A Continuation Method for a Class of periodic Evolution Variational Inequalities

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Outline of the talk

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1. Position of the Problem
2. Some background
   - Continuation methods for ODE's
   - The degree theory
3. Existence results for VI
4. Existence of periodic solutions
   - Showalter's Theorem and the Poincaré operator
   - Existence and unicity of periodic solutions
5. Second order periodic dynamical system with friction
6. Some Numerical experiments
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Position of the problem

Find a T-periodic function $u \in C^0([0,T];\mathbb{R}^n)$ such that :

- $\dfrac{du}{dt} \in L^\infty(0,T;\mathbb{R}^n)$;
- u is right-differentiable on $[0,T)$, $\quad u(0) = u(T)$;
- $\langle \dfrac{du}{dt}(t) + F(u(t)) - f(t), v - u(t) \rangle + \varphi(v) - \varphi(u(t)) \geq 0,$

$$\forall v \in \mathbb{R}^n, \ \text{a.e. } t \in [0,T].$$

Position of the problem

where,

- $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map ;
- $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a convex function ;
- $f \in C^0([0, +\infty[; \mathbb{R}^n)$ is such that : $\dfrac{df}{dt} \in L^1_{loc}(0, +\infty; \mathbb{R}^n)$;
- $T > 0$ is a prescribed period.

Remark

The variational inequality

$$\langle \frac{du}{dt}(t) + F(u(t)) - f(t), v - u(t)\rangle + \varphi(v) - \varphi(u(t)) \geq 0,$$

$$\forall v \in \mathbb{R}^n,\ a.e.\ t \in [0, T]$$

is equivalent to the differential inclusion

$$\frac{du}{dt}(t) + F(u(t)) - f(t) \in -\partial\varphi(u(t)),\ a.e.\ t \in [0, T].$$

Outline

Continuation methods for ODE

Continuation methods of the Leray-Schauder type play an important role in the Theory of Differential Equations.
M.A. Krasnosel'skii and H. Amann developed a continuation method to compute the Brouwer topological degree associated to some gradient mapping (method of "guiding functions"). This approach was useful for the study of the existence of periodic solutions for ODE's.

M.A. Krasonel'skii, The Operator of Translation along the Trajectories of Differential Equations, Nauka, Moscow (1966) (Russian) ; english translation : American Math. Soc., Translations of math. Monographs, vol. 19, Providence, 1968.

Roughly speaking, if on some ball of $\mathbb{R}^n$ the Brouwer topological degree of the Poincaré translation operator associated to the ODE is different from zero, the problem has at least one periodic solution.

Krasnol'skii's original approach for ODE's has known some extensions in order to obtain continuation methods for differential inclusions (see for instance, L. Gorniewicz and the reference therein).

L. Gorniewicz, Topological Approach to Differential Inclusions, in Topological Methods in Differential Equations and Inclusions (ed. A. Granas and M. Frigon), NATO ASI Series, Mathematical and Physical Sciences, Vol. 472 (1994), Kluwer Academic Publishers.

See also an expository paper by J. Mawhin.

Outline

The degree Theory

In the sequel, the scalar product on $\mathbb{R}^n$ is denoted as usual by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the associated norm. For $r > 0$, we note

$$\mathbb{B}_r := \{x \in \mathbb{R}^n : \|x\| < r\}$$

$$\bar{\mathbb{B}}_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$$

$$\partial \mathbb{B}_r := \bar{\mathbb{B}}_r \setminus \overset{o}{\mathbb{B}}_r = \{x \in \mathbb{R}^n : \|x\| = r\}.$$

Motivation

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded subset with boundary $\partial\Omega$ and $f \in C^1(\Omega; \mathbb{R}^n) \cap C^0(\overline{\Omega}, \mathbb{R}^n)$.

We note $f'(x) = (\partial_{x_i} f_j(x))_{1 \leq i,j \leq n}$ the Jacobian matrix of f at $x \in \Omega$

$J_f(x) = \det(f'(x))$, the Jacobian determinant of f at $x \in \Omega$.

We set,

$$A_f(\Omega) = \{x \in \Omega \ : \ J_f(x) = 0\}.$$

If $f^{-1}(0) \cap A_f(\Omega) = \emptyset$ and $0 \notin f(\partial\Omega)$, then $f^{-1}(0)$ is a finite set.

The Brouwer topological degree of f with respect to Ω and 0 is well defined by the following formula

$$\deg(f, \Omega, 0) = \sum_{x \in f^{-1}(0)} \operatorname{sign}\left(J_f(x)\right).$$

More generally, if

$$f \in C^0(\overline{\Omega}; \mathbb{R}^n) \text{ and } 0 \notin f(\partial\Omega),$$

then the Brouwer topological degree of f with respect to Ω and 0, denoted by $\deg(f, \Omega, 0)$, is well defined (see Lloyd for more details).

Let us now recall some properties of the topological degree that we will use later.

- If $0 \notin f(\partial \mathbb{B}_r)$ and $\deg(f, \mathbb{B}_r, 0) \neq 0$, then there exist $x \in \mathbb{B}_r$ such that $f(x) = 0$.
- Let $\varphi : [0,1] \times \bar{\mathbb{B}}_r \to \mathbb{R}^n; (\lambda, x) \to \varphi(\lambda, x)$, be continuous such that, for each $\lambda \in [0,1]$, one has $0 \notin \varphi(\lambda, \partial \mathbb{B}_r)$, then the map $\lambda \to \deg(\varphi(\lambda, .), \mathbb{B}_r, 0)$ is constant on $[0,1]$.
- Let us denote by $id_{\mathbb{R}^n}$ the identity mapping on $\mathbb{R}^n$. We have

$$\deg(id_{\mathbb{R}^n}, \mathbb{B}_r, 0) = 1.$$

- If $0 \notin f(\partial \mathbb{B}_r)$ and $\alpha > 0$, then

$$\deg(\alpha f, \mathbb{B}_r, 0) = \deg(f, \mathbb{B}_r, 0)$$

and

$$\deg(-\alpha f, \mathbb{B}_r, 0) = (-1)^n \deg(f, \mathbb{B}_r, 0).$$

- If $0 \notin f(\partial \mathbb{B}_r)$ and f is odd on $\mathbb{B}_r$, then $\deg(f, \mathbb{B}_r, 0)$ is odd.
- Let $f(x) = Ax - b$, with $A \in \mathbb{R}^{n \times n}$ a nonsingular matrix and $b \in \mathbb{R}^n$. Then

$$\deg(f, A^{-1}b, \mathbb{B}_r, 0) = \operatorname{sign}(\det A) = \pm 1.$$

Let $V \in C^1(\mathbb{R}^n;\mathbb{R})$ and suppose that there exists $r_0 > 0$ such that for every $r \geq r_0$, $0 \notin \nabla V(\partial \mathbb{B}_r)$.

Then $\deg(\nabla V, \mathbb{B}_r, 0)$ is constant for $r \geq r_0$ and one defines the index of V at infinity "$\mathrm{ind}(V,\infty)$" by

$$\mathrm{ind}(V,\infty) := \deg(\nabla V, \mathbb{B}_r, 0), \quad \forall r \geq r_0.$$

Existence results for VI

We first study the following variational inequality :

$$\mathrm{VI}(\Lambda, \varphi) \begin{cases} \text{Find } \overline{x} \in \mathbb{R}^n \text{ such that} \\ \langle \Lambda(\overline{x}), v - \overline{x} \rangle + \varphi(v) - \varphi(\overline{x}) \geq 0, \quad \forall v \in \mathbb{R}^n, \end{cases}$$

where $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map, $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $\langle \cdot, \cdot \rangle$ denotes the euclidean scalar product in $\mathbb{R}^n$. Using the Brouwer topological degree, we prove in an original way some results related to the existence of a solution to problem $\mathrm{VI}(\Lambda, \varphi)$.

Let denote by $P_{\lambda,\varphi}(y)$, the resolvent operator

$$P_{\lambda,\varphi} : \mathbb{R}^n \to \mathbb{R}^n; y \to P_{\lambda,\varphi}(y) = (I + \lambda\partial\varphi)^{-1}(y).$$

which is a contraction on $\mathbb{R}^n$, i.e.,

$$\|P_{\lambda,\varphi}(x) - P_{\lambda,\varphi}(y)\| \leq \|x - y\|, \quad \forall x,\ y \in \mathbb{R}^n$$

and therefore a continuous operator on $\mathbb{R}^n$. It is well known that

$$\lim_{\lambda \to 0^+} P_{\lambda,\varphi}(x) = \mathrm{Proj}_{\overline{D(\partial\varphi)}}(x).$$

For simplicity, we note P_φ instead of $P_{1,\varphi}$.

Let $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping and consider the inequality problem : Find $\bar{x} \in \mathbb{R}^n$ such that

$$\langle \Lambda(\bar{x}), v - \bar{x} \rangle + \varphi(v) - \varphi(\bar{x}) \geq 0, \quad \forall v \in \mathbb{R}^n. \tag{1}$$

Clearly the variational inequality (1) is equivalent to the nonlinear equation : Find $\bar{x} \in \mathbb{R}^n$ such that

$$\bar{x} - P_\varphi(\bar{x} - \Lambda(\bar{x})) = 0. \tag{2}$$

Importance of computing the degree of the operator $id_{\mathbb{R}^n} - P_\varphi \circ (id_{\mathbb{R}^n} - \Lambda)$**.**

Example

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\varphi(x) = |x|, \quad \forall x \in \mathbb{R}.$$

We have

$$\partial\varphi(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ [-1, 1] & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

Example

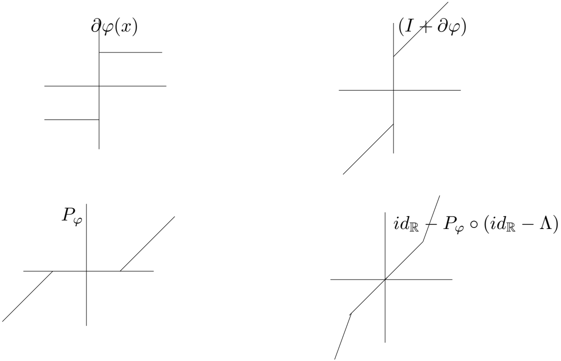
and

$$P_\varphi(x) = (I + \partial\varphi)^{-1}(x) = \begin{cases} x - 1 & \text{if} \quad x \geq 1 \\ 0 & \text{if} \quad x \in [-1, 1] \\ x + 1 & \text{if} \quad x \leq -1 \end{cases}$$

Setting $\Lambda(x) = 2x$, we get

$$x - P_\varphi(x - \Lambda(x)) = \begin{cases} x & \text{if} \quad |x| \leq 1 \\ 2x - 1 & \text{if} \quad x \geq 1 \\ 2x + 1 & \text{if} \quad x \leq -1 \end{cases}$$

We see that the operator $id_{\mathbb{R}} - P_{\varphi} \circ (id_{\mathbb{R}} - \Lambda)$ has a unique zero



on $\mathbb{R}$.

Proposition (♋)

Suppose that $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a convex function. If there exists a continuous mapping $H : \mathbb{R}^n \to \mathbb{R}^n$ and $r > 0$ such that

$$\langle \Lambda(x), H(x) \rangle + \varphi'(x; H(x)) < 0, \quad \forall x \in \partial \mathbb{B}_r. \tag{3}$$

Then

$$\deg(H, \mathbb{B}_r, 0) = (-1)^n \deg(id_{\mathbb{R}^n} - P_\varphi(id_{\mathbb{R}^n} - \Lambda), \mathbb{B}_r, 0).$$

Theorem (✪)

Theorem

Suppose that

1) $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous operator ;*

2) $\varphi : \mathbb{R}^n \to \mathbb{R}$ *is a convex function ;*

3) there exists $r > 0$ *such that*

$$\langle \Lambda(x), x \rangle - \varphi'(x; -x) > 0, \quad \forall x \in \partial \mathbb{B}_r.$$

Then there exists $\bar{x} \in \mathbb{B}_r$ *such that*

$$\langle \Lambda(\bar{x}), v - \bar{x} \rangle + \varphi(v) - \varphi(\bar{x}) \geq 0, \quad \forall v \in \mathbb{R}^n.$$

Proof. The result follows from the last Proposition (♋) with $H := -id_{\mathbb{R}^n}$. Indeed, here we have

$$\langle \Lambda(x), H(x) \rangle + \varphi'(x; H(x)) = -\langle \Lambda(x), x \rangle + \varphi'(x; -x).$$

Theorem (☼)

Theorem

Suppose that

1) $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ *is continuous ;*

2) $\varphi : \mathbb{R}^n \to \mathbb{R}$ *is convex and Lipschitz continuous with Lipschitz constant* $K > 0$
3) there exists $r > 0$ *such that*

$$\|\Lambda(x)\| > K, \quad \forall x \in \partial\mathbb{B}_r, \;\; \deg(\Lambda, \mathbb{B}_r, 0) \neq 0.$$

Then there exists $\bar{x} \in \mathbb{B}_r$ *such that*

$$\langle \Lambda(\bar{x}), v - \bar{x} \rangle + \varphi(v) - \varphi(\bar{x}) \geq 0, \quad \forall v \in \mathbb{R}^n.$$

Proof. Take in Proposition (♋) $H := -\Lambda$.

$$\begin{array}{rcl} \langle \Lambda(x), H(x)\rangle + \varphi'(x; H(x)) & = & -\|\Lambda(x)\|^2 + \varphi'(x; -\Lambda(x)) \\ & \leq & -\|\Lambda(x)\|^2 + K\|\Lambda(x)\| \\ & = & \|\Lambda(x)\|(K - \|\Lambda(x)\|). \end{array}$$

Therefore,

$$\langle \Lambda(x), H(x)\rangle + \varphi'(x; H(x)) < 0, \quad \forall x \in \partial\mathbb{B}_r.$$

Proposition (♋) ensures that

$$\deg(id_{\mathbb{R}^n} - P_\varphi(id_{\mathbb{R}^n} - \Lambda), \mathbb{B}_r, 0) = (-1)^n \deg(H, \mathbb{B}_r, 0)$$

$$= \deg(\Lambda, \mathbb{B}_r, 0) \neq 0.$$

Hence, there exists $\bar{x} \in \mathbb{B}_r$ such that $\bar{x} = P_\varphi(\bar{x} - \Lambda(\bar{x}))$.

Theorem (☺)

Suppose that

1. $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and that there exists $r > 0$ such that
$$\langle \Lambda x, x \rangle > 0, \quad \forall x \in \partial \mathbb{B}_r \text{ and } \deg(\mathrm{id}_{\mathbb{R}^n} + \Lambda, \mathbb{B}_r, 0) \neq 0.$$

2. $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a convex function satisfying,
$$\varphi'(x; -x - \Lambda x) \leq 0, \quad \forall x \in \partial \mathbb{B}_r;$$

Then there exists $\bar{x} \in \mathbb{B}_r$ such that

$$\langle \Lambda(\bar{x}), v - \bar{x} \rangle + \varphi(v) - \varphi(\bar{x}) \geq 0, \quad \forall v \in \mathbb{R}^n.$$

Proof. Take $H := -\mathrm{id}_{\mathbb{R}^n} - \Lambda$. Indeed, we have

$$\langle \Lambda(x), H(x)\rangle + \varphi'(x; H(x)) = -\|\Lambda(x)\|^2 - \langle \Lambda(x), x\rangle +$$

$$\varphi'(x; -x - \Lambda(x)) < 0, \quad \forall x \in \partial\mathbb{B}_r.$$

By Proposition (♋), we have

$$\deg(id_{\mathbb{R}^n} - P_\varphi(id_{\mathbb{R}^n} - \Lambda), \mathbb{B}_r, 0) = (-1)^n \deg(H, \mathbb{B}_r, 0) =$$

$$\deg(id_{\mathbb{R}^n} + \Lambda, \mathbb{B}_r, 0) \neq 0.$$

Hence, there exists $\bar{x} \in \mathbb{B}_r$ such that $\bar{x} = P_\varphi(\bar{x} - \Lambda(\bar{x}))$.

Corollary

Let $f \in \mathbb{R}^n$ be given. Suppose that

1) $A \in \mathbb{R}^{n \times n}$ is a real nonsingular matrix ;

2) $\varphi : \mathbb{R}^n \to \mathbb{R}$ is convex and Lipschitz continuous with Lipschitz constant $K > 0$.

Then there exists $\bar{x} \in \mathbb{R}^n$ such that

$$\langle A\bar{x} - f, v - \bar{x} \rangle + \varphi(v) - \varphi(\bar{x}) \geq 0, \quad \forall v \in \mathbb{R}^n.$$

Outline

Existence and uniqueness of periodic solutions

Let us recall the following theorem (Showalter)

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous such that for some $\omega \in \mathbb{R}$, $F + \omega I$ is monotone. Let $u_0 \in \mathbb{R}^n$ be given. There exists a unique $u \in C^0([0, T]; \mathbb{R}^n)$ such that

$$\frac{du}{dt} \in L^\infty(0, T; \mathbb{R}^n);$$

$$u \text{ is right-differentiable on } [0, T), \quad u(0) = u_0;$$

$$\left\langle \frac{du}{dt}(t) + F(u(t)) - f(t), v - u(t) \right\rangle + \varphi(v) - \varphi(u(t)) \geq 0,$$

$$\forall v \in \mathbb{R}^n, \text{ a.e. } t \in [0, T].$$

Let $T > 0$ be given. The preceding Theorem enables us to define the one parameter family $\{S(t) : 0 \leq t \leq T\}$ of operators from $\mathbb{R}^n$ into $\mathbb{R}^n$, as follows :

$$\forall y \in \mathbb{R}^n, \quad S(t)y = u(t), \tag{4}$$

u being the unique solution on $[0, T]$ of the evolution variational inequality. Note that

$$\forall y \in \mathbb{R}^n, \quad S(0)y = y.$$

Theorem

Suppose that the assumptions of the preceding Theorem hold. Then

$$\|S(t)y - S(t)z\| \leq e^{\omega t}\|y - z\|, \forall y, z \in \mathbb{R}^n, t \in [0, T].$$

Remark

i) Note that if F is continuous and monotone, then the preceding Theorem holds with $\omega = 0$. In this case, the Poincaré operator $S(T)$ is nonexpansive, i.e.,

$$\|S(T)y - S(T)z\| \leq \|y - z\|, \quad \forall y, z \in \mathbb{R}^n.$$

ii) If F is continuous and strongly monotone, i.e., there exists $\alpha > 0$ such that

$$\langle F(x) - F(y), x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in \mathbb{R}^n,$$

then the preceding Theorem holds with $\omega = -\alpha < 0$ and the Poincaré operator $S(T)$ is a contraction.

Remark

According to (4), the unique solution of the evolution VI satisfies, in addition, the periodicity condition

$$u(0) = u(T)$$

if and only if y is a **fixed point** *of $S(T)$, that is*

$$S(T)y = y.$$

Thus the problem of the **existence of a periodic solution** *for the evolution periodic VI* **reduces to the existence of a fixed point for $S(T)$.**

Outline

Theorem

Suppose $\exists C_1 \geq 0, C_2 \geq 0$ *such that*

$$\langle F(x), x\rangle + \varphi'(x; x) \leq C_1\|x\|^2 + C_2\|x\|, \quad \forall x \in \mathbb{R}^n.$$

Let $T > 0$ *be given. Assume* $\exists V \in C^1(\mathbb{R}^n; \mathbb{R})$ *and* $R > 0$ *such that*

$$\langle F(x) - f(t), \nabla V(x)\rangle + \varphi'(x; \nabla V(x)) < 0,$$

$$\forall x \in \mathbb{R}^n, \|x\| \geq R, t \in [0, T].$$

and

$$\mathrm{ind}(V, \infty) \neq 0.$$

Theorem

Then, there exists at least one $u \in C^0([0,T];\mathbb{R}^n)$ *such that* $\dfrac{du}{dt} \in L^\infty(0,T;\mathbb{R}^n)$,

$$u(0) = u(T);$$

$$\left\langle \frac{du}{dt}(t) + F(u(t)) - f(t), v - u(t) \right\rangle + \varphi(v) - \varphi(u(t)) \geq 0,$$

$$\forall v \in \mathbb{R}^n, \; a.e. \; t \in [0,T].$$

Corollary

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex and Lipschitz continuous function and let $T > 0$ be given. Let $A \in \mathbb{R}^{n\times n}$ be a real matrix and denote by $\sigma(A)$ the set of eigenvalues of A. If

$$\mathrm{Re}(\lambda) > 0, \quad \forall \lambda \in \sigma(A)$$

then there exists at least one $u \in C^0([0,T];\mathbb{R}^n)$ such that $\dfrac{du}{dt} \in L^\infty(0,T;\mathbb{R}^n)$,

$$u(0) = u(T);$$

$$\left\langle \frac{du}{dt}(t) + Au(t) - f(t), v - u(t) \right\rangle + \varphi(v) - \varphi(u(t)) \geq 0,$$

$$\forall v \in \mathbb{R}^n, \ a.e.\ t \in [0,T].$$

Examples

i) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$ defined by

$$F(x) = x \text{ and } \varphi(x) = \|x\|, x \in \mathbb{R}^n.$$

We have

$$\langle F(x), x\rangle + \varphi'(x; x) \leq \|x\|^2 + \|x\|.$$

Hence $C_1 = C_2 = 1$.

ii) If we take $F(x) = -x$ and $\varphi(x) = \|x\|$, then

$$\langle F(x), x\rangle + \varphi'(x; x) \leq -\|x\|^2 + \|x\| \leq \|x\|$$

Hence $C_1 = 0$ and $C_2 = 1$.

Second order periodic dynamical system with friction

For $(q_0, \dot{q}_0) \in \mathbb{R}^m \times \mathbb{R}^m$, we consider the problem $P(q_0, \dot{q}_0)$ of finding a function $t \mapsto q(t)$ ($t \in [0, T]$) with $q \in C^1([0, T]; \mathbb{R}^m)$, such that :

$$\frac{d^2q}{dt^2} \in L^\infty(0, T; \mathbb{R}^m),$$

$$\frac{dq}{dt} \text{ is right-differentiable on } [0, T],$$

$$q(0) = q(T) \text{ and } \dot{q}(0) = \dot{q}(T),$$

$$M\frac{d^2q}{dt^2}(t) + C\frac{dq}{dt}(t) + K(q(t)) - F(t) \in -H_1\partial\Phi(H_1^T\frac{dq}{dt}(t)),$$

$$\text{a.e. } t \in [0, T].$$

- $\Phi : \mathbb{R}^l \to \mathbb{R}$ is a convex function,
- $M \in \mathbb{R}^{m\times m}$ is a symmetric and positive definite matrix,
- $C \in \mathbb{R}^{m\times m}$ and $K \in \mathbb{R}^{m\times m}$ are given matrices
- $H_1 \in \mathbb{R}^{m\times l}$ is a given matrix whose coefficients are related to the directions of friction forces,
- $F \in C^0([0,+\infty);\mathbb{R}^m)$ is such that $\dfrac{dF}{dt} \in L^1_{loc}(([0,+\infty);\mathbb{R}^m)$.

Many problems in unilateral mechanics involves a second order dynamical system.

For instance, the motion of various mechanical systems with frictional contact can be studied within the framework of this equation.

For such problems m is the number of degrees of freedom, M is the mass matrix, C is the viscous damping matrix and K is the stiffness matrix. The vector $q \in \mathbb{R}^m$ is the vector of generalised coordinates. The term $H_1 \partial \Phi(H_1^T \cdot)$ is used to modelize the unilaterality of the contact induced by the friction forces.

This problem is equivalent to the first order variational inclusion :

$$\begin{cases} \dot{x}(t) + A(x(t)) - f(t) \in -\partial\varphi(x(t)) \\ x(0) = x(T) \end{cases}$$

where the vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^n$ $(n = 2m)$ and the matrix $A \in \mathbb{R}^{n\times n}$ is defined by

$$A = \begin{pmatrix} 0_{m\times m} & -I_{m\times m} \\ M^{-\frac{1}{2}}KM^{-\frac{1}{2}} & M^{-\frac{1}{2}}CM^{-\frac{1}{2}} \end{pmatrix},$$

with

$$x(0) = \begin{pmatrix} M^{-\frac{1}{2}} q(0) \\ M^{-\frac{1}{2}} \dot{q}(0) \end{pmatrix}, \quad x(T) = \begin{pmatrix} M^{-\frac{1}{2}} q(T) \\ M^{-\frac{1}{2}} \dot{q}(T) \end{pmatrix},$$

$$f(t) = \begin{pmatrix} 0_{m\times m} \\ F(t) \end{pmatrix}$$

and the convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\varphi(x) = (\Phi \circ H_1^T M^{-\frac{1}{2}})(x_2).$$

In this case, let us observe that the subdifferential of φ is given by :

$$\partial\varphi(x) = \begin{pmatrix} 0_{m\times 1} \\ M^{-\frac{1}{2}} H_1 \partial\Phi(H_1^T M^{-\frac{1}{2}} x_2) \end{pmatrix}$$

Theorem

If the function Φ is convex and Lipschitz continuous and $Re(\sigma(A)) \subset]0, +\infty[$, then there exists at least one $q \in C^1(0, T; \mathbb{R}^n)$ such that $\frac{d^2q}{dt^2} \in L^\infty(0, T; \mathbb{R}^n)$ satisfying

$$\frac{dq}{dt} \text{ is right-differentiable on } [0, T],$$

$$q(0) = q(T) \text{ and } \dot{q}(0) = \dot{q}(T),$$

$$M\frac{d^2q}{dt^2}(t) + C\frac{dq}{dt}(t) + K(q(t)) - F(t) \in -H_1\partial\Phi(H_1^T\frac{dq}{dt}(t)),$$

$$\text{a.e. } t \in [0, T].$$

Remark

We note that the conclusions of the preceding Theorem hold under the key assumption Φ *is convex and Lipschitz continuous.* **If Φ is convex and not Lipschitz continuous** *: Take for example* $m = 1$, $\Phi : \mathbb{R} \to \mathbb{R},\ x \mapsto \Phi(x) = x^2$, $M = C = K = H_1 = 1$ *and* $F(t) = -t, \quad \forall t \in [0, 1]$. *The differential inclusion then reduces to*

$$\begin{cases} \ddot{q}(t) + 2\dot{q}(t) + q(t) = t. \\ q(0) = q(1). \\ \dot{q}(0) = \dot{q}(1). \end{cases} \tag{5}$$

It can be checked that Problem (5) *has no solutions.*

Example

Let us take $m = 1$, $\Phi : \mathbb{R} \to \mathbb{R};\ x \mapsto \Phi(x) = |x|$. In this case, we have

$$\partial\Phi(x) = \mathrm{Sign}(x)$$

where

$$\mathrm{Sign}(x) := \begin{cases} -1 & \text{if} \quad x < 0, \\ [-1, +1] & \text{if} \quad x = 0, \\ +1 & \text{if} \quad x > 0. \end{cases}$$

Example

We consider the following problem

$$\begin{cases} m\ddot{q}(t) + c\dot{q}(t) + kq(t) - F(t) \in -\mathrm{Sign}(\dot{q}(t)), \; t \in [0, T], \\ q(0) = q(T), \\ \dot{q}(0) = \dot{q}(T), \end{cases} \tag{6}$$

with $F \in C^0([0, +\infty[; \mathbb{R})$ such that $\dfrac{dF}{dt} \in L^1_{\mathrm{loc}}([0, +\infty[; \mathbb{R})$. The matrix A previously defined is given by

$$A = \begin{pmatrix} 0 & -1 \\ \dfrac{k}{m} & \dfrac{c}{m} \end{pmatrix}$$

Example

We suppose that $m,\ c,\ k > 0$ and we set $\Delta = \dfrac{c^2}{m^2} - 4\dfrac{k}{m}$ and we have

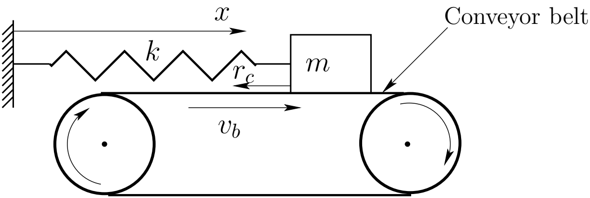
$$\sigma(A) = \begin{cases} \left\{ \dfrac{c}{2m} - \dfrac{\sqrt{\Delta}}{2}, \dfrac{c}{2m} + \dfrac{\sqrt{\Delta}}{2} \right\} & \text{if} \quad c \geq 2k\sqrt{m} \\ \left\{ \dfrac{c}{2m} - i\dfrac{\sqrt{-\Delta}}{2}, \dfrac{c}{2m} + i\dfrac{\sqrt{-\Delta}}{2} \right\} & \text{if} \quad c < 2k\sqrt{m} \end{cases}$$

We note that in both cases $\mathrm{Re}(\sigma(A)) \subset]0, +\infty[$ and hence our theorem applies and Problem (6) has at least a solution.

Some Numerical experiments

We can simulate the bowing of a violin string by the mass-spring system as follow

Let us consider a mass m attached to inertial space by a spring $k > 0$ where $m = 1$ Kg and $k = 1$ N/m. The mass is riding on a driving belt, that is moving at a constant velocity $v_b = 10$ m/s



We suppose that between the mass and the belt, a dry friction of Coulomb type occurs. The state equation is then of the form

$$m\ddot{x}(t) + kx(t) \in -\partial\Phi\,(\dot{x} - v_b), \tag{7}$$

where $\Phi : \mathbb{R} \to \mathbb{R},\ x \mapsto \gamma|x|$ and $\gamma > 0$ is the coefficient of friction.

Let q denotes the relative displacement of the conveyor belt,

$$q = x - v_b t,$$

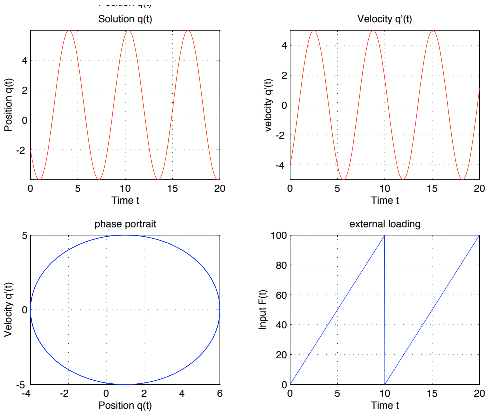
and $\dot{q}$ the relative velocity, i.e.,

$$\dot{q} = \dot{x} - v_b.$$

In terms of the relative displacement q, the last equation is recast to

$$m\ddot{q}(t) + kq(t) - F(t) \in -\partial\Phi(\dot{q}(t)),$$

with $F(t) = k\, v_b\, t$. By Application of the last Theorem, clearly this problem has a periodic solution. We give a period $T > 0$ (in the example $T = 10$ s), we compute the Poincaré operator S_T associated to the inclusion (7). We use the Picard-successive iteration in order to compute a fixed point of the Poincaré operator S_T, which is a T-periodic solution to the Problem.


Solution q(t)
Position q(t)
Time t
Velocity q'(t)
velocity q'(t)
Time t
phase portrait
Velocity q'(t)
Position q(t)
external loading
Input F(t)
Time t

References

The talk is based on the following paper :

- S. ADLY, D. GOELEVEN, M. THÉRA, *Existence results for a class of periodic evolution variational inequalities*, Chinese Annals of Mathematics, 28B(6), 2007, 629-650.

Thank you for your attention