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PARAREAL EXPONENTIAL θ-SCHEME FOR LONGTIME SIMULATION OF STOCHASTIC SCHRÖDINGER EQUATIONS WITH WEAK DAMPING*

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Abstract. A parareal algorithm based on an exponential θ-scheme is proposed for the stochastic Schrödinger equation with weak damping and additive noise. It proceeds as a two-level temporal parallelizable integrator with the exponential θ-scheme as the integrator on the coarse grid. The proposed algorithm in the linear case increases the convergence order from one to k for $\theta \in [0,1]\backslash\{\frac{1}{2}\}$. In particular, the convergence order increases to $2k$ when $\theta = \frac{1}{2}$ due to the symmetry of the algorithm. The convergence condition for longtime simulation is also established for the proposed algorithm in the nonlinear case, which indicates the superiority of implicit schemes. Numerical experiments are dedicated to illustrating the convergence order of the algorithm for different choices of θ, as well as the error for different iterated number k.

Key words. stochastic Schrödinger equation, parareal algorithm, exponential θ-scheme, invariant measure

AMS subject classifications. 60H35, 65M12, 65W05

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1. Introduction.

In the numerical approximation for both deterministic and stochastic evolution equations, several methods have been developed to improve the convergence order of classical schemes, such as (partitioned) Runge–Kutta methods, schemes via modified equations, predictor-corrector schemes, and so on (see [4, 18, 23, 24] and references therein). For high order numerical approximations of stochastic partial differential equations (SPDEs), the computing cost can be prohibitively large due to the high dimension in space, especially for longtime simulations. It motivates us to study algorithms allowing for parallel implementations to obtain a significant improvement of efficiency.

The parareal algorithm was pioneered in [20] as a time discretization of a deterministic partial differential evolution equation on finite time intervals and was then modified in [22] to tackle nondifferential evolution equations. This algorithm is described through a coarse integrator calculated on a coarse grid with step size δT and a fine integrator calculated in parallel on each subinterval with step size $\delta t = \delta T/J$, where $J \in \mathbb{N}_+$ denotes the number of available processors. It is pointed out in [20] and [22] that the error caused by the parareal architecture after a few iterations is comparable to the error caused by a global use of the fine integrator without iteration.

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More specifically, for a fixed iterated step $k \in \mathbb{N}_+$, the parareal algorithm could show order kp with respect to δT if a scheme with local truncation error $O(\delta T^{p+1})$ is chosen as the coarse integrator and the exact flow is chosen as the fine integrator. Over the past few years, the parareal algorithm has been further studied by [2, 14, 17, 22, 25] on its stability and convergence, by [15, 16] on the potential of longtime simulation, by [3, 5, 12, 26] on the application to stochastic problems, and by [13] as a review.

When exploring parareal algorithms for stochastic differential equations (SDEs) driven by standard Brownian motions, one of the main differences from the deterministic case is that the stochastic systems may be less regular than the deterministic ones. Bal [3] deals with this problem for SDEs adding assumptions on drift and diffusion coefficients as well as their derivatives and considers the parareal algorithm when the explicit Euler scheme is chosen as the coarse integrator. The optimal rate $\frac{k}{2}(\alpha \wedge 3 - 1)$ is deduced taking advantage of the independency between the increments of Brownian motions, where α varies for different drift and diffusion coefficients and $\alpha = 2$ in general. For SPDEs of parabolic type, recently, [5] studies the influence on the convergence of parareal algorithms of the regularity of the noise, choice of coarse integrator, and iterated number.

For the stochastic nonlinear Schrödinger equation considered in this paper, there are two main obstacles when establishing implementable parareal algorithms for longtime simulation. One is that the stiffness caused by the noise makes it unavailable to construct parareal algorithms based on existing stable schemes (see, e.g., [6]). It may require higher regularity assumptions due to the iteration adopted in parareal algorithms (see Remark 4), which are unusual for SPDEs. The other one is that the $\mathbb{C}$-valued nonlinear coefficient (i.e., $\mathbf{i}F(u)$ in (2.1)) does not satisfy one-sided Lipschitz type conditions in general. It leads to strict restrictions on the scale of the coarse grid, especially for explicit numerical schemes, when one wishes to get a uniform convergence rate.

In this paper, we propose an exponential θ-scheme based parareal algorithm with $\theta \in [0, 1]$. It allows us to perform the iteration without high regularity assumptions on the noise and the initial condition taking advantage of the semigroup generated by the linear operator of the considered model. For the linear case with $\theta \in [\frac{1}{2}, 1]$, the exponential θ-scheme possesses a unique invariant Gaussian distribution, which converges to the invariant measure of the exact solution due to the existence of the damping term $-\alpha u$ with $\alpha > 0$ (see (2.1)). This type of absolute stability ensures the uniform convergence of the proposed parareal algorithm with order k for $\theta > \frac{1}{2}$ and $2k$ for $\theta = \frac{1}{2}$. If $\theta \in [0, \frac{1}{2})$ and the damping parameter α is large enough, the uniform convergence still holds. Otherwise, the algorithm is only suitable for simulation over finite time intervals, which coincides with the fact that the distribution of the exponential θ scheme diverges over a long time if $\theta \in [0, \frac{1}{2})$ and α is not large enough (see section 3.2). The damping term ensures that the semigroup generated by the linear operator in (2.1) is exponentially stable, which makes the proposed scheme valid. For the nonlinear case, we take the proposed algorithm with $\theta = 0$ as a keystone to illustrate the convergence analysis for fully discrete schemes with the fine integrator being a numerical solver as well. This result is only available for bounded time intervals. To get a time-uniform estimate, internal stage values are utilized in the analysis for the nonlinear case with general $\theta \in [0, 1]$. The results give the convergence condition on θ, L_F, α, and δT.

The paper is organized as follows. Section 2 introduces some notation and assumptions used in the subsequent sections and gives a brief recall about parareal algorithms. Section 3 is dedicated to analyzing the stability of the parareal exponential θ-scheme by investigating the distribution of the exponential θ-scheme over a long time. The rate of convergence for both unbounded and bounded intervals is given for the linear case. Section 4 focuses on the application of the proposed parareal algorithm to the nonlinear case as well as the fully discrete scheme based on the parareal algorithm. These results are illustrated through numerical experiments in section 5.

2. Preliminaries. We consider the following initial-boundary problem of the stochastic nonlinear Schrödinger equation driven by additive noise:

$$\begin{aligned} du &= (\mathbf{i}\Delta u - \alpha u + \mathbf{i}F(u))\,dt + dW^Q(t), \\ u(t,0) &= u(t,1) = 0, \qquad t \in (0,T], \\ u(0,x) &= u_0(x), \qquad x \in (0,1). \end{aligned} \tag{2.1}$$

Here, $\alpha \geq 0$ is the damping coefficient, and W^Q is a Q-Wiener process defined on the filtered complete probability space $(\Omega, \mathcal{B}, \mathbb{P}, \{\mathcal{B}_t\}_{t\geq 0})$ with a self-adjoint covariance operator Q. The assumptions on F and Q will be given in the following section.

2.1. Notation and assumptions. Throughout this paper, we denote by $H := L^2(0,1)$ the square integrable space, and by $\{\lambda_m\}_{m\in\mathbb{N}} := \left\{\mathbf{i}(m\pi)^2 + \alpha\right\}_{m\in\mathbb{N}}$ the eigenvalues of the linear operator $\Lambda := -\mathbf{i}\Delta + \alpha$ with associated eigenvectors $\{e_m\}_{m\in\mathbb{N}}$. Denote the complex inner product in H by

$$\langle v_1, v_2\rangle := \int_0^1 v_1(x)\overline{v_2(x)}dx, \quad v_1, v_2 \in H.$$

In what follows, we will use the space

$$\dot{H}^s := D(\Lambda^{\frac{s}{2}}) = \left\{ u = \sum_{m=1}^{\infty} \langle u, e_m\rangle e_m \,\middle|\, \sum_{m=1}^{\infty} |\langle u, e_m\rangle|^2 |\lambda_m|^s < \infty \right\}$$

equipped with the norm

$$\|u\|^2_{\dot{H}^s} = \sum_{m=1}^{\infty} |\langle u, e_m\rangle|^2 |\lambda_m|^s,$$

which is equivalent to the Sobolev norm $\|\cdot\|_{H^s}$ when $s = 0, 1, 2$. For $s = 0$, $\|\cdot\|_{\dot{H}^0} = \|\cdot\|_H$, and we use the notation $\|\cdot\|$ instead of $\|\cdot\|_H$ for convenience.

Let $\{\tilde{e}_m\}_{m\in\mathbb{N}}$ be a complete orthonormal basis in H, which may be different from the eigenvectors $\{e_m\}_{m\in\mathbb{N}}$ of the Laplacian. Then for the self-adjoint operator Q, there exists a set of nonnegative constants $\{\gamma_m\}_{m\in\mathbb{N}}$ such that

$$Q^{\frac{1}{2}} v = \sum_{m=1}^{\infty} \sqrt{\gamma_m}\langle v, \tilde{e}_m\rangle \tilde{e}_m \quad \forall\ v \in H.$$

The Q-Wiener process is then defined by

$$W^Q(t) = \sum_{m=1}^{\infty} Q^{\frac{1}{2}} \beta_m(t)\tilde{e}_m, \quad t \geq 0,$$

with a family of mutually independent identically distributed $\mathbb{C}$-valued Brownian motions $\{\beta_m\}_{m\in\mathbb{N}}$. It can also be written as

$$W^Q(t) = Q^{\frac{1}{2}} W(t),$$

where

$$W(t) = \sum_{m=1}^{\infty} \beta_m(t)\tilde{e}_m, \quad t \geq 0,$$

is a cylindrical Wiener process.

For the nonlinear function F and the covariance operator Q, the following assumptions are needed to ensure the well-posedness of (2.1).

Assumption 1. There exists a positive constant L_F such that

$$\|F(v) - F(w)\| \leq L_F \|v - w\| \quad \forall\, v, w \in H.$$

In addition, $F(0) = 0$ and

$$\Im\langle \overline{v}, F(v)\rangle = 0 \quad \forall\, v \in H.$$

For any $s \geq 0$, the Hilbert–Schmidt norm of operator $Q^{\frac{1}{2}}$ is defined as

$$\|Q^{\frac{1}{2}}\|^2_{\mathcal{HS}(H,\dot{H}^s)} := \|(-\Delta)^{\frac{s}{2}} Q^{\frac{1}{2}}\|^2_{\mathcal{HS}(H,H)}.$$

Assumption 2. Assume that Q is a self-adjoint operator on H, and there exists some $s \geq 0$ such that

$$\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(H,\dot{H}^s)} < \infty.$$

Let $S(t) := e^{-t\Lambda}$ be the semigroup generated by operator Λ. The mild solution of (2.1) exists globally under Assumptions 1 and 2 with the following form (see [21, Theorem 6.2.3]):

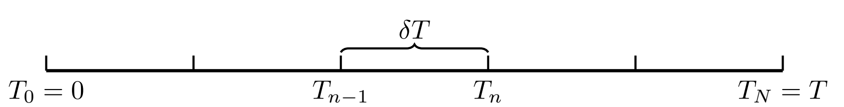
$$u(t) = S(t)u_0 + \mathbf{i}\int_0^t S(t-s)F(u)ds + \int_0^t S(t-s)Q^{\frac{1}{2}}dW(s). \tag{2.2}$$

For any $0 \leq r \leq l$, it holds that

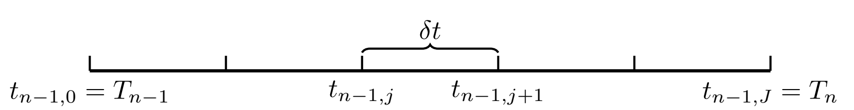
$$\|S(t)\|_{\mathcal{L}(\dot{H}^l,\dot{H}^r)} := \sup_{v\in\dot{H}^l} \frac{\|S(t)v\|_{\dot{H}^r}}{\|v\|_{\dot{H}^l}} \leq e^{-\alpha t}.$$

2.2. Framework of parallelization in time. In this section, we briefly recall the procedure of parareal algorithms, which are constructed through the interaction of two different level integrators. The parareal algorithm, known as a kind of "parallel-in-time" algorithm, consists of four parts in general: interval partition, initialization, time-parallel computation, and correction.

2.2.1. Interval partition. The considered interval $[0, T]$ is first divided into N parts with a uniform coarse step size $\delta T = T_n - T_{n-1}$ for any $n = 1, \ldots, N$ as follows:



Each subinterval is further divided into J parts with a uniform fine step size $\delta t = t_{n,j+1} - t_{n,j} = \frac{\delta T}{J}$ for any $n = 0, \ldots, N-1$ and $j = 1, \ldots, J-1$. It satisfies that $t_{n-1,0} = T_{n-1}$ and $t_{n-1,J} =: t_{n,0}$:



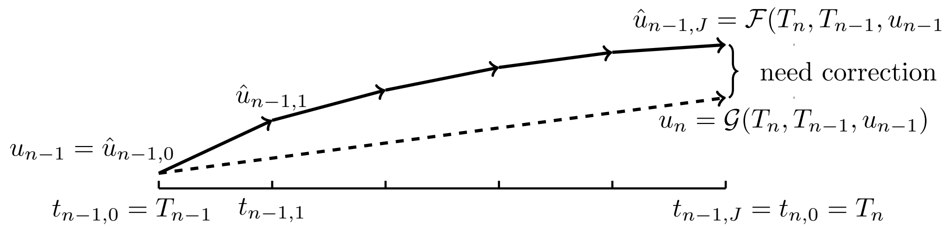


FIG. 1. *Numerical solutions obtained by $\mathcal{F}$ and $\mathcal{G}$ on $[T_{n-1}, T_n]$.*

If the value at the coarse grid $\{T_n\}_{n=0}^N$ is given, denoted by $\{u_n\}_{n=0}^N$, the numerical solutions at the fine grid $\{t_{n-1,j}\}_{j=1}^J$ on each subinterval $[T_{n-1}, T_n]$ can be calculated in parallel.

2.2.2. Initialization. The coarse integrator $\mathcal{G}$ is defined by

$$u_n = \mathcal{G}(T_n, T_{n-1}, u_{n-1}) \tag{2.3}$$

based on some specific scheme to gain a numerical solution $\{u_n\}_{n=0}^N$ at the coarse grid $\{T_n\}_{n=0}^N$.

The coarse integrator $\mathcal{G}$ gives a rough approximation on the coarse grid $\{T_n\}_{n=0}^N$, which makes it possible to calculate the numerical solutions on each subinterval parallel to one another. In general, $\mathcal{G}$ is required to be easy to calculate and need not to be of high accuracy. On the other hand, the fine integrator $\mathcal{F}$ defined on each subinterval is assumed to be more accurate than $\mathcal{G}$ to ensure that the proposed parareal algorithm is accurate enough.

2.2.3. Time-parallel computation. We consider the subinterval $[T_{n-1}, T_n]$ with initial value u_{n-1} at T_{n-1} and apply a fine integrator $\mathcal{F}$ over this subinterval. More precisely, we denote by

$$\hat{u}_{n-1,1} := \mathcal{F}(t_{n-1,1}, t_{n-1,0}, \hat{u}_{n-1,0})$$

the one-step approximation obtained by $\mathcal{F}$ starting from $\hat{u}_{n-1,0} := u_{n-1}$ at time $t_{n-1,0} := T_{n-1}$ (see Figure 1). Thus, the numerical solution at time $t_{n-1,j}$ can be expressed as

$$\begin{aligned}\hat{u}_{n-1,j} &= \mathcal{F}(t_{n-1,j}, t_{n-1,j-1}, \hat{u}_{n-1,j-1})\\ &= \mathcal{F}(t_{n-1,j}, t_{n-1,0}, \hat{u}_{n-1,0})\end{aligned}$$

for $j = 1, \ldots, J$. Hence, we derive $\hat{u}_{n-1,J} = \mathcal{F}(T_n, T_{n-1}, u_{n-1})$ which is $\mathcal{B}_{T_n}$-adapted.

2.2.4. Correction. Note that we get two numerical solutions u_n and $\hat{u}_{n-1,J}$ at time T_n through the initialization and parallelization, respectively (see Figure 1). Some corrections should be applied to get the final numerical solution on the coarse grid $\{T_n\}_{n=0}^N$. The correction iteration (see also [3, 15, 16]) is defined as

$$\begin{aligned}u_n^{(0)} &= \mathcal{G}(T_n, T_{n-1}, u_{n-1}^{(0)}),\\ u_n^{(k)} &= \mathcal{G}(T_n, T_{n-1}, u_{n-1}^{(k)}) + \mathcal{F}(T_n, T_{n-1}, u_{n-1}^{(k-1)}) - \mathcal{G}(T_n, T_{n-1}, u_{n-1}^{(k-1)})\end{aligned} \tag{2.4}$$

starting from $u_0^{(k)} = u_0$ for all $k \in \mathbb{N}$. The solution $\{u_n^{(k)}\}_{0\le n\le N} \subset H$ of (2.4) is obtained after the calculation of $\{u_n^{(k-1)}\}_{0\le n\le N}$, and is $\{\mathcal{B}_{T_n}\}_{0\le n\le N}$-adapted for any $k \in \mathbb{N}$.

3. Parareal exponential θ-scheme for the linear case. In this section, we focus on the linear case

$$du = (\mathbf{i}\Delta u - \alpha u + \mathbf{i}\lambda u)dt + Q^{\frac{1}{2}}dW, \tag{3.1}$$

that is, $F(u) = \lambda u$ in (2.1) for some constant $\lambda \in \mathbb{R}$, and investigate the behavior of the parareal algorithm obtained by choosing the exponential θ-scheme as the coarse integrator and the exact solver as the fine integrator. We show that the proposed parareal algorithms are valid for longtime simulation with a unique invariant Gaussian distribution under some restrictions on $\theta \in [0, 1]$.

Rewriting the above equation through its components $u^m := \langle u, e_m\rangle$, we obtain

$$du^m = (-\lambda_m + \mathbf{i}\lambda)u^m dt + \sum_{i=1}^{\infty}\langle Q^{\frac{1}{2}}\tilde{e}_i, e_m\rangle d\beta_i, \quad m \in \mathbb{N}.$$

Its solution is given by an Ornstein–Uhlenbeck process

$$u^m(t) = e^{(-\lambda_m+\mathbf{i}\lambda)t}u^m(0) + \sum_{i=1}^{\infty}\int_0^t e^{(-\lambda_m+\mathbf{i}\lambda)(t-s)}\langle Q^{\frac{1}{2}}\tilde{e}_i, e_m\rangle d\beta_i(s)$$

with $u^m(0) = \langle u_0, e_m\rangle$.

3.1. Complex invariant Gaussian measure. Note that $\{u^m(t)\}_{t\geq 0}$ satisfies a complex Gaussian distribution $\mathcal{N}(\mathbf{m}, \mathbf{C}, \mathbf{R})$ defined by its mean $\mathbf{m}$, covariance $\mathbf{C}$, and relation $\mathbf{R}$:

$$\begin{aligned}
&\mathbf{m}\left(u^m(t)\right) := \mathbb{E}\left[u^m(t)\right] = e^{(-\lambda_m+\mathbf{i}\lambda)t}\mathbf{m}\left[u^m(0)\right],\\
&\mathbf{C}\left(u^m(t)\right) := \mathbb{E}\left|u^m(t) - \mathbf{m}\left(u^m(t)\right)\right|^2 = e^{-2\alpha t}\mathbf{C}\left(u^m(0)\right) + \frac{1-e^{-2\alpha t}}{\alpha}\|Q^{\frac{1}{2}}e_m\|^2,\\
&\mathbf{R}\left(u^m(t)\right) := \mathbb{E}\left(u^m(t) - \mathbf{m}\left(u^m(t)\right)\right)^2 = e^{2(-\lambda_m+\mathbf{i}\lambda)t}\mathbf{R}\left(u^m(0)\right).
\end{aligned}$$

We use the notation $\mu_t^m := \mathcal{N}(\mathbf{m}(u^m(t)), \mathbf{C}(u^m(t)), \mathbf{R}(u^m(t)))$ for simplicity.

Remark 1. Consider a one-dimensional $\mathbb{C}$-valued Gaussian random variable $Z = \mathbf{a} + \mathbf{i}\mathbf{b}$ with $\mathbf{a}$ and $\mathbf{b}$ being two $\mathbb{R}$-valued Gaussian random variables. If its relation vanishes, i.e.,

$$\mathbf{R}(Z) = \mathbb{E}|\mathbf{a} - \mathbb{E}\mathbf{a}|^2 - \mathbb{E}|\mathbf{b} - \mathbb{E}\mathbf{b}|^2 + 2\mathbf{i}(\mathbb{E}[\mathbf{a}\mathbf{b}] - \mathbb{E}\mathbf{a}\mathbb{E}\mathbf{b}) = 0,$$

it implies $\mathbb{E}|\mathbf{a} - \mathbb{E}\mathbf{a}|^2 = \mathbb{E}|\mathbf{b} - \mathbb{E}\mathbf{b}|^2$ and $\mathbb{E}[\mathbf{a}\mathbf{b}] = \mathbb{E}\mathbf{a}\mathbb{E}\mathbf{b}$. Since $\mathbf{a}$ and $\mathbf{b}$ are both Gaussian, we obtain equivalently that $\mathbf{a}$ and $\mathbf{b}$ are independent with the same covariance.

Remark 2. The characteristic function of a one-dimensional complex Gaussian variable Z with distribution $\nu = \mathcal{N}(\mathbf{m}, \mathbf{C}, \mathbf{R})$ reads (see, e.g., [1])

$$\begin{aligned}
\hat{\nu}(c) &:= \mathbb{E}[\exp\{\mathbf{i}\Re(\bar{c}Z)\}] = \int_{\mathbb{C}}\exp\{\mathbf{i}\Re(\bar{c}z)\}\nu(dz)\\
&= \exp\left\{\mathbf{i}\Re(\bar{c}\mathbf{m}) - \frac{1}{4}(\bar{c}\mathbf{C}c + \Re(\bar{c}\mathbf{R}\bar{c}))\right\}, \quad c \in \mathbb{C}.
\end{aligned}$$

It can be generalized for the infinite-dimensional case utilizing an inner product in H:

$$\hat{\nu}(w) := \exp\left\{\mathbf{i}\Re\langle\bar{w}, \mathbf{m}\rangle - \frac{1}{4}\left(\langle\mathbf{C}\bar{w}, w\rangle + \Re\langle\mathbf{R}\bar{w}, \bar{w}\rangle\right)\right\}, \quad w \in H.$$

Hence, we get that the unique invariant measure of (3.1) is a complex Gaussian distribution, which is stated in the following theorem. We refer to [10, 11] and references therein for the existence of invariant measures for the nonlinear case, and refer to [4, 7] and references therein for other types of SPDEs.

THEOREM 3.1. *Assume that Assumption 2 holds with $s = 0$. The solution u in* (3.1) *possesses a unique invariant measure*

$$\mu_\infty = \mathcal{N}\left(0, \frac{1}{\alpha}Q, \mathbf{0}\right).$$

Proof. Based on Remark 1, we define

$$u_\infty^m = \frac{\|Q^{\frac{1}{2}}e_m\|}{\sqrt{2\alpha}}(\xi_m + \mathbf{i}r_m)$$

with $\{\xi_m, r_m\}_{m\in\mathbb{N}}$ being independent standard $\mathbb{R}$-valued normal random variables, i.e., $\xi_m, r_m \sim \mathcal{N}(0,1)$. Apparently,

$$u_\infty^m \sim \mathcal{N}\left(0, \frac{\|Q^{\frac{1}{2}}e_m\|^2}{\alpha}, 0\right) =: \mu_\infty^m.$$

We claim that the following random variable has the distribution μ_∞:

$$u_\infty := \sum_{m=1}^{\infty} u_\infty^m e_m = \sum_{m=1}^{\infty} \frac{\|Q^{\frac{1}{2}}e_m\|}{\sqrt{2\alpha}}(\xi_m + \mathbf{i}r_m)e_m.$$

Compared with $u(t) = \sum_{m=1}^{\infty} u^m(t)e_m$, it then suffices to show that the distribution μ_t^m of $u^m(t)$ converges to μ_∞^m. As a result of Remark 2, the characteristic function of μ_t^m is

$$\begin{aligned}\hat{\mu}_t^m(c) = \exp\Bigg\{&\mathbf{i}\Re(\bar{c}e^{(-\lambda_m+\mathbf{i}\lambda)t}\mathbb{E}\left[u^m(0)\right]) - \frac{1}{4}\Re\left(e^{2(-\lambda_m+\mathbf{i}\lambda)t}\mathbf{R}\left(u^m(0)\right)\bar{c}^2\right)\\ &- \frac{1}{4}\left(e^{-2\alpha t}\mathbf{C}\left(u^m(0)\right) + \frac{1-e^{-2\alpha t}}{\alpha}\|Q^{\frac{1}{2}}e_m\|^2|c|^2\right)\Bigg\}\end{aligned}$$

and $\hat{\mu}_t^m(c) \to \exp\{-\frac{\|Q^{\frac{1}{2}}e_m\|^2}{4\alpha}|c|^2\} = \hat{\mu}_\infty^m(c)$. ∎

3.2. Parareal exponential $\boldsymbol{\theta}$-scheme. In this section, we construct a parareal algorithm based on the exponential θ-scheme as the coarse integrator. We show that the proposed parareal algorithm converges to the solution generated by the fine integrator $\mathcal{F}$ as $k \to \infty$.

We first define the exponential θ-scheme, denoted by $\mathcal{G}_\theta$, applied to (3.1),

$$u_n = S(\delta T)u_{n-1} + \mathbf{i}(1-\theta)\lambda\delta T S(\delta T)u_{n-1} + \mathbf{i}\theta\lambda\delta T u_n + S(\delta T)Q^{\frac{1}{2}}\delta_n W,$$

and equivalently,

$$\begin{aligned}u_n &= (1+\mathbf{i}(1-\theta)\lambda\delta T)S_\theta S(\delta T)u_{n-1} + S_\theta S(\delta T)Q^{\frac{1}{2}}\delta_n W\\ &=: \mathcal{G}_\theta(T_n, T_{n-1}, u_{n-1})\end{aligned} \tag{3.2}$$

with $S_\theta := (1-\mathbf{i}\theta\lambda\delta T)^{-1}$, $\theta \in [0,1]$ and $\delta_n W := W(T_n) - W(T_{n-1})$. The initial value of the numerical solution is the same as the initial value of the exact solution, and apparently $\{u_n\}_{n=0}^N$ is $\{\mathcal{B}_{T_n}\}_{n=0}^N$-adapted.

The distribution of $\{u_n\}_{n=0}^N$ can also be calculated in the same procedure as Theorem 3.1 by rewriting the Fourier components $u_n^m := \langle u_n, e_m\rangle$ of u_n as

$$\begin{aligned} u_n^m &= (1+\mathbf{i}(1-\theta)\lambda\delta T)S_\theta e^{-\lambda_m\delta T}u_{n-1}^m + S_\theta e^{-\lambda_m \delta T}\sum_{i=1}^{\infty}\langle Q^{\frac12}\tilde e_i, e_m\rangle \delta_n\beta_i \\ &= \eta^n e^{-\lambda_m\delta T n}u_0^m + S_\theta e^{-\lambda_m\delta T}\sum_{j=0}^{n-1}\eta^j e^{-\lambda_m\delta T j}\sum_{i=1}^{\infty}\langle Q^{\frac12}\tilde e_i, e_m\rangle\delta_{n-j}\beta_i \end{aligned}$$

with

$$\eta := (1+\mathbf{i}(1-\theta)\lambda\delta T)S_\theta = \frac{1+\mathbf{i}(1-\theta)\lambda\delta T}{1-\mathbf{i}\theta\lambda\delta T}.$$

Then according to the independence of $\{\delta_{n-j}\beta_i\}_{1\le j\le n-1, i\ge 1}$ and $\mathbb{E}|\delta_{n-j}\beta_i|^2 = 2\delta T$, we derive the distribution of u_n^m defined by its mean, covariance, and relation:

$$\begin{aligned} \mathbf{m}(u_n^m) &= \eta^n e^{-\lambda_m\delta T n}\mathbb{E}[u_0^m], \\ \mathbf{C}(u_n^m) &= |\eta|^{2n}e^{-2\alpha\delta T n}\mathbf{C}(u_0^m) \\ &\quad + \left(\left(1+\theta^2\lambda^2\delta T^2\right)e^{2\alpha\delta T}\right)^{-1}\frac{1-\tilde\eta^n}{1-\tilde\eta}\|Q^{\frac12}e_m\|^2(2\delta T), \\ \mathbf{R}(u_n^m) &= \eta^{2n}e^{-2\lambda_m\delta T n}\mathbf{R}(u_0^m), \end{aligned}$$

where

$$\tilde\eta := \frac{1+(1-\theta)^2\lambda^2\delta T^2}{\left(1+\theta^2\lambda^2\delta T^2\right)e^{2\alpha\delta T}} = |\eta|^2 e^{-2\alpha\delta T}$$

is called the *stability function* here.

The distribution of u_n^m converges to μ_∞^m as $n\to\infty$ and $\delta T\to 0$ for any $\alpha > 0$ if and only if $|\eta| < 1$, or equivalently, $\theta\in[\frac12, 1]$ (see Figure 2). The surface in each subfigure in Figure 2 denotes the stability function for different $\theta = 0, 0.3, 0.5$ and $\delta T = 0.1, 0.005$. This condition also leads to the time-independent error analysis of the parareal algorithm (see Theorem 3.2).

By choosing the exponential θ-scheme $\mathcal{G}_\theta$ defined in (3.2) as the coarse integrator, and the exact solver as the fine integrator, i.e.,

$$\mathcal{F}(T_n, T_{n-1}, u_{n-1}^{(k-1)}) := e^{(\mathbf{i}\Delta-\alpha+\mathbf{i}\lambda)\delta T}u_{n-1}^{(k-1)} + \int_{T_{n-1}}^{T_n} e^{(\mathbf{i}\Delta-\alpha+\mathbf{i}\lambda)(T_n-s)}Q^{\frac12}dW(s),$$

the parareal algorithm (2.4) has the form

$$\begin{aligned} u_n^{(k)} &= \mathcal{G}_\theta(T_n, T_{n-1}, u_{n-1}^{(k)}) + \mathcal{F}(T_n, T_{n-1}, u_{n-1}^{(k-1)}) - \mathcal{G}_\theta(T_n, T_{n-1}, u_{n-1}^{(k-1)}) \\ &= (1+\mathbf{i}(1-\theta)\lambda\delta T)S_\theta S(\delta T)u_{n-1}^{(k)} \\ &\quad - (1+\mathbf{i}(1-\theta)\lambda\delta T)S_\theta S(\delta T)u_{n-1}^{(k-1)} + \mathcal{F}(T_n, T_{n-1}, u_{n-1}^{(k-1)}) \\ &= \eta S(\delta T)u_{n-1}^{(k)} - \eta S(\delta T)u_{n-1}^{(k-1)} + \mathcal{F}(T_n, T_{n-1}, u_{n-1}^{(k-1)}). \end{aligned} \tag{3.3}$$

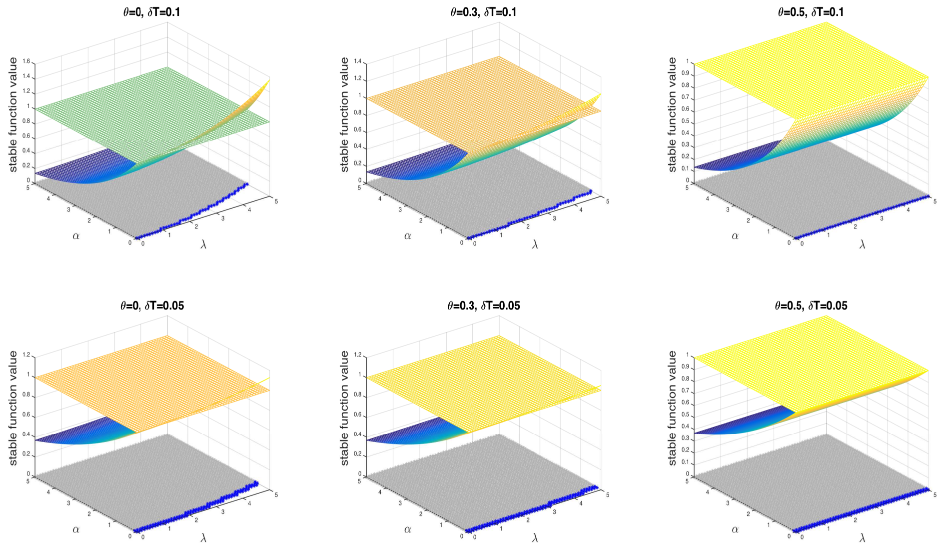


FIG. 2. *Convergence area (gray) vs. α and λ.*

3.3. Convergence error. The following result gives the convergence error in the strong sense caused by the parareal algorithms, which shows order k with respect to the coarse step size δT.

THEOREM 3.2. *Let Assumptions* 1 *and* 2 *hold with $s = 0$, and $\{u_n^{(k)}\}_{0\le n\le N, k\in\mathbb{N}}$ be the solution of* (3.3) *with $\mathcal{F}$ being the exact integrator. For a fixed iterated number $k \in \mathbb{N}$, $u_n^{(k)}$ is an approximation of $u(T_n)$ with order k. More precisely,*

(i) *if $\alpha > |\lambda|\sqrt{\left(\frac{1}{2}-\theta\right)\vee 0}$, then* (3.3) *converges for long times*

$$\sup_{n\in\mathbb{N}}\left\|u(T_n)-u_n^{(k)}\right\|_{L^2(\Omega;H)} \le C\left(|2\theta-1|\delta T^k+\delta T^{2k}\right)\sup_{n\in\mathbb{N}}\left\|u(T_n)-u_n^{(0)}\right\|_{L^2(\Omega;H)}$$

with $C = C(k,\alpha,\theta,\lambda)$ independent of time interval;

(ii) *if $0 < \alpha \le |\lambda|\sqrt{\left(\frac{1}{2}-\theta\right)\vee 0}$, that is, $\theta\in[0,\frac{1}{2})$ and $\alpha\le|\lambda|\sqrt{\left(\frac{1}{2}-\theta\right)}$, then*

$$\sup_{0\le n\le N}\left\|u(T_n)-u_n^{(k)}\right\|_{L^2(\Omega;H)} \le C\delta T^k \sup_{0\le n\le N}\left\|u(T_n)-u_n^{(0)}\right\|_{L^2(\Omega;H)}$$

with $C = C(T,k)$ and $T = \delta T N$ for some fixed $N\in\mathbb{N}$.

Proof. The parareal algorithm (2.4) based on $\mathcal{G} = \mathcal{G}_\theta$ defined in (3.2) and $\mathcal{F}$ being the exact integrator yields

$$\begin{aligned} u_n^{(k)} &= \eta S(\delta T)u_{n-1}^{(k)} - \eta S(\delta T)u_{n-1}^{(k-1)} + S(\delta T)u_{n-1}^{(k-1)} \\ &\quad + \mathbf{i}\lambda\int_{T_{n-1}}^{T_n} S(T_n-s)u_{u_{n-1}^{(k-1)}}(s)ds + \int_{T_{n-1}}^{T_n} S(T_n-s)Q^{\frac{1}{2}}dW \end{aligned}$$

for $n\ge 1$ and $k\ge 1$, where $u_{u_{n-1}^{(k-1)}}(s)$ denotes the exact solution at time s starting from $u_{n-1}^{(k-1)}$ at time T_{n-1}.

Denoting the error $\epsilon_n^{(k)} := u_n^{(k)} - u(T_n)$, we obtain

$$\begin{aligned}\epsilon_n^{(k)} &= \eta S(\delta T)\epsilon_{n-1}^{(k)} - \eta S(\delta T)\epsilon_{n-1}^{(k-1)} + S(\delta T)\epsilon_{n-1}^{(k-1)} \\ &\quad + \mathbf{i}\lambda \int_{T_{n-1}}^{T_n} S(T_n - s)\left[u_{u_{n-1}^{(k-1)}}(s) - u_{u(T_{n-1})}(s)\right] ds \\ &= \eta S(\delta T)\epsilon_{n-1}^{(k)} + \left[e^{\mathbf{i}\lambda\delta T} - \eta\right] S(\delta T)\epsilon_{n-1}^{(k-1)},\end{aligned}$$

where

$$\eta = \frac{1 + \mathbf{i}(1-\theta)\lambda\delta T}{1 - \mathbf{i}\theta\delta T},$$

and in the last step we have used the following fact:

$$\begin{aligned}&u_{u_{n-1}^{(k-1)}}(s) - u_{u(T_{n-1})}(s) \\ &\quad = e^{(\mathbf{i}\Delta - \alpha + \mathbf{i}\lambda)(s - T_{n-1})} u_{n-1}^{(k-1)} + \int_{T_{n-1}}^{s} e^{(\mathbf{i}\Delta - \alpha + \mathbf{i}\lambda)(s-r)} Q^{\frac{1}{2}} dW(r) \\ &\qquad - e^{(\mathbf{i}\Delta - \alpha + \mathbf{i}\lambda)(s - T_{n-1})} u(T_{n-1}) - \int_{T_{n-1}}^{s} e^{(\mathbf{i}\Delta - \alpha + \mathbf{i}\lambda)(s-r)} Q^{\frac{1}{2}} dW(r) \\ &\quad = S(s - T_{n-1}) e^{\mathbf{i}\lambda(s - T_{n-1})} \epsilon_{n-1}^{(k-1)}.\end{aligned}$$

Hence, we get

$$\begin{aligned}\|\epsilon_n^{(k)}\|_{L^2(\Omega;H)} &\le |\eta| e^{-\alpha\delta T}\|\epsilon_{n-1}^{(k)}\|_{L^2(\Omega;H)} + |e^{\mathbf{i}\lambda\delta T} - \eta| e^{-\alpha\delta T}\|\epsilon_{n-1}^{(k-1)}\|_{L^2(\Omega;H)} \\ &\le \left(|\eta| e^{-\alpha\delta T}\right)^n \|\epsilon_0^{(k)}\|_{L^2(\Omega;H)} \\ &\quad + |e^{\mathbf{i}\lambda\delta T} - \eta| e^{-\alpha\delta T} \sum_{j=0}^{n-1} \left(|\eta| e^{-\alpha\delta T}\right)^{n-1-j} \|\epsilon_j^{(k-1)}\|_{L^2(\Omega;H)} \\ &= |e^{\mathbf{i}\lambda\delta T} - \eta| e^{-\alpha\delta T} \sum_{j=1}^{n-1} \left(|\eta| e^{-\alpha\delta T}\right)^{n-1-j} \|\epsilon_j^{(k-1)}\|_{L^2(\Omega;H)} \qquad (3.4)\end{aligned}$$

based on the fact $\epsilon_0^{(k)} = 0$ for any $k \in \mathbb{N}$. Denoting the error vector

$$\varepsilon^{(k)} := \left(\|\epsilon_1^{(k)}\|_{L^2(\Omega;H)}, \ldots, \|\epsilon_n^{(k)}\|_{L^2(\Omega;H)}\right)^\top$$

and the n-dimensional matrix (see also [16])

$$M(\beta) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \beta & 1 & \cdots & 0 & 0 \\ \beta^2 & \beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta^{n-2} & \beta^{n-3} & \cdots & 1 & 0 \end{pmatrix},$$

we rewrite (3.4) as

$$\begin{aligned}\varepsilon^{(k)} &\le |e^{\mathbf{i}\lambda\delta T} - \eta| e^{-\alpha\delta T} M(|\eta| e^{-\alpha\delta T})\varepsilon^{(k-1)} \\ &\le |e^{\mathbf{i}\lambda\delta T} - \eta|^k e^{-\alpha\delta T k} M^k(|\eta| e^{-\alpha\delta T})\varepsilon^{(0)}. \qquad (3.5)\end{aligned}$$

It is shown in [16] that

$$\|M^k(\beta)\|_\infty \le \begin{cases} \min\left\{\left(\frac{1-\beta^{n-1}}{1-\beta}\right)^k, \begin{pmatrix} n-1 \\ k \end{pmatrix}\right\} & \text{if } \beta < 1, \\ \beta^{n-1-k}\begin{pmatrix} n-1 \\ k \end{pmatrix} & \text{if } \beta \ge 1, \end{cases}$$

where

$$\begin{pmatrix} n-1 \\ k \end{pmatrix} = \frac{(n-1)(n-2)\cdots(n-k)}{k!} \le \frac{n^k}{k!}.$$

If $\alpha > |\lambda|\sqrt{(\frac{1}{2}-\theta)\vee 0}$, we get

$$e^{2\alpha\delta T} > 1 + 2\alpha^2\delta T^2 > 1 + (1-2\theta)\lambda^2\delta T^2 > 1 + \frac{(1-2\theta)\lambda^2\delta T^2}{1+\theta^2\lambda^2\delta T^2} = |\eta|^2,$$

which then yields $|\eta|e^{-\alpha\delta T} < 1$. It is apparent that this condition holds for all $\alpha > 0$ if $\theta \in [\frac{1}{2}, 1]$. We conclude under this condition that

$$\|\varepsilon^{(k)}\|_\infty \le \left(\frac{|e^{\mathbf{i}\lambda\delta T} - \eta|e^{-\alpha\delta T}}{1-|\eta|e^{-\alpha\delta T}}\right)^k \|\varepsilon^{(0)}\|_\infty. \tag{3.6}$$

Note that

$$\begin{aligned} |e^{\mathbf{i}\lambda\delta T} - \eta|^2 &= \left|\frac{(1-\mathbf{i}\theta\lambda\delta T)(\cos(\lambda\delta T) + \mathbf{i}\sin(\lambda\delta T)) - (1+\mathbf{i}(1-\theta)\lambda\delta T)}{1-\mathbf{i}\theta\lambda\delta T}\right|^2 \\ &\le 2 - 2\cos(\lambda\delta T) - 2\lambda\delta T\sin(\lambda\delta T) \\ &\quad + (\theta^2 + (1-\theta)^2)\lambda^2\delta T^2 + 2\theta(1-\theta)\lambda^2\delta T^2\cos(\lambda\delta T) \\ &\le \frac{(2\theta-1)^2}{4}\lambda^4\delta T^4 + C\delta T^6 \end{aligned}$$

with $C = C(\theta,\lambda)$ utilizing the Taylor expansion for $\delta T < 1$, which leads to

$$\left(|e^{\mathbf{i}\lambda\delta T} - \eta|e^{-\alpha\delta T}\right)^k \le \left(\frac{|2\theta-1|}{2}\lambda^2\delta T^2 + C\delta T^3\right)^k e^{-\alpha\delta T k}. \tag{3.7}$$

Noticing that

$$|\eta| = \sqrt{1 + \frac{(1-2\theta)\lambda^2\delta T^2}{1+\theta^2\lambda^2\delta T^2}} \le \sqrt{1+(1-2\theta)\lambda^2\delta T^2} \le e^{|\lambda|\sqrt{(\frac{1}{2}-\theta)\vee 0}\,\delta T}$$

one derives

$$\left(1-|\eta|e^{-\alpha\delta T}\right)^{-k} \le \left(1 - e^{\left(|\lambda|\sqrt{(\frac{1}{2}-\theta)\vee 0} - \alpha\right)\delta T}\right)^{-k} \le (C\delta T^{-1})^k \tag{3.8}$$

with constant $C = C(\alpha - |\lambda|\sqrt{(\frac{1}{2}-\theta)\vee 0})$ decreasing as $\alpha - |\lambda|\sqrt{(\frac{1}{2}-\theta)\vee 0}$ becomes larger. Combining (3.6)–(3.8), we finally conclude

$$\|\varepsilon^{(k)}\|_\infty \le (C|2\theta-1|\delta T + C\delta T^2)^k \|\varepsilon^{(0)}\|_\infty.$$

If $\theta \in [0, \frac{1}{2})$ and $\alpha \leq |\lambda|\sqrt{(\frac{1}{2}-\theta)}$, according to (3.5), we get

$$\begin{aligned}\|\varepsilon^{(k)}\|_\infty &\leq \left(|e^{\mathbf{i}\lambda\delta T} - \eta|e^{-\alpha\delta T}\right)^k (|\eta|e^{-\alpha\delta T} \vee 1)^{n-1-k} \frac{n^k}{k!}\|\varepsilon^{(0)}\|_\infty \\ &\leq \left(C\delta T^2 e^{-\alpha\delta T}\right)^k e^{\left(\sqrt{2(1-2\theta)}|\lambda|-\alpha\right)T_n} \frac{n^k}{k!}\|\varepsilon^{(0)}\|_\infty \\ &\leq \frac{(CT_n e^{-\alpha\delta T})^k}{k!} e^{\left(\sqrt{2(1-2\theta)}|\lambda|-\alpha\right)T_n} \delta T^k \|\varepsilon^{(0)}\|_\infty,\end{aligned}$$

which converges as $k \to \infty$ and shows order k only on finite time intervals. ∎

Remark 3. Note that for the additive noise case, the discretization of the noise term will not affect the rate of convergence of the parareal algorithm (2.4) due to the correction technique. More precisely, if the coarse integrator $\mathcal{G}_\theta$ in (3.2) is replaced by

$$u_n = (1 + \mathbf{i}(1-\theta)\lambda\delta T)S_\theta S(\delta T)u_{n-1} + S_\theta \int_{T_{n-1}}^{T_n} S(T_n - s)Q^{\frac{1}{2}} dW(s),$$

the accuracy of parareal algorithm (3.3) remains the same.

Remark 4. If instead the implicit Euler scheme is considered as the coarse integrator $\mathcal{G}$, the parareal algorithm (2.4) with $\mathcal{F}$ being the exact integrator turns out to be

$$u_n^{(k)} = \breve{S}_{\delta T} u_{n-1}^{(k)} - \breve{S}_{\delta T} u_{n-1}^{(k-1)} + \breve{S}(\delta T) u_{n-1}^{(k-1)} + \int_{T_{n-1}}^{T_n} \breve{S}(T_n - s)Q^{\frac{1}{2}} dW(s),$$

where $\breve{S}_{\delta T} = (1 + \alpha\delta T - \mathbf{i}\lambda\delta T - \mathbf{i}\delta T\Delta)^{-1}$ and $\breve{S}(\delta T) = e^{(\mathbf{i}\Delta - \alpha + \mathbf{i}\lambda)\delta T}$.

In this case, the error between $u_n^{(k)}$ and $u(T_n)$ shows

$$\epsilon_n^{(k)} = \breve{S}_{\delta T}\epsilon_{n-1}^{(k)} + \left(\breve{S}(\delta T) - \breve{S}_{\delta T}\right)\epsilon_{n-1}^{(k-1)}.$$

To gain a convergence order, the estimations of $\|\breve{S}(\delta T) - \breve{S}_{\delta T}\|_{\mathcal{L}(\dot{H}^s, H)}$ and $\|\epsilon_n^{(0)}\|_{\dot{H}^{ks}}$ will be needed. It then requires an extremely high regularity of both $u(t)$ and $u_n^{(0)}$ and requires that parameter s in Assumption 2 is large enough.

4. Application to the nonlinear case. For the nonlinear case, parareal algorithms, with the exponential θ-scheme being the coarse integrator, are also suitable for longtime simulation with some restriction on δT and α.

In section 4.1, the case $\theta = 0$ is considered and the parareal algorithm is in explicit form. The convergence of scheme (2.4) is studied for $\mathcal{F}$ being both the exact solver and the exponential Euler scheme.

In section 4.2, for $\theta \in [0, 1]$, the long time convergence of scheme (2.4) is studied with $\mathcal{F}$ being the exact solver.

4.1. Parareal exponential Euler scheme ($\boldsymbol{\theta = 0}$). The exponential Euler scheme applied to the nonlinear equation (2.1), denoted by $\mathcal{G}_I$, is in the form

$$\begin{aligned}u_{n+1} &= S(\delta T)u_n + \mathbf{i}S(\delta T)F(u_n)\delta T + S(\delta T)Q^{\frac{1}{2}}\delta_{n+1}W \\ &=: \mathcal{G}_I(T_{n+1}, T_n, u_n)\end{aligned} \tag{4.1}$$

with $\delta_{n+1}W := W(T_{n+1}) - W(T_n)$ and $u_0 = u(0)$. Then apparently, $\{u_n\}_{n=1}^N$ is $\{\mathcal{B}_{T_n}\}_{n=1}^N$-adapted.

The following result gives the error caused by the parareal algorithms. When the coarse step size δT is not extremely small, the convergence shows order k with respect to δT in a strong sense. Its proof is quite similar to that of Theorem 3.2 and is given in the Appendix A.

THEOREM 4.1. *Let Assumptions* 1 *and* 2 *hold with* $s = 0$, *and* $\{u_n^{(k)}\}_{0\le n\le N, k\in\mathbb{N}}$ *be the solution of* (2.4) *with* $\mathcal{F}$ *being the exact solver and* $\mathcal{G} = \mathcal{G}_I$ *defined in* (4.1). *For* $\alpha \ge 0$, *scheme* (2.4) *converges as* $k \to \infty$:

$$\sup_{0\le n\le N}\left\|u(T_n) - u_n^{(k)}\right\|_{L^2(\Omega;H)} \le \left(e^{-\alpha\delta T}\right)^k \frac{(CT)^k}{k!}\left(e^{(L_F-\alpha)T} \vee 1\right)\sup_{0\le n\le N}\left\|u(T_n) - u_n^{(0)}\right\|_{L^2(\Omega;H)}$$

for any $k \in \mathbb{N}$ *with* $C = C(L_F, \alpha) > 0$.

If $\alpha > 0$, *there exists some* $\delta T_* = \delta T_*(\alpha) \in (0,1)$ *satisfying* $\delta T_*^{-1}\ln \delta T_*^{-1} = \alpha$ *such that the error above shows order* k *with respect to* δT *when* $\delta T \in [\delta T_*, 1)$:

$$\sup_{0\le n\le N}\left\|u(T_n) - u_n^{(k)}\right\|_{L^2(\Omega;H)} \le (\delta T)^k \frac{(CT)^k}{k!}\left(e^{(L_F-\alpha)T} \vee 1\right)\sup_{0\le n\le N}\left\|u(T_n) - u_n^{(0)}\right\|_{L^2(\Omega;H)}.$$

To obtain an implementable numerical method, the fine integrator $\mathcal{F}$ needs to be chosen as a specific numerical method instead of the exact integrator. In this case, it is called a fully discrete scheme, which does not mean the discretization in both space and time direction as it usually does. We refer to [6] for the discretization in space of the stochastic cubic nonlinear Schrödinger equation, which is also available for the model considered in the present paper.

In particular, we choose $\mathcal{F}$ as an integrator obtained by applying the exponential Euler integrator repeatedly on the fine grid with step size δt:

$$\mathcal{F}_I(t_{n,j}, t_{n,j-1}, v) := S(\delta t)v + \mathbf{i}S(\delta t)F(v)\delta t + S(\delta t)Q^{\frac{1}{2}}\delta_{n,j}W \quad \forall\, v \in H$$

with $\delta_{n,j}W := W(t_{n,j}) - W(t_{n,j-1})$. Hence, we get the following fully discrete scheme:

$$\begin{aligned} u_{n+1}^{(0)} &= \mathcal{G}_I(T_{n+1}, T_n, u_n^{(0)}), \quad u_0^{(0)} = u_0,\\ \hat{u}_{n,j}^{(k-1)} &= \mathcal{F}_I(t_{n,j}, t_{n,j-1}, \hat{u}_{n,j-1}^{(k-1)}), \quad \hat{u}_{n,0}^{(k-1)} = u_n^{(k-1)},\\ u_{n+1}^{(k)} &= \mathcal{G}_I(T_{n+1}, T_n, u_n^{(k)}) + \hat{u}_{n,J}^{(k-1)} - \mathcal{G}_I(T_{n+1}, T_n, u_n^{(k-1)}), \end{aligned} \tag{4.2}$$

where $n = 0, \ldots, N$, $j = 1, \ldots, J$, and $t_{n,j}$ is defined in section 2.

The approximate error of the fully discrete scheme (4.2) comes from two parts: the parareal technique based on a coarse integrator and the approximate error of the fine integrator. In fact, the second part is exactly the approximate error of a specific serial scheme without iteration and depends heavily on the regularity of the noise given in Assumption 2, which will not be studied in this article. Readers are referred to [6, 8, 9] and references therein for the study on accuracy of serial schemes for stochastic Schrödinger equations.

We mainly focus on the error caused by the former part and aim to show that the solution of (4.2) converges to the solution of the fine integrator $\mathcal{F}$ as k goes to infinity. To this end, we denote by

$$v_{n,j} = \mathcal{F}_I(t_{n,j}, t_{n,j-1}, v_{n,j-1}), \quad n = 0, \ldots, N,\ j = 1, \ldots, J$$

the solution of $\mathcal{F}$ on fine gird $\{t_{n,j}\}_{n\in\{0,\ldots,N\},j\in\{0,\ldots,J\}}$ starting from $v_{0,0}=u_0$, where $t_{n+1,0}=T_{n+1}=t_{n,J}$ and $v_{n+1,0}:=v_{n,J}$.

THEOREM 4.2. *Let Assumptions* 1 *and* 2 *hold with* $s=0$ *and* $\{u_n^{(k)}\}_{0\le n\le N,k\in\mathbb{N}}$ *being the solution of* (4.2). *For any* $k\in\mathbb{N}$, *it holds that*

$$\sup_{0\le n\le N}\left\|u_n^{(k)}-v_{n,0}\right\|_{L^2(\Omega;H)} \le \left(e^{-\alpha\delta T}\right)^k \frac{(CT)^k}{k!}\left(e^{(L_F-\alpha)T}\vee 1\right)\sup_{0\le n\le N}\left\|u_n^{(0)}-v_{n,0}\right\|_{L^2(\Omega;H)}.$$

In addition, if $\delta T\in[\delta T_*,1)$ *with* δT_* *being defined as in Theorem* 4.1, *the error shows order* k *with respect to* δT *similar to that in Theorem* 4.1.

The proof of this theorem follows the same procedure as that of Theorem 4.1 and is given in the Appendix B for the readers' convenience.

4.2. Parareal exponential θ-scheme for long times. We now consider the exponential θ-scheme applied to the nonlinear equation (2.1)

$$u_n=S(\delta T)u_{n-1}+\mathbf{i}(1-\theta)\delta TS(\delta T)F(u_{n-1})+\mathbf{i}\theta\delta TF(u_n)+S(\delta T)Q^{\frac12}\delta_nW.$$

The existence and uniqueness for the numerical solution of the above implicit scheme is obtained under Assumptions 1 and 2 through the same procedure as those in [6, 8]. So we denote the unique solution of the above scheme by

$$u_n=\tilde{\mathcal{G}}_\theta(T_n,T_{n-1},u_{n-1}).$$

The parareal algorithm based on $\tilde{\mathcal{G}}_\theta$ and the exact solver $\mathcal{F}$ is in the form

$$\begin{aligned}u_n^{(k)}&=\tilde{\mathcal{G}}_\theta(T_n,T_{n-1},u_{n-1}^{(k)})+\mathcal{F}(T_n,T_{n-1},u_{n-1}^{(k-1)})-\tilde{\mathcal{G}}_\theta(T_n,T_{n-1},u_{n-1}^{(k-1)})\\&=:a_k+b_{k-1}-a_{k-1},\end{aligned}\tag{4.3}$$

where

$$a_k=S(\delta T)u_{n-1}^{(k)}+\mathbf{i}(1-\theta)\delta TS(\delta T)F(u_{n-1}^{(k)})+\mathbf{i}\theta\delta TF(a_k)+S(\delta T)Q^{\frac12}\delta_nW,$$

$$b_{k-1}=S(\delta T)u_{n-1}^{(k-1)}+\mathbf{i}\int_{T_{n-1}}^{T_n}S(T_n-s)F\left(u_{u_{n-1}^{(k-1)}}(s)\right)ds+\int_{T_{n-1}}^{T_n}S(T_n-s)Q^{\frac12}dW.$$

If F is differentiable, based on the Taylor expansion of $F(a_k)=F(a_{k-1})+F'(\tau_k)(a_k-a_{k-1})$ with τ_k being determined by a_k and a_{k-1}, we derive

$$\begin{aligned}a_k-a_{k-1}&=S(\delta T)\left(u_{n-1}^{(k)}-u_{n-1}^{(k-1)}\right)+\mathbf{i}(1-\theta)\delta TS(\delta T)\left(F(u_{n-1}^{(k)})-F(u_{n-1}^{(k-1)})\right)\\&\quad+\mathbf{i}\theta\delta T\left(F(a_k)-F(a_{k-1})\right)\\&=S(\delta T)\left(u_{n-1}^{(k)}-u_{n-1}^{(k-1)}\right)+\mathbf{i}(1-\theta)\delta TS(\delta T)\left(F(u_{n-1}^{(k)})-F(u_{n-1}^{(k-1)})\right)\\&\quad+\mathbf{i}\theta\delta TF'(\tau_k)(a_k-a_{k-1}).\end{aligned}$$

Hence, scheme (4.3) is rewritten equivalently as

$$\begin{aligned} u_n^{(k)} &= S_{\theta,k} S(\delta T) u_{n-1}^{(k)} + (1 - S_{\theta,k})\, S(\delta T) u_{n-1}^{(k-1)} \\ &\quad + \mathbf{i}(1-\theta)\delta T S_{\theta,k} S(\delta T) \left(F(u_{n-1}^{(k)}) - F(u_{n-1}^{(k-1)}) \right) \\ &\quad + \mathbf{i} \int_{T_{n-1}}^{T_n} S(T_n - s) F\left(u_{u_{n-1}^{(k-1)}}(s) \right) ds + \int_{T_{n-1}}^{T_n} S(T_n - s) Q^{\frac{1}{2}} dW, \end{aligned}$$

where $S_{\theta,k} := (1 - \mathbf{i}\theta\delta T F'(\tau_k))^{-1}$.

THEOREM 4.3. *Let Assumptions* 1 *and* 2 *hold with* $s = 0$ *and* F *being differentiable. Denote by* $\{u_n^{(k)}\}_{0 \le n \le N, k \in \mathbb{N}}$ *the solution of* (4.3). *Algorithm* (4.3) *converges to the exact solution as* $k \to \infty$ *over unbounded time domain if*

$$f(\theta, \delta T) := \left((2-\theta) L_F \delta T + e^{L_F \delta T} \right) e^{-\alpha \delta T} < 1. \tag{4.4}$$

More precisely,

$$\sup_{n \in \mathbb{N}} \left\| u(T_n) - u_n^{(k)} \right\|_{L^2(\Omega;H)} \le [f(\theta, \delta T)]^k \sup_{n \in \mathbb{N}} \left\| u(T_n) - u_n^{(0)} \right\|_{L^2(\Omega;H)}.$$

Proof. Based on the notation $\epsilon_n^{(k)} := u_n^{(k)} - u(T_n)$ again, we derive

$$\begin{aligned} \epsilon_n^{(k)} &= S_{\theta,k} S(\delta T) \epsilon_{n-1}^{(k)} + (1 - S_{\theta,k})\, S(\delta T) \epsilon_{n-1}^{(k-1)} \\ &\quad + \mathbf{i}(1-\theta)\delta T S_{\theta,k} S(\delta T) \left(F(u_{n-1}^{(k)}) - F(u_{n-1}^{(k-1)}) \right) \\ &\quad + \mathbf{i} \int_{T_{n-1}}^{T_n} S(T_n - s) \left[F\left(u_{u_{n-1}^{(k-1)}}(s) \right) - F\left(u_{u(T_{n-1})}(s) \right) \right] ds. \end{aligned}$$

It then leads to

$$\begin{aligned} &\|\epsilon_n^{(k)}\|_{L^2(\Omega;H)} \\ &\quad \le (1 + (1-\theta) L_F \delta T)\, \|S_{\theta,k}\|_{\mathcal{L}(H)} e^{-\alpha\delta T} \|\epsilon_{n-1}^{(k)}\|_{L^2(\Omega;H)} \\ &\qquad + \left(\|1 - S_{\theta,k}\|_{\mathcal{L}(H)} + (1-\theta) L_F \delta T \|S_{\theta,k}\|_{\mathcal{L}(H)} \right) e^{-\alpha\delta T} \|\epsilon_{n-1}^{(k-1)}\|_{L^2(\Omega;H)} \\ &\qquad + L_F \int_{T_{n-1}}^{T_n} e^{-\alpha(T_n - s)} \|G(s)\|_{L^2(\Omega;H)} ds \end{aligned}$$

with the notation $G(s) := u_{u_{n-1}^{(k-1)}}(s) - u_{u(T_{n-1})}(s)$. For operator $1 - S_{\theta,k}$, we deduce

$$\|1 - S_{\theta,k}\|_{\mathcal{L}(H)} = \|S_{\theta,k}\|_{\mathcal{L}(H)} \|\mathbf{i}\theta\delta T F'(\tau_k)\|_{\mathcal{L}(H)} \le \theta L_F \delta T$$

due to the fact $\|S_{\theta,k}\|_{\mathcal{L}(H)} < 1$. Moreover, according to the mild solution (2.2), we get for any $s \in [T_{n-1}, T_n]$ that

$$\begin{aligned} \|G(s)\|_{L^2(\Omega;H)} &= \|u_{u_{n-1}^{(k-1)}}(s) - u_{u(T_{n-1})}(s)\|_{L^2(\Omega;H)} \\ &\le e^{-\alpha(s - T_{n-1})} \|\epsilon_{n-1}^{(k-1)}\|_{L^2(\Omega;H)} + L_F \int_{T_{n-1}}^{s} e^{-\alpha(s-r)} \|G(r)\|_{L^2(\Omega;H)} dr. \end{aligned}$$

Then the Gronwall inequality yields

$$\|G(s)\|_{L^2(\Omega;H)} \le e^{(L_F - \alpha)(s - T_{n-1})} \|\epsilon_{n-1}^{(k-1)}\|_{L^2(\Omega;H)}.$$

The above estimations finally lead to

$$\begin{aligned}\|\epsilon_n^{(k)}\|_{L^2(\Omega;H)} &\le (1+(1-\theta)L_F\delta T)\, e^{-\alpha\delta T}\|\epsilon_{n-1}^{(k)}\|_{L^2(\Omega;H)}\\ &\quad + \left(L_F\delta T + e^{L_F\delta T}-1\right) e^{-\alpha\delta T}\|\epsilon_{n-1}^{(k-1)}\|_{L^2(\Omega;H)}\\ &=: \gamma_1\|\epsilon_{n-1}^{(k)}\|_{L^2(\Omega;H)} + \gamma_2\|\epsilon_{n-1}^{(k-1)}\|_{L^2(\Omega;H)},\end{aligned}$$

where we have used the following estimation:

$$L_F\int_{T_{n-1}}^{T_n} e^{L_F(s-T_{n-1})}ds = e^{L_F\delta T}-1.$$

Based on the arguments in Theorem 3.2, the error converges to zero as $k\to\infty$ if

$$f(\theta,\delta T)=\gamma_1+\gamma_2=\left((2-\theta)L_F\delta T+e^{L_F\delta T}\right)e^{-\alpha\delta T}<1,$$

and the rate of convergence turns out to be

$$\|\varepsilon^{(k)}\|_\infty \le \left(\frac{\gamma_2}{1-\gamma_1}\right)^k\|\varepsilon^{(0)}\|_\infty = \left(\frac{f(\theta,\delta T)-\gamma_1}{1-\gamma_1}\right)^k\|\varepsilon^{(0)}\|_\infty < [f(\theta,\delta T)]^k\,\|\varepsilon^{(0)}\|_\infty$$

with $\varepsilon^{(k)} := (\|\epsilon_1^{(k)}\|_{L^2(\Omega;H)},\dots,\|\epsilon_N^{(k)}\|_{L^2(\Omega;H)})^\top$. □

Remark 5. For the case $\alpha=0$, (2.1) turns to be a stochastic Hamiltonian partial differential equation (see [19]). It does not possess an invariant measure as the solution will blow-up over long times, and the condition (4.4) in Theorem 4.3 is not satisfied for all L_F and δT. It implies that the parareal algorithms perform worse over long times for nonlinear Hamiltonian systems, which is also pointed out in [15].

Remark 6. Note that the condition $f(\theta,\delta T)<1$ in Theorem 4.3 gives only a sufficient condition for the parareal algorithm to converge for long times, which may not be optimal: for the linear case $F(u)=\lambda u$, the error is estimated in Theorem 3.2 with

$$f(\theta,\delta T)=|e^{\mathbf{i}\lambda\delta T}-\eta|e^{-\alpha\delta T}+|\eta|e^{-\alpha\delta T},\quad \eta=\frac{1+\mathbf{i}(1-\theta)\lambda\delta T}{1-\mathbf{i}\theta\lambda\delta T}$$

according to (3.6). Different from the condition obtained in Theorem 4.3, the optimal choice of θ in the linear case is $\frac{1}{2}$, which is also illustrated through numerical experiments in section 5.

5. Numerical experiments.

This section is devoted to investigating the relationship between the convergence error and several parameters, i.e., α, λ, and θ, based on which we can find a proper iteration number k as the terminate iterated number for different cases.

We consider the linear equation (3.1) with initial value $u_0=0$. Throughout the numerical experiments, we use the average of 1000 sample paths as an approximation of the expectation and choose dimension $M=10$ for the spectral Galerkin approximation in a spatial direction.

We get from Theorem 3.2 that the time-uniform convergence holds for all $\lambda\in\mathbb{R}$ and $\alpha>0$ if $\theta\in[\frac{1}{2},1]$, which is illustrated in Figure 3 for $\theta=0.5,1$ and time interval $T=1,20$. Figure 3 shows the evolution of the mean square error $(\sup_{1\le n\le N}\mathbb{E}\|u_n^{(k)}-v_n\|^2)^{\frac{1}{2}}$ with iterated number k. For $T=1$, the iterated number can be chosen as $k=4$ for $\theta=\frac{1}{2}$ and $k=7$ when $\theta=1$, which coincides with the result that the convergence order is $2k$ instead of k when $\theta=\frac{1}{2}$. For larger time $T=20$, since the constant C in Theorem 3.2 is negatively correlated with α for $\theta\in[\frac{1}{2},1]$, the proposed algorithm also converges but with different iterated number k.

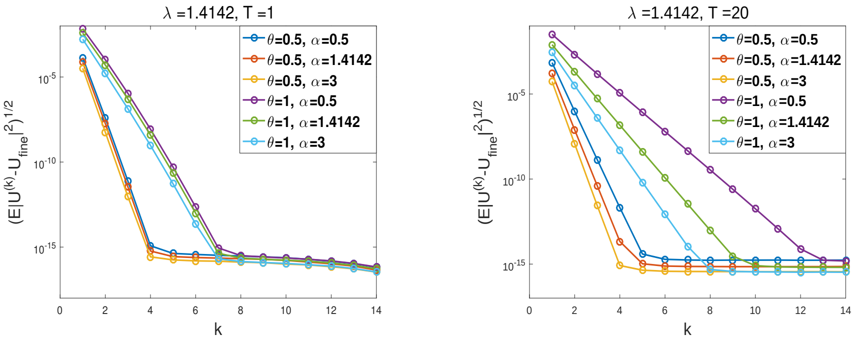


FIG. 3. *Mean square error* $(\sup_{1\le n\le N}\mathbb{E}\|u_n^{(k)}-v_n\|^2)^{\frac{1}{2}}$ *vs. iterated number* k *(*$\lambda=\sqrt{2},\delta t=2^{-6},J=4$*).*

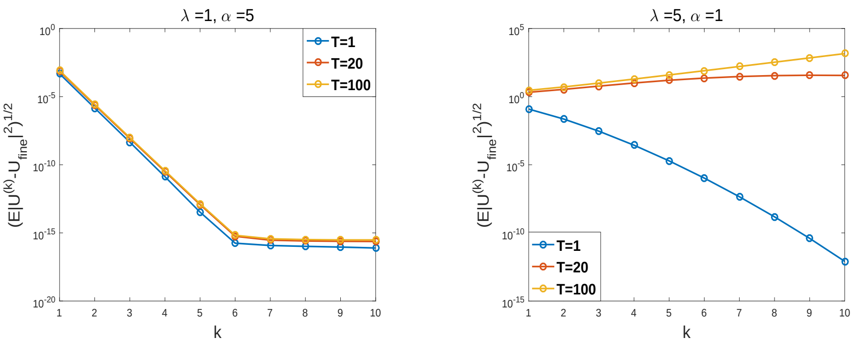


FIG. 4. *Mean square error* $(\sup_{1\le n\le N}\mathbb{E}\|u_n^{(k)}-v_n\|^2)^{\frac{1}{2}}$ *vs. iterated number* k*.*

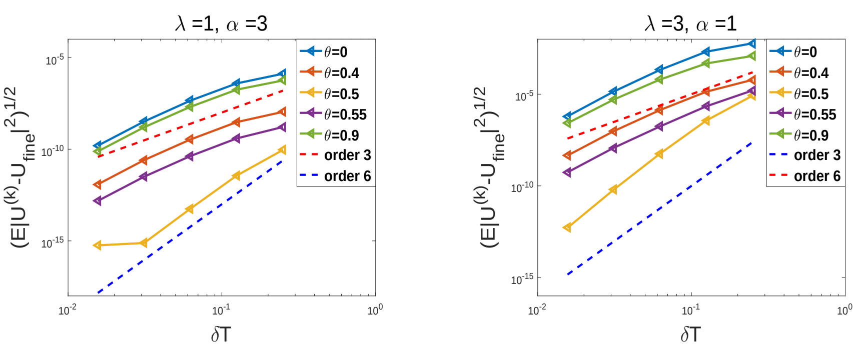


FIG. 5. *Mean square order with respect to* $\delta T=2^{-i}$*,* $i=2,\dots,6$*.*

When $\theta\in[0,\frac{1}{2})$, the convergence result holds uniformly if $\alpha>\sqrt{(\frac{1}{2}-\theta)|\lambda|}$ as stated in Theorem 3.2. Figure 4 also shows the evolution of the mean square error with respect to k for $\theta=0$ and $T=1,20,100$. It can be found that if the condition $\alpha>\sqrt{(\frac{1}{2}-\theta)|\lambda|}$ is not satisfied, e.g., $\lambda=5$, $\alpha=1$, the proposed algorithm diverges as the time grows larger.

In particular, based on numerical experiments above, we now fix $k=3$ to verify the convergence order of the proposed scheme for different $\theta\in[0,1]$. Figure 5 considers the convergence order of the proposed parareal algorithm for different λ and α with fine step size $\delta t = 2^{-8}$. The order turns out to be k for $\theta = 0, 0.4, 0.55, 0.9$, but increases to $2k$ when $\theta = \frac{1}{2}$, which coincides with the result in Theorem 3.2.

Appendix A. Proof of Theorem 4.1. Since $\mathcal{F}$ is the exact integrator, it has the following expression:

$$\begin{aligned}\mathcal{F}(T_{n+1}, T_n, u_n^{(k-1)}) = S(\delta T)u_n^{(k-1)} &+ \mathbf{i}\int_{T_n}^{T_{n+1}} S(T_{n+1}-s)F(u_{u_n^{(k-1)}}(s))ds \\ &+ \int_{T_n}^{T_{n+1}} S(T_{n+1}-s)Q^{\frac{1}{2}}dW(s),\end{aligned}$$

where $u_{u_n^{(k-1)}}(s)$ denotes the exact solution at time s starting from $u_n^{(k-1)}$ at T_n. Then algorithm (2.4) yields

$$\begin{aligned}u_{n+1}^{(k)} &= S(\delta T)u_n^{(k)} + \mathbf{i}S(\delta T)F(u_n^{(k)})\delta T - \mathbf{i}S(\delta T)F(u_n^{(k-1)})\delta T \\ &\quad + \mathbf{i}\int_{T_n}^{T_{n+1}} S(T_{n+1}-s)F(u_{u_n^{(k-1)}}(s))ds + \int_{T_n}^{T_{n+1}} S(T_{n+1}-s)Q^{\frac{1}{2}}dW(s),\end{aligned} \tag{A.1}$$

compared with the exact solution

$$u(T_{n+1}) = \mathcal{F}(T_{n+1}, T_n, u(T_n)).$$

Denoting the error $\epsilon_n^{(k)} := u(T_n) - u_n^{(k)}$, we get

$$\begin{aligned}\epsilon_{n+1}^{(k)} &= S(\delta T)\epsilon_n^{(k)} - \mathbf{i}S(\delta T)F(u_n^{(k)})\delta T + \mathbf{i}S(\delta T)F(u_n^{(k-1)})\delta T \\ &\quad + \mathbf{i}\int_{T_n}^{T_{n+1}} S(T_{n+1}-s)F(u_{u(T_n)}(s))ds \\ &\quad - \mathbf{i}\int_{T_n}^{T_{n+1}} S(T_{n+1}-s)F(u_{u_n^{(k-1)}}(s))ds \\ &= S(\delta T)\epsilon_n^{(k)} + \mathbf{i}S(\delta T)\left[F(u(T_n)) - F(u_n^{(k)})\right]\delta T \\ &\quad - \mathbf{i}S(\delta T)\left[F(u(T_n)) - F(u_n^{(k-1)})\right]\delta T \\ &\quad + \mathbf{i}\int_{T_n}^{T_{n+1}} S(T_{n+1}-s)\left[F(u_{u(T_n)}(s)) - F(u_{u_n^{(k-1)}}(s))\right]ds \\ &=: \mathrm{I} + \mathrm{II} - \mathrm{III} + \mathrm{IV}.\end{aligned}$$

Thus, the mean square error reads

$$\|\epsilon_{n+1}^{(k)}\|_{L^2(\Omega;H)} \le \|\mathrm{I}\|_{L^2(\Omega;H)} + \|\mathrm{II}\|_{L^2(\Omega;H)} + \|\mathrm{III}\|_{L^2(\Omega;H)} + \|\mathrm{IV}\|_{L^2(\Omega;H)},$$

where

$$\|\mathrm{I}\|_{L^2(\Omega;H)} \le e^{-\alpha\delta T}\|\epsilon_n^{(k)}\|_{L^2(\Omega;H)}, \tag{A.2}$$

$$\|\mathrm{II}\|_{L^2(\Omega;H)} \le L_F\delta T e^{-\alpha\delta T}\|\epsilon_n^{(k)}\|_{L^2(\Omega;H)}, \tag{A.3}$$

and

$$\|\mathrm{III}\|_{L^2(\Omega;H)} \le L_F\delta T e^{-\alpha\delta T}\|\epsilon_n^{(k-1)}\|_{L^2(\Omega;H)}. \tag{A.4}$$

It then suffices to estimate term IV. In fact, denoting $G(s) := u_{u(T_n)}(s) - u_{u_n^{(k-1)}}(s)$ and according to the mild solution (2.2), we obtain for any $s \in [T_n, T_{n+1}]$ that

$$\begin{aligned}\|G(s)\|_{L^2(\Omega;H)} &= \|u_{u(T_n)}(s) - u_{u_n^{(k-1)}}(s)\|_{L^2(\Omega;H)}\\ &\le e^{-\alpha(s-T_n)}\|\epsilon_n^{(k-1)}\|_{L^2(\Omega;H)}\\ &\quad + \left\|\int_{T_n}^{s} S(s-r)\left[F(u_{u(T_n)}(r)) - F(u_{u_n^{(k-1)}}(r))\right]dr\right\|_{L^2(\Omega;H)}\\ &\le e^{-\alpha(s-T_n)}\|\epsilon_n^{(k-1)}\|_{L^2(\Omega;H)} + L_F\int_{T_n}^{s} e^{-\alpha(s-r)}\|G(r)\|_{L^2(\Omega;H)}dr.\end{aligned}$$

Then the Gronwall inequality yields

$$\|G(s)\|_{L^2(\Omega;H)} \le e^{(L_F-\alpha)(s-T_n)}\|\epsilon_n^{(k-1)}\|_{L^2(\Omega;H)}.$$

As a result,

$$\begin{aligned}\|IV\|_{L^2(\Omega;H)} &\le L_F\int_{T_n}^{T_{n+1}} e^{-\alpha(T_{n+1}-s)}\|G(s)\|_{L^2(\Omega;H)}ds\\ &\le \left(e^{L_F\delta T} - 1\right)e^{-\alpha\delta T}\|\epsilon_n^{(k-1)}\|_{L^2(\Omega;H)}. \quad \text{(A.5)}\end{aligned}$$

Based on estimations (A.2)–(A.5) and the fact that $\epsilon_0^{(k)} = 0$ for all $k \in \mathbb{N}$, we derive for $n = 1, \ldots, N-1$ that

$$\begin{aligned}\|\epsilon_{n+1}^{(k)}\|_{L^2(\Omega;H)} &\le (1 + L_F\delta T)e^{-\alpha\delta T}\|\epsilon_n^{(k)}\|_{L^2(\Omega;H)}\\ &\quad + 2L_F\delta T e^{-\alpha\delta T}\|\epsilon_n^{(k-1)}\|_{L^2(\Omega;H)}\\ &\quad + \left(e^{L_F\delta T} - 1\right)e^{-\alpha\delta T}\|\epsilon_n^{(k-1)}\|_{L^2(\Omega;H)} \quad \text{(A.6)}\\ &\le \left(2L_F\delta T + e^{L_F\delta T} - 1\right)e^{-\alpha\delta T}\sum_{j=1}^{n}\left(\beta^{n-j}\|\epsilon_j^{(k-1)}\|_{L^2(\Omega;H)}\right) \quad \text{(A.7)}\end{aligned}$$

with the notation $\beta := (1 + L_F\delta T)e^{-\alpha\delta T} > 0$. Denoting the error vector

$$\varepsilon^{(k)} := \left(\|\epsilon_1^{(k)}\|_{L^2(\Omega;H)}, \ldots, \|\epsilon_N^{(k)}\|_{L^2(\Omega;H)}\right)^\top$$

and the N-dimensional matrix (see also [16])

$$M(\beta) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0\\ 1 & 0 & \cdots & 0 & 0\\ \beta & 1 & \cdots & 0 & 0\\ \beta^2 & \beta & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \beta^{N-2} & \beta^{N-3} & \cdots & 1 & 0 \end{pmatrix},$$

we can rewrite (A.6) as

$$\varepsilon^{(k)} \le C\delta T e^{-\alpha\delta T}M(\beta)\varepsilon^{(k-1)} \le (C\delta T e^{-\alpha\delta T})^k M^k(\beta)\varepsilon^{(0)}.$$

It is shown in [16] that

$$\|M^k(\beta)\|_\infty \le \frac{(N-1)(N-2)\cdots(N-k)}{k!}(\beta\vee 1)^{N-k-1} \le \frac{N^k}{k!}\left(\beta^N \vee 1\right),$$

which leads to the first result in the theorem:

$$\begin{aligned}\|\varepsilon^{(k)}\|_\infty &\le \left(C\delta T e^{-\alpha\delta T}\right)^k \frac{N^k}{k!}\left(\beta^N\vee 1\right)\|\varepsilon^{(0)}\|_\infty\\ &\le \left(e^{-\alpha\delta T}\right)^k \frac{(CT)^k}{k!}\left(e^{(L_F-\alpha)T}\vee 1\right)\|\varepsilon^{(0)}\|_\infty.\end{aligned}$$

Note that the function $f(\delta T) := e^{-\alpha\delta T} - \delta T$ is continuous and takes a value in $(e^{-\alpha}-1, 1]$ for $\delta T \in [0,1)$. Hence, there exists some $\delta T_* = \delta T_*(\alpha) \in (0,1)$ such that $f(\delta T) \le 0$ for any $\delta T \in [\delta T_*, 1)$. In fact, δT_* satisfies that $\delta T_*^{-1}\ln \delta T_*^{-1} = \alpha$, which decreases when α increases.

Appendix B. Proof of Theorem 4.2. Note that

$$\begin{aligned}v_{n,0} &= v_{n-1,J} = \mathcal{F}_I(t_{n-1,J}, t_{n-1,J-1}, v_{n-1,J-1})\\ &= S(\delta T)v_{n-1,0} + \mathbf{i}\sum_{l=1}^{J} S(l\delta t)F(v_{n-1,J-l})\delta t + \sum_{l=1}^{J} S(l\delta t)Q^{\frac12}\delta_{n,J+1-l}W. \qquad \text{(B.1)}\end{aligned}$$

Similarly, we get

$$\begin{aligned}\hat u^{(k-1)}_{n-1,J} &= \mathcal{F}_I(t_{n-1,J}, t_{n-1,J-1}, \hat u^{(k-1)}_{n-1,J-1})\\ &= S(\delta T)u^{(k-1)}_{n-1} + \mathbf{i}\sum_{l=1}^{J} S(l\delta t)F(\hat u^{(k-1)}_{n-1,J-l})\delta t + \sum_{l=1}^{J} S(l\delta t)Q^{\frac12}\delta_{n,J+1-l}W. \qquad \text{(B.2)}\end{aligned}$$

In the following, we still denote the above error by $\epsilon_n^{(k)} := u_n^{(k)} - v_{n,0}$ for convenience, which has the same symbol as in the proof of Theorem 4.1 but with different meaning. Then we can decompose the error into several parts

$$\begin{aligned}\epsilon_n^{(k)} &= \Big(\mathcal{G}_I(T_n, T_{n-1}, u^{(k)}_{n-1}) - v_{n,0}\Big) - \Big(\mathcal{G}_I(T_n, T_{n-1}, u^{(k-1)}_{n-1}) - v_{n,0}\Big) + \hat u^{(k-1)}_{n-1,J} - v_{n,0}\\ &= S(\delta T)\epsilon^{(k)}_{n-1} + \mathbf{i}\Big(S(\delta T)F(u^{(k)}_{n-1})\delta T - S(\delta T)F(v_{n-1,0})\delta T\Big)\\ &\quad - \mathbf{i}\Big(S(\delta T)F(u^{(k-1)}_{n-1})\delta T - S(\delta T)F(v_{n-1,0})\delta T\Big)\\ &\quad + \mathbf{i}\left(\sum_{l=1}^{J} S(l\delta t)F(\hat u^{(k-1)}_{n-1,J-l})\delta t - \sum_{l=1}^{J} S(l\delta t)F(v_{n-1,J-l})\delta t\right)\\ &=: \tilde{\mathrm{I}} + \tilde{\mathrm{II}} - \tilde{\mathrm{III}} + \tilde{\mathrm{IV}}\end{aligned}$$

according to (B.1) and (B.2). For the first three terms, we derive

$$\begin{aligned}\|\tilde{\mathrm{I}}\|_{L^2(\Omega;H)} &\le e^{-\alpha\delta T}\|\epsilon^{(k)}_{n-1}\|_{L^2(\Omega;H)},\\ \|\tilde{\mathrm{II}}\|_{L^2(\Omega;H)} &\le L_F\delta T e^{-\alpha\delta T}\|\epsilon^{(k)}_{n-1}\|_{L^2(\Omega;H)},\end{aligned}$$

and

$$\|\tilde{\mathrm{III}}\|_{L^2(\Omega;H)} \le C\delta T e^{-\alpha\delta T}\|\epsilon^{(k-1)}_{n-1}\|_{L^2(\Omega;H)}.$$

To get the estimation of term $\tilde{IV}$, we define $\tilde{G}_j := \hat{u}_{n-1,j}^{(k-1)} - v_{n-1,j}$ for any $j = 0, \dots, J$, and then (B.1) and (B.2) yields

$$\begin{aligned}\|\tilde{G}_j\|^2_{L^2(\Omega;H)} &= \Big\| S(\delta T)u_{n-1}^{(k-1)} + \mathbf{i}\sum_{l=1}^{j} S(l\delta t)F(\hat{u}_{n-1,j-l}^{(k-1)})\delta t \\ &\quad - \Big(S(\delta T)v_{n-1,0} + \mathbf{i}\sum_{l=1}^{j} S(l\delta t)F(v_{n-1,j-l})\delta t\Big)\Big\|^2_{L^2(\Omega;H)} \\ &\le 2e^{-2\alpha\delta T}\|\epsilon_{n-1}^{(k-1)}\|^2_{L^2(\Omega;H)} + 2j\delta t^2 L_F^2 \sum_{l=1}^{j} e^{-2\alpha l\delta t}\|\tilde{G}_{j-l}\|^2_{L^2(\Omega;H)} \\ &\le 2e^{-2\alpha\delta T}\|\epsilon_{n-1}^{(k-1)}\|^2_{L^2(\Omega;H)} + 2\delta T\delta t L_F^2 \sum_{m=0}^{j-1} e^{-2\alpha(j-m)\delta t}\|\tilde{G}_m\|^2_{L^2(\Omega;H)}.\end{aligned}$$

Equivalently, it can be written as

$$\begin{aligned}&e^{2\alpha j\delta t}\|\tilde{G}_j\|^2_{L^2(\Omega;H)} \\ &\quad \le 2e^{-2\alpha(\delta T - j\delta t)}\|\epsilon_{n-1}^{(k-1)}\|^2_{L^2(\Omega;H)} + 2\delta T\delta t L_F^2 \sum_{m=0}^{j-1} e^{2\alpha m\delta t}\|\tilde{G}_m\|^2_{L^2(\Omega;H)}.\end{aligned}$$

According to the discrete Gronwall inequality, we get

$$\begin{aligned}&\|\tilde{G}_j\|^2_{L^2(\Omega;H)} \\ &\quad \le 2e^{-2\alpha\delta T}\|\epsilon_{n-1}^{(k-1)}\|^2_{L^2(\Omega;H)}\Big(1 + e^{-2\alpha j\delta t}\sum_{0\le m<j} e^{2\alpha m\delta t} 2\delta T\delta t L_F^2 (1 + 2\delta T\delta t L_F^2)^{j-m-2}\Big) \\ &\quad \le Ce^{-2\alpha\delta T}\|\epsilon_{n-1}^{(k-1)}\|^2_{L^2(\Omega;H)}\end{aligned}$$

with C independent of j. Hence,

$$\begin{aligned}\|\tilde{IV}\|^2_{L^2(\Omega;H)} &\le \delta T\delta t L_F^2 \sum_{l=1}^{J} e^{-2\alpha l\delta t}\|\tilde{G}_{J-l}\|^2_{L^2(\Omega;H)} \\ &\le C\delta T^2 e^{-2\alpha\delta T}\|\epsilon_{n-1}^{(k-1)}\|^2_{L^2(\Omega;H)}.\end{aligned}$$

In conclusion, we get

$$\|\epsilon_n^{(k)}\|^2_{L^2(\Omega;H)} \le (1 + L_F\delta T)e^{-\alpha\delta T}\|\epsilon_{n-1}^{(k)}\|^2_{L^2(\Omega;H)} + C\delta T e^{-\alpha\delta T}\|\epsilon_{n-1}^{(k-1)}\|^2_{L^2(\Omega;H)},$$

which leads to the final results based on the procedure in the proof of Theorem 4.1.

REFERENCES

[1] H. H. Andersen, M. Højbjerre, D. Sørensen, and P. S. Eriksen, *Linear and Graphical Models for the Multivariate Complex Normal Distribution*, Lecture Notes in Statist. 101, Springer-Verlag, Berlin, 1995, https://doi.org/10.1007/978-1-4612-4240-6.

[2] G. Bal, *On the convergence and the stability of the parareal algorithm to solve partial differential equations*, in Domain Decomposition Methods in Science and Engineering, Lect. Notes Comput. Sci. Eng. 40, Springer, Berlin, 2005, pp. 425–432, https://doi.org/10.1007/3-540-26825-1_43.

[3] G. Bal, *Parallelization in Time of (Stochastic) Ordinary Differential Equations*, preprint, 2006.
[4] C.-E. Bréhier and G. Vilmart, *High order integrator for sampling the invariant distribution of a class of parabolic stochastic PDEs with additive space-time noise*, SIAM J. Sci. Comput., 38 (2016), pp. A2283–A2306, https://doi.org/10.1137/15M1021088.
[5] C.-E. Bréhier and X. Wang, *On Parareal Algorithms for Semilinear Parabolic Stochastic PDEs*, preprint, arXiv:1902.05370.
[6] C. Chen, J. Hong, and X. Wang, *Approximation of invariant measure for damped stochastic nonlinear Schrödinger equation via an ergodic numerical scheme*, Potential Anal., 46 (2017), pp. 323–367, https://doi.org/10.1007/s11118-016-9583-9.
[7] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite-Dimensional Systems*, London Math. Soc. Lecture Note Ser. 229, Cambridge University Press, Cambridge, 1996, https://doi.org/10.1017/CBO9780511662829.
[8] A. De Bouard and A. Debussche, *A semi-discrete scheme for the stochastic nonlinear Schrödinger equation*, Numer. Math., 96 (2004), pp. 733–770, https://doi.org/10.1007/s00211-003-0494-5.
[9] A. De Bouard and A. Debussche, *Weak and strong order of convergence of a semidiscrete scheme for the stochastic nonlinear Schrödinger equation*, Appl. Math. Optim., 54 (2006), pp. 369–399, https://doi.org/10.1007/s00245-006-0875-0.
[10] A. Debussche and C. Odasso, *Ergodicity for a weakly damped stochastic non-linear Schrödinger equation*, J. Evol. Equ., 5 (2005), pp. 317–356, https://doi.org/10.1007/s00028-005-0195-x.
[11] I. Ekren, I. Kukavica, and M. Ziane, *Existence of invariant measures for the stochastic damped Schrödinger equation*, Stoch. Partial Differ. Equ. Anal. Comput., 5 (2017), pp. 343–367, https://doi.org/10.1007/s40072-016-0090-1.
[12] S. Engblom, *Parallel in time simulation of multiscale stochastic chemical kinetics*, Multiscale Model. Simul., 8 (2009), pp. 46–68, https://doi.org/10.1137/080733723.
[13] M. J. Gander, 50 *years of time parallel time integration*, in Multiple Shooting and Time Domain Decomposition Methods, Contrib. Math. Comput. Sci. 9, Springer, Berlin, 2015, pp. 69–113.
[14] M. J. Gander and E. Hairer, *Nonlinear convergence analysis for the parareal algorithm*, in Domain Decomposition Methods in Science and Engineering XVII, Lect. Notes Comput. Sci. Eng. 60, Springer, Berlin, 2008, pp. 45–56, https://doi.org/10.1007/978-3-540-75199-1_4.
[15] M. J. Gander and E. Hairer, *Analysis for parareal algorithms applied to Hamiltonian differential equations*, J. Comput. Appl. Math., 259 (2014), pp. 2–13, https://doi.org/10.1016/j.cam.2013.01.011.
[16] M. J. Gander and S. Vandewalle, *Analysis of the parareal time-parallel time-integration method*, SIAM J. Sci. Comput., 29 (2007), pp. 556–578, https://doi.org/10.1137/05064607X.
[17] M. J. Gander and S. Vandewalle, *On the superlinear and linear convergence of the parareal algorithm*, in Domain Decomposition Methods in Science and Engineering XVI, Lect. Notes Comput. Sci. Eng. 55, Springer, Berlin, 2007, pp. 291–298, https://doi.org/10.1007/978-3-540-34469-8_34.
[18] J. Hong, L. Sun, and X. Wang, *High order conformal symplectic and ergodic schemes for the stochastic Langevin equation via generating functions*, SIAM J. Numer. Anal., 55 (2017), pp. 3006–3029, https://doi.org/10.1137/17M111691X.
[19] J. Hong, X. Wang, and L. Zhang, *Numerical analysis on ergodic limit of approximations for stochastic NLS equation via multi-symplectic scheme*, SIAM J. Numer. Anal., 55 (2017), pp. 305–327, https://doi.org/10.1137/16M1079099.
[20] J.-L. Lions, Y. Maday, and G. Turinici, *Résolution d'EDP par un schéma en temps "pararéel,"* C. R. Acad. Sci. Paris Sér. I Math., 332 (2001), pp. 661–668, https://doi.org/10.1016/S0764-4442(00)01793-6.
[21] W. Liu and M. Röckner, *Stochastic partial differential equations: An introduction*, Universitext, Springer, Berlin, 2015, https://doi.org/10.1007/978-3-319-22354-4.
[22] Y. Maday and G. Turinici, *A parareal in time procedure for the control of partial differential equations*, C. R. Math. Acad. Sci. Paris, 335 (2002), pp. 387–392, https://doi.org/10.1016/S1631-073X(02)02467-6.
[23] W. L. Miranker and W. Liniger, *Parallel methods for the numerical integration of ordinary differential equations*, Math. Comp., 21 (1967), pp. 303–320, https://doi.org/10.2307/2003233.

[24] A. Rößler, *Second order Runge–Kutta methods for Itô stochastic differential equations*, SIAM J. Numer. Anal., 47 (2009), pp. 1713–1738, https://doi.org/10.1137/060673308.

[25] G. A. Staff and E. M. Rønquist, *Stability of the parareal algorithm*, in Domain Decomposition Methods in Science and Engineering, Lect. Notes Comput. Sci. Eng. 40, Springer, Berlin, 2005, pp. 449–456, https://doi.org/10.1007/3-540-26825-1_46.

[26] L. Zhang, W. Zhou, and L. Ji, *Parareal algorithms applied to stochastic differential equations with conserved quantities*, J. Comput. Math., 37 (2019), pp. 48–60, https://doi.org/10.4208/jcm.1708-m2017-0089.