ has been studied in [30]. We investigate the more general case $\alpha \in (0, 1]$ and $H \in (0, 1) \setminus \{\frac{1}{2}\}$, in which Itô's isometry is not available and properties of fractional Brownian motions will be used instead.

Next, we discuss the cases $H \in (0, \frac{1}{2})$ and $H \in (\frac{1}{2}, 1)$ separately since the covariance operator of B^H in $I_{k,2}$ has different forms in these two cases.

3.1. The case $H \in (0, \frac{1}{2})$

It follows from appendix on the fractional Brownian motion B^H that the stochastic integral in (3.5) with respect to B^H satisfies

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\tau)^{\alpha})\mathrm{d}B^H(\tau)\right)^2\right] \\
&= \int_0^t \left[K_{H,t}^{*}\left((t-\cdot)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\cdot)^{\alpha})\right)\right]^2(\tau)\mathrm{d}\tau \\
&= \int_0^t \Bigg[K_H(t,\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\tau)^{\alpha}) \\
&\quad + \int_\tau^t \left[(t-u)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-u)^{\alpha}) - (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\tau)^{\alpha})\right]\frac{\partial K_H(u,\tau)}{\partial u}\mathrm{d}u\Bigg]^2\mathrm{d}\tau \\
&\lesssim \int_0^t \left[\left(\frac{t}{\tau}\right)^{H-\frac{1}{2}}(t-\tau)^{\alpha+H-\frac{3}{2}}E_{\alpha,\alpha}(-\lambda_k(t-\tau)^{\alpha})\right]^2\mathrm{d}\tau \\
&\quad + \int_0^t \tau^{1-2H}\left[\left(\int_\tau^t u^{H-\frac{3}{2}}(u-\tau)^{H-\frac{1}{2}}\mathrm{d}u\right)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\tau)^{\alpha})\right]^2\mathrm{d}\tau \\
&\quad + \int_0^t \left[\int_\tau^t \left[(t-u)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-u)^{\alpha}) - (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\tau)^{\alpha})\right]\frac{\partial K_H(u,\tau)}{\partial u}\mathrm{d}u\right]^2\mathrm{d}\tau \\
&:= I_1(t) + I_2(t) + I_3(t),
\end{aligned} \tag{3.6}
$$

where $K_H(u,\tau)$ denotes the kernel of B^H given by (A.5).

For $I_1(t)$, according to lemma 2.2, we get

$$
\begin{aligned}
I_1(t) &= \int_0^t \left(\frac{t}{\tau}\right)^{2H-1}(t-\tau)^{2\alpha+2H-3}E_{\alpha,\alpha}^2(-\lambda_k(t-\tau)^{\alpha})\mathrm{d}\tau \\
&\lesssim t^{2H-1}\int_0^t \tau^{1-2H}(t-\tau)^{2\alpha+2H-3}\mathrm{d}\tau \\
&\leqslant \frac{t^{2\alpha+2H-2}}{2\alpha+2H-2}
\end{aligned} \tag{3.7}
$$

under conditions $H \in (0, \frac{1}{2})$ and $\alpha + H > 1$.

For $I_2(t)$, using lemma 2.2, we have

$$
\begin{aligned}
I_2(t) &= \int_0^t \tau^{1-2H}\left(\int_\tau^t u^{H-\frac{3}{2}}(u-\tau)^{H-\frac{1}{2}}\mathrm{d}u\right)^2(t-\tau)^{2\alpha-2}E_{\alpha,\alpha}^2(-\lambda_k(t-\tau)^{\alpha})\mathrm{d}\tau \\
&\lesssim \int_0^t \tau^{1-2H}\left(\int_\tau^t u^{H-\frac{3}{2}}(u-\tau)^{H-\frac{1}{2}}\mathrm{d}u\right)^2(t-\tau)^{2\alpha-2}\mathrm{d}\tau.
\end{aligned} \tag{3.8}
$$

Since $H > 0$, the integral $\int_\tau^t u^{H-\frac{3}{2}}(u-\tau)^{H-\frac{1}{2}}\mathrm{d}u$ is well-defined. Furthermore, we have from the binomial expansion that

$$\begin{aligned}
&\int_\tau^t u^{H-\frac{3}{2}}(u-\tau)^{H-\frac{1}{2}}\mathrm{d}u \\
&= \int_\tau^t u^{2H-2}\left[\sum_{n=0}^{\infty}\binom{H-\frac{1}{2}}{n}\left(-\frac{\tau}{u}\right)^n\right]\mathrm{d}u \\
&= \sum_{n=0}^{\infty}\binom{H-\frac{1}{2}}{n}(-1)^n\tau^n\frac{t^{2H-1-n}-\tau^{2H-1-n}}{2H-1-n} \\
&\leqslant (t^{2H-1}-\tau^{2H-1})\sum_{n=0}^{\infty}\binom{H-\frac{1}{2}}{n}\frac{(-1)^n}{2H-1-n} \\
&\lesssim t^{2H-1}-\tau^{2H-1}.
\end{aligned}$$

Combing the above estimate, we obtain from (3.8) that

$$\begin{aligned}
I_2(t) &\lesssim \int_0^t \tau^{1-2H}\left(t^{4H-2}+\tau^{4H-2}\right)(t-\tau)^{2\alpha-2}\mathrm{d}\tau \\
&= \frac{t^{2H+2\alpha-2}}{2\alpha-1}+t^{2\alpha-2}\int_0^t \tau^{2H-1}\left[\sum_{n=0}^{\infty}\binom{2\alpha-2}{n}\left(-\frac{\tau}{t}\right)^n\right]\mathrm{d}\tau \\
&\lesssim t^{2H+2\alpha-2}+t^{2H+2\alpha-2}\sum_{n=0}^{\infty}\binom{2\alpha-2}{n}\frac{(-1)^n}{2H+n} \\
&\lesssim t^{2H+2\alpha-2},
\end{aligned} \tag{3.9}$$

where we have used conditions $H\in(0,\frac{1}{2})$ and $\alpha\in(\frac{1}{2},1)$ due to $\alpha+H>1$.

For $I_3(t)$, based on lemma 2.5, there holds

$$\begin{aligned}
&|(t-u)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-u)^\alpha)-(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\tau)^\alpha)| \\
&\lesssim \int_{t-u}^{t-\tau} r^{\alpha-2}\mathrm{d}r \lesssim (t-u)^{\alpha-\frac{3}{2}}(u-\tau)^{\frac{1}{2}}, \quad 0<\tau<u<t.
\end{aligned}$$

Hence,

$$I_3(t)\lesssim\int_0^t\left[\int_\tau^t(t-u)^{\alpha-\frac{3}{2}}(u-\tau)^{H-1}\left(\frac{u}{\tau}\right)^{H-\frac{1}{2}}\mathrm{d}u\right]^2\mathrm{d}\tau,$$

which is well-defined under conditions $H\in(0,\frac{1}{2})$ and $\alpha+H>1$. In addition, we have $\left(\frac{u}{\tau}\right)^{H-\frac{1}{2}}<1$ for $0<\tau<u<t$, which leads to

$$\begin{aligned}
I_3(t) &\lesssim \int_0^t\left[\int_0^{t-\tau}(t-\tau-r)^{\alpha-\frac{3}{2}}r^{H-1}\mathrm{d}r\right]^2\mathrm{d}\tau \\
&= \left[\sum_{n=0}^{\infty}\binom{\alpha-\frac{3}{2}}{n}\frac{(-1)^n}{n+H}\right]^2\int_0^t(t-\tau)^{2\alpha+2H-3}\mathrm{d}\tau \\
&\lesssim t^{2\alpha+2H-2}.
\end{aligned} \tag{3.10}$$

Plugging (3.7), (3.9), and (3.10) into (3.6), we obtain for $H\in(0,\frac{1}{2})$ that

$$\mathbb{E}\left|\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\tau)^\alpha)\mathrm{d}B^H(\tau)\right|^2\lesssim t^{2\alpha+2H-2}. \tag{3.11}$$

3.2. The case $H \in (\frac{1}{2}, 1)$

It follows from appendix again that the stochastic integral in (3.5) with respect to B^H satisfies

$$\mathbb{E}\left[\left(\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\tau)^{\alpha})\mathrm{d}B^H(\tau)\right)^2\right]$$
$$= \alpha_H \int_0^t \int_0^t (t-p)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-p)^{\alpha})(t-q)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-q)^{\alpha})|p-q|^{2H-2}\mathrm{d}p\mathrm{d}q$$

By lemma 2.2, we have

$$\mathbb{E}\left[\left(\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\tau)^{\alpha})\mathrm{d}B^H(\tau)\right)^2\right]$$
$$\lesssim \alpha_H \int_0^t \int_0^t (t-p)^{\alpha-1}(t-q)^{\alpha-1}|p-q|^{2H-2}\mathrm{d}p\mathrm{d}q$$
$$= 2\alpha_H \int_0^t \int_q^t p^{\alpha-1}q^{\alpha-1}(p-q)^{2H-2}\mathrm{d}p\mathrm{d}q$$
$$= 2\alpha_H \int_0^t q^{\alpha-1}\sum_{n=0}^{\infty}\binom{2H-2}{n}(-1)^n q^n\left(\int_q^t p^{\alpha+2H-3-n}\mathrm{d}p\right)\mathrm{d}q,$$

where we used the binomial expansion since $|\frac{q}{p}| < 1$.

To deal with the singularity of the above integral, we consider the following two cases, respectively.

Case I: $2H-2=-\alpha$. It follows from the straightforward calculations that

$$\int_0^t q^{\alpha-1}\sum_{n=0}^{\infty}\binom{2H-2}{n}(-1)^n q^n\left(\int_q^t p^{\alpha+2H-3-n}\mathrm{d}p\right)\mathrm{d}q$$
$$= \int_0^t\left[q^{\alpha-1}\int_q^t p^{-1}\mathrm{d}p + q^{\alpha-1}\sum_{n=1}^{\infty}\binom{2H-2}{n}(-1)^n q^n\left(\int_q^t p^{-1-n}\mathrm{d}p\right)\right]\mathrm{d}q$$
$$= \ln t\int_0^t q^{\alpha-1}\mathrm{d}q - \lim_{\epsilon\to 0+}\int_{\epsilon}^t q^{\alpha-1}\ln q\mathrm{d}q + \sum_{n=1}^{\infty}\binom{2H-2}{n}\frac{(-1)^n}{n}\left(\int_0^t q^{\alpha-1}\mathrm{d}q - t^{-n}\int_0^t q^{\alpha+n-1}\mathrm{d}q\right)$$
$$= \frac{t^{\alpha}}{\alpha}\sum_{n=0}^{\infty}\binom{2H-2}{n}(-1)^n\frac{1}{\alpha+n}.$$

Moreover, the condition $\alpha > 0$ can guarantee the convergence of the singular integrals.

Case II: $2H-2 \neq -\alpha$. Similarly, we have from straightforward calculations that

$$\int_0^t q^{\alpha-1}\sum_{n=0}^{\infty}\binom{2H-2}{n}(-1)^n q^n\left(\int_q^t p^{\alpha+2H-3-n}\mathrm{d}p\right)\mathrm{d}q$$
$$= \sum_{n=0}^{\infty}\binom{2H-2}{n}(-1)^n\frac{1}{\alpha+2H-2-n}\left(t^{\alpha+2H-2-n}\int_0^t q^{n+\alpha-1}\mathrm{d}q - \int_0^t q^{2\alpha+2H-3}\mathrm{d}q\right)$$
$$= \frac{t^{2\alpha+2H-2}}{2\alpha+2H-2}\sum_{n=0}^{\infty}\binom{2H-2}{n}(-1)^n\frac{1}{\alpha+n},$$

where the conditions $\alpha > 0$ and $\alpha + H > 1$ are needed to ensure the convergence of the singular integrals.

Combining Case I and Case II, we get

$$\int_0^t q^{\alpha-1} \sum_{n=0}^{\infty} \binom{2H-2}{n} (-1)^n q^n \left(\int_q^t p^{\alpha+2H-3-n} \mathrm{d}p \right) \mathrm{d}q$$
$$= \frac{t^{2\alpha+2H-2}}{2\alpha+2H-2} \sum_{n=0}^{\infty} \binom{2H-2}{n} (-1)^n \frac{1}{\alpha+n}.$$

It is easy to know from the asymptotic expansion for the binomial coefficients that the series in above formula is convergent. Therefore,

$$\mathbb{E}\left[\left(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (t-\tau)^\alpha) \mathrm{d}B^H(\tau) \right)^2 \right] \lesssim t^{2\alpha+2H-2}. \tag{3.12}$$

3.3. Estimates of the solution

In this section, we discuss the stability of the solution. From (3.11) and (3.12) and the analysis for $H = \frac{1}{2}$ in [30], it holds

$$\mathbb{E}\left[\left(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (t-\tau)^\alpha) \mathrm{d}B^H(\tau) \right)^2 \right] \lesssim t^{2\alpha+2H-2} \tag{3.13}$$

for H, α satisfying assumption 1. With the help of (3.13), we obtain the stability estimates for the mild solution (2.2).

Theorem 3.1. *Let assumption 1 hold. Then the stochastic process u given in (2.2) satisfies*

$$\mathbb{E}\left[\|u\|^2_{L^2(D\times[0,T])} \right] \lesssim \|h\|^2_{L^\infty(0,T)} \|f\|^2_{L^2(D)} + T^{2\alpha+2H-1} \|g\|^2_{L^2(D)}.$$

Proof. The proof follows easily from (3.1), (3.4), (3.5) and (3.13). Especially, one can check it is also true for $\alpha = 1$. □

Moreover, boundedness of the solution u in stronger norms can also be obtained.

Theorem 3.2. *Let assumption 1 hold. The supremum of the expected norm of the solution satisfies*

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\left[\|u(\cdot,t)\|^2_{L^2(D)} \right] \lesssim \|h\|^2_{L^\infty(0,T)} \|f\|^2_{L^2(D)} + T^{2\alpha+2H-2} \|g\|^2_{L^2(D)}.$$

Moreover, if in addition $g \in H^2(D)$, there also holds

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\left[\|u(\cdot,t)\|^2_{H^2(D)} \right] \lesssim \|h\|^2_{L^\infty(0,T)} \|f\|^2_{L^2(D)} + T^{2\alpha+2H-2} \|g\|^2_{H^2(D)}.$$

Proof. The theorem can be proved following similar arguments for the case $H = \frac{1}{2}, \alpha \in (\frac{1}{2}, 1)$ in [30, lemma 3.5]. The details are omitted for brevity. □

We would like to mention that the procedure above is applicable to a more general case:

$$\begin{cases} \partial_t^\alpha u(x,t) + (-\Delta)^s u(x,t) = f(x)h(t) + g(x)\dot{B}^H(t), & (x,t) \in D \times (0,T), \\ u(x,t) = 0, & (x,t) \in \mathbb{R}^n \backslash D \times [0,T], \\ u(x,0) = 0, & x \in D, \end{cases}$$

where $(-\Delta)^s$, $0 < s \leqslant 1$ is the fractional Laplacian, and H, α satisfying assumption 1. The fractional Laplacian operator $(-\Delta)^s$ is defined by (see e.g. [29, formula (3.1)]):

$$(-\Delta)^s u(x,t) = C_{n,s}\text{p.v.} \int_{\mathbb{R}^n} \frac{u(x,t) - u(y,t)}{|x-y|^{n+2s}} \mathrm{d}y,$$

where $C_{n,s}$ is a positive constant depending on n and s. Using the properties of the eigensystem for the fractional Laplacian operator $(-\Delta)^s$ given in [46, proposition 2.1], one can also use the method of separation of variables to obtain a mild solution. Then all the results are the same except the second result in theorem 3.2. But it can be easily shown that if $g \in H^s(D)$, then there holds

$$\sup_{0 \leqslant t \leqslant T} \mathbb{E}\left[\|u(\cdot,t)\|^2_{H^s(D)}\right] \lesssim \|h\|^2_{L^\infty(0,T)} \|f\|^2_{L^2(D)} + T^{2\alpha+2H-2} \|g\|^2_{H^s(D)}.$$

We refer to [29] and references therein for more details about the fractional Sobolev space $H^s(D)$.

4. The inverse problem

In this section, we consider the inverse problem of reconstructing f and $|g|$ from the empirical expectation and correlations of the final time data $u(x,T)$. More specifically, the data may be assumed to be given by

$$u_k(T) := (u(\cdot,T), \varphi_k(\cdot))_{L^2(D)}.$$

We shall discuss the uniqueness and the issue of instability, separately.

4.1. The uniqueness of reconstruction

It follows from (2.2) and (3.5) that

$$\mathbb{E}(u_k(T)) = f_k \int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (T-\tau)^\alpha) h(\tau) \mathrm{d}\tau, \tag{4.1}$$

$$\mathbb{V}(u_k(T)) = g_k^2\, \mathbb{E}\left[\left(\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (T-\tau)^\alpha) \mathrm{d}B^H(\tau)\right)^2\right] \tag{4.2}$$

and

$$\begin{aligned} \mathrm{Cov}(u_k(T), u_l(T)) = g_k g_l\, \mathbb{E}\Bigg[&\left(\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (T-\tau)^\alpha) \mathrm{d}B^H(\tau)\right) \\ &\cdot \left(\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_l (T-\tau)^\alpha) \mathrm{d}B^H(\tau)\right)\Bigg]. \end{aligned} \tag{4.3}$$

Lemma 4.1. *Let assumption 1 hold. For each fixed $k \in \mathbb{N}$, there exists a constant $C_1 > 0$ such that*

$$\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (T-\tau)^\alpha) h(\tau) \mathrm{d}\tau \geqslant C_1 > 0.$$

Proof. Letting $\tilde{\tau} = T - \tau$, we have from lemma 2.7 and assumption 1 that

$$\begin{aligned}
&\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (T-\tau)^\alpha) h(\tau) \mathrm{d}\tau \\
&= \int_0^T \tilde{\tau}^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k \tilde{\tau}^\alpha) h(T-\tilde{\tau}) \mathrm{d}\tilde{\tau} \\
&\geqslant T^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k T^\alpha) \int_0^T h(T-\tilde{\tau}) \mathrm{d}\tilde{\tau} \\
&\geqslant c_h T^\alpha E_{\alpha,\alpha}(-\lambda_k T^\alpha) := C_1 > 0,
\end{aligned}$$

here $c_h := \int_0^T h(T-\tilde{\tau}) \mathrm{d}\tilde{\tau}$. It completes the proof. □

Lemma 4.2. *Let assumption 1 hold. For each fixed $k, l \in \mathbb{N}$, there exists a constant $C_2 > 0$ such that*

$$\begin{aligned}
&\mathbb{E}\Bigg[\left(\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (T-\tau)^\alpha) \mathrm{d}B^H(\tau) \right) \\
&\quad \cdot \left(\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_l (T-\tau)^\alpha) \mathrm{d}B^H(\tau) \right) \Bigg] \geqslant C_2 > 0.
\end{aligned}$$

Proof. Denote $\phi_k(s) = (T-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (T-s)^\alpha)$ and

$$I_{kl} = \mathbb{E}\left[\left(\int_0^T \phi_k(\tau) \mathrm{d}B^H(\tau) \right) \left(\int_0^T \phi_l(\tau) \mathrm{d}B^H(\tau) \right) \right].$$

We estimate I_{kl} for $H \in (\frac{1}{2}, 1)$ and $H \in (0, \frac{1}{2})$, separately.

For $H \in (\frac{1}{2}, 1)$, we have from (A.4) that

$$\begin{aligned}
I_{kl} &= \alpha_H \int_0^T \int_0^T \phi_k(r) \phi_l(u) |r-u|^{2H-2} \mathrm{d}u \mathrm{d}r \\
&= \alpha_H \int_0^T \int_0^T (T-r)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (T-r)^\alpha) (T-u)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_l (T-u)^\alpha) |r-u|^{2H-2} \mathrm{d}u \mathrm{d}r \\
&= \alpha_H \int_0^T \int_0^T r^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k r^\alpha) u^{\alpha-1} E_{\alpha,\alpha}(-\lambda_l u^\alpha) |u-r|^{2H-2} \mathrm{d}u \mathrm{d}r.
\end{aligned}$$

By lemma 2.7, the function $\tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k \tau^\alpha)$ is a monotonically decreasing function with respect to τ. Hence

$$\begin{aligned} I_{kl} &\geqslant \alpha_H \int_0^T \int_0^T T^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k T^\alpha) T^{\alpha-1} E_{\alpha,\alpha}(-\lambda_l T^\alpha) |u-r|^{2H-2} \mathrm{d}u \mathrm{d}r \\ &= \frac{\alpha_H T^{2(\alpha+H-1)}}{H(2H-1)} E_{\alpha,\alpha}(-\lambda_k T^\alpha) E_{\alpha,\alpha}(-\lambda_l T^\alpha) =: C_2 > 0. \end{aligned}$$

For $H \in (0, \frac{1}{2})$, by (A.6), we have

$$I_{kl} = \langle K^*_{H,T}\phi_k, K^*_{H,T}\phi_l \rangle_{L^2(0,T)},$$

where

$$(K^*_{H,T}\phi_k)(s) = K_H(T,s)\phi_k(s) + \int_s^T (\phi_k(u) - \phi_k(s)) \frac{\partial K_H(u,s)}{\partial u} \mathrm{d}u,$$

$$K_H(T,s) = c_H \left[\left(\frac{T}{s} \right)^{H-\frac{1}{2}} (T-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2} \right) s^{\frac{1}{2}-H} \int_s^T u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} \mathrm{d}u \right],$$

$$\frac{\partial K_H(u,s)}{\partial u} = c_H \left(\frac{u}{s} \right)^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}}.$$

Obviously, $K_H(T,s) > 0$ since $H \in (0, \frac{1}{2}), 0 < s < T$. It follows from the mean value theorem that

$$\begin{aligned} &\int_s^T (\phi_k(u) - \phi_k(s)) \frac{\partial K_H(u,s)}{\partial u} \mathrm{d}u \\ &= c_H \int_s^T (\phi_k(u) - \phi_k(s)) \left(\frac{u}{s} \right)^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} \mathrm{d}u \\ &= c_H \int_s^T \phi_k'(u_k^*) \left(\frac{u}{s} \right)^{H-\frac{1}{2}} (u-s)^{H-\frac{1}{2}} \mathrm{d}u \qquad (s < u_k^* < u < T) \\ &= c_H \phi_k'(u_k^{**}) \int_s^T \left(\frac{u}{s} \right)^{H-\frac{1}{2}} (u-s)^{H-\frac{1}{2}} \mathrm{d}u \qquad (s < u_k^{**} < T) \\ &= M_H(s) \phi_k'(u_k^{**}), \end{aligned}$$

where

$$M_H(s) = c_H \int_s^T \left(\frac{u}{s} \right)^{H-\frac{1}{2}} (u-s)^{H-\frac{1}{2}} \mathrm{d}u > 0.$$

A simple calculation gives that

$$\begin{aligned} I_{kl} &= \langle K_H(T,s)\phi_k(s) + M_H(s)\phi_k'(u_k^{**}), K_H(T,s)\phi_l(s) + M_H(s)\phi_l'(u_l^{**}) \rangle_{L^2(0,T)} \\ &= \langle K_H(T,s)\phi_k(s), K_H(T,s)\phi_l(s) \rangle_{L^2(0,T)} + \langle M_H(s)\phi_k'(u_k^{**}), M_H(s)\phi_l'(u_l^{**}) \rangle_{L^2(0,T)} \\ &\quad + \langle K_H(T,s)\phi_k(s), M_H(s)\phi_l'(u_l^{**}) \rangle_{L^2(0,T)} + \langle M_H(s)\phi_k'(u_k^{**}), K_H(T,s)\phi_l(s) \rangle_{L^2(0,T)}. \end{aligned}$$

It follows lemma 2.7 again that there holds

$$\begin{aligned}
&\langle K_H(T,s)\phi_k(s), K_H(T,s)\phi_l(s)\rangle_{L^2(0,T)} \\
&= \int_0^T K_H^2(T,s)\phi_k(s)\phi_l(s)\mathrm{d}s \\
&\geqslant T^{2\alpha-2}E_{\alpha,\alpha}(-\lambda_k T^\alpha)E_{\alpha,\alpha}(-\lambda_l T^\alpha)\int_0^T K_H^2(T,s)\mathrm{d}s := \tilde{c}_1 > 0.
\end{aligned}$$

Using lemmas 2.3 and 2.6, and noting $T-s=t$, we obtain that $\phi_k(s)>0$ and $\phi_k(s)$ is a monotonically increasing function; $\phi_k'(s)>0$ and $\phi_k'(s)$ is a monotonically increasing function, which imply $\phi_k(s)\geqslant\phi_k(0)>0$ and $\phi_k'(s)\geqslant\phi_k'(0)>0$ for $0<s<T$. Hence

$$\langle M_H(s)\phi_k'(u_k^{**}), M_H(s)\phi_l'(u_l^{**})\rangle_{L^2(0,T)} \geqslant \phi_k'(0)\phi_l'(0)\int_0^T M_H^2(s)\mathrm{d}s =: \tilde{c}_2 > 0.$$

Similarly, we have

$$\langle K_H(T,s)\phi_k(s), M_H(s)\phi_l'(u_l^{**})\rangle_{L^2(0,T)} \geqslant \phi_k(0)\phi_l'(0)\int_0^T K_H(T,s)M_H(s)\mathrm{d}s =: \tilde{c}_3 > 0$$

and

$$\langle M_H(s)\phi_k'(u_k^{**}), K_H(T,s)\phi_l(s)\rangle_{L^2(0,T)} \geqslant \phi_k'(0)\phi_l(0)\int_0^T K_H(T,s)M_H(s)\mathrm{d}s =: \tilde{c}_4 > 0.$$

Combining the above estimates gives

$$I_{kl} \geqslant \sum_{j=1}^4 \tilde{c}_j =: C_2 > 0,$$

which completes the proof. □

Combining (4.1)–(4.3) and lemmas 4.1 and 4.2, we obtain the uniqueness of the inverse problem.

Theorem 4.3. *Let assumption 1 hold. Then the source terms f and g up to sign, i.e.* $\pm g$*, can be uniquely determined by the data set*

$$\{\mathbb{E}(u_k(T)), \mathrm{Cov}(u_k(T), u_l(T)) : k, l \in \mathbb{N}\}.$$

Proof. Since $f, g \in L^2(D)$, we have

$$f(x) = \sum_{k=1}^\infty f_k\varphi_k(x), \quad g(x) = \sum_{k=1}^\infty g_k\varphi_k(x),$$

which gives that

$$g^2(x) = \left(\sum_{k=1}^\infty g_k\varphi_k(x)\right)\left(\sum_{l=1}^\infty g_l\varphi_l(x)\right) = \sum_{k,l\in\mathbb{N}} g_k g_l \varphi_k(x)\varphi_l(x).$$

By lemmas 4.1 and 4.2, the proof is completed by noting (4.1) and (4.3). □

4.2. *Instability*

Unfortunately, the inverse source problem is unstable. In [30, lemma 4.4], the authors have explained the instability for $H = \frac{1}{2}, \alpha \in (\frac{1}{2}, 1)$. Since the formula (4.1) does not involve the Brownian motion, the instability of recovering f is the same. Therefore, we shall only discuss the instability of recovering g up to sign. It suffices to show that it is unstable to recover g_k^2 in (4.3) when $k = l$, i.e. we shall focus on the estimate of (4.2).

First, we choose t_* small enough such that

$$\frac{1}{1+\lambda_k r^\alpha} \leqslant \begin{cases} 1 & \text{if } r < t_*, \\ \frac{1}{\lambda_k r^\alpha} & \text{if } r > t_*. \end{cases} \tag{4.4}$$

Below we discuss the two different cases $H \in (0, \frac{1}{2})$ and $H \in (\frac{1}{2}, 1)$, separately.

4.2.1. *The case* $H \in (0, \frac{1}{2})$.

We consider the estimate (3.6) with $t = T$ and estimate $I_j(T), j = 1,2,3$.

For $I_1(T)$, a simple calculation yields

$$\begin{aligned} I_1(T) &= \int_0^T \left[\left(\frac{T}{\tau}\right)^{H-\frac{1}{2}} (T-\tau)^{\alpha+H-\frac{3}{2}} E_{\alpha,\alpha}(-\lambda_k (T-\tau)^\alpha) \right]^2 \mathrm{d}\tau \\ &= \left(\int_0^{T-t_*} + \int_{T-t_*}^T \right) \left(\left(\frac{T}{\tau}\right)^{2H-1} (T-\tau)^{2\alpha+2H-3} E_{\alpha,\alpha}^2(-\lambda_k (T-\tau)^\alpha) \right) \mathrm{d}\tau. \end{aligned}$$

We have from (4.4) that

$$\begin{aligned} &\int_0^{T-t_*} \left(\frac{T}{\tau}\right)^{2H-1} (T-\tau)^{2\alpha+2H-3} E_{\alpha,\alpha}^2(-\lambda_k (T-\tau)^\alpha) \,\mathrm{d}\tau \\ &\leqslant \int_0^{T-t_*} \left(\frac{T}{\tau}\right)^{2H-1} (T-\tau)^{2\alpha+2H-3} \frac{1}{\lambda_k^2 (T-\tau)^{2\alpha}} \,\mathrm{d}\tau \\ &= \frac{1}{\lambda_k^2} \int_0^{T-t_*} \left(\frac{T}{\tau}\right)^{2H-1} (T-\tau)^{2H-3} \,\mathrm{d}\tau. \end{aligned}$$

The condition $H > 0$ is enough to ensure the convergence of the above singular integral. Moreover, it follows from the binomial expansion that we obtain

$$\begin{aligned} &\frac{1}{\lambda_k^2} \int_0^{T-t_*} \left(\frac{T}{\tau}\right)^{2H-1} (T-\tau)^{2H-3} \,\mathrm{d}\tau \\ &= \frac{T^{4H-4}}{\lambda_k^2} (T-t_*)^{2-2H} \sum_{n=0}^\infty \binom{2H-3}{n} \frac{(-1)^n}{n+2-2H} \left(\frac{T-t_*}{T}\right)^n \\ &\lesssim \frac{1}{\lambda_k^2} T^{4H-4} (T-t_*)^{2-2H} \leqslant \frac{1}{\lambda_k^2} T^{2H-2}. \end{aligned}$$

On the other hand, by (4.4), there holds

$$\int_{T-t_*}^{T}\left(\frac{T}{\tau}\right)^{2H-1}(T-\tau)^{2\alpha+2H-3}E_{\alpha,\alpha}^{2}(-\lambda_k(T-\tau)^{\alpha})\mathrm{d}\tau.$$
$$\lesssim\int_{T-t_*}^{T}\left(\frac{T}{\tau}\right)^{2H-1}(T-\tau)^{2\alpha+2H-3}\mathrm{d}\tau.$$

Clearly, the condition $\alpha+H>1$ is essential to ensure the convergence of the singular integral. By the mean value theorem for the definite integral, there exists $T-t_*<\xi<T$ such that

$$\begin{aligned}&\int_{T-t_*}^{T}\left(\frac{T}{\tau}\right)^{2H-1}(T-\tau)^{2\alpha+2H-3}\mathrm{d}\tau\\ &=\left(\frac{T}{\xi}\right)^{2H-1}\int_{T-t_*}^{T}(T-\tau)^{2\alpha+2H-3}\mathrm{d}\tau\\ &=\left(\frac{T}{\xi}\right)^{2H-1}\frac{t_*^{2\alpha+2H-2}}{2\alpha+2H-2}\lesssim t_*^{2\alpha+2H-2}.\end{aligned}$$

Combining the above estimate leads to

$$I_1(T)\lesssim\frac{1}{\lambda_k^2}T^{2H-2}+t_*^{2\alpha+2H-2}. \tag{4.5}$$

For $I_2(T)$, using lemma 2.2 and formula (4.4), we have

$$\begin{aligned}I_2(T)&=\int_0^T\tau^{1-2H}\left[\left(\int_\tau^T u^{H-\frac{3}{2}}(u-\tau)^{H-\frac{1}{2}}\mathrm{d}u\right)(T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(T-\tau)^{\alpha})\right]^2\mathrm{d}\tau\\
&\lesssim\int_0^T\tau^{1-2H}\left(\int_\tau^T u^{H-\frac{3}{2}}(u-\tau)^{H-\frac{1}{2}}\mathrm{d}u\right)^2(T-\tau)^{2\alpha-2}\left(\frac{1}{1+\lambda_k(T-\tau)^{\alpha}}\right)^2\mathrm{d}\tau\\
&\lesssim\int_0^{T-t_*}\tau^{1-2H}\left(\int_\tau^{T-t_*} u^{H-\frac{3}{2}}(u-\tau)^{H-\frac{1}{2}}\mathrm{d}u\right)^2(T-\tau)^{2\alpha-2}\left(\frac{1}{1+\lambda_k(T-\tau)^{\alpha}}\right)^2\mathrm{d}\tau\\
&\quad+\int_0^{T-t_*}\tau^{1-2H}\left(\int_{T-t_*}^T u^{H-\frac{3}{2}}(u-\tau)^{H-\frac{1}{2}}\mathrm{d}u\right)^2(T-\tau)^{2\alpha-2}\left(\frac{1}{1+\lambda_k(T-\tau)^{\alpha}}\right)^2\mathrm{d}\tau\\
&\quad+\int_{T-t_*}^T\tau^{1-2H}\left(\int_\tau^T u^{H-\frac{3}{2}}(u-\tau)^{H-\frac{1}{2}}\mathrm{d}u\right)^2(T-\tau)^{2\alpha-2}\left(\frac{1}{1+\lambda_k(T-\tau)^{\alpha}}\right)^2\mathrm{d}\tau\\
&\lesssim\int_0^{T-t_*}\tau^{1-2H}\left(\int_\tau^{T-t_*} u^{H-\frac{3}{2}}(u-\tau)^{H-\frac{1}{2}}\mathrm{d}u\right)^2(T-\tau)^{2\alpha-2}\frac{1}{\lambda_k^2(T-\tau)^{2\alpha}}\mathrm{d}\tau\\
&\quad+\int_0^{T-t_*}\tau^{1-2H}\left(\int_{T-t_*}^T u^{H-\frac{3}{2}}(u-\tau)^{H-\frac{1}{2}}\mathrm{d}u\right)^2(T-\tau)^{2\alpha-2}\frac{1}{\lambda_k^2(T-\tau)^{2\alpha}}\mathrm{d}\tau\\
&\quad+\int_{T-t_*}^T\tau^{1-2H}\left(\int_\tau^T u^{H-\frac{3}{2}}(u-\tau)^{H-\frac{1}{2}}\mathrm{d}u\right)^2(T-\tau)^{2\alpha-2}\mathrm{d}\tau\\
&:=J_1(T)+J_2(T)+J_3(T).\end{aligned} \tag{4.6}$$

Next we estimate $J_j(T), j=1,2,3$, respectively.

For the term $J_1(T)$, we get

$$
\begin{aligned}
J_1(T) &= \frac{1}{\lambda_k^2}\int_0^{T-t_*}\tau^{1-2H}(T-\tau)^{-2}\left(\int_\tau^{T-t_*}u^{H-\frac{3}{2}}(u-\tau)^{H-\frac{1}{2}}\mathrm{d}u\right)^2\mathrm{d}\tau\\
&= \frac{1}{\lambda_k^2}\int_0^{T-t_*}\tau^{1-2H}(T-\tau)^{-2}\Bigg[(T-t_*)^{2H-1}\sum_{n=0}^{\infty}\binom{H-\frac{1}{2}}{n}\frac{(-1)^n}{2H-1-n}\left(\frac{\tau}{T-t_*}\right)^n\\
&\quad-\tau^{2H-1}\sum_{n=0}^{\infty}\binom{H-\frac{1}{2}}{n}\frac{(-1)^n}{2H-1-n}\Bigg]^2\mathrm{d}\tau\\
&\leqslant \frac{1}{\lambda_k^2}\left((T-t_*)^{4H-2}\int_0^{T-t_*}\tau^{1-2H}(T-\tau)^{-2}\mathrm{d}\tau+\int_0^{T-t_*}\tau^{2H-1}(T-\tau)^{-2}\mathrm{d}\tau\right),\\
&= \frac{1}{\lambda_k^2}\Bigg[T^{-2}(T-t_*)^{2H}\sum_{n=0}^{\infty}\binom{-2}{n}\frac{(-1)^n}{2-2H+n}\left(\frac{T-t_*}{T}\right)^n\\
&\quad+T^{-2}(T-t_*)^{2H}\sum_{n=0}^{\infty}\binom{-2}{n}\frac{(-1)^n}{2H+n}\left(\frac{T-t_*}{T}\right)^n\Bigg]\lesssim\frac{1}{\lambda_k^2}T^{2H-2},
\end{aligned}
\tag{4.7}
$$

where the condition $H\in(0,\frac{1}{2})$ is used.

For the second term $J_2(T)$, similar to the estimate of $J_1(T)$, we have

$$
\begin{aligned}
J_2(T) &= \frac{1}{\lambda_k^2}\int_0^{T-t_*}\tau^{1-2H}(T-\tau)^{-2}\left(\int_{T-t_*}^{T}u^{H-\frac{3}{2}}(u-\tau)^{H-\frac{1}{2}}\mathrm{d}u\right)^2\mathrm{d}\tau\\
&= \frac{1}{\lambda_k^2}\int_0^{T-t_*}\tau^{1-2H}(T-\tau)^{-2}\Bigg[T^{2H-1}\sum_{n=0}^{\infty}\binom{H-\frac{1}{2}}{n}\frac{(-1)^n}{2H-1-n}\left(\frac{\tau}{T}\right)^n\\
&\quad-(T-t_*)^{2H-1}\sum_{n=0}^{\infty}\binom{H-\frac{1}{2}}{n}\frac{(-1)^n}{2H-1-n}\left(\frac{\tau}{T-t_*}\right)^n\Bigg]^2\mathrm{d}\tau\\
&\lesssim \frac{1}{\lambda_k^2}\int_0^{T-t_*}\tau^{1-2H}(T-\tau)^{-2}\left[T^{4H-2}+(T-t_*)^{4H-2}\right]\mathrm{d}\tau\\
&= \frac{1}{\lambda_k^2}\left[T^{4H-2}+(T-t_*)^{4H-2}\right]T^{-2}(T-t_*)^{2-2H}\sum_{n=0}^{\infty}\binom{-2}{n}\frac{(-1)^n}{2-2H+n}\left(\frac{T-t_*}{T}\right)^n\\
&\lesssim \frac{1}{\lambda_k^2}T^{2H-2}.
\end{aligned}
\tag{4.8}
$$

For the third term $J_3(T)$, we obtain

$$
\begin{aligned}
J_3(T) &= \int_{T-t_*}^{T}\tau^{1-2H}(T-\tau)^{2\alpha-2}\left(\int_\tau^{T}u^{H-\frac{3}{2}}(u-\tau)^{H-\frac{1}{2}}\mathrm{d}u\right)^2\mathrm{d}\tau\\
&\lesssim \int_{T-t_*}^{T}\tau^{1-2H}(T-\tau)^{2\alpha-2}\left(T^{4H-2}+\tau^{4H-2}\right)\mathrm{d}\tau\\
&= T^{4H+2\alpha-4}\sum_{n=0}^{\infty}\binom{2\alpha-2}{n}(-1)^nT^{-n}\frac{T^{2-2H+n}-(T-t_*)^{2-2H+n}}{2-2H+n}\\
&\quad+T^{2\alpha-2}\sum_{n=0}^{\infty}\binom{2\alpha-2}{n}(-1)^nT^{-n}\frac{T^{2H+n}-(T-t_*)^{2H+n}}{2H+n}.
\end{aligned}
$$

The differential mean value theorem and the Hölder continuity of x^{2H} are used to obtain

$$J_3(T) \lesssim T^{2H+2\alpha-3}t_* + T^{2\alpha-2}(t_*^{2H} \vee t_*(T-t_*)^{2H-1}). \tag{4.9}$$

Here $a \vee b = \max\{a, b\}$. Combining (4.7), (4.8) and (4.9), we have

$$I_2(T) \lesssim \lambda_k^{-2} + (t_*^{2H} \vee t_*). \tag{4.10}$$

For $I_3(T)$, according to lemma 2.5,

$$\begin{aligned}
I_3(T) \lesssim & \int_0^T \left[\int_\tau^T \left[\int_{T-u}^{T-\tau} \frac{r^{\alpha-2}}{1+\lambda_k r^\alpha}\mathrm{d}r\right] \left(\frac{u}{\tau}\right)^{H-\frac{1}{2}} (u-\tau)^{H-\frac{3}{2}}\mathrm{d}u\right]^2 \mathrm{d}\tau \\
\lesssim & \int_0^{T-t_*} \left[\int_\tau^{T-t_*} \left[\int_{T-u}^{T-\tau} \frac{r^{\alpha-2}}{1+\lambda_k r^\alpha}\mathrm{d}r\right] \left(\frac{u}{\tau}\right)^{H-\frac{1}{2}} (u-\tau)^{H-\frac{3}{2}}\mathrm{d}u\right]^2 \mathrm{d}\tau \\
& + \int_0^{T-t_*} \left[\int_{T-t_*}^{T} \left[\int_{T-u}^{T-\tau} \frac{r^{\alpha-2}}{1+\lambda_k r^\alpha}\mathrm{d}r\right] \left(\frac{u}{\tau}\right)^{H-\frac{1}{2}} (u-\tau)^{H-\frac{3}{2}}\mathrm{d}u\right]^2 \mathrm{d}\tau \\
& + \int_{T-t_*}^{T} \left[\int_\tau^{T} \left[\int_{T-u}^{T-\tau} \frac{r^{\alpha-2}}{1+\lambda_k r^\alpha}\mathrm{d}r\right] \left(\frac{u}{\tau}\right)^{H-\frac{1}{2}} (u-\tau)^{H-\frac{3}{2}}\mathrm{d}u\right]^2 \mathrm{d}\tau \\
:= & K_1(T) + K_2(T) + K_3(T).
\end{aligned}$$

For $K_1(T)$, a simple calculation gives

$$\begin{aligned}
K_1(T) \lesssim & \int_0^{T-t_*} \left[\int_\tau^{T-t_*} \left[\frac{1}{\lambda_k}\int_{T-u}^{T-\tau} r^{-2}\mathrm{d}r\right] \left(\frac{u}{\tau}\right)^{H-\frac{1}{2}} (u-\tau)^{H-\frac{3}{2}}\mathrm{d}u\right]^2 \mathrm{d}\tau \\
\lesssim & \frac{1}{\lambda_k^2}\int_0^{T-t_*} \left[\int_\tau^{T-t_*} (T-u)^{-2}(u-\tau)^{H-\frac{1}{2}} \left(\frac{u}{\tau}\right)^{H-\frac{1}{2}} \mathrm{d}u\right]^2 \mathrm{d}\tau. \\
\lesssim & \frac{1}{\lambda_k^2}\int_0^{T-t_*} (T-\tau)^{-4} \left[\sum_{n=0}^{\infty} \binom{-2}{n} (-1)^n (T-\tau)^{-n} \int_0^{T-t_*-\tau} r^{n+H-\frac{1}{2}}\mathrm{d}r\right]^2 \mathrm{d}\tau \\
\lesssim & \frac{1}{\lambda_k^2}\int_0^{T-t_*} (T-\tau)^{2H-3}\mathrm{d}\tau \lesssim \frac{1}{\lambda_k^2 t_*^{2-2H}}.
\end{aligned}$$

For $K_2(T)$, noting that $t_* < T-\tau < T$ and $0 < T-u < t_*$ since $0 < \tau < T-t_*$ and $T-t_* < u < T$, we get

$$\begin{aligned}
\int_{T-u}^{T-\tau} \frac{r^{\alpha-2}}{1+\lambda_k r^\alpha}\mathrm{d}r \leqslant & \int_{T-u}^{t_*} r^{\alpha-2}\mathrm{d}r + \int_{t_*}^{T-\tau} \frac{1}{\lambda_k} r^{-2}\mathrm{d}r \\
\leqslant & \left(t_*^\alpha \vee \frac{1}{\lambda_k}\right) \int_{T-u}^{T-\tau} r^{-2}\mathrm{d}r \\
\leqslant & \left(t_*^\alpha \vee \frac{1}{\lambda_k}\right) (T-u)^{-2}(u-\tau).
\end{aligned}$$

As a result, we have for $H \in (0, \frac{1}{2})$ that

$$K_2(T) \lesssim \int_0^{T-t_*} \left[\int_{T-t_*}^T \left(t_*^\alpha \vee \frac{1}{\lambda_k}\right)(T-u)^{-2}(u-\tau)^{H-\frac{1}{2}}\left(\frac{u}{\tau}\right)^{H-\frac{1}{2}}\mathrm{d}u\right]^2 \mathrm{d}\tau$$
$$\lesssim \left(t_*^{2\alpha} \vee \frac{1}{\lambda_k^2}\right)\int_0^{T-t_*}(T-\tau)^{-4}\left[\sum_{n=0}^{\infty}\binom{-2}{n}(-1)^n(T-\tau)^{-n}\int_{T-t_*-\tau}^{T-\tau} r^{n+H-\frac{1}{2}}\mathrm{d}r\right]^2\mathrm{d}\tau$$
$$\lesssim \left(t_*^{2\alpha+2H-2} \vee \frac{1}{\lambda_k^2 t_*^{2-2H}}\right).$$

For $K_3(T)$, since $0 < T-u < T-\tau < t_*$ for $T-t_* < \tau < u < T$,

$$K_3(T) \lesssim \int_{T-t_*}^T \left[\int_\tau^T \left[\int_{T-u}^{T-\tau} r^{\alpha-2}\mathrm{d}r\right]\left(\frac{u}{\tau}\right)^{H-\frac{1}{2}}(u-\tau)^{H-\frac{3}{2}}\mathrm{d}u\right]^2\mathrm{d}\tau$$
$$\lesssim \int_{T-t_*}^T \left[\int_\tau^T (T-u)^{\alpha-2}(u-\tau)^{H-\frac{1}{2}}\left(\frac{u}{\tau}\right)^{H-\frac{1}{2}}\mathrm{d}u\right]^2\mathrm{d}\tau$$
$$\lesssim \int_{T-t_*}^T \left[\int_0^{T-\tau}(T-\tau-r)^{\alpha-2}r^{H-\frac{1}{2}}\mathrm{d}r\right]^2\mathrm{d}\tau$$
$$\lesssim \int_{T-t_*}^T (T-\tau)^{2\alpha+2H-3}\mathrm{d}\tau \lesssim t_*^{2\alpha+2H-2},$$

where the condition $\alpha + H > 1$ is used.

Combining the above estimates, we conclude that

$$I_3(T) \lesssim \left(t_*^{2\alpha+2H-2} \vee \frac{1}{\lambda_k^2 t_*^{2-2H}}\right). \tag{4.11}$$

Finally, it follows from (4.5), (4.10) and (4.11) that

$$\mathbb{E}\left|\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(T-\tau)^\alpha)\mathrm{d}B^H(\tau)\right|^2$$
$$\lesssim \max\left\{t_*^{2\alpha+2H-2}, \lambda_k^{-2}t_*^{2H-2}, \lambda_k^{-2}, t_*^{2H}, t_*\right\}. \tag{4.12}$$

4.2.2. The case $H \in (\frac{1}{2}, 1)$. From (4.2) and (A.4), we have

$$\mathbb{E}\left|\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(T-\tau)^\alpha)\mathrm{d}B^H(\tau)\right|^2$$
$$= \alpha_H \int_0^T\int_0^T (T-p)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(T-p)^\alpha)(T-q)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(T-q)^\alpha)|p-q|^{2H-2}\mathrm{d}p\mathrm{d}q$$
$$= \alpha_H\left(\int_0^{t_*}\int_0^{t_*} + \int_{t_*}^T\int_{t_*}^T + \int_0^{t_*}\int_{t_*}^T + \int_{t_*}^T\int_0^{t_*}\right)p^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k p^\alpha)q^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k q^\alpha)|q-p|^{2H-2}\mathrm{d}p\mathrm{d}q$$
$$:= \alpha_H(M_1(T)+M_2(T)+M_3(T)+M_4(T)). \tag{4.13}$$

We choose t_* as that in (4.4). It is easy to see that $M_3(T) = M_4(T)$. Then we only need to discuss $M_j(T), j = 1,2,3$.

For the term $M_1(T)$, similar to the estimate in section 3.2, we get

$$\begin{aligned} M_1(T) &= \int_0^{t_*}\int_0^{t_*} p^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k p^\alpha)q^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k q^\alpha)|q-p|^{2H-2}\mathrm{d}p\mathrm{d}q \\ &\lesssim \int_0^{t_*}\int_0^{t_*} p^{\alpha-1}q^{\alpha-1}|q-p|^{2H-2}\mathrm{d}p\mathrm{d}q \\ &\lesssim t_*^{2\alpha+2H-2}. \end{aligned} \tag{4.14}$$

For the term $M_2(T)$, it is easy to verify that

$$\begin{aligned} M_2(T) &= \int_{t_*}^{T}\int_{t_*}^{T} p^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k p^\alpha)q^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k q^\alpha)|q-p|^{2H-2}\mathrm{d}p\mathrm{d}q \\ &\lesssim \frac{1}{\lambda_k^2}\int_{t_*}^{T}\int_{t_*}^{T} \frac{1}{p}\frac{1}{q}|q-p|^{2H-2}\mathrm{d}p\mathrm{d}q \\ &\leqslant \frac{1}{\lambda_k^2 t_*^2}\int_{t_*}^{T}\int_{t_*}^{T} |q-p|^{2H-2}\mathrm{d}p\mathrm{d}q \\ &= \frac{2}{\lambda_k^2 t_*^2}\int_{t_*}^{T}\int_{t_*}^{q} (q-p)^{2H-2}\mathrm{d}p\mathrm{d}q \\ &= \frac{2}{\lambda_k^2 t_*^2}\frac{(T-t_*)^{2H}}{2H(2H-1)} \lesssim \frac{1}{\lambda_k^2 t_*^2}. \end{aligned} \tag{4.15}$$

For the term $M_3(T)$, we may similarly have

$$\begin{aligned} M_3(T) &= \int_0^{t_*}\int_{t_*}^{T} p^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k p^\alpha)q^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k q^\alpha)|q-p|^{2H-2}\mathrm{d}p\mathrm{d}q \\ &\lesssim \frac{1}{\lambda_k}\int_0^{t_*}\int_{t_*}^{T} \frac{1}{p}q^{\alpha-1}(p-q)^{2H-2}\mathrm{d}p\mathrm{d}q \\ &\leqslant \frac{1}{\lambda_k t_*}\int_0^{t_*}\int_{t_*}^{T} q^{\alpha-1}(p-q)^{2H-2}\mathrm{d}p\mathrm{d}q \\ &\lesssim \frac{1}{\lambda_k t_*}\int_0^{t_*} q^{\alpha-1}\mathrm{d}q \lesssim \frac{1}{\lambda_k}t_*^{\alpha-1}. \end{aligned} \tag{4.16}$$

It follows from (4.13)–(4.16) that

$$\mathbb{E}\left|\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(T-\tau)^\alpha)\mathrm{d}B^H(\tau)\right|^2 \lesssim \max\left\{t_*^{2\alpha+2H-2}, \frac{1}{\lambda_k^2 t_*^2}, \frac{1}{\lambda_k}t_*^{\alpha-1}\right\}, \tag{4.17}$$

which is crucial to explain the instability of the inverse problem.

Based on the analysis above, we can obtain the following theorem which shows that it is unstable to reconstruct f and $\pm g$.

Theorem 4.4. *The problem of recovering the source terms f and $\pm g$ is unstable. Moreover, the following estimates hold*

$$\left|\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(T-\tau)^\alpha)h(\tau)\mathrm{d}\tau\right| \lesssim \lambda_k^{-1} \tag{4.18}$$

and

$$\mathbb{E}\left|\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(T-\tau)^{\alpha})\mathrm{d}B^H(\tau)\right|^2 \lesssim \lambda_k^{-\beta}, \tag{4.19}$$

where

$$\beta = \begin{cases} \min\{2\gamma(\alpha+H-1), 2+2\gamma(H-1), 2H\gamma\} & \text{if} \quad H\in(0,\frac{1}{2}), \\ \min\{2\gamma(\alpha+H-1), 2(1-\gamma), 1-\gamma(1-\alpha)\} & \text{if} \quad H\in(\frac{1}{2},1), \\ \min\{\gamma(2\alpha-1), 1-\gamma\} & \text{if} \quad H=\frac{1}{2}. \end{cases}$$

Here $\gamma\in(0,1)$ *and* $\alpha+H>1$.

Proof. For (4.18), one can refer to [30, lemma 4.4]. For (4.19), one can obtain it by choosing $t_* = \lambda_k^{-\gamma}, 0<\gamma<1$ in (4.12) and (4.17). Here the case $H=\frac{1}{2}$ can be seen in [30, lemma 4.4]. For $\alpha=1$, one can use $\mathrm{e}^{-x}<\frac{1}{1+x}, x\geqslant 0$ to obtain the same results. Since $\lambda_k\to\infty$ as $k\to\infty$, the instability follows easily from the estimates (4.18)–(4.19) and the reconstruction formulas (4.1) and (4.2). □

5. Numerical experiments

In this section, we present some numerical results of a one-dimensional problem with $D=[0,\pi]$. The eigen-system of the operator $-\Delta$ is

$$\lambda_k = k^2, \quad \varphi_k = \sqrt{\frac{2}{\pi}}\sin(kx), \quad k=1,\ldots,\infty.$$

For some fixed integers N_t and N_x, we define the time and space step-sizes $h_t = T/N_t$, $h_x = \pi/N_x$ and nodes

$$t_n = nh_t,\ n\in\{0,1,2,\ldots,N_t\}, \quad x_i = ih_x,\ i\in\{0,1,2,\ldots,N_x\}.$$

5.1. Numerical scheme

We introduce the numerical discretization of the direct problem, which is used to generate the synthetic data for the inverse problem.

To discretize the Caputo derivative, we use the method proposed in [27, 49]:

$$\partial_t^{\alpha} u(x_i,t_n) = \sigma_{\alpha,h_t}\sum_{j=1}^{n}(u(x_i,t_j)-u(x_i,t_{j-1}))[(n-j+1)^{1-\alpha}-(n-j)^{1-\alpha}] + O(h_t),$$

where

$$\sigma_{\alpha,h_t} := \frac{1}{\Gamma(1-\alpha)}\frac{1}{1-\alpha}h_t^{-\alpha}.$$

The central difference is utilized for the spatial discretization:

$$u_{xx}(x_i,t_n) = \frac{u(x_{i-1},t_n)-2u(x_i,t_n)+u(x_{i+1},t_n)}{h_x^2} + O(h_x^2).$$

Let u_i^n be the numerical approximation to $u(x_i, t_n)$. Then we obtain the following implicit scheme:

$$\sigma_{\alpha,h_t}\sum_{j=1}^{n}(u_i^j-u_i^{j-1})[(n-j+1)^{1-\alpha}-(n-j)^{1-\alpha}]-\frac{u_{i-1}^n-2u_i^n+u_{i+1}^n}{h_x^2}$$
$$=f(x_i)h(t_n)+g(x_i)\frac{B^H(t_n)-B^H(t_{n-1})}{t_n-t_{n-1}}.$$

Remark 5.1. For the two-dimensional case, one may similarly get the following semi-discrete scheme:

$$\begin{aligned}&-\Delta u^n(x)+\sigma_{\alpha,h_t}u^n(x)\\ =&f(x)h(t_n)+g(x)\frac{B^H(t_{n+1})-B^H(t_n)}{t_{n+1}-t_n}-\sigma_{\alpha,h_t}u^0(x)[(n-1)^{1-\alpha}-n^{1-\alpha}]\\ &-\sigma_{\alpha,h_t}\sum_{k=1}^{n-1}u^k[(n-k-1)^{1-\alpha}-2(n-k)^{1-\alpha}+(n-k+1)^{1-\alpha}]\\ &=:F^n(x)\qquad n=1,\dots,N_t.\end{aligned}$$

Based on the homogeneous Dirichlet condition, we have the following variational problem:

$$(\nabla u^n(x),\nabla v(x))+\sigma_{\alpha,h_t}(u^n(x),v(x))=(F^n(x),v(x)),\quad v(x)\in H^1(D).$$

Then we can use the finite element method to solve the above variational problem and obtain $u^n(x)$. The final approximated data is $u^{N_t}(x)\approx u(x,T)$.

5.2. Numerical results

For the inverse problem, coefficients f_k and g_kg_l can be recovered by (4.3) and (4.1), respectively, based on which one get

$$f=\sum_{k=1}^{\infty}f_k\varphi_k,\quad g^2=\sum_{k,l=1}^{\infty}g_kg_l\varphi_k\varphi_l.$$

Since the inverse problem is ill-posed, we truncate above series by the first N terms as a regularization.

In the numerical experiments, we choose $N=3$, $N_t=2^{13}, N_x=100$, and $T=1$. Functions in (2.1) are chosen as

$$h(t)=1,\quad f(x)=\sin(3x),\quad g(x)=\exp(-(x-\frac{\pi}{2})^2).$$

The total number of 1000 sample paths are used when simulating the covariance of the solution. In addition, the data is assumed to be polluted by a uniformly distributed noise with the noise level δ. The code given by [33] is adopted to evaluate the Mittag–Leffler function $E_{\alpha,\alpha}$. To compute the singular integral in (4.3), we use the global adaptive quadrature.

We present the results for three different sets of parameters (α, H). The results of $\{\alpha=0.9, H=0.9\}, \{\alpha=0.9, H=0.4\}$, and $\{\alpha=0.4, H=0.9\}$ are given in figures 1–3, respectively. Based on the numerical experiments, it can be observed that the recovery would

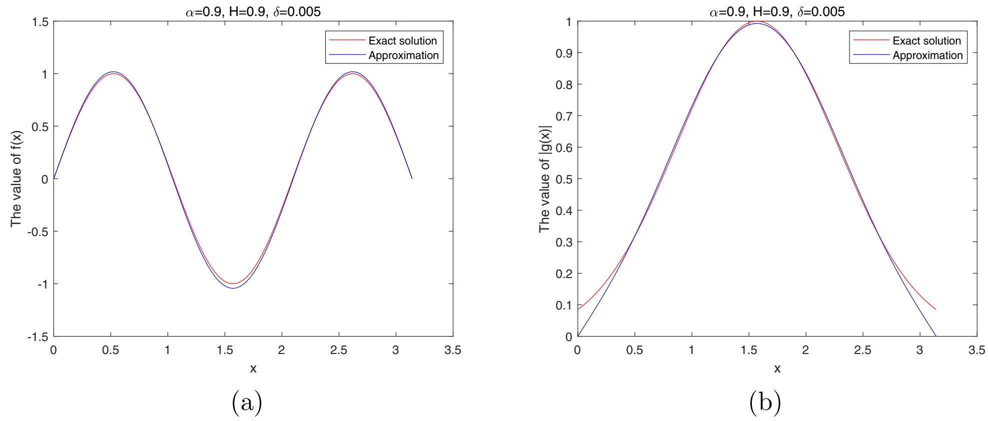


Figure 1. The exact solutions and the reconstructed solutions for $\alpha = 0.9$ and $H = 0.9$ with $\delta = 0.005$. (a) $f(x)$; (b) $|g(x)|$.

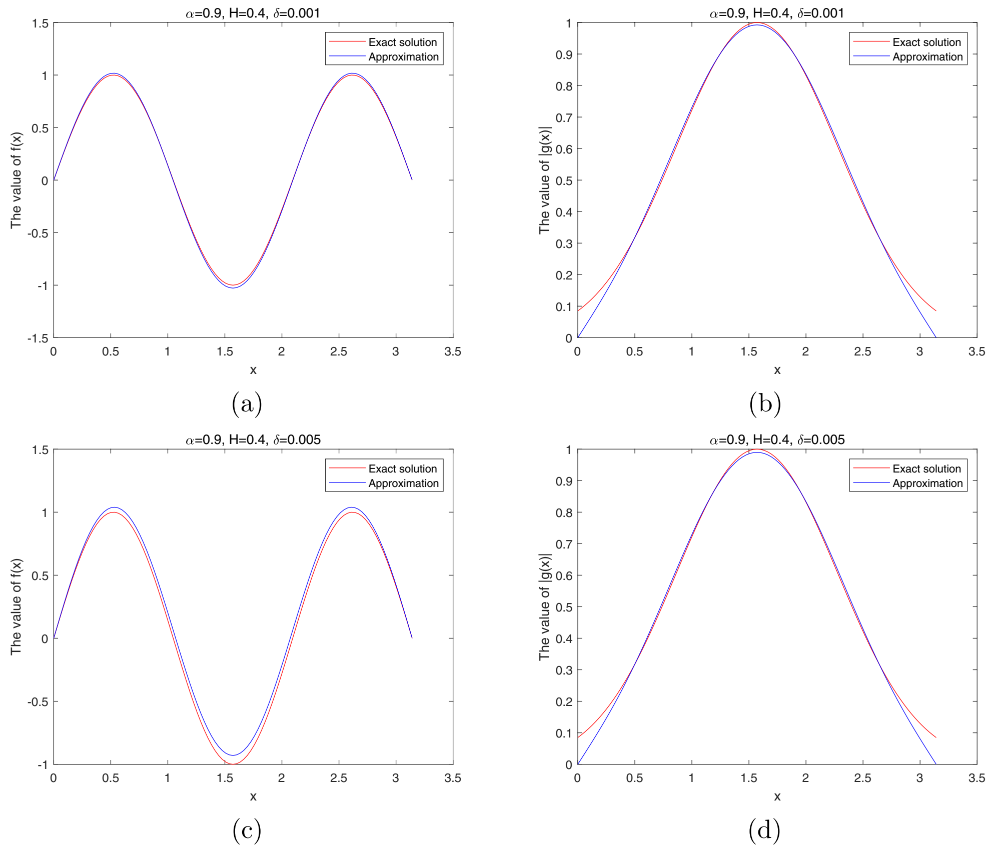
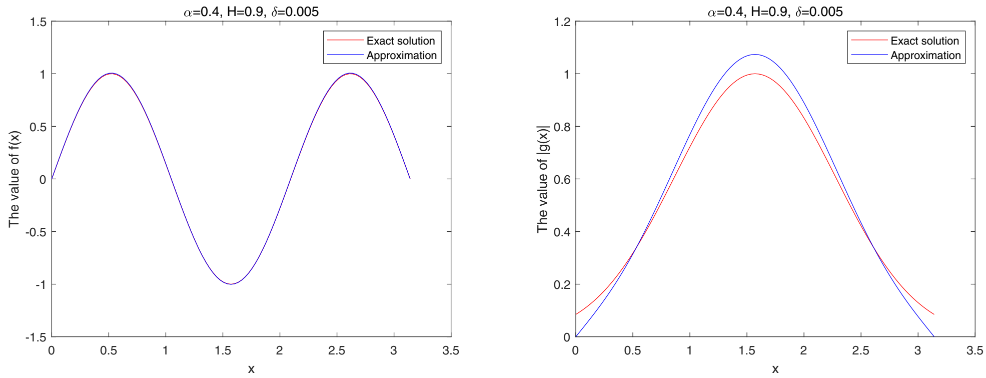


Figure 2. The exact solutions and the reconstructed solutions for $\alpha = 0.9$ and $H = 0.4$. (a) $f(x)$ with $\delta = 0.001$; (b) $|g(x)|$ with $\delta = 0.001$; (c) $f(x)$ with $\delta = 0.005$; (d) $|g(x)|$ with $\delta = 0.005$.



(a) (b)

Figure 3. The exact solutions and the reconstructed for $\alpha = 0.4$ and $H = 0.9$ with $\delta = 0.005$. (a) $f(x)$; (b) $|g(x)|$.

be more accurate if the problem is more regular, i.e. α or H is larger; if the noise level δ is smaller, the result would also be better, which exactly implies the ill-posedness of the inverse problem.

6. Conclusion

In this paper, we have studied an inverse random source problem for the time fractional diffusion equation driven by fractional Brownian motions. By the analysis, we deduce the relationship of the time fractional order α and the Hurst index H in the fractional Brownian motion to ensure that the solution is well-defined for the stochastic time fractional diffusion equation. We show that the direct problem is well-posed when $\alpha + H > 1$ and the inverse source problem has a unique solution. But the inverse problem is ill-posed in the sense that a small deviation of the data may lead to a huge error in the reconstruction.

There are a couple of interesting observations. First, if the Laplacian operator is also fractional, the method can be directly applied and all the results can be similarly proved. Second, for $1 < \alpha \leqslant 2$, the direct problem can be shown to be well-posed since lemma 2.2 is still valid. However, the inverse problem may not have a unique solution. The reason is that lemma 2.7 is not true any more for $1 < \alpha \leqslant 2$. We will investigate the case $1 < \alpha \leqslant 2$ and report the numerical results elsewhere in the future.

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Appendix. Fractional Brownian motion

In the appendix, we briefly introduce the fractional Brownian motion (fBm) and present some results which are used in this work.

A.1. Definition and Hölder continuity

A one dimensional fBm B^H with the Hurst parameter $H \in (0,1)$ is a centered Gaussian process (i.e. $B^H(0)=0$) determined by its covariance function

$$R_H(t,s) = \mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

for any $s,t \geqslant 0$. In particular, if $H=\frac{1}{2}$, B^H turns to be the standard Brownian motion, which is usually denoted by W, with covariance function $R_H(t,s) = t \wedge s$.

The increments of fBms satisfies

$$\mathbb{E}\left[\left(B^H(t) - B^H(s)\right)\left(B^H(s) - B^H(r)\right)\right] = \frac{1}{2}\left[(t-r)^{2H} - (t-s)^{2H} - (r-s)^{2H}\right]$$

and

$$\mathbb{E}\left[\left(B^H(t) - B^H(s)\right)^2\right] = (t-s)^{2H}$$

for any $0<r<s<t$. It then indicates that the increments of B^H in disjoint intervals are linearly dependent except for the case $H=\frac{1}{2}$, and the increments are stationary since its moment depends only on the length of the interval.

Based on the moment estimates and the Kolmogorov continuity criterion, it holds for any $\epsilon>0$ and $s,t \in [0,T]$ that

$$|B^H(t) - B^H(s)| \leqslant C|t-s|^{H-\epsilon}$$

almost surely with constant C depending on ϵ and T. That is, H represents the regularity of B^H: the trajectories of fBm B^H with Hurst parameter $H \in (0,1)$ are $(H-\epsilon)$-Hölder continuous.

A.2. Representation of fBm and integration

For a fBm B^H with $H \in (0,1)$, it has the following Wiener integral representation

$$B^H(t) = \int_0^t K_H(t,s)\mathrm{d}W(s)$$

with K_H being a square integrable kernel and W being the standard Brownian motion (i.e. $H=\frac{1}{2}$).

For a fixed interval $[0,T]$, denote by $\mathcal{E}$ the space of step functions on $[0,T]$ and by $\mathcal{H}$ the closure of $\mathcal{E}$ with respect to the product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s),$$

where $\mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}$ are the characteristic functions. Define the linear operator $K^*_{H,T} : \mathcal{E} \to L^2(0,T)$ by

$$(K^*_{H,T}\psi)(s) = K_H(T,s)\psi(s) + \int_s^T (\psi(u) - \psi(s))\frac{\partial K_H(u,s)}{\partial u}\mathrm{d}u, \tag{A.1}$$

where

$$\frac{\partial K_H(u,s)}{\partial u} = c_H\left(\frac{u}{s}\right)^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}$$

and c_H is a constant given below depending on H. Then $K^*_{H,T}$ is an isometry from $\mathcal{E}$ to $L^2(0,T)$ (see e.g. [31, 39]), and the integral with respect to B^H can be defined for functions φ satisfying

$$\|\psi\|^2_{|\mathcal{H}|} := \langle \psi, \psi \rangle_{\mathcal{H}} < \infty,$$

and (see e.g. [31, 39])

$$\int_0^t \psi(s)\mathrm{d}B^H(s) = \int_0^T \psi(s)\mathbf{1}_{[0,t]}(s)\mathrm{d}B^H(s) = \int_0^T [K^*_{H,T}(\psi\mathbf{1}_{[0,t]})](s)\mathrm{d}W(s)$$

for any $t \in [0, T]$. Hence, according to the Itô isometry,

$$\mathbb{E}\left[\int_0^t \psi(s)\mathrm{d}B^H(s) \int_0^t \phi(s)\mathrm{d}B^H(s)\right] = \langle K^*_{H,T}(\psi\mathbf{1}_{[0,t]}), K^*_{H,T}(\phi\mathbf{1}_{[0,t]})\rangle_{L^2(0,T)}. \tag{A.2}$$

A.2.1. The case $H \in (\frac{1}{2}, 1)$. For the case $H \in (\frac{1}{2}, 1)$, the covariance function R_H of B^H satisfies

$$\begin{aligned} R_H(t,s) =& \alpha_H \int_0^t \int_0^s |r-u|^{2H-2}\mathrm{d}u\mathrm{d}r \\ =& \alpha_H \int_0^T \int_0^T \mathbf{1}_{[0,t]}(r)\mathbf{1}_{[0,s]}(u)|r-u|^{2H-2}\mathrm{d}u\mathrm{d}r \end{aligned}$$

with $\alpha_H = H(2H-1)$. The square integrable kernel has form

$$K_H(t,s) = c_H \int_s^t \left(\frac{u}{s}\right)^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}}\mathrm{d}u$$

with $c_H = \left(\frac{\alpha_H}{\beta(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}}$ such that

$$\begin{aligned} \langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\rangle_{\mathcal{H}} = R_H(t,s) =& \alpha_H \int_0^T \int_0^T \mathbf{1}_{[0,t]}(r)\mathbf{1}_{[0,s]}(u)|r-u|^{2H-2}\mathrm{d}u\mathrm{d}r \\ =& \int_0^T \mathbf{1}_{[0,t]}(u)\mathbf{1}_{[0,s]}(u)K_H(t,u)K_H(s,u)\mathrm{d}u, \end{aligned} \tag{A.3}$$

and $K^*_{H,T}$ in (A.1) turns to be

$$(K^*_{H,T}\psi)(s) = \int_s^T \psi(u)\frac{\partial K_H(u,s)}{\partial u}\mathrm{d}u.$$

By noting that

$$(K^*_{H,T}\mathbf{1}_{[0,t]})(s) = \int_s^T \mathbf{1}_{[0,t]}(u)\frac{\partial K_H(u,s)}{\partial u}\mathrm{d}u = \mathbf{1}_{[0,t]}(s)\int_s^t \frac{\partial K_H(u,s)}{\partial u}\mathrm{d}u = \mathbf{1}_{[0,t]}(s)K_H(t,s),$$

one get

$$\begin{aligned} \langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\rangle_{\mathcal{H}} =& \int_0^T \mathbf{1}_{[0,t]}(u)\mathbf{1}_{[0,s]}(u)K_H(t,u)K_H(s,u)\mathrm{d}u \\ =& \int_0^T (K^*_{H,T}\mathbf{1}_{[0,t]})(u)(K^*_{H,T}\mathbf{1}_{[0,s]})(u)\mathrm{d}u \\ =& \langle K^*_{H,T}\mathbf{1}_{[0,t]}, K^*_{H,T}\mathbf{1}_{[0,s]}\rangle_{L^2(0,T)}. \end{aligned}$$

In this case, (A.2) can be calculated as follows

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \psi(s)\mathrm{d}B^H(s) \int_0^t \phi(s)\mathrm{d}B^H(s)\right] \\
=&\langle K_{H,T}^*(\psi\mathbf{1}_{[0,t]}), K_{H,T}^*(\phi\mathbf{1}_{[0,t]})\rangle_{L^2(0,T)} \\
=&\langle \psi\mathbf{1}_{[0,t]}, \phi\mathbf{1}_{[0,t]}\rangle_{\mathcal{H}} \\
=&\alpha_H \int_0^t \int_0^t \psi(r)\phi(u)|r-u|^{2H-2}\mathrm{d}u\mathrm{d}r
\end{aligned} \tag{A.4}$$

according to (A.3), which is used in (3.6).

A.2.2. The case $H \in (0, \frac{1}{2})$. If the trajectories of B^H is less regular than the case above with $H \in (0, \frac{1}{2})$, the square integrable kernel K_H has the following form instead

$$K_H(t,s) = c_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}\mathrm{d}u\right] \tag{A.5}$$

with $c_H = \left(\frac{2H}{(1-2H)\beta(1-2H,H+\frac{1}{2})}\right)^{\frac{1}{2}}$ such that

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,u)K_H(s,u)\mathrm{d}u$$

similar to (A.3). Utilizing the fact (see [39])

$$[K_{H,T}^*(\psi\mathbf{1}_{[0,t]})](s) = [(K_{H,t}^*\psi)(s)]\mathbf{1}_{[0,t]}(s), \quad \forall\, t \in [0,T],$$

where $K_{H,t}^*$ is defined in (A.1), we may rewrite (A.2) into

$$\mathbb{E}\left[\int_0^t \psi(s)\mathrm{d}B^H(s) \int_0^t \phi(s)\mathrm{d}B^H(s)\right] = \langle K_{H,T}^*(\psi\mathbf{1}_{[0,t]}), K_{H,T}^*(\phi\mathbf{1}_{[0,t]})\rangle_{L^2(0,T)} \tag{A.6}$$

$$= \langle K_{H,t}^*\psi, K_{H,t}^*\phi\rangle_{L^2(0,t)}, \tag{A.7}$$

which is used in section 3.2 and (4.13).

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