A new look at nonnegativity and polynomial optimization

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Obtaining *tractable characterizations* of functions which are nonnegative on a set $\mathbf{K} \subset \mathbb{R}^n$ is a topic of primary importance. Indeed, such characterizations are highly desirable to help solve (or at least approximate) many important problem in various areas, and in particular, the global optimization problem:

$$\mathbf{P}: \quad f^* = \min_{\mathbf{x}} \{ f(\mathbf{x}) : \mathbf{x} \in \mathbf{K} \},$$

because solving $\mathbf{P}$ is equivalent to solving $f^* = \max\{\lambda : f - f^* \geq 0 \text{ on } \mathbf{K}\}$. When f is a polynomial and $\mathbf{K}$ a basic semi-algebraic set, we have seen in the previous talk that Putinar's Positivstellensatz provides such tractable characterizations. Those characterizations depend on the representation of $\mathbf{K}$ through its defining polynomials.

In this talk we consider another way to look at continuous functions that are non-negative on a (non necessarily compact basic semi-algebraic) set $\mathbf{K} \subseteq \mathbb{R}^n$. This time, knowledge on $\mathbf{K}$ is through a finite Borel measure μ with support $\sup \mu = \mathbf{K}$, and whose all moments $\mathbf{y} = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, are available. This new characterization permits to define convergent *outer* approximations of the convex cone $C_d(\mathbf{K})$ of polynomials of degree at most d, nonnegative on $\mathbf{K}$, by a hierarchy of spectrahedra (convex sets defined by linear matrix inequalities) defined uniquely in terms of the coefficients of f. Important examples of cones $C_d(\mathbf{K})$ are the cone of nonnegative polynomials on $\mathbb{R}^n$ and the cone of copositve matrices. Checking whether a fixed and known polynomial f is nonnegative on $\mathbf{K}$ reduces to solving a sequence of *generalized eigenvalue* problems for real symmetric matrices of increasing size.