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NUMERICAL ANALYSIS ON ERGODIC LIMIT OF APPROXIMATIONS FOR STOCHASTIC NLS EQUATION VIA MULTI-SYMPLECTIC SCHEME*

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Abstract. We consider a finite dimensional approximation of the stochastic nonlinear Schrödinger equation driven by multiplicative noise, which is derived by applying a symplectic method to the original equation in spatial direction. Both the unique ergodicity and the charge conservation law for this finite dimensional approximation are obtained on the unit sphere. To simulate the ergodic limit over long time for the finite dimensional approximation, we discretize it further in the temporal direction to obtain a fully discrete scheme, which inherits not only the stochastic multi-symplecticity and charge conservation law of the original equation but also the unique ergodicity of the finite dimensional approximation. The temporal average of the fully discrete numerical solution is proved to converge to the ergodic limit with order 1 with respect to the time step for a fixed spatial step. Numerical experiments verify our theoretical results on charge conservation, ergodicity, and weak convergence.

Key words. stochastic Schrödinger equation, multiplicative noise, unique ergodicity, multi-symplectic scheme, weak error

AMS subject classifications. 37M25, 60H35, 65C30, 65P10

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1. Introduction.

For the stochastic nonlinear Schrödinger (NLS) equation with a multiplicative noise in Stratonovich sense,

$$\begin{cases} du = \mathbf{i}\big(\Delta u + \lambda |u|^2 u\big)dt + \mathbf{i}u \circ dW, \\ u(t,0) = u(t,1) = 0, \ t \geq 0, \\ u(0,x) = u_0(x), \ x \in [0,1] \end{cases} \tag{1}$$

with $\lambda = \pm 1$, we consider the case that W is a real valued Q-Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with paths in $H_0^1 := H_0^1(0,1)$ with Dirichlet boundary condition. The Karhunen–Loève expansion of W is as follows:

$$W(t,x,\omega) = \sum_{k=0}^{\infty} \beta_k(t,\omega) Q^{\frac{1}{2}} e_k(x), \quad t \geq 0, \quad x \in [0,1], \quad \omega \in \Omega,$$

where $(e_k = \sqrt{2}\sin(k\pi x))_{k\geq 1}$ is an eigenbasis of the Dirichlet Laplacian Δ in $L^2 := L^2(0,1)$ and $(\beta_k)_{k\geq 1}$ is a sequence of independent real valued Brownian motions

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associated with the filtration $(\mathcal{F}_t)_{t\geq 0}$. In addition, the covariance operator Q is assumed to commute with the Laplacian and satisfies

$$Qe_k = \eta_k e_k, \quad \eta_k > 0, \quad \forall k \in \mathbb{N}, \quad \eta := \sum_{k=1}^{\infty} \eta_k < \infty.$$

We refer to [9] for additional assumptions on the well-posedness of (1). It is shown that (1) is a Hamiltonian system with stochastic multi-symplectic structure and charge conservation law (see [6, 9, 11] and references therein). Structure-preserving numerical schemes have remarkable superiority over conventional schemes to numerically solve Hamiltonian systems over long time. As another kind of long-time behavior, the ergodicity for this kind of conservative system is an important and difficult problem which is still open. Motivated by [10], we study the ergodicity for a finite dimensional approximation (FDA) of the original equation instead.

In this paper, we investigate the ergodicity for a symplectic FDA of (1) and approximate its ergodic limit via a multi-symplectic and ergodic scheme. As we show that the FDA is charge conserved, without loss of generality, we consider the ergodicity in the finite dimensional unit sphere $\mathcal{S}$. There have been some papers considering the additive noise case with dissipative assumptions, and also some papers requiring a uniformly elliptic assumption on the whole space to ensure unique ergodicity (see, e.g., [3, 12, 13, 15, 16]). A conservative FDA with linear multiplicative noise has an uncertain nondegeneracy, which relies heavily on the solution. To overcome this difficulty, we construct an invariant control set $\mathcal{M}_0 \subset \mathcal{S}$, in which the FDA is shown to be nondegenerate. Together with the Krylov–Bogoliubov theorem and the Hörmander condition, we prove that the solution U possesses a unique invariant measure μ_h (i.e., U is uniquely ergodic) with

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{E} f(U(t))dt = \int_{\mathcal{M}_0} f d\mu_h = \int_{\mathcal{S}} f d\mu_h.$$

For many physical applications, the approximation of the invariant measure is of fundamental importance, especially when the invariant measure is unknown (see, e.g., [1, 3, 4, 5, 7, 13, 14, 15, 16]). Some papers construct numerical schemes which also possess unique invariant measures, and then show the approximate error between invariant measures. For example, [7, 15] work with dissipative systems driven by additive noise, and [16] considers elliptic stochastic differential equations (SDEs) with bounded coefficients and dissipative type condition. There is also some work concentrating on the approximation of the invariant measure, i.e., the approximation of the ergodic limit $\int_{\mathcal{S}} f d\mu_h$, in which case the numerical schemes may not be uniquely ergodic. For instance, [3] approximates the invariant measure of stochastic partial differential equations with an additive noise based on Kolmogorov equation. Reference [13] gives error estimates for time-averaging estimators of numerical schemes based on the associated Poisson equation and the assumption of local weak convergence order. Authors in [14] calculate the ergodic limit for Langevin equations with dissipations via quasi-symplectic integrators. There have been few results on constructing conservative and uniquely ergodic schemes to calculate the ergodic limit for conservative systems, to our knowledge. We focus on the approximation of the ergodic limit via a multi-symplectic scheme, which is also shown to be uniquely ergodic. For a fixed spacial dimension, the local weak error of this fully discrete scheme (FDS) in temporal

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direction is of order 2, which yields order 1 for the approximate error of the ergodic limit based on the associated Poisson equation (see also [4, 13]) and a priori estimates of the numerical solutions. That is,

$$\left|\mathbb{E}\left[\frac{1}{N}\sum_{n=0}^{N-1} f(U^n) - \int_{\mathcal{S}} f d\mu_h\right]\right| \leq C_h\left(\frac{1}{T}+\tau\right).$$

The paper is organized as follows. In section 2, we apply a symplectic semidiscrete scheme to the original equation to get the FDA, and show the unique ergodicity as well as the charge conservation law for the FDA. In section 3, we present a multi-symplectic and ergodic FDS to approximate the ergodic limit, and show the approximate error based on a priori estimates and local weak error. In section 4, the discrete charge evolution compared with those of the Euler–Maruyama scheme and implicit Euler scheme, the ergodic limit, and global weak convergence order are tested numerically. Section 5 is the appendix containing proofs of some a priori estimates.

2. Unique ergodicity. In this section, we first apply the central finite difference scheme to (1) in the spatial direction to obtain a FDA, which is also a Hamiltonian system. To investigate the ergodicity of this conservative system, we then construct an invariant control set $\mathcal{M}_0 \subset \mathcal{S}$ with respect to a control function introduced in section 2.2. The FDA is proved to be ergodic in $\mathcal{M}_0$ based on the Krylov–Bogoliubov theorem and the Hörmander condition.

2.1. Finite dimensional approximation (FDA). Based on the central finite difference scheme and the notation $u_j := u_j(t)$, $j = 1, \ldots, M$, we consider the following spatial semidiscretization:

$$du_j = \mathbf{i}\left[\frac{u_{j+1}-2u_j+u_{j-1}}{h^2} + \lambda|u_j|^2 u_j\right]dt + \mathbf{i}u_j\sum_{k=1}^{K}\sqrt{\eta_k}e_k(x_j)\circ d\beta_k(t)$$

with a truncated noise $\sum_{k=1}^{K}\sqrt{\eta_k}e_k(x)\beta_k(t)$, $K \in \mathbb{N}$, a given uniform step size $h = \frac{1}{M+1}$ for some $M \leq K$, and $x_j = jh$, $j = 1, \ldots, M$. The condition $M \leq K$ here ensures the existence of the solution for the control function. Denoting vectors $U := U(t) = (u_1, \ldots, u_M)^T \in \mathbb{C}^M$, $\beta(t) = (\beta_1(t), \ldots, \beta_K(t))^T \in \mathbb{R}^K$, and matrices $F(U) = \mathrm{diag}\{|u_1|^2, \ldots, |u_M|^2\}$, $E_k = \mathrm{diag}\{e_k(x_1), \ldots, e_k(x_M)\}$, $\Lambda = \mathrm{diag}\{\sqrt{\eta_1}, \ldots, \sqrt{\eta_K}\}$, $Z(U) = \mathrm{diag}\{u_1, \ldots, u_M\}E_{MK}\Lambda$,

$$A = \begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2 \end{pmatrix} \in \mathbb{R}^{M\times M}, \quad E_{MK} = \begin{pmatrix} e_1(x_1) & \cdots & e_K(x_1) \\ \vdots & & \vdots \\ e_1(x_M) & \cdots & e_K(x_M) \end{pmatrix}_{M\times K},$$

then the FDA is in the following form:

$$\begin{cases} dU = \mathbf{i}\left[\frac{1}{h^2}AU + \lambda F(U)U\right]dt + \mathbf{i}Z(U)\circ d\beta(t), \\ U(0) = c_*\left(u_0(x_1), \ldots, u_0(x_M)\right)^T, \end{cases} \tag{2}$$

where c_* is a normalized constant. The noise term in (2) has an equivalent Itô form

$$\mathbf{i}Z(U)\circ d\beta(t) = \mathbf{i}\sum_{k=1}^{K}\sqrt{\eta_k}E_kU\circ d\beta_k(t) = -\frac{1}{2}\sum_{k=1}^{K}\eta_kE_k^2Udt + \mathbf{i}\sum_{k=1}^{K}\sqrt{\eta_k}E_kUd\beta_k(t)$$

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$$=: -\hat{E}Udt + \mathbf{i}\sum_{k=1}^{K}\sqrt{\eta_k}E_kUd\beta_k(t) \tag{3}$$

with $\hat{E} = \frac{1}{2}\sum_{k=1}^{K}\eta_k E_k^2$. In the following, $\|\cdot\|$ denotes the 2-norm for both matrices and vectors, which satisfies $\|BV\| \le \|B\|\|V\|$ for any matrices $B \in \mathbb{C}^{m\times n}$ and vectors $V \in \mathbb{C}^n$, $m, n \in \mathbb{N}$. It is then easy to show that $\|A\| \le 4$, which is independent of the dimension M.

PROPOSITION 2.1. *The FDA* (2) *possesses the charge conservation law, i.e.,*

$$\|U(t)\|^2 = \|U(0)\|^2, \quad \forall\, t \ge 0, \quad \mathbb{P}\text{-a.s.}$$

where $\|U(t)\| = (\|P(t)\|^2 + \|Q(t)\|^2)^{\frac{1}{2}} = \left(\sum_{m=1}^{M}(|p_m(t)|^2 + |q_m(t)|^2)\right)^{\frac{1}{2}}$, *and* $P(t) = (p_1(t), \ldots, p_M(t))^T$ *and* $Q(t) = (q_1(t), \ldots, q_M(t))^T$ *are the real and imaginary parts of* $U(t)$ *respectively.*

Proof. Noticing that matrices A and $F(U)$ are symmetric and the linear function $Z(U)$ satisfies

$$\begin{aligned}\overline{U}^T Z(U) &= (\overline{u_1}, \ldots, \overline{u_M})\begin{pmatrix} u_1 & & \\ & \ddots & \\ & & u_M\end{pmatrix} E_{MK}\begin{pmatrix}\sqrt{\eta_1} & & \\ & \ddots & \\ & & \sqrt{\eta_K}\end{pmatrix}\\ &= (|u_1|^2, \ldots, |u_M|^2)E_{MK}\begin{pmatrix}\sqrt{\eta_1} & & \\ & \ddots & \\ & & \sqrt{\eta_K}\end{pmatrix} \in \mathbb{R}^K,\end{aligned} \tag{4}$$

where $\overline{U}$ denotes the conjugate of U, we multiply (2) by $\overline{U}^T$, take the real part, and then get the charge conservation law for U. ∎

In the following, without pointing out, all equations hold in the sense $\mathbb{P}$-a.s.

Remark 2.2. Equation. (1) can be rewritten into an infinite dimensional Hamiltonian system (see [11]). It is easy to verify that the central finite difference scheme (2) applied to (1) is equivalent to the symplectic Euler scheme applied to the infinite dimensional Hamiltonian form of (1), which implies the symplecticity of (2).

2.2. Unique ergodicity. As the charge of (2) is conserved shown in Proposition 2.1, without loss of generality, we assume that $U(0) \in \mathcal{S}$ and investigate the unique ergodicity of (2) on $\mathcal{S}$. As the nondegeneracy for (2) relies on the solution U as a result of the multiplicative noise, the standard procedure to show the irreducibility and strong Feller property on the whole $\mathcal{S}$ do not apply. So we need to construct an invariant control set.

DEFINITION 2.3. *(see, e.g.,* [2]*) A subset* $\mathcal{M} \neq \emptyset$ *of* $\mathcal{S}$ *is called an invariant control set for the control system*

$$d\phi = \mathbf{i}\left[\frac{1}{h^2}A\phi + \lambda F(\phi)\phi\right]dt + \mathbf{i}Z(\phi)d\Psi(t) \tag{5}$$

of (2) *with a differentiable deterministic function* Ψ, *if* $\overline{\mathcal{O}^+(x)} = \overline{\mathcal{M}}$, $\forall x \in \mathcal{M}$, *and* $\mathcal{M}$ *is maximal with respect to inclusion, where* $\mathcal{O}^+(x)$ *denotes the set of points reachable from* x *(i.e., connected with* x*) in any finite time and* $\overline{\mathcal{M}}$ *denotes the closure of* $\mathcal{M}$.

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We state one of our main results in the following theorem.

THEOREM 2.4. *The FDA* (2) *possesses a unique invariant probability measure* μ_h *on an invariant control set* $\mathcal{M}_0$*, which implies the unique ergodicity of* (2)*. Moreover,*

$$\operatorname{supp}(\mu_h) = \mathcal{S} \text{ and } \mu_h(\mathcal{S}) = \mu_h(\mathcal{M}_0) = 1.$$

Proof.

Step 1. *Existence of invariant measures.*

From Proposition 2.1, we find $\pi_t(U(0), \mathcal{S}) = 1, \forall\, t \geq 0$, where $\pi_t(U(0), \cdot)$ denotes the transition probability (probability kernel) of $U(t)$. As the finite dimensional unit sphere $\mathcal{S}$ is tight, the family of measures $\pi_t(U(0), \cdot)$ is tight, which implies the existence of invariant measures by the Krylov–Bogoliubov theorem [8].

Step 2. *Invariant control set.*

Denoting $U = P + \mathbf{i}Q$ with P and Q being the real and imaginary parts of U respectively, we first consider the following subset of $\mathcal{S}$:

$$\mathcal{S}_1 = \{U = P + \mathbf{i}Q \in \mathcal{S} : \ P > 0\}.$$

For any $t > 0$, $y, z \in \mathcal{S}_1$, there exists a differentiable function ϕ satisfying $\phi(s) = (\phi_1(s), \ldots, \phi_M(s))^T \in \mathcal{S}_1$, $s \in [0, t]$, $\phi(0) = y$, and $\phi(t) = z$ by polynomial interpolation argument. As $\operatorname{rank}(Z(\phi(s))) = M$ for $\phi(s) \in \mathcal{S}_1$ and $M \leq K$, the linear equations

$$Z(\phi(s))X = -\mathbf{i}\phi'(s) - \left[\frac{1}{h^2}A\phi(s) + \lambda F(\phi(s))\phi(s)\right]$$

possess a solution $X \in \mathbb{C}^M$. As, in addition, $Z(\phi(s)) = \operatorname{diag}\{\phi_1(s), \ldots, \phi_M(s)\}E_{MK}\Lambda$, where $\operatorname{diag}\{\phi_1(s), \ldots, \phi_M(s)\}$ is invertible for $\phi(s) \in \mathcal{S}_1$, the solution X depends continuously on s and is denoted by $X(s)$. Thus, there exists a differentiable function $\Psi(\cdot) := \int_0^{\cdot} X(s)ds$ which, together with ϕ defined above, satisfies the control function (5) with initial data $\Psi(0) = 0$. That is, for any $y, z \in \mathcal{S}_1$, y and z are connected, denoted by $y \leftrightarrow z$. The above argument also holds for the following subsets:

$$\begin{aligned}
\mathcal{S}_2 &= \{U = P + \mathbf{i}Q \in \mathcal{S} : \ P < 0\},\\
\mathcal{S}_3 &= \{U = P + \mathbf{i}Q \in \mathcal{S} : \ Q > 0\},\\
\mathcal{S}_4 &= \{U = P + \mathbf{i}Q \in \mathcal{S} : \ Q < 0\}.
\end{aligned}$$

For any $y \in \mathcal{S}_i$, $z \in \mathcal{S}_j$ with $i \neq j$ and $i, j \in \{1, 2, 3, 4\}$, there must exist $\mathcal{S}_l$, r_i, and r_j, satisfying $r_i \in \mathcal{S}_i \cap \mathcal{S}_l \neq \emptyset$ and $r_j \in \mathcal{S}_j \cap \mathcal{S}_l \neq \emptyset$ for some $l \in \{1, 2, 3, 4\}$, such that $y \leftrightarrow r_i \leftrightarrow r_j \leftrightarrow z$. Thus,

$$\mathcal{M}_0 := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4 = \{U = P + \mathbf{i}Q \in \mathcal{S} : P \neq 0 \text{ or } Q \neq 0\},$$

with $\overline{\mathcal{M}_0} = \mathcal{S}$, is an invariant control set for (5).

Step 3. *Uniqueness of the invariant measure.*

We rewrite (2) with P and Q according to its equivalent form in the Itô sense and obtain

$$\begin{aligned}
d\begin{pmatrix} P \\ Q \end{pmatrix} &= \begin{pmatrix} -\hat{E} & -\frac{1}{h^2}A - \lambda F(P,Q) \\ \frac{1}{h^2}A + \lambda F(P,Q) & -\hat{E} \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} dt\\
&\quad + \sum_{k=1}^{K} \sqrt{\eta_k} \begin{pmatrix} 0 & -E_k \\ E_k & 0 \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} d\beta_k(t)
\end{aligned}$$

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$$=: X_0(P,Q)dt + \sum_{k=1}^{K} X_k(P,Q)d\beta_k(t). \tag{6}$$

To derive the uniqueness of the invariant measure, we consider the Lie algebra generated by the diffusions of (6),

$$L(X_0, X_1, \ldots, X_K) = span\Big\{X_l, [X_i, X_j], [X_l, [X_i, X_j]], \ldots, 0 \le l, i, j \le K\Big\}.$$

Choosing $p_* = 0$ and $q_* = \frac{-1}{\sqrt{M}}(1, \ldots, 1)^T$ such that $z_* := p_* + \mathbf{i}q_* \in \mathcal{S}_4 \subset \mathcal{M}_0$, we derive that the vectors

$$X_k(p_*, q_*) = \sqrt{\frac{\eta_k}{M}} \begin{pmatrix} e_k(x_1) \\ \vdots \\ e_k(x_M) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ [X_0, X_k](p_*, q_*) = \sqrt{\frac{\eta_k}{M}} \begin{pmatrix} -\hat{E}\begin{pmatrix} e_k(x_1) \\ \vdots \\ e_k(x_M) \end{pmatrix} \\ (\frac{1}{h^2}A + \frac{1}{M}I)\begin{pmatrix} e_k(x_1) \\ \vdots \\ e_k(x_M) \end{pmatrix} \end{pmatrix}$$

are independent of each other for $k = 1, \ldots, M$, which hence implies the following Hörmander condition:

$$\dim L(X_0, X_1, \ldots, X_K)(z_*) = 2M.$$

Then there is at most one invariant measure with $\mathrm{supp}(\mu_h) = \mathcal{S}$ according to [2]. Actually, according to the above procedure, we obtain that the Hörmander condition holds uniformly for any $z \in \mathcal{M}_0$.

Combining the three steps above, we conclude that there exists a unique invariant measure μ_h on $\mathcal{M}_0$ for the FDA, with $\mu_h(\mathcal{S}) = \mu_h(\mathcal{M}_0) = 1$. ∎

From the theorem above, we can find out that for some other nonlinearities, e.g., $iF(x, |u|)u$ with F being some potential function, such that the equation still possesses the charge conservation law, we can still get the ergodicity of the finite dimensional approximation of the original equation through the procedure above. The procedure could also applied to higher dimensional Schrödinger equations with proper well-posed assumptions, but it may be more technical to verify the Hörmander condition.

Remark 2.5. According to the ergodicity of (2), we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}f(U(t))dt = \int_{\mathcal{S}} f d\mu_h, \quad \forall\ f \in B_b(\mathcal{S}), \quad \text{in } L^2(\mathcal{S}, \mu_h),$$

where $B_b(\mathcal{S})$ denotes the set of bounded and measurable functions and $\int_{\mathcal{S}} f d\mu_h$ is known as the ergodic limit with respect to the invariant measure μ_h.

For more details, we refer to [8] and references therein.

3. Approximation of ergodic limit. A fully discrete scheme (FDS) with the discrete multi-symplectic structure and the discrete charge conservation law is constructed in this section, which could also inherit the unique ergodicity of the FDA. In addition, we prove that the time average of the FDS can approximate the ergodic limit $\int_{\mathcal{S}} f d\mu_h$ with order 1 with respect to the time step.

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3.1. Fully discrete scheme (FDS). We apply the midpoint scheme to (2), and obtain the following FDS:

$$\begin{cases} U^{n+1} - U^n = \mathbf{i}\dfrac{\tau}{h^2} A U^{n+\frac{1}{2}} + \mathbf{i}\lambda\tau F(U^{n+\frac{1}{2}})U^{n+\frac{1}{2}} + \mathbf{i}Z(U^{n+\frac{1}{2}})\delta_{n+1}\beta, \\ U^0 = U(0) \in \mathcal{S}, \end{cases} \tag{7}$$

where τ denotes the uniform time step, $t_n = n\tau$, $U^n = (u_1^n, \ldots, u_M^n) \in \mathbb{C}^M$, $U^{n+\frac{1}{2}} = \frac{U^{n+1}+U^n}{2}$, and $\delta_{n+1}\beta = \beta(t_{n+1}) - \beta(t_n)$. For the FDS (7), which is implicit in both deterministic and stochastic terms, its well-posedness is stated in the following proposition.

PROPOSITION 3.1. *For any initial value $U^0 = U(0) \in \mathcal{S}$, there exists a unique solution $(U^n)_{n\in\mathbb{N}}$ of (7), and it possesses the discrete charge conservation law, i.e.,*

$$\|U^{n+1}\|^2 = \|U^n\|^2 = 1, \quad \forall\, n \in \mathbb{N}.$$

Proof. We multiply both sides of (7) by $\overline{U^{n+\frac{1}{2}}}$, take the real part, and obtain the existence of the numerical solution by the Brouwer fixed-point theorem as well as the discrete charge conservation law.

For the uniqueness, we assume that $X = (X_1, \ldots, X_M)^T$ and $Y = (Y_1, \ldots, Y_M)^T$ are two solutions of (7) with $U^n = z = (z_1, \ldots, z_M)^T \in \mathcal{S}$. It follows that $X, Y \in \mathcal{S}$ and

$$X - Y = \mathbf{i}\frac{\tau}{h^2} A \frac{X-Y}{2} + \frac{\mathbf{i}\lambda\tau}{8} H(X,Y,z) + \mathbf{i}Z\left(\frac{X-Y}{2}\right)\delta_{n+1}\beta, \tag{8}$$

where

$$H(X,Y,z) = \begin{pmatrix} |X_1+z_1|^2(X_1+z_1) - |Y_1+z_1|^2(Y_1+z_1) \\ \vdots \\ |X_M+z_M|^2(X_M+z_M) - |Y_M+z_M|^2(Y_M+z_M) \end{pmatrix}.$$

Based on the fact that $|a|^2a - |b|^2b = |a|^2(a-b) + |b|^2(a-b) + ab(\overline{a} - \overline{b})$ for any $a, b \in \mathbb{C}$, we have

$$\Im\left[(\overline{X} - \overline{Y})^T H(X,Y,z)\right] = \Im\left[\sum_{m=1}^{M} (X_m + z_m)(Y_m + z_m)(\overline{X_m} - \overline{Y_m})^2\right]$$

with $\Im[V]$ denoting the imaginary part of V. Multiplying (8) by $(\overline{X} - \overline{Y})^T$ and taking the real part, we get

$$\begin{aligned} \|X - Y\|^2 &= -\frac{\lambda\tau}{8}\Im\left[(\overline{X} - \overline{Y})^T H(X,Y,z)\right] \\ &\le \frac{\tau}{8}\left(\max_{1\le m\le M} |X_m + z_m||Y_m + z_m|\right)\|X - Y\|^2 \le \frac{\tau}{2}\|X - Y\|^2, \end{aligned}$$

where we have used the fact $X, Y, z \in \mathcal{S}$ and (4). For $\tau < 1$, we get $X = Y$ and complete the proof. ∎

The proposition above shows that (7) possesses the discrete charge conservation law. Furthermore, (7) also inherits the unique ergodicity of the FDA and the stochastic multi-symplecticity of the original equation, which are stated in the following two theorems.

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THEOREM 3.2. *The FDS* (7) *is also ergodic with a unique invariant measure* μ_h^τ *on the control set* $\mathcal{M}_0$*, such that* $\mu_h^\tau(\mathcal{S}) = \mu_h^\tau(\mathcal{M}_0) = 1$*. Also,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(U^n) = \int_{\mathcal{S}} f d\mu_h^\tau, \quad \forall\ f \in B_b(\mathcal{S}), \quad in\ \ L^2(\mathcal{S}, \mu_h^\tau).$$

Proof. Based on the charge conservation law for $\{U^n\}_{n\geq 1}$, we obtain the existence of the invariant measure similar to the proof of Theorem 2.4.

To obtain the uniqueness of the invariant measure, we show that the Markov chain $\{U^{3n}\}_{n\geq 1}$ satisfies the minorization condition (see, e.g., [12]). First, Proposition 3.1 implies that for a given $U^n \in \mathcal{S}$, solution U^{n+1} can be defined through a continuous function $U^{n+1} = \kappa(U^n, \delta_{n+1}\beta)$. As $\delta_{n+1}\beta$ has a C^∞ density, we get a jointly continuous density for U^{n+1}. Second, similar to Theorem 2.4, for any given $y, z \in \mathcal{M}_0$, there must exist $i, j, k \in \{1, 2, 3, 4\}$ and $r_i, r_j \in \mathcal{M}_0$, such that $y \in \mathcal{S}_i, z \in \mathcal{S}_j, r_i \in \mathcal{S}_i \cap \mathcal{S}_k$, and $r_j \in \mathcal{S}_j \cap \mathcal{S}_k$. As $\frac{y+r_i}{2} \in \mathcal{S}_i$ and $Z(\frac{y+r_i}{2})$ is invertible, $\delta_{3n+1}\beta$ can be chosen to ensure that

$$r_i - y = \mathbf{i}\frac{\tau}{h^2} A \frac{y + r_i}{2} + \mathbf{i}\lambda\tau F(\frac{y + r_i}{2})\frac{y + r_i}{2} + \mathbf{i}Z(\frac{y + r_i}{2})\delta_{3n+1}\beta$$

holds, i.e., $r_i = \kappa(y, \delta_{3n+1}\beta)$. Similarly, based on the fact $\frac{r_i+r_j}{2} \in \mathcal{S}_k$ and $\frac{r_j+z}{2} \in \mathcal{S}_j$, we have $r_j = \kappa(r_i, \delta_{3n+2}\beta)$ and $z = \kappa(r_j, \delta_{3n+3}\beta)$. That is, for any given $y, z \in \mathcal{M}_0$, $\delta_{3n+1}\beta, \delta_{3n+2}\beta, \delta_{3n+3}\beta$ can be chosen to ensure that $U^{3n} = y$ and $U^{3(n+1)} = z$. Finally we obtain that, for any $\delta > 0$,

$$\mathbb{P}_3\left(y, B(z, \delta)\right) := \mathbb{P}\left(U^3 \in B(z, \delta)\middle| U^0 = y\right) > 0,$$

where $B(z, \delta)$ denotes the open ball of radius δ centered at z. ∎

The infinite dimensional system (1) has been shown to preserve the stochastic multi-symplectic conservation law locally (see, i.e., [11]):

$$d_t(dp \wedge dq) - \partial_x(dp \wedge dv + dq \wedge dw)dt = 0,$$

with p, q denoting the real and imaginary parts of solution u respectively and $v = p_x$, $w = q_x$ being the derivatives of p and q with respect to variable x. We now show that this ergodic FDS (7) not only possesses the discrete charge conservation law as shown in Proposition 3.1 but also preserves the discrete stochastic multi-symplectic structure.

THEOREM 3.3. *The implicit FDS* (7) *preserves the discrete multi-symplectic structure*

$$\begin{aligned}&\frac{1}{\tau}(dp_j^{n+1} \wedge dq_j^{n+1} - dp_j^n \wedge dq_j^n) - \frac{1}{h}(dp_j^{n+\frac{1}{2}} \wedge dv_{j+1}^{n+\frac{1}{2}} - dp_{j-1}^{n+\frac{1}{2}} \wedge dv_j^{n+\frac{1}{2}})\\ &\quad - \frac{1}{h}(dq_j^{n+\frac{1}{2}} \wedge dw_{j+1}^{n+\frac{1}{2}} - dq_{j-1}^{n+\frac{1}{2}} \wedge dw_j^{n+\frac{1}{2}}) = 0,\end{aligned}$$

where p_j^n, q_j^n *denote the real and imaginary parts of* u_j^n*,* $v_j = \frac{1}{h}(p_j^n - p_{j-1}^n)$*, and* $w_j = \frac{1}{h}(q_j^n - q_{j-1}^n)$*.*

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Proof. Rewriting (7) with the real and imaginary parts of the components u_j^n of U^n, we get

(9)
$$\begin{cases} \frac{1}{\tau}(q_j^{n+1}-q_j^n)-\frac{1}{h}(v_{j+1}^{n+\frac{1}{2}}-v_j^{n+\frac{1}{2}})=\left((p_j^{n+\frac{1}{2}})^2+(q_j^{n+\frac{1}{2}})^2\right)p_j^{n+\frac{1}{2}}+p_j^{n+\frac{1}{2}}\zeta_j^K, \\ -\frac{1}{\tau}(p_j^{n+1}-p_j^n)-\frac{1}{h}(w_{j+1}^{n+\frac{1}{2}}-w_j^{n+\frac{1}{2}})=\left((p_j^{n+\frac{1}{2}})^2+(q_j^{n+\frac{1}{2}})^2\right)q_j^{n+\frac{1}{2}}+q_j^{n+\frac{1}{2}}\zeta_j^K, \\ \frac{1}{h}(p_j^{n+\frac{1}{2}}-p_{j-1}^{n+\frac{1}{2}})=v_j^{n+\frac{1}{2}}, \\ \frac{1}{h}(q_j^{n+\frac{1}{2}}-q_{j-1}^{n+\frac{1}{2}})=w_j^{n+\frac{1}{2}}, \end{cases}$$

where $\zeta_j^K=\sum_{k=1}^K\sqrt{\eta_k}e_k(x_j)\circ d\beta_k(t)$. Denoting $z_j^{n+\frac{1}{2}}=(p_j^{n+\frac{1}{2}},q_j^{n+\frac{1}{2}},v_j^{n+\frac{1}{2}},w_j^{n+\frac{1}{2}})^T$ and taking the differential in the phase space on both sides of (9), we obtain

$$\frac{1}{\tau}d\begin{pmatrix} q_j^{n+1}-q_j^n \\ -(p_j^{n+1}-p_j^n) \\ 0 \\ 0 \end{pmatrix}+\frac{1}{h}d\begin{pmatrix} -(v_{j+1}^{n+\frac{1}{2}}-v_j^{n+\frac{1}{2}}) \\ -(w_{j+1}^{n+\frac{1}{2}}-w_j^{n+\frac{1}{2}}) \\ p_j^{n+\frac{1}{2}}-p_{j-1}^{n+\frac{1}{2}} \\ q_j^{n+\frac{1}{2}}-q_{j-1}^{n+\frac{1}{2}} \end{pmatrix}$$

$$=\nabla^2S_1(z_j^{n+\frac{1}{2}})dz_j^{n+\frac{1}{2}}+\nabla^2S_2(z_j^{n+\frac{1}{2}})dz_j^{n+\frac{1}{2}}\zeta_j^K, \tag{10}$$

where

$$S_1(z_j^{n+\frac{1}{2}})=\frac{1}{4}\left((p_j^{n+\frac{1}{2}})^2+(q_j^{n+\frac{1}{2}})^2\right)^2+\frac{1}{2}\left(v_j^{n+\frac{1}{2}}\right)^2+\frac{1}{2}\left(w_j^{n+\frac{1}{2}}\right)^2$$

and

$$S_2(z_j^{n+\frac{1}{2}})=\frac{1}{2}\left(p_j^{n+\frac{1}{2}}\right)^2+\frac{1}{2}\left(q_j^{n+\frac{1}{2}}\right)^2.$$

Then the wedge product between $dz_j^{n+\frac{1}{2}}$ and (10) concludes the proof based on the symmetry of ∇^2S_1 and ∇^2S_2. ∎

Before giving the approximate error of the ergodic limit, we give some essential a priori estimates about the stability of (7) and (2). In the following, C denotes a generic constant independent of T, N, τ, and h while C_h denotes a constant depending also on h, whose value may be different from line to line.

LEMMA 3.4. *For any initial value $U^0\in\mathcal{S}$ and $\gamma\geq 1$, if $Q\in\mathcal{HS}(L^2,H^{\frac{3}{2}-\frac{1}{\gamma}})$, then there exists a constant C such that the solution $(U^n)_{n\in\mathbb{N}}$ of (7) satisfies*

$$\mathbb{E}\left\|U^{n+1}-U^n\right\|^{2\gamma}\leq C(\tau^{2\gamma}h^{-4\gamma}+\tau^\gamma),\quad\forall\, n\in\mathbb{N},$$

where $\mathcal{HS}(L^{\gamma_1},H^{\gamma_2})$ denotes the space of Hilbert–Schmidt operators from L^{γ_1} to H^{γ_2}.

LEMMA 3.5. *For any initial value $U(0)\in\mathcal{S}$ and $\gamma\geq 1$, there exists a constant C such that the solution $U(t)$ of (2) satisfies*

$$\mathbb{E}\|U(t_{n+1})-U(t_n)\|^{2\gamma}\leq C(\tau^{2\gamma}h^{-4\gamma}+\tau^\gamma),\quad\forall\, n\in\mathbb{N}.$$

The proofs of the lemmas above are given in the Appendix for the readers' convenience.

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3.2. Approximation of ergodic limit. To approximate the ergodic limit of (2) and get the approximate error, we give an estimate of the local weak convergence between $U(\tau)$ and U^1, and the Poisson equation associated with (2) is also used (see [13]). Recall that the SDE (2) in the Stratonovich sense has an equivalent Itô form

$$\begin{aligned} dU &= \left[\mathbf{i}\frac{1}{h^2}AU + \mathbf{i}\lambda F(U)U - \hat{E}U\right]dt + \mathbf{i}Z(U)d\beta(t) \\ &=: b(U)dt + \sigma(U)d\beta(t) \end{aligned} \tag{11}$$

based on (3). For any fixed $f \in W^{4,\infty}(\mathcal{S})$, let $\hat{f} := \int_{\mathcal{S}} f d\mu_h$ and φ be the unique solution of the Poisson equation $\mathcal{L}\varphi = f - \hat{f}$, where

$$\mathcal{L} := b \cdot \nabla + \frac{1}{2}\sigma\sigma^T : \nabla^2$$

denotes the generator of (11). It is easy to find out that (11) satisfies the hypoelliptic setting (see, e.g., [13]) according to the Hörmander condition in Theorem 2.4. Thus, $\varphi \in W^{4,\infty}(\mathcal{S})$ according to Theorem 4.1 in [13]. Based on the well-posedness of the numerical solution $(U^n)_{n\in\mathbb{N}}$ and the implicit function theorem, (7) can be rewritten in the form

$$U^{n+1} = U^n + \tau\Phi(U^n, \tau, h, \delta_{n+1}\beta) \tag{12}$$

for some function Φ. Denoting by $D\varphi(u)\Phi_1$ and $D^k\varphi(u)(\Phi_1, \ldots, \Phi_k)$ the first and kth order weak derivatives evaluated in the directions Φ_j, $j = 1, \ldots, k$, with $D^k\varphi(u)(\Phi)^k$ for shorthand if all the directions are the same in the kth derivatives, then we have

$$\begin{aligned} \varphi(U^{n+1}) &= \varphi(U^n) + \tau\left[D\varphi(U^n)\Phi^n + \frac{1}{2}\tau D^2\varphi(U^n)(\Phi^n)^2\right] + \frac{1}{6}D^3\varphi(U^n)(\tau\Phi^n)^3 + R_n^\Phi \\ &=: \varphi(U^n) + \tau\mathcal{L}^\Phi\varphi(U^n) + \frac{1}{6}D^3\varphi(U^n)(\tau\Phi^n)^3 + R_n^\Phi, \end{aligned} \tag{13}$$

where $\Phi^n := \Phi(U^n, \tau, h, \delta_{n+1}\beta)$,

$$\mathcal{L}^\Phi\varphi(U^n) = D\varphi(U^n)\Phi^n + \frac{1}{2}\tau D^2\varphi(U^n)(\Phi^n)^2,$$

and

$$R_n^\Phi = \frac{1}{4!}D^4\varphi(\theta_n)(\tau\Phi^n)^4$$

for some $\theta_n \in [U^n, U^{n+1}] := [u_1^n, u_1^{n+1}] \times \ldots \times [u_M^n, u_M^{n+1}]$. Adding (13) together from $n = 0$ to $n = N - 1$ for some fixed $N \in \mathbb{N}$, then dividing the result by $T = N\tau$, and noticing that $\mathcal{L}\varphi(U^n) = f(U^n) - \hat{f}$, we obtain

$$\begin{aligned} \frac{\varphi(U^N) - \varphi(U^0)}{N\tau} =& \frac{1}{N}\left(\sum_{n=0}^{N-1}\left[\mathcal{L}^\Phi\varphi(U^n) - \mathcal{L}\varphi(U^n)\right] + \sum_{n=0}^{N-1}\mathcal{L}\varphi(U^n)\right. \\ &\left. + \frac{1}{\tau}\sum_{n=0}^{N-1}\frac{1}{6}D^3\varphi(U^n)(\tau\Phi^n)^3 + \frac{1}{\tau}\sum_{n=0}^{N-1}R_n^\Phi\right) \\ =& \frac{1}{N}\sum_{n=0}^{N-1}\left[\mathcal{L}^\Phi\varphi(U^n) - \mathcal{L}\varphi(U^n) + \frac{1}{6\tau}D^3\varphi(U^n)(\tau\Phi^n)^3\right] \end{aligned}$$

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$$+\left(\frac{1}{N}\sum_{n=0}^{N-1} f(U^n)-\hat{f}\right)+\frac{1}{N\tau}\sum_{n=0}^{N-1} R_n^{\Phi},$$

which shows

$$\left|\mathbb{E}\left[\frac{1}{N}\sum_{n=0}^{N-1} f(U^n)-\hat{f}\right]\right| \le \left|\frac{1}{N\tau}\mathbb{E}\left[\varphi(U^N)-\varphi(U^0)\right]\right| + \left|\frac{1}{N\tau}\sum_{n=0}^{N-1}\mathbb{E}R_n^{\Phi}\right|$$

$$+\left|\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\left[\mathcal{L}^{\Phi}\varphi(U^n)-\mathcal{L}\varphi(U^n)+\frac{1}{6\tau}D^3\varphi(U^n)(\tau\Phi^n)^3\right]\right| =: I+II+III. \tag{14}$$

The average $\frac{1}{N}\sum_{n=0}^{N-1} f(U^n)$ is regarded as an approximation of $\hat{f}$. We next begin to investigate the approximate error by estimating I, II, and III respectively.

According to the fact that $\varphi \in W^{4,\infty}(\mathcal{S})$ and Lemma 3.4, we have

$$I \le \frac{2\|\varphi\|_{0,\infty}}{N\tau} \le \frac{C}{T} \tag{15}$$

and

$$II \le \frac{1}{N\tau}\sum_{n=0}^{N-1}\mathbb{E}\left[\|\tau\Phi^n\|^4\,\|D^4\varphi\|_{L^\infty}\right] \le \frac{C}{N\tau}\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|U^{n+1}-U^n\right\|^4\right]$$

$$\le \frac{C}{N\tau}\sum_{n=0}^{N-1}\left(\tau^4h^{-8}+\tau^2\right) \le C\left(\tau^3h^{-8}+\tau\right), \tag{16}$$

where $\|\varphi\|_{\gamma,\infty} := \sup_{|\alpha|\le\gamma, u\in\mathcal{S}}|D^\alpha\varphi(u)|$, $\gamma\in\mathbb{N}$.

It then remains to estimate the term III. To this end, we need the estimate of the local weak convergence, which is stated in the following theorem. The proof of the following theorem is also given in the Appendix.

THEOREM 3.6. *For a fixed spatial approximation* (2)*, and for any initial value* $U^0\in\mathcal{S}$ *and* $\varphi\in W^{4,\infty}(\mathcal{S})$*, it holds under the condition* $Q\in\mathcal{HS}(L^2,H^{\frac{5}{4}})$ *and* $\tau = O(h^4)$ *that*

$$\left|\mathbb{E}\left[\varphi(U(\tau))-\varphi(U^1)\right]\right| \le C_h\tau^2$$

for some constant $C_h = C(\varphi,\eta,h)$.

Now we are in the position of showing the approximation error between the time average of FDS and the ergodic limit of FDA.

THEOREM 3.7. *Under the assumptions in Theorem* 3.6 *and for any* $f\in W^{4,\infty}(\mathcal{S})$*, there exists a positive constant* $C_h = C(f,\eta,h)$ *such that*

$$\left|\mathbb{E}\left[\frac{1}{N}\sum_{n=0}^{N-1} f(U^n)-\hat{f}\right]\right| \le C_h\left(\frac{1}{T}+\tau\right).$$

Proof. Based on (14)–(16), it suffices to estimate term III. For any $f\in W^{4,\infty}(\mathcal{S})$, we know from the statement above that the solution to the Poisson equation $\mathcal{L}\varphi = f-\hat{f}$ satisfies $\varphi\in W^{4,\infty}(\mathcal{S})$. Based on (13), Lemma 3.4, and the condition $\tau = O(h^4)$, we have

$$\varphi(U^1) \overset{\mathbb{E}}{=} \varphi(U^0)+\tau\mathcal{L}^{\Phi}\varphi(U^0)+\frac{1}{6}D^3\varphi(U^0)(U^1-U^0)^3+O(\tau^2)$$

$$\overset{\mathbb{E}}{=} \varphi(U^0)+\tau\mathcal{L}^{\Phi}\varphi(U^0)+O(\tau^2), \tag{17}$$

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where $\overset{\mathbb{E}}{=}$ means that the equation holds in expectation sense, and in the last step we have used the fact that

$$\begin{aligned} D^3\varphi(U^0)(U^1-U^0)^3 &= D^3\varphi(U^0)\left(\mathbf{i}\frac{\tau}{h^2}AU^{\frac{1}{2}}+\mathbf{i}\lambda\tau F(U^{\frac{1}{2}})U^{\frac{1}{2}}+\mathbf{i}Z(U^{\frac{1}{2}})\delta_1\beta\right)^3 \\ &\overset{\mathbb{E}}{=} D^3\varphi(U^0)\left(\mathbf{i}Z(U^{\frac{1}{2}})\delta_1\beta\right)^3+O(\tau^2h^{-2}+\tau^2) \\ &\overset{\mathbb{E}}{=} D^3\varphi(U^0)\left(\frac{\mathbf{i}}{2}Z(U^1-U^0)\delta_1\beta+\mathbf{i}Z(U^0)\delta_1\beta\right)^3+O(\tau^2h^{-2}+\tau^2) \\ &\overset{\mathbb{E}}{=} O(\tau^2h^{-2}+\tau^2) \end{aligned} \tag{18}$$

based on the linearity of Z, Lemma 3.4, and that $\mathbb{E}\left(\mathbf{i}Z(U^0)\delta_1\beta\right)^3=0$. We can also get the following expression similar to (17) based on Taylor expansion and Lemma 3.5:

$$\begin{aligned} \varphi(U(\tau)) &\overset{\mathbb{E}}{=} \varphi(U^0)+\int_0^\tau\left(D\varphi(U^0)b(U(t))+\frac{1}{2}D^2\varphi(U^0)\left(\sigma(U(t))\right)^2\right)dt \\ &\quad+\int_0^\tau D\varphi(U^0)\sigma(U(t))d\beta(t)+\frac{1}{6}D^3\varphi(U^0)(U(\tau)-U^0)^3+O(\tau^2) \\ &\overset{\mathbb{E}}{=} \varphi(U^0)+\int_0^\tau\tilde{\mathcal{L}}_t\varphi(U^0)dt+O(\tau^2), \end{aligned} \tag{19}$$

where

$$\tilde{\mathcal{L}}_t\varphi(U^0):=D\varphi(U^0)b(U(t))+\frac{1}{2}D^2\varphi(U^0)\left(\sigma(U(t))\right)^2$$

and $\mathbb{E}\left[\int_0^\tau D\varphi(U^0)\sigma(U(t))d\beta(t)\right]=0$. Thus, subtracting (17) from (19), we derive

$$\left|\mathbb{E}\left[\tau\mathcal{L}^\Phi\varphi(U^0)-\int_0^\tau\tilde{\mathcal{L}}_t\varphi(U^0)dt\right]\right|\le\left|\mathbb{E}\left[\varphi(U(\tau))-\varphi(U^1)\right]\right|+C\tau^2. \tag{20}$$

We notice that

$$\begin{aligned} \left|\int_0^\tau\mathbb{E}\left[\tilde{\mathcal{L}}_t\varphi(U^0)-\mathcal{L}\varphi(U^0)\right]dt\right| &\le\left|\int_0^\tau\mathbb{E}\left[D\varphi(U^0)\left(b(U(t))-b(U^0)\right)\right]dt\right| \\ &\quad+\left|\frac{1}{2}\int_0^\tau\mathbb{E}\left[D^2\varphi(U^0)\left(\sigma(U(t))-\sigma(U^0),\sigma(U(t))+\sigma(U^0)\right)\right]dt\right|, \end{aligned} \tag{21}$$

in which we have

$$\begin{aligned} &\left|\mathbb{E}\left[D\varphi(U^0)\left(b(U(t))-b(U^0)\right)\right]\right|=\Big|\mathbb{E}\Big[D^2\varphi(U^0)\Big(\mathbf{i}\frac{1}{h^2}A\left(U(t)-U^0\right) \\ &\quad+\mathbf{i}\lambda\Big(F(U(t))U(t)-F(U^0)U^0\Big)-\hat{E}(U(t)-U^0)\Big)\Big]\Big|\le C(th^{-2}+t) \end{aligned}$$

for the first term in (21). In the last step, we have used the fact that $g(V):=F(V)V$, $\forall\ V\in\mathcal{S}$, is a continuous differentiable function which satisfies $|D^kg(V)|\le C$ for $\|V\|\le 1$ and $k\in\mathbb{N}$, and then replaced $U(t)-U^0$ by the integral form of (2) to get the result. The second term in (21) can be estimated in the same way. Thus, we have

$$\left|\int_0^\tau\mathbb{E}\left[\tilde{\mathcal{L}}_t\varphi(U^0)-\mathcal{L}\varphi(U^0)\right]dt\right|\le C(\tau^2h^{-2}+\tau^2). \tag{22}$$

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We hence conclude based on (18), (20), (22) and Theorem 3.6 that

$$
\begin{aligned}
III &= \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\left[\mathcal{L}^{\Phi}\varphi(U^n) - \mathcal{L}\varphi(U^n) + \frac{1}{6\tau} D^3\varphi(U^n)(U^{n+1}-U^n)^3 \right] \right| \\
&\le \frac{1}{\tau} \sup_{U^0\in\mathcal{S}} \left\{ \left| \mathbb{E}\left[\tau\mathcal{L}^{\Phi}\varphi(U^0) - \int_0^{\tau} \tilde{\mathcal{L}}_t\varphi(U^0)dt \right] \right| + \left| \int_0^{\tau} \mathbb{E}\left[\tilde{\mathcal{L}}_t\varphi(U^0) - \mathcal{L}\varphi(U^0) \right] dt \right| \right\} \\
&\quad + C(\tau h^{-2} + \tau) \le C_h \tau.
\end{aligned} \tag{23}
$$

Noticing that $\tau^3 h^{-8} = O(\tau)$ under the condition $\tau = O(h^4)$, from (15), (16), and (23), we finally obtain

$$
\left| \mathbb{E}\left[\frac{1}{N} \sum_{n=0}^{N-1} f(U^n) - \hat{f} \right] \right| \le C_h \left(\frac{1}{T} + \tau \right). \qquad \square
$$

Remark 3.8. Based on the theorem above and the ergodicity of (2), for a fixed h, we obtain

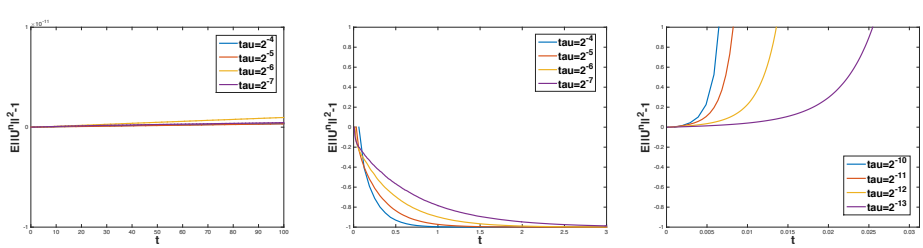
$$
\left| \mathbb{E}\left[\frac{1}{N} \sum_{n=0}^{N-1} f(U^n) - \frac{1}{T}\int_0^T f(U(t))dt \right] \right| \le C_h(B(T) + \tau),
$$

which implies that the global weak error is of order 1, i.e.,

$$
\left| \mathbb{E}\Big[f(U^n) - f(U(t)) \Big] \right| \le C_h(\tilde{B}(t) + \tau), \quad t \in [n\tau, (n+1)\tau],
$$

where $B(T) \to 0$ and $\tilde{B}(T) \to 0$ as $T \to \infty$. On the other hand, a time independent weak error in turn leads to the result stated in Theorem 3.7.

4. Numerical experiments. In this section, numerical experiments are given to test several properties of scheme (7) with $\lambda = 1$, i.e., the focusing case. In the following experiments, we simulate the noise $\delta_{n+1}\beta$ by $\sqrt{\tau}\xi_n$ with ξ_n being independent K-dimensional $N(0,1)$-random variables, and choose $\eta_k = k^{-4}$, $k = 1, \ldots, K$. In addition, we approximate the expectation by taking averaged value over 500 paths, and the proposed scheme, which is implicit, is numerically solved utilizing the fixed point iteration. In the following, we will use the notation $\|U\|_\gamma^\gamma := \sum_{m=1}^{M} (|p_m|^\gamma + |q_m|^\gamma)$ for $U \in \mathbb{C}^M$ and $\gamma \in \mathbb{N}$ with $P = (p_1, \ldots, p_M)^T$, and $Q = (q_1, \ldots, p_M)^T$ being the real and imaginary parts of U. Notice that $\|\cdot\|_2 = \|\cdot\|$.



(a) Proposed scheme (b) IME scheme (c) EM scheme

FIG. 1. *Charge evolution* $\mathbb{E}\|U^n\|^2 - 1$ *for* (a) *the proposed scheme with* $T = 100$ *under steps* $\tau = 2^{-i}$ $(i = 4, 5, 6, 7)$, (b) *IME scheme with* $T = 3$ *under steps* $\tau = 2^{-i}$ $(i = 4, 5, 6, 7)$, *and* (c) *EM scheme with* $T = 2^{-5}$ *under steps* $\tau = 2^{-i}$ $(i = 10, 11, 12, 13)$ $(h = 0.05,\ K = 30)$.

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We omit the boundary nodes in the simulation; as a result, we may choose the normalized initial value $U^0 = c_*(U^0(1), \ldots, U^0(M))^T$ based on function $u_0(x)$ satisfying $U^0(m) = u_0(mh)$, $m = 1, \ldots, M$, in which $u_0(x)$ need not to satisfy the boundary condition in (1). Let $u_0(x) = 1$, and we get the normalized initial value U^0 satisfying $\|U^0\| = 1$, which is used in Figures 1, 3, and 4. We first simulate the discrete charge for the proposed scheme compared with the Euler–Maruyama (EM) scheme and implicit Euler (IE) scheme, respectively. Figure 1 shows that the proposed scheme possesses the discrete charge conservation law $\mathbb{E}\|U^n\|^2 = 1$, which coincides with Proposition 3.1, while both the EM scheme and the IE scheme do not. As the EM scheme is not stable—its solution will blow up in a short time—we choose the time step τ small enough for the EM scheme in the experiments.

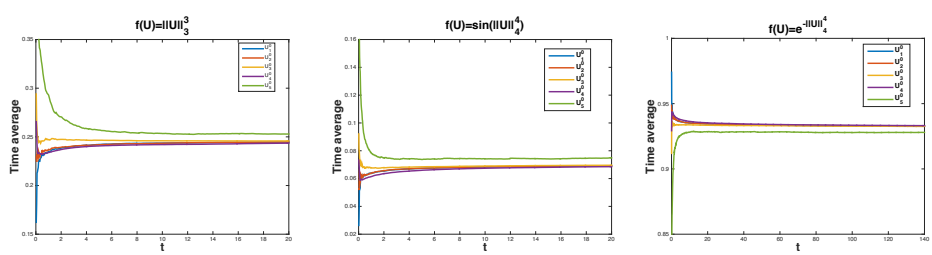
As the ergodic limit $\int_{\mathcal{S}} f d\mu_h$ is unknown, to verify the ergodicity of the numerical solution, we simulate the time averages $\frac{1}{N}\sum_{n=1}^{N} \mathbb{E}[f(U^n)]$ for the proposed scheme with the bounded function $f \in C_b(\mathcal{S})$ being (a) $f(U) = \|U\|_3^3$, (b) $f(U) = \sin(\|U\|_4^4)$, and (c) $f(U) = e^{-\|U\|_4^4}$ in Figure 2, starting from five different initial values U_l^0, $1 \le l \le 5$. It is known from Theorem 3.2 that, for almost every initial value $U^0 \in \mathcal{S}$, the time averages will converge to the same value, i.e., the ergodic limit. Thus, we choose five initial values

$$U_l^0 = c_*(U_l^0(1), \ldots, U_l^0(M))^T, \ l = 1, \ldots, 5$$

based on the following five functions:

$$\begin{aligned}
u_{0,1}(x) &= \frac{1}{\sqrt{2}} + \frac{\mathbf{i}}{\sqrt{2}}, \quad u_{0,2}(x) = 1, \quad u_{0,3}(x) = 2x, \\
u_{0,4}(x) &= \left(1 - \sqrt{\frac{\pi}{2}(\exp\frac{1}{4} - 1)}\right)(1 - \exp(x(1-x))), \\
u_{0,5}(x) &= c_* \operatorname{sech}\left(\frac{x}{\sqrt{2}}\right) \exp\left(\mathbf{i}\frac{x}{2}\right)
\end{aligned}$$

with $U_l^0(m) = u_{0,l}(hm)$, $1 \le m \le M$, and c_* being normalized constants. The charge of all the initial functions equals 1, and $u_{0,4}(x)$ even satisfies the boundary condition in (1). Figure 2 shows that the proposed scheme starting from different initial values converges to the same value with error no more than $O(\tau)$ with $h = 0.05$ and $\tau = 2^{-6}$, which coincides with Theorem 3.7.



(a) $f(U) = \|U\|_3^3$, $T = 20$ (b) $f(U) = \sin(\|U\|_4^4)$, $T = 20$ (c) $f(U) = e^{-\|U\|_4^4}$, $T = 140$

FIG. 2. *The time averages* $\frac{1}{N}\sum_{n=1}^{N} \mathbb{E}[f(U^n)]$ *for the proposed scheme with* (a) $f(U) = \|U\|_3^3$, (b) $f(U) = \sin(\|U\|_4^4)$, *and* (c) $f(U) = e^{-\|U\|_4^4}$ ($\tau = 2^{-6}$, $h = 0.05$, $K = 30$).

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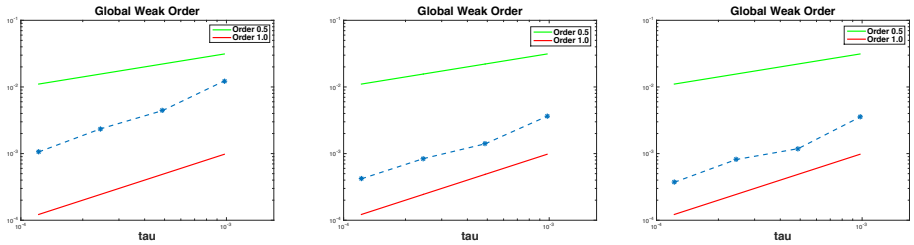
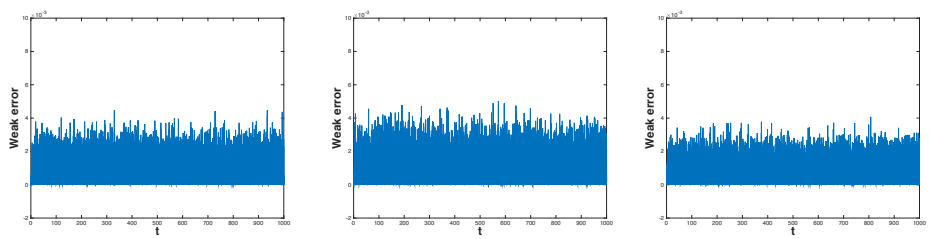


(a) $f(U) = \|U\|_3^3$ (b) $f(U) = \sin(\|U\|_4^4)$ (c) $f(U) = e^{-\|U\|_4^4}$

FIG. 3. *The weak convergence order of* $|\mathbb{E}[f(U^n) - f(U(T))]|$ *with* (a) $f(U) = \|U\|_3^3$, (b) $f(U) = \sin(\|U\|_4^4)$, *and* (c) $f(U) = e^{-\|U\|_4^4}$ ($\tau = 2^{-i}, 10 \le i \le 13$, $h = 0.05$, $T = 2^{-1}$, $K = 30$).

For a fixed h, Figures 3 and 4 show the weak convergence order in the temporal direction and the weak error over long time, respectively. Figure 3 shows that the proposed scheme is of order 1 in the weak sense for (a) $f(U) = \|U\|_3^3$, (b) $f(U) = \sin(\|U\|_4^4)$, and (c) $f(U) = e^{-\|U\|_4^4}$, which coincides with the statement in Remark 3.8. Furthermore, based on the ergodicity for both FDS and FDA, the weak error is supposed to be independent of time interval when time is large enough. To verify this property, we simulate the weak error over long time in Figure 4 for (a) $f(U) = \|U\|_3^3$, (b) $f(U) = \sin(\|U\|_4^4)$, and (c) $f(U) = e^{-\|U\|_4^4}$; it shows that the weak error for the proposed scheme would not increase before $T = 1000$ while the weak error for the EM scheme would increase with time.



(a) $f(U) = \|U\|_3^3$ (b) $f(U) = \sin(\|U\|_4^4)$ (c) $f(U) = e^{-\|U\|_4^4}$

FIG. 4. *The weak error* $|\mathbb{E}[f(U^n) - f(U(T))]|$ *for* (a) $f(U) = \|U\|_3^3$, (b) $f(U) = \sin(\|U\|_4^4)$, *and* (c) $f(U) = e^{-\|U\|_4^4}$ ($\tau = 2^{-12}$, $h = 0.05$, $T = 10^3$, $K = 30$).

5. Appendix.

5.1. Proof of Lemma 3.4. As proved in Proposition 3.1 that $\|U^n\| = 1$, $\forall\, n \in \mathbb{N}$, for the nonlinear term, we have

$$\mathbb{E}\left\|F(U^{n+\frac{1}{2}})U^{n+\frac{1}{2}}\right\|^{2\gamma} = \mathbb{E}\sum_{m=1}^{M}\left|u_m^{n+\frac{1}{2}}\right|^{6\gamma} \le \mathbb{E}\left(\sum_{m=1}^{M}\left|u_m^{n+\frac{1}{2}}\right|^2\right)^{3\gamma} \le \mathbb{E}\left\|U^{n+\frac{1}{2}}\right\|^{6\gamma} \le 1$$

by the convexity of $\mathcal{S}$, i.e., $\|U^{n+\frac{1}{2}}\| \le 1$, a.s. The noise term can be estimated as

$$\mathbb{E}\left\|Z(U^{n+\frac{1}{2}})\delta_{n+1}\beta\right\|^{2\gamma} = \mathbb{E}\left(\sum_{m=1}^{M}\left|\sum_{k=1}^{K}u_m^{n+\frac{1}{2}}e_k(x_m)\sqrt{\eta_k}\delta_{n+1}\beta_k\right|^2\right)^{\gamma}$$

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$$\leq \mathbb{E}\left(2\sum_{m=1}^{M}\left|u_m^{n+\frac{1}{2}}\right|^2\left(\sum_{k=1}^{K}\sqrt{\eta_k}|\delta_{n+1}\beta_k|\right)^2\right)^{\gamma} = \mathbb{E}\left(2\left\|U^{n+\frac{1}{2}}\right\|^2\left(\sum_{k=1}^{K}\sqrt{\eta_k}|\delta_{n+1}\beta_k|\right)^2\right)^{\gamma}$$

(24)

$$\leq C\mathbb{E}\left(\sum_{k=1}^{K}\eta_k^{\frac{1}{4}}\eta_k^{\frac{1}{4}}|\delta_{n+1}\beta_k|\right)^{2\gamma} \leq C\mathbb{E}\left[\left(\sum_{k=1}^{K}\eta_k^{\frac{\gamma}{2(2\gamma-1)}}\right)^{2\gamma-1}\left(\sum_{k=1}^{K}\eta_k^{\frac{\gamma}{2}}|\delta_{n+1}\beta_k|^{2\gamma}\right)\right] \leq C\tau^{\gamma}$$

by $|e_k(x_m)| \leq \sqrt{2}$ and Hölder's inequality. In the last step of (24) we notice that, as $Q \in \mathcal{HS}(L^2, H^{\frac{3}{2}-\frac{1}{\gamma}})$, $\sum_{k=1}^{\infty} k^{3-\frac{2}{\gamma}}\eta_k < \infty$, so $\eta_k = O(k^{-(4-\frac{2}{\gamma}+\epsilon)})$ for any $\epsilon > 0$. Thus,

$$\sum_{k=1}^{\infty}\eta_k^{\frac{\gamma}{2(2\gamma-1)}} \leq C\sum_{k=1}^{\infty}k^{-(4-\frac{2}{\gamma}+\epsilon)\frac{\gamma}{2(2\gamma-1)}} = C\sum_{k=1}^{\infty}k^{-\left(1+\frac{\epsilon\gamma}{2(2\gamma-1)}\right)} < \infty.$$

In conclusion,

$$\begin{aligned}
&\mathbb{E}\left\|U^{n+1}-U^n\right\|^{2\gamma}\\
&\quad\leq C\left(\mathbb{E}\left\|\frac{\tau}{h^2}AU^{n+\frac{1}{2}}\right\|^{2\gamma} + \mathbb{E}\left\|\lambda\tau F(U^{n+\frac{1}{2}})U^{n+\frac{1}{2}}\right\|^{2\gamma} + \mathbb{E}\left\|Z(U^{n+\frac{1}{2}})\delta_{n+1}\beta\right\|^{2\gamma}\right)\\
&\quad\leq \frac{C\tau^{2\gamma}}{h^{4\gamma}}\mathbb{E}\left\|U^{n+\frac{1}{2}}\right\|^{2\gamma} + C\tau^{2\gamma} + C\tau^{\gamma} \leq C\left(\tau^{2\gamma}h^{-4\gamma}+\tau^{\gamma}\right),
\end{aligned}$$

where we have used the fact that $\|A\| \leq 4$.

5.2. Proof of Lemma 3.5. From (2) and (3), based on Hölder's inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\|U(t_{n+1})-U(t_n)\|^{2\gamma}\\
&\quad= \mathbb{E}\left\|\int_{t_n}^{t_{n+1}}\left[\mathbf{i}\frac{1}{h^2}AU + \mathbf{i}\lambda F(U)U - \hat{E}U\right]dt + \int_{t_n}^{t_{n+1}}\mathbf{i}Z(U)d\beta(t)\right\|^{2\gamma}\\
&\quad\leq C\left(\int_{t_n}^{t_{n+1}}\mathbb{E}\left\|\mathbf{i}\frac{1}{h^2}AU + \mathbf{i}\lambda F(U)U - \hat{E}U\right\|^{2\gamma}dt\left(\int_{t_n}^{t_{n+1}}1^{\frac{2\gamma}{2\gamma-1}}dt\right)^{2\gamma-1}\right.\\
&\qquad\left.+\mathbb{E}\left\|\int_{t_n}^{t_{n+1}}\mathbf{i}Z(U)d\beta(t)\right\|^{2\gamma}\right)\\
&\quad\leq C\tau^{2\gamma-1}\left\|\frac{1}{h^2}A\right\|^{2\gamma}\int_{t_n}^{t_{n+1}}\mathbb{E}\left\|U\right\|^{2\gamma}dt + C\tau^{2\gamma} + C\tau^{\gamma}\\
&\quad\leq C(\tau^{2\gamma}h^{-4\gamma}+\tau^{\gamma}),
\end{aligned}$$

where we have used the boundedness of $F(U)U$ in $\mathcal{S}$ similar to that in Lemma 3.4. In the third step of the equation above, we also used

$$\begin{aligned}
\mathbb{E}\|\hat{E}U\|^{2\gamma} &\leq C\mathbb{E}\left(\sum_{m=1}^{M}\left|\sum_{k=1}^{K}\eta_k e_k^2(x_m)u_m\right|^2\right)^{\gamma}\\
&\leq C\mathbb{E}\left(\sum_{m=1}^{M}|u_m|^2\left(\sum_{k=1}^{K}\eta_k\right)^2\right)^{\gamma} \leq C\eta^{2\gamma}\mathbb{E}\|U\|^{2\gamma} \leq C
\end{aligned}$$

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and

$$\begin{aligned}\mathbb{E}\left\|\int_{t_n}^{t_{n+1}} \mathbf{i}Z(U)d\beta(t)\right\|^{2\gamma} &\le C\left(\int_{t_n}^{t_{n+1}}\left(\mathbb{E}\|Z(U)\|_{\mathcal{HS}}^{2\gamma}\right)^{\frac{1}{\gamma}}dt\right)^{\gamma}\\ &\le C\left(\int_{t_n}^{t_{n+1}}\left(\mathbb{E}\left(\sum_{m=1}^{M}\sum_{k=1}^{K}|u_m e_k(x_m)\sqrt{\eta_k}|^2\right)^{\gamma}\right)^{\frac{1}{\gamma}}dt\right)^{\gamma}\\ &\le C\left(\int_{t_n}^{t_{n+1}}\left(\mathbb{E}\left(2\eta\|U\|^2\right)^{\gamma}\right)^{\frac{1}{\gamma}}dt\right)^{\gamma} \le C\tau^{\gamma}\end{aligned}$$

according to the Burkholder–Davis–Gundy inequality and the fact that the Hilbert–Schmidt operater norm $\|Z(U)\|_{\mathcal{HS}} = \|Z(U)\|_F$, with $\|\cdot\|_F$ denoting the Frobenius norm.

5.3. Proof of Theorem 3.6. Based on Taylor expansion and Lemmas 3.4 and 3.5, we obtain

$$\begin{aligned}\mathbb{E}\left[\varphi(U(\tau))-\varphi(U^1)\right] &= \mathbb{E}\left[D\varphi(U^1)\big(U(\tau)-U^1\big)+O\big(\|U(\tau)-U^1\|^2\big)\right]\\ &= \mathbb{E}\left[D\varphi(U^0)\big(U(\tau)-U^1\big)\right]+\mathbb{E}\left[D^2\varphi(U^0)(U^1-U^0,U(\tau)-U^1)\right]\\ &\quad + O\Big(\mathbb{E}\left[\|U^1-U^0\|^2\|U(\tau)-U^1\|\right]+\mathbb{E}\|U(\tau)-U^1\|^2\Big)\\ &=: \mathcal{A}+\mathcal{B}+\mathcal{C}.\end{aligned}$$

We give the mild solution and discrete mild solution of (2) and (7) respectively:

$$\begin{aligned}U(\tau) = e^{\mathbf{i}\frac{1}{h^2}A\tau}U^0 &+ \int_0^{\tau} e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\Big(\mathbf{i}\lambda F(U(s))U(s)-\hat{E}U(s)\Big)ds\\ &+ \int_0^{\tau} e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\mathbf{i}Z(U(s))d\beta(s),\end{aligned}$$

$$\begin{aligned}U^1 = &\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\left(I+\frac{\mathbf{i}\tau}{2h^2}A\right)U^0+\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\mathbf{i}\lambda\tau F\left(U^{\frac{1}{2}}\right)U^{\frac{1}{2}}\\ &+\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\mathbf{i}Z\left(U^{\frac{1}{2}}\right)\delta_1\beta.\end{aligned}$$

Estimation of $\mathcal{A}$. Considering the difference between the above equations, we have

$$\begin{aligned}U(\tau)-U^1 = &\left(e^{\mathbf{i}\frac{1}{h^2}A\tau}-\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\left(I+\frac{\mathbf{i}\tau}{2h^2}A\right)\right)U^0\\ &+\mathbf{i}\int_0^{\tau}\left[e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}-\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\right]\lambda F(U(s))U(s)ds\\ &+\mathbf{i}\int_0^{\tau}\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\lambda\left[F(U(s))U(s)-F\left(U^{\frac{1}{2}}\right)U^{\frac{1}{2}}\right]ds\\ &+\mathbf{i}\int_0^{\tau}\left[e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}-\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\right]Z(U(s))d\beta(s)\end{aligned}$$

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$$
\begin{aligned}
&+\mathbf{i}\int_0^\tau \left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} Z(U(s)-U^0)d\beta(s)\\
&-\left[\frac{\mathbf{i}}{2}\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} Z(U^1-U^0)\delta_1\beta+\int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\hat{E}U(s)ds\right],\\
=:&\ \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}+\mathbf{e}+\mathbf{f},
\end{aligned}
$$

which, together with the fact that $\mathbb{E}[D\varphi(U^0)\mathbf{d}]=\mathbb{E}[D\varphi(U^0)\mathbf{e}]=0$, yields that

$$
\begin{aligned}
\mathcal{A}&=\mathbb{E}\left[D\varphi(U^0)\mathbf{a}\right]+\mathbb{E}\left[D\varphi(U^0)\mathbf{b}\right]+\mathbb{E}\left[D\varphi(U^0)\mathbf{c}\right]+\mathbb{E}\left[D\varphi(U^0)\mathbf{f}\right]\\
&=:A_1+A_2+A_3+A_4.
\end{aligned}
$$

Based on the estimates $e^x-(1-\frac{x}{2})^{-1}(1+\frac{x}{2})=O(x^3)$ for $\|x\|<1$, and

$$
\left\|e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}-(I-\frac{\mathbf{i}\tau}{2h^2}A)^{-1}\right\|\le C\left(\frac{\tau}{h^2}\|A\|\right)\le C\tau h^{-2},\quad \forall\ s\in[0,\tau], \tag{25}
$$

we have

$$
|A_1|\le C\|\varphi\|_{1,\infty}\|\tau h^{-2}A\|^3\mathbb{E}\|U^0\|\le C\tau^3h^{-6}\le C\tau^2h^{-2} \tag{26}
$$

under the condition $\tau=O(h^4)$, and

$$
|A_2|\le C\|\varphi\|_{1,\infty}\int_0^\tau \|\tau h^{-2}A\|\|F(U(s))U(s)\|ds\le C\tau^2h^{-2}. \tag{27}
$$

Term A_3 can be estimated based on Lemmas 3.4 and 3.5:

$$
\begin{aligned}
|A_3|=&\left|\mathbb{E}\left[D\varphi(U^0)\int_0^\tau\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\Big[\left(F(U(s))U(s)-F(U^0)U^0\right)\right.\right.\\
&\left.\left.-\left(F\left(U^{\frac12}\right)U^{\frac12}-F(U^0)U^0\right)\Big]ds\right]\right|,
\end{aligned}
$$

in which we have known from the proof of Theorem 3.7 that

$$
\begin{aligned}
&F(U(s))U(s)-F(U^0)U^0=g(U(s))-g(U^0)\\
&=Dg(U^0)(U(s)-U^0)+\frac12D^2g(\theta(s))(U(s)-U^0)^2\\
&=Dg(U^0)\left(\int_0^s\frac{\mathbf{i}}{h^2}AU(r)+\mathbf{i}\lambda F(U(r))U(r)-\hat{E}U(r)dr+\int_0^s Z(U(r))d\beta(r)\right)\\
&\quad+\frac12D^2g(\theta(s))(U(s)-U^0)^2
\end{aligned}
$$

for some $\theta(s)\in[U^0,U(s)]$ and $s\in[0,\tau]$, and the same for the term $F\left(U^{\frac12}\right)U^{\frac12}-F(U^0)U^0$. Based on the fact that $\mathbb{E}\left[Dg(U^0)\int_0^s Z(U(r))d\beta(r)\right]=0$, we hence get

$$
|A_3|\le C(\tau^2h^{-2}+\tau^2) \tag{28}
$$

similar to the proof of Lemma 3.5. We rewrite

$$
Z(U^1-U^0)\delta_1\beta=\begin{pmatrix}u_1^1-u_1^0 & & \\ & \ddots & \\ & & u_M^1-u_M^0\end{pmatrix}E_{MK}\Lambda\delta_1\beta
$$

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$$
\begin{aligned}
&= \begin{pmatrix} \sum_{k=1}^{K} e_k(x_1)\sqrt{\eta_k}\delta_1\beta_k & & \\ & \ddots & \\ & & \sum_{k=1}^{K} e_k(x_M)\sqrt{\eta_k}\delta_1\beta_k \end{pmatrix} (U^1 - U^0) \\
&=: G(U^1 - U^0),
\end{aligned}
$$

where G satisfies that $\mathbb{E}[GU^0] = 0$. Utilizing that $\mathbb{E}[GF(U^0)U^0] = 0$, we can rewrite term A_4 as

$$
\begin{aligned}
A_4 &= -\mathbb{E}\left[D\varphi(U^0)\left(\frac{\mathbf{i}}{2}\left(I - \frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} G(U^1 - U^0) + \int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau - s)}\hat{E}U(s)ds\right)\right] \\
&= -\frac{\mathbf{i}}{2}\mathbb{E}\left[D\varphi(U^0)\left(I - \frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} G\left(\mathbf{i}\frac{\tau}{h^2}AU^{\frac{1}{2}} + \mathbf{i}\lambda\tau F(U^{\frac{1}{2}})U^{\frac{1}{2}} + \mathbf{i}GU^{\frac{1}{2}}\right)\right] \\
&\quad - \mathbb{E}\left[D\varphi(U^0)\int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau - s)}\hat{E}U(s)ds\right] \\
&= \frac{\tau}{4h^2}\mathbb{E}\left[D\varphi(U^0)\left(I - \frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} GA(U^1 - U^0)\right] \\
&\quad + \frac{1}{2}\lambda\tau\mathbb{E}\left[D\varphi(U^0)\left(I - \frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} G\left(F(U^{\frac{1}{2}})U^{\frac{1}{2}} - F(U^0)U^0\right)\right] \\
&\quad + \frac{1}{4}\mathbb{E}\left[D\varphi(U^0)\left(I - \frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} G^2(U^1 - U^0)\right] \\
&\quad + \mathbb{E}\left[D\varphi(U^0)\left(\left(I - \frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\frac{1}{2}G^2U^0 - \int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau - s)}\hat{E}U(s)ds\right)\right] \\
&=: A_{4,1} + A_{4,2} + A_{4,3} + A_{4,4},
\end{aligned}
$$

in which, based on $\mathbb{E}[G^3U^0] = 0$, $A_{4,3}$ can be expressed as

$$
\frac{1}{4}\mathbb{E}\left[D\varphi(U^0)\left(I - \frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} G^2\left(\mathbf{i}\frac{\tau}{h^2}AU^{\frac{1}{2}} + \mathbf{i}\tau\lambda F(U^{\frac{1}{2}})U^{\frac{1}{2}} + \frac{\mathbf{i}}{2}G(U^1 - U^0)\right)\right].
$$

For any $U \in \mathbb{C}^M$, we have

$$
\mathbb{E}\|GU\| = \mathbb{E}\|Z(U)\delta_1\beta\| \le C\mathbb{E}\left(\|U\|^2\Big(\sum_{k=1}^{K}\sqrt{\eta_k}|\delta_1\beta_k|\Big)^2\right)^{\frac{1}{2}} \le C\tau^{\frac{1}{2}}\left(\mathbb{E}\|U\|^2\right)^{\frac{1}{2}}.
$$

Hence $\mathbb{E}\|G^3(U^1 - U^0)\| \le C\tau^{\frac{1}{2}}(\mathbb{E}\|G^2(U^1 - U^0)\|^2)^{\frac{1}{2}}$ can be further estimated based on (24) with $\gamma = 4$ under the condition $Q \in \mathcal{HS}(L^2, H^{\frac{5}{4}})$, which together with Lemma 3.4 and $\|U^{\frac{1}{2}}\| \le 1$ yields

$$
|A_{4,1} + A_{4,2} + A_{4,3}| \le C(\tau^{\frac{5}{2}}h^{-4} + \tau^2h^{-2} + \tau^2) \le C(\tau^2h^{-2} + \tau^2). \tag{29}
$$

For the term $A_{4,4}$, we have

$$
\frac{1}{2}G^2U^0 \overset{\mathbb{E}}{=} \frac{1}{2}\begin{pmatrix} \sum_{k=1}^{K} e_k^2(x_1)\eta_k(\delta_1\beta_k)^2u_1^0 \\ \vdots \\ \sum_{k=1}^{K} e_k^2(x_M)\eta_k(\delta_1\beta_k)^2u_M^0 \end{pmatrix}, \ \hat{E}U(s) = \frac{1}{2}\begin{pmatrix} \sum_{k=1}^{K} e_k^2(x_1)\eta_k u_1(s) \\ \vdots \\ \sum_{k=1}^{K} e_k^2(x_M)\eta_k u_M(s) \end{pmatrix}.
$$

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Thus, we obtain

$$
\begin{aligned}
A_{4,4} &= \frac{1}{2}\mathbb{E}\left[D\varphi(U^0)\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\begin{pmatrix}\sum_{k=1}^K e_k^2(x_1)\eta_k(\delta_1\beta_k)^2u_1^0\\ \vdots\\ \sum_{k=1}^K e_k^2(x_M)\eta_k(\delta_1\beta_k)^2u_M^0\end{pmatrix}\right]\\
&\quad -\frac{1}{2}\mathbb{E}\left[D\varphi(U^0)\int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\begin{pmatrix}\sum_{k=1}^K e_k^2(x_1)\eta_k u_1(s)\\ \vdots\\ \sum_{k=1}^K e_k^2(x_M)\eta_k u_M(s)\end{pmatrix}ds\right]\\
&= \frac{1}{2}\mathbb{E}\left[D\varphi(U^0)\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\begin{pmatrix}\sum_{k=1}^K e_k^2(x_1)\eta_k((\delta_1\beta_k)^2-\tau)u_1^0\\ \vdots\\ \sum_{k=1}^K e_k^2(x_M)\eta_k((\delta_1\beta_k)^2-\tau)u_M^0\end{pmatrix}\right]\\
&\quad +\frac{1}{2}\mathbb{E}\left[D\varphi(U^0)\int_0^\tau\left(\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}-e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\right)\begin{pmatrix}\sum_{k=1}^K e_k^2(x_1)\eta_k u_1^0\\ \vdots\\ \sum_{k=1}^K e_k^2(x_M)\eta_k u_M^0\end{pmatrix}ds\right]\\
&\quad -\frac{1}{2}\mathbb{E}\left[D\varphi(U^0)\int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\begin{pmatrix}\sum_{k=1}^K e_k^2(x_1)\eta_k\left(u_1(s)-u_1^0\right)\\ \vdots\\ \sum_{k=1}^K e_k^2(x_M)\eta_k\left(u_M(s)-u_M^0\right)\end{pmatrix}ds\right],
\end{aligned}
\tag{30}
$$

where in the last step we have used the fact that

$$
\begin{pmatrix}\sum_{k=1}^K e_k^2(x_1)\eta_k\tau u_1^0\\ \vdots\\ \sum_{k=1}^K e_k^2(x_M)\eta_k\tau u_M^0\end{pmatrix}=\int_0^\tau\begin{pmatrix}\sum_{k=1}^K e_k^2(x_1)\eta_k u_1^0\\ \vdots\\ \sum_{k=1}^K e_k^2(x_M)\eta_k u_M^0\end{pmatrix}ds.
$$

Noticing that the first term in (30) vanishes as $\mathbb{E}(\delta_1\beta_k)^2=\tau$ and replacing $U(s)-U^0$ by the integral type of (2), then further calculation shows that

$$
|A_{4,4}|\le C(\tau^2h^{-2}+\tau^2) \tag{31}
$$

based on (25) and the technique used in (28). We then conclude from (26)–(31) that

$$
|\mathcal{A}|\le C\big(\tau^2h^{-2}+\tau^2\big)\le C_h\tau^2. \tag{32}
$$

Estimation of $\mathcal{C}$. Estimations of A_1 and A_2 show that

$$
\mathbb{E}\|\mathbf{a}+\mathbf{b}\|^2\le C\big(\tau^6h^{-12}+\tau^4h^{-4}\big)\le C\tau^3. \tag{33}
$$

Based on Hölder's inequality, Itô isometry, and Lemmas 3.4 and 3.5, we have

$$
\mathbb{E}\|\mathbf{c}+\mathbf{d}\|^2\le C\tau\int_0^\tau\mathbb{E}\|U(s)-U^{\frac{1}{2}}\|^2ds+\int_0^\tau C\tau^2h^{-4}ds\le C(\tau^3h^{-4}+\tau^3) \tag{34}
$$

and

$$
\mathbb{E}\|\mathbf{e}\|^2\le C\mathbb{E}\left[\int_0^\tau\left\|\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}Z\left(U(s)-U^0\right)\right\|_{\mathcal{HS}}^2ds\right]\le C\tau^2. \tag{35}
$$

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Rewriting $Z(U^1-U^0)\delta_1\beta = G\left(\mathbf{i}\frac{\tau}{h^2}AU^{\frac{1}{2}} + \mathbf{i}\lambda\tau F(U^{\frac{1}{2}})U^{\frac{1}{2}} + \mathbf{i}GU^{\frac{1}{2}}\right)$ together with the Hölder's inequality and (24) yields

$$\mathbb{E}\|\mathbf{f}\|^2 \le C(\tau^3 h^{-4} + \tau^2). \tag{36}$$

We then conclude from (33)–(36) and the condition $\tau = O(h^4)$ that

$$\mathbb{E}\|U(\tau) - U^1\|^2 \le C\tau^2, \tag{37}$$

which yields

$$|\mathcal{C}| = O\left(\left(\mathbb{E}\|U^1-U^0\|^4\right)^{\frac{1}{2}}\left(\mathbb{E}\|U(\tau)-U^1\|^2\right)^{\frac{1}{2}} + \mathbb{E}\|U(\tau)-U^1\|^2\right) \le C\tau^2. \tag{38}$$

Estimation of $\mathcal{B}$. As for $\mathcal{B} = \mathbb{E}\left[D^2\varphi(U^0)\left(U^1-U^0, \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}+\mathbf{e}+\mathbf{f}\right)\right]$, according to Hölder's inequality, (33), and (34), we have

$$\begin{aligned}&\left|\mathbb{E}\left[D^2\varphi(U^0)\left(U^1-U^0, \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}\right)\right]\right| \\ &\le C\left(\mathbb{E}\|U^1-U^0\|^2\right)^{\frac{1}{2}}\left(\mathbb{E}\|\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}\|^2\right)^{\frac{1}{2}} \le C(\tau^2h^{-2}+\tau^2).\end{aligned}$$

Noticing that

$$\begin{aligned}&\mathbb{E}\left[D^2\varphi(U^0)\left(U^1-U^0, \mathbf{e}+\mathbf{f}\right)\right] \\ &= \mathbb{E}\left[D^2\varphi(U^0)\left(U^1-U^0, \mathbf{i}\int_0^\tau\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}Z(U(s)-U^1)d\beta(s)\right)\right] \\ &\quad + \frac{1}{2}\mathbb{E}\left[D^2\varphi(U^0)\left(U^1-U^0, \mathbf{i}\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}Z(U^1-U^0)\delta_1\beta\right)\right] \\ &\quad - \mathbb{E}\left[D^2\varphi(U^0)\left(U^1-U^0, \int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\hat{E}U(s)ds\right)\right] \\ &=: B_1 + B_2 + B_3,\end{aligned}$$

where $|B_1| \le C\tau^2$ according to (37) and Lemma 3.4. Furthermore,

$$\begin{aligned}B_2 &= \frac{1}{2}\mathbb{E}\left[D^2\varphi(U^0)\left(\mathbf{i}\frac{\tau}{h^2}AU^{\frac{1}{2}} + \mathbf{i}\tau\lambda F(U^{\frac{1}{2}})U^{\frac{1}{2}}, \mathbf{i}\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}Z(U^1-U^0)\delta_1\beta\right)\right] \\ &\quad + \frac{1}{2}\mathbb{E}\left[D^2\varphi(U^0)\left(\mathbf{i}Z\left(\frac{U^1-U^0}{2}\right)\delta_1\beta, \mathbf{i}\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}Z(U^1-U^0)\delta_1\beta\right)\right] \\ &\quad + \frac{1}{2}\mathbb{E}\left[D^2\varphi(U^0)\left(\mathbf{i}Z(U^0)\delta_1\beta, \mathbf{i}\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}Z(U^1-U^0)\delta_1\beta\right)\right] \\ &=: B_{2,1} + B_{2,2} + B_{2,3}\end{aligned}$$

with $|B_{2,1} + B_{2,2}| \le C(\tau^2h^{-2}+\tau^2)$. Replacing U^1-U^0 again by (7), we obtain

$$|B_{2,3}| \le \left|\frac{1}{2}\mathbb{E}\left[D^2\varphi(U^0)\left(\mathbf{i}Z(U^0)\delta_1\beta, \mathbf{i}\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}Z\left(\mathbf{i}Z(U^{\frac{1}{2}})\delta_1\beta\right)\delta_1\beta\right)\right]\right|$$

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$$
\begin{aligned}
&+ C(\tau^2 h^{-2} + \tau^2) \\
&\le \left| \frac{1}{2}\mathbb{E}\left[D^2\varphi(U^0)\left(\mathbf{i}Z(U^0)\delta_1\beta, \mathbf{i}\left(I - \frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} Z\left(\mathbf{i}Z(U^0)\delta_1\beta\right)\delta_1\beta \right)\right]\right| \\
&+ C(\tau^2 h^{-2} + \tau^2) \\
&= C(\tau^2 h^{-2} + \tau^2),
\end{aligned}
$$

where in the last step we used the fact $\mathbb{E}[(\delta_1\beta)^3] = 0$ and U^0 is $\mathcal{F}_0$-adapted. Also,

$$
\begin{aligned}
|B_3| &\le \left| \mathbb{E}\left[D^2\varphi(U^0)\left(\mathbf{i}\frac{\tau}{h^2}AU^{\frac{1}{2}} + \mathbf{i}\tau\lambda F(U^{\frac{1}{2}})U^{\frac{1}{2}}, \int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\hat{E}U(s)ds \right)\right]\right| \\
&\quad + \left| \mathbb{E}\left[D^2\varphi(U^0)\left(\mathbf{i}Z(U^{\frac{1}{2}})\delta_1\beta, \int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\hat{E}\left(U(s) - U^0\right)ds \right)\right]\right| \\
&\quad + \left| \mathbb{E}\left[D^2\varphi(U^0)\left(\mathbf{i}Z(U^{\frac{1}{2}})\delta_1\beta, \int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\hat{E}U^0 ds \right)\right]\right| \\
&\le C\left(\tau^2 h^{-2} + \tau^2\right) + \frac{1}{2}\left| \mathbb{E}\left[D^2\varphi(U^0)\left(\mathbf{i}Z(U^1 - U^0)\delta_1\beta, \int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\hat{E}U^0 ds \right)\right]\right| \\
&\le C(\tau^2 h^{-2} + \tau^2),
\end{aligned}
$$

so we finally obtain

$$
|\mathcal{B}| \le C(\tau^2 h^{-2} + \tau^2) \le C_h \tau^2,
$$

which, together with (32) and (38), completes the proof.

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