More Applications of LP and SDP

Yinyu Ye
Department of Management Science and Engineering
Stanford University

http://www.stanford.edu/~yyye

Max-Cut Problem

Consider the Max Cut problem on an undirected graph $G = (V, E)$ with non-negative weights w_{ij} for each edge in E (and $w_{ij} = 0$ if $(i, j) \notin E$), which is the problem of partitioning the nodes of V into two sets S and $V \setminus S$ so that

$$w(S) := \sum_{i \in S,\, j \in V \setminus S} w_{ij}$$

is maximized. A problem of this type arises from many network planning, circuit design, and scheduling applications.

Quadratic Optimization Formulation

This problem can be formulated by assigning each node a binary variable x_j:

$$z^* = \quad \text{Maximize} \quad w(x) := \frac{1}{4}\sum_{i,j} w_{ij}(1 - x_i x_j) = x^T W x$$

(MC)

$$\text{Subject to} \quad x_i^2 = 1, \quad i = 1, ..., n.$$

The Coin-Toss Method

Let each node be selected to one side, or x_i be 1, independently with probability $.5$. $\blacksquare$

$$\mathsf{E}[w(x)] \geq 0.5 \cdot z^*.$$

Semi-definite relaxation

$$
\begin{array}{rll}
z^{SDP} := & \text{minimize} & W \bullet X \\
& \text{s.t.} & I_j \bullet X = 1,\ j = 1, ..., n, \\
& & X \succeq 0.
\end{array} \tag{1}
$$

The dual is

$$
\begin{array}{rll}
z^{SDP} = & \text{maximize} & e^T y \\
& \text{s.t.} & W \succeq D(y).
\end{array} \tag{2}
$$

Randomized Rounding of Goemans and Williamson

Let $V = (v_1, \ldots, v_n) \in \mathcal{R}^{n \times n}$, i.e., v_j is the jth column of V, such that $X^* = V^T V$.

Generate a random vector $u \in N(0, I)$:

$$\hat{x} = \text{sign}(V^T u),$$

$$\text{sign}(x_j) = \begin{cases} 1 & \text{if } x_j \geq 0 \\ -1 & \text{otherwise.} \end{cases}$$

Approximation analysis

Then, one can prove from Sheppard (1900):

$$\mathsf{E}[\hat{x}_i \hat{x}_j] = \frac{2}{\pi}\arcsin(\bar{X}_{ij}), \quad i, j = 1, 2, \ldots, n.$$

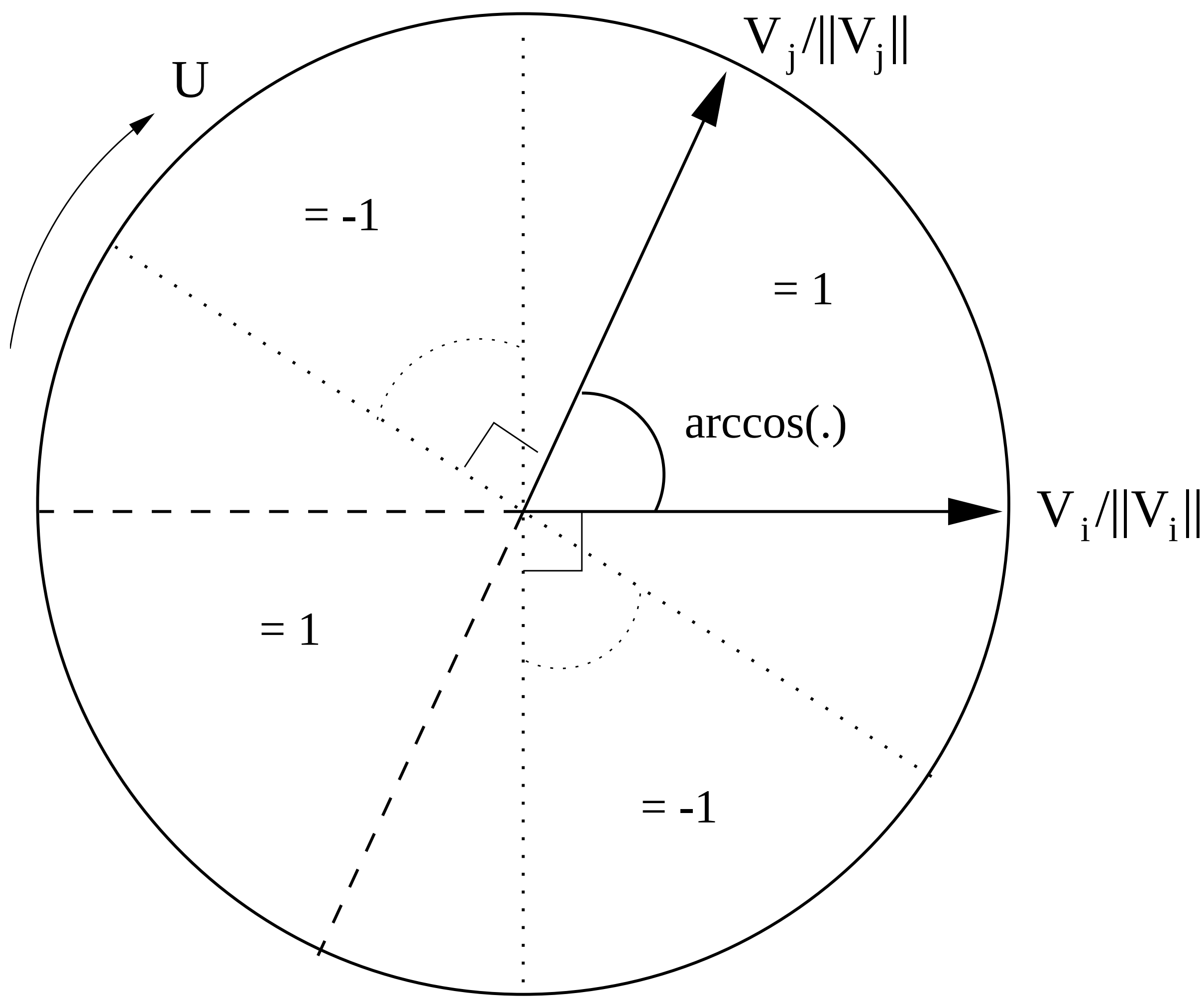


Figure 1: Illustration of the product $\sigma(\frac{v_i^T u}{\|v_i\|}) \cdot \sigma(\frac{v_j^T u}{\|v_j\|})$ on the 2-dimensional unit circle, where u is uniformly generated along the circle.

Analyses

Lemma 1 *For* $x \in [-1, 1)$

$$\frac{1 - (2/\pi) \cdot \arcsin(x)}{1 - x} \geq .878.$$

Final Results

Theorem 1 *We have*

$$\mathsf{E}(\hat{x}^T W \hat{x}) \geq .878 z^{SDP} \geq .878 z^*,$$

so that

$$z^* \geq .878 z^{SDP}.$$

Generalized Max-Cut

$$z^* := \quad \text{Maximize} \quad x^T Q x$$

(GMC)

$$\text{Subject to} \quad x_j^2 \{= \text{ or } \leq\} 1, \ \forall j,$$

where Q is positive semidefinite.

Generalized Max-Cut SDP Relaxation

$$
\begin{array}{lll}
z^{SDP} := & \text{Maximize} & Q \bullet X \\
& \text{s.t.} & I_j \bullet X \{= \text{or} \leq\} 1, \ \forall j, \\
& & X \succeq 0.
\end{array}
$$

Generalized Max-Cut Appriximation Result

Homework 13:

Lemma 2 *Let* $X \succeq 0$ *and* $d(X) \leq 1$*. Then* $\arcsin[X] \succeq X$*.*

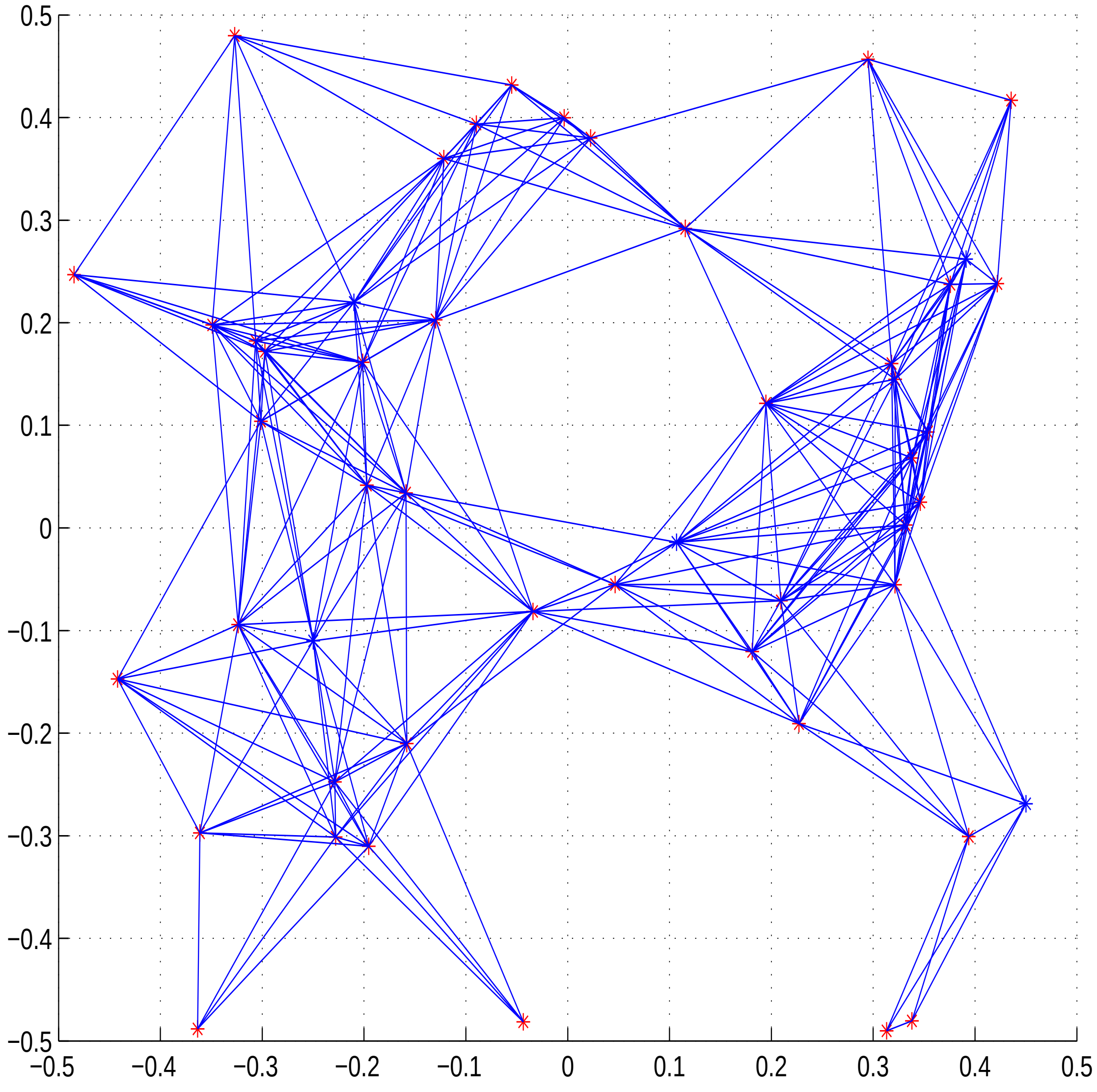
Theorem 2 *(Nesterov (1998)) We have*

$$\mathsf{E}(\hat{x}^T Q \hat{x}) \geq \frac{2}{\pi} z^{SDP} \geq \frac{2}{\pi} z^*.$$

Ad Hoc Wireless Sensor Network Localization

- **Input** m known points (anchors) $a_k \in \mathbf{R}^2$, $k = 1, ..., m$, and n unknown points (sensors or targets) $x_j \in \mathbf{R}^2$, $j = 1, ..., n$. For some pair of two points, we have a Euclidean distance measure $\hat{d}_{kj}$ between a_k and x_j, or distance measure $\hat{d}_{ij}$ between x_i and x_j.
- **Output** Position estimation for all unknown points, and confidence measures on reliability of each position estimation.
- **Objective** Robust, fast and accurate.

Figure 2: 50-Sensor Network with Radio Range .3



Related Work

- FCC requires wireless carriers to provide far more precise location information, within 50 to 100 meters in most cases, of a wireless 911 caller by December 31, 2005.
- A great deal of research has been done on the topic of position estimation in ad-hoc networks, see Hightower and Boriello (2001) and Ganesan et al. (2002); Beacon grid: e.g., Bulusu and Heidemann (2000) and Howard et al. (2001); Distance measurement: e.g., Doherty et al. (2001), Niculescu and Nath (2001), Savarese et al. (2002), Savvides et al. (2001, 2002), Shang et al. (2003), Eren et al. (2004).
- Metric embeddings and Distance geometry problems: Johnson and Lindenstrauss (1984), Bourgain (1985), Barvinok (1995), Moré and Wu (1997), Alfakih et al. (1999), Laurent (2001), etc.

Euclidean Distance Geometry Model

$$\begin{aligned}
\|x_i - x_j\|^2 &= d_{ij}^2, \ \forall\,(i,j) \in N_x, \ i < j, \\
\|a_k - x_j\|^2 &= d_{kj}^2, \ \forall\,(k,j) \in N_a, \\
\|x_i - x_j\|^2 &\ge R_{ij}^2, \ \forall(i,j) \notin N_x, \ i < j, \\
\|a_k - x_j\|^2 &\ge R_{kj}^2, \ \forall(k,j) \notin N_a.
\end{aligned}$$

d_{ij}^2 (d_{kj}^2) connects x_i to x_j (a_k to x_j) with an edge whose length is d_{ij} (d_{kj}).

Does the system has a localization or realization of all x_j's? Is the localization unique? Is the localization reliable or trustworthy? Is the system partially localizable?

Euclidean Distance Geometry Model

Consider a simpler Euclidean Distance Geometry Model:

$$\|x_i - x_j\|^2 = d_{ij}^2, \ \forall\,(i,j) \in N_x, \ i < j,$$

$$\|a_k - x_j\|^2 = d_{kj}^2, \ \forall\,(k,j) \in N_a.$$

Convex Optimization Method

$$\|x_i - x_j\|^2 \le d_{ij}^2,\ \forall\,(i,j) \in N_x,\ i < j,$$
$$\|a_k - x_j\|^2 \le d_{kj}^2,\ \forall\,(k,j) \in N_a.$$

Doherty et al. (2001)

Global and Nonlinear Least Errors Method

$$\min \quad \sum_{i,j \in N_x} |\|x_i - x_j\|^2 - d_{ij}^2)| + \sum_{k,j \in N_a} |\|a_k - x_j\|^2 - d_{kj}^2)|$$

$$\min \quad \sum_{i,j \in N_x} (\|x_i - x_j\|^2 - d_{ij}^2)^2 + \sum_{k,j \in N_a} (\|a_k - x_j\|^2 - d_{kj}^2)^2$$

Matrix Representation

Let $X = [x_1\ x_2\ ...\ x_n]$ be the $2 \times n$ matrix that needs to be determined. Then

$\|x_i - x_j\|^2 = (e_i - e_j)^T X^T X (e_i - e_j)$ and $\|a_k - x_j\|^2 = (a_k; -e_j)^T [I\ \ X]^T [I\ \ X] (a_k; -e_j)$,

where e_i is the vector of all zero except 1 at the ith position.

$$
\begin{aligned}
&(e_i - e_j)^T Y (e_i - e_j) = d_{ij}^2, \ \forall\, i, j \in N_x,\ i < j, \\
&(a_k; -e_j)^T \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} (a_k; -e_j) = d_{kj}^2, \ \forall\, k, j \in N_a, \\
&Y = X^T X.
\end{aligned}
$$

where Y denotes the Gram matrix $X^T X$.

SDP Relaxation

$$(e_i - e_j)^T Y (e_i - e_j) = d_{ij}^2, \text{ for } i, j \in N_x,\ i < j,$$

$$(a_k; -e_j)^T \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} (a_k; -e_j) = d_{kj}^2, \text{ for } k, j \in N_a,$$

$$Y \succeq X^T X.$$

Let

$$Z = \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix}$$

SDP Standard Form

$$
\begin{aligned}
&(1;0;\mathbf{0})(1;0;\mathbf{0})^T \bullet Z = 1,\\
&(0;1;\mathbf{0})(0;1;\mathbf{0})^T \bullet Z = 1,\\
&(1;1;\mathbf{0})(1;1;\mathbf{0})^T \bullet Z = 2,\\
&(\mathbf{0};(e_i-e_j))(\mathbf{0};(e_i-e_j))^T \bullet Y = d_{ij}^2, \text{ for } i,j \in N_x,\ i<j,\\
&(a_k;-e_j)(a_k;-e_j)^T \bullet Z = d_{kj}^2, \text{ for } k,j \in N_a,\\
&Z \succeq 0.
\end{aligned}
$$

$$
\bar{Z} = \begin{pmatrix} I & \bar{X} \\ \bar{X}^T & \bar{X}^T\bar{X} \end{pmatrix} = (I,\ \bar{X})^T(I,\ \bar{X})
$$

is a feasible solution for the relaxation, where $\bar{X} = [\bar{x}_1\ \bar{x}_2\ \ldots\ \bar{x}_n]$.

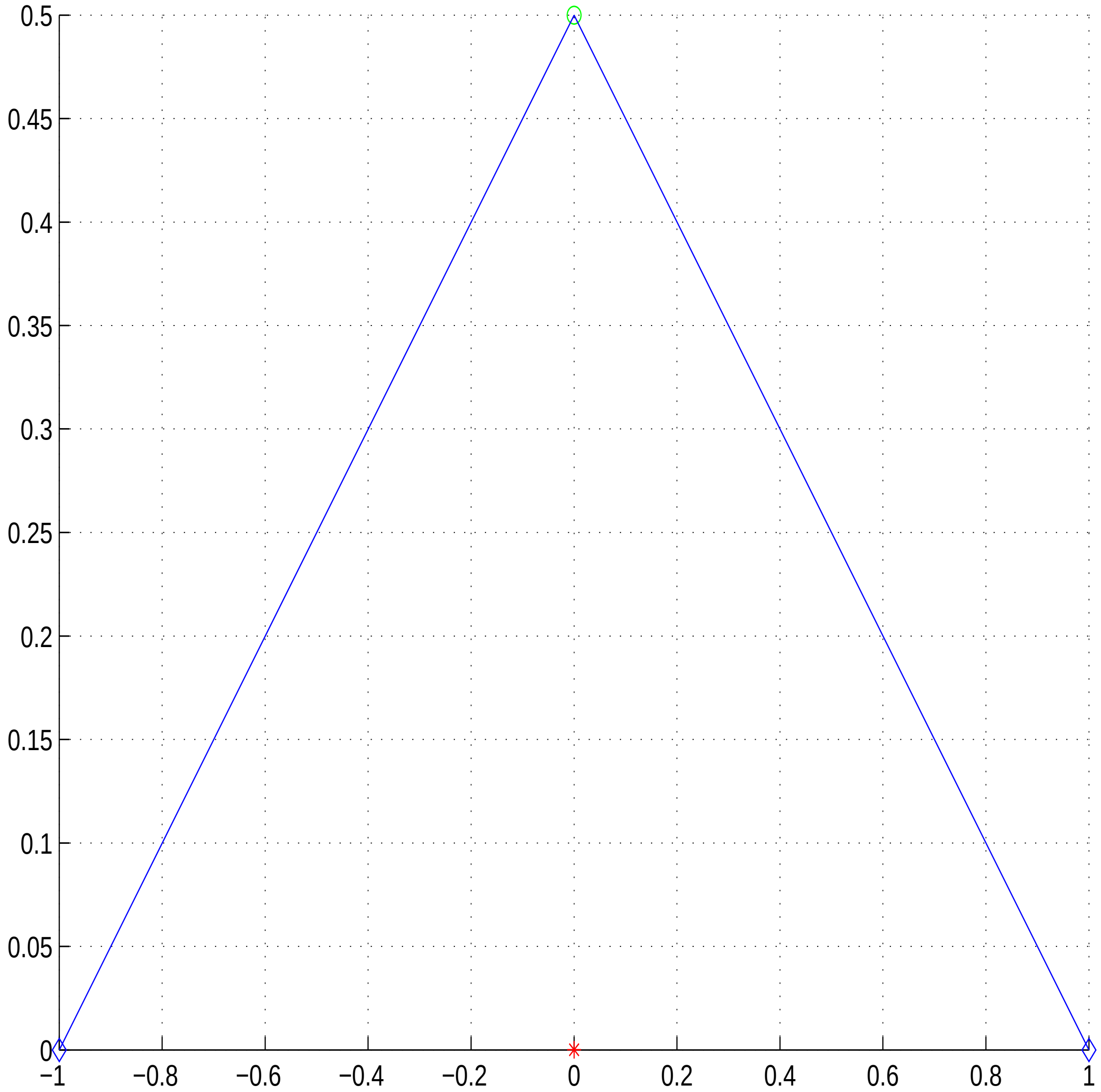
The dual of the SDP relaxation

$$\begin{array}{ll} \text{min} & w_1 + w_2 + w_3 + \sum_{i<j \in N_x} w_{ij} d_{ij}^2 + \sum_{k,j \in N_a} w_{kj} d_{kj}^2 \\ \text{s.t.} & w_1 (1; 0; \mathbf{0})(1; 0; \mathbf{0})^T + w_2 (0; 1; \mathbf{0})(0; 1; \mathbf{0})^T + w_3 (1; 1; \mathbf{0})(1; 1; \mathbf{0})^T + \\ & \sum_{i<j \in N_x} w_{ij} (\mathbf{0}; (e_i - e_j))(\mathbf{0}; (e_i - e_j))^T + \sum_{k,j \in N_a} w_{kj} (a_k; -e_j)(a_k; -e_j)^T \succeq \end{array}$$

Since the primal is feasible, the minimal value of the dual is 0. Note that all 0 is an minimal solution.

If an optimal dual slack matrix, call it U, has rank n, then every primal matrix solution has rank no more than 2, that is, we have $Y = X^T X$ in Z.

Figure 3: One sensor-Two anchors: Not localizable



One sensor and three anchors

Find $x_1 \in \mathbf{R}^2$ such that

$$\|a_k - x_1\|^2 = d_{kj}^2, \text{ for } k = 1, 2, 3,$$

Let $\bar{x}_1$ be the true position of x_1.

SDP Standard Form

$$
\begin{aligned}
&(1;0;0)(1;0;0)^T \bullet Z = 1,\\
&(0;1;0)(0;1;0)^T \bullet Z = 1,\\
&(1;1;0)(1;1;0)^T \bullet Z = 2,\\
&(a_k;-1)(a_k;-1)^T \bullet Z = d_{k1}^2, \text{ for } k = 1,2,3,\\
&Z \succeq 0.
\end{aligned}
$$

$$
\bar{Z} = \begin{pmatrix} I & \bar{x}_1 \\ \bar{x}_1^T & \bar{x}_1^T\bar{x}_1 \end{pmatrix} = (I,\ \bar{x}_1)^T(I,\ \bar{x}_1)
$$

is a feasible solution for the relaxation.

The dual slack matrix

$$\begin{pmatrix} \begin{pmatrix} w_1 + w_3 & w_3 \\ w_3 & w_2 + w_3 \end{pmatrix} + \sum_{k=1}^{3} w_{k1} a_k a_k^T & -\sum_{k=1}^{3} w_{k1} a_k \\ -(\sum_{k=1}^{3} w_{k1} a_k)^T & w_{11} + w_{21} + w_{31} \end{pmatrix} \succeq 0.$$

Does an optimal matrix U have rank 1 with

$$w_1 + w_2 + w_3 + \sum_{k=1}^{3} w_{k1} d_{k1}^2 = 0$$

An optimal dual slack matrix

If we choose $w_{\bullet}$'s such that

$$\bar{U} = (-\bar{x}_1; 1)(-\bar{x}_1; 1)^T,$$

then, $\bar{U} \succeq 0$ and $\bar{U} \bullet \bar{X} = 0$ so that $\bar{U}$ is an optimal slack matrix for the dual and its rank is 1.

How to select w's

Let

$$\sum_{k=1}^{3} w_{k1} a_k = \bar{x}_1 \qquad \text{or} \qquad \sum_{k=1}^{3} w_{k1}(a_k - \bar{x}_1) = 0$$

$$w_{11} + w_{21} + w_{31} = 1. \qquad\qquad w_{11} + w_{21} + w_{31} = 1.$$

This system always has a solution if a_k is not co-linear.

Then, select

$$\begin{pmatrix} w_1 + w_3 & w_3 \\ w_3 & w_2 + w_3 \end{pmatrix} = \bar{x}_1 \bar{x}_1^T - \sum_{k=1}^{3} w_{k1} a_k a_k^T$$

Other conditions?

Even if a_k is co-linear, the system

$$\sum_{k=1}^{3} w_{k1}(a_k - \bar{x}_1) = 0$$

$$w_{11} + w_{21} + w_{31} = 1$$

has a solution $w_\bullet$ if $\bar{x}_1$ on the same line.

Physical interpretation: w_{ij} is a force on the edge and all forces are balanced.

The objective represents the work of the system.

Localizable problem

A sensor network is localizable if there is a unique localization in $\mathbf{R}^2$ and there is no $x_j \in \mathbf{R}^h,\ j = 1, ..., n$, where $h > 2$, such that

$$\|x_i - x_j\|^2 = d_{ij}^2,\ \forall\, i, j \in N_x,\ i < j,$$
$$\|(a_k; \mathbf{0}) - x_j\|^2 = d_{kj}^2,\ \forall\, k, j \in N_a.$$

The latter says that the problem cannot be localized in a higher dimension space where anchor points are augmented to $(a_k; \mathbf{0}) \in \mathbf{R}^h$, $k = 1, ..., m$.

When is the problem localizable?

Theorem 3 *The following statements are equivalent:*

1. *The sensor network is localizable;*
2. *The max-rank solution of the SDP relaxaion has rank* 2*;*
3. *The solution matrix has* $Y = X^T X$ *or* Trace$(Y - X^T X) = 0$ *.*

Strongly localizable

When the dual has a solution with rank n, then the problem is strongly localizable.

If we can choose $w_\bullet$'s such that the dual slack matrix

$$\bar{U} = (-\bar{X}; I)\bar{V}(-\bar{X}; I)^T$$

where n-dimension matrix $\bar{V}$ is positive definite, then, $\bar{U} \succeq 0$ and $\bar{U} \bullet \bar{X} = 0$ so that $\bar{U}$ is an optimal slack matrix for the dual and its rank is n.

Figure 4: Two sensor-Three anchors: Strongly Localizable

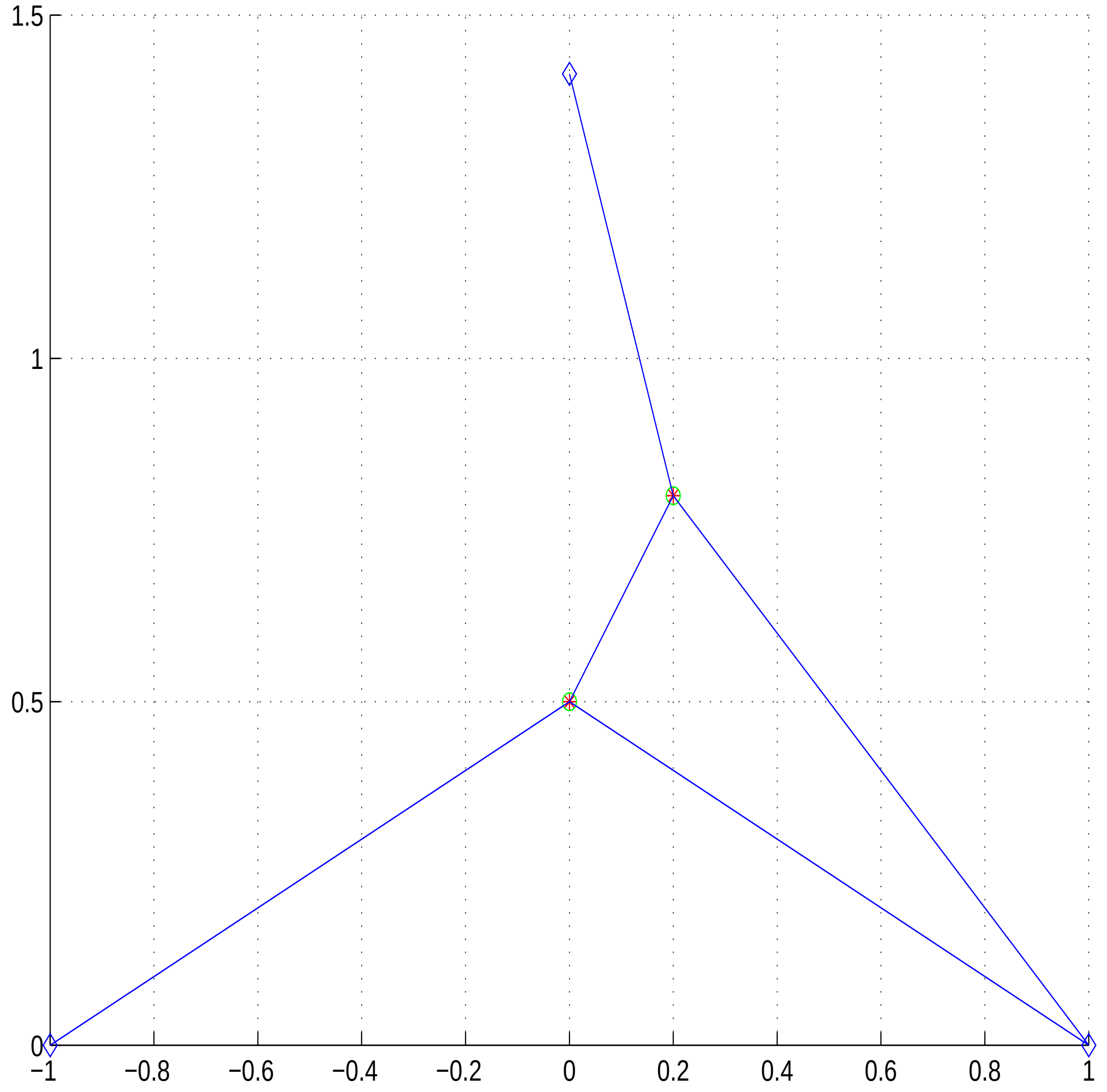


Figure 5: Two sensor-Three anchors: Localizable but not Strongly

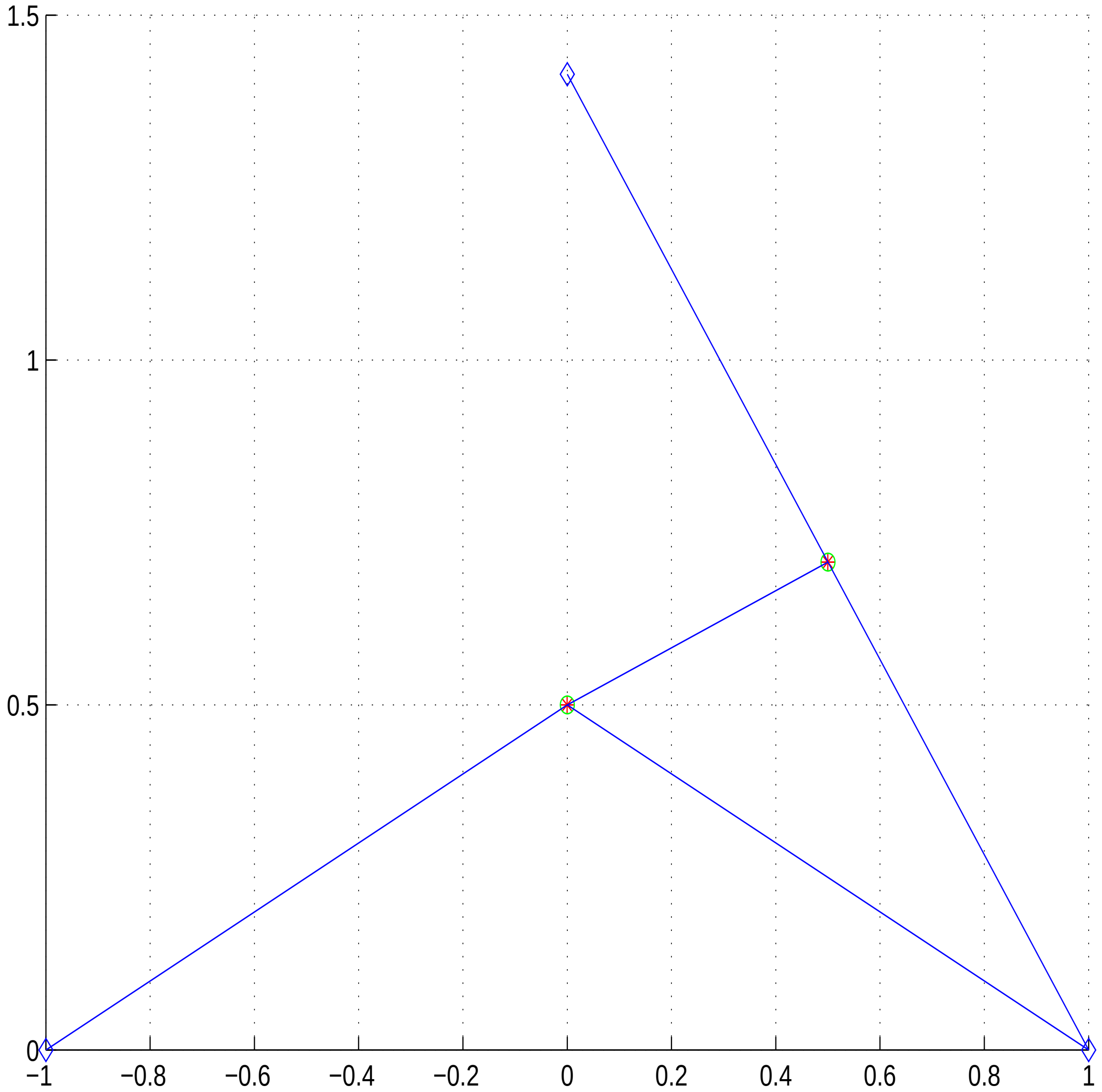


Figure 6: Two sensor-Three anchors: Not localizable

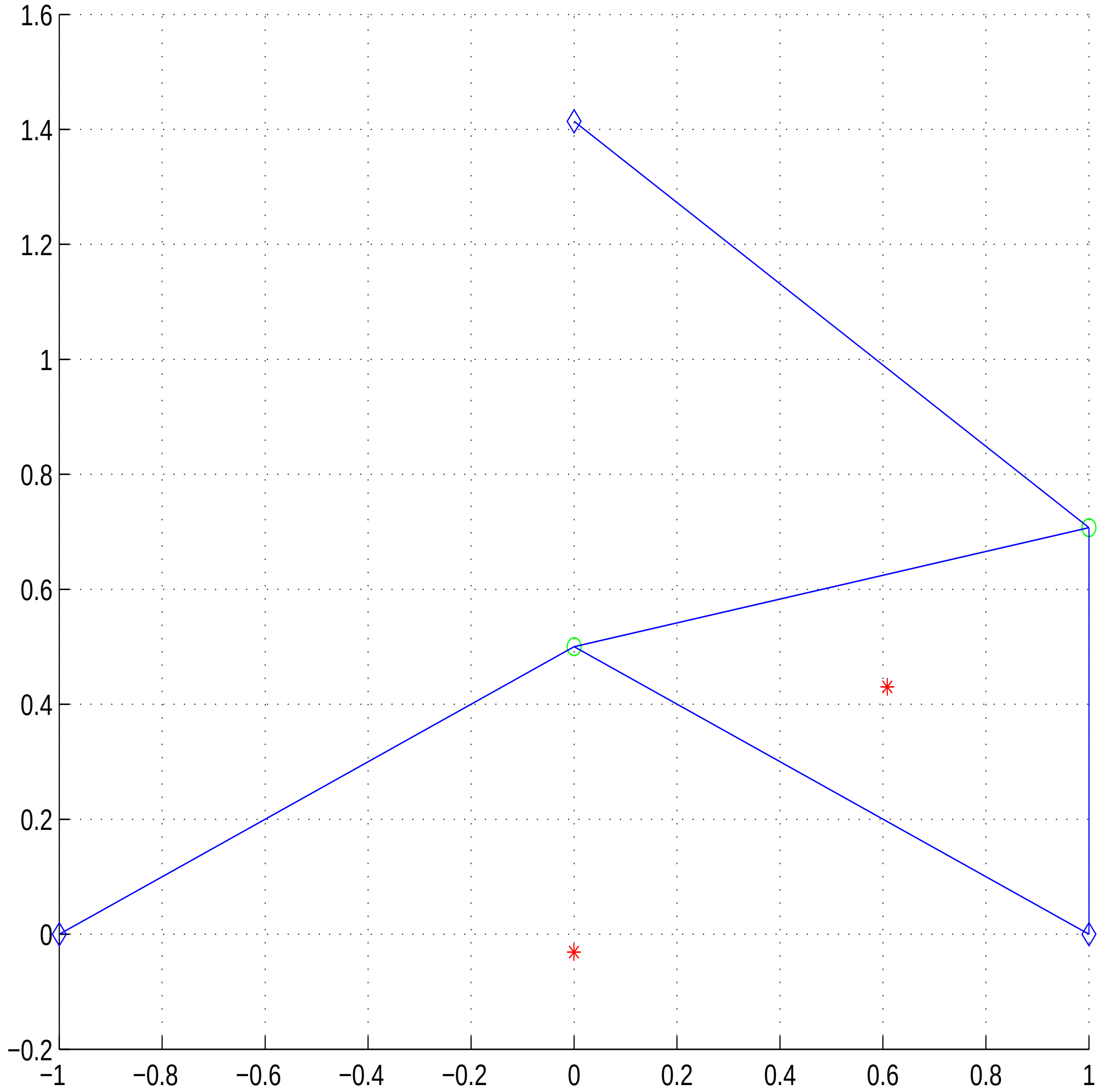
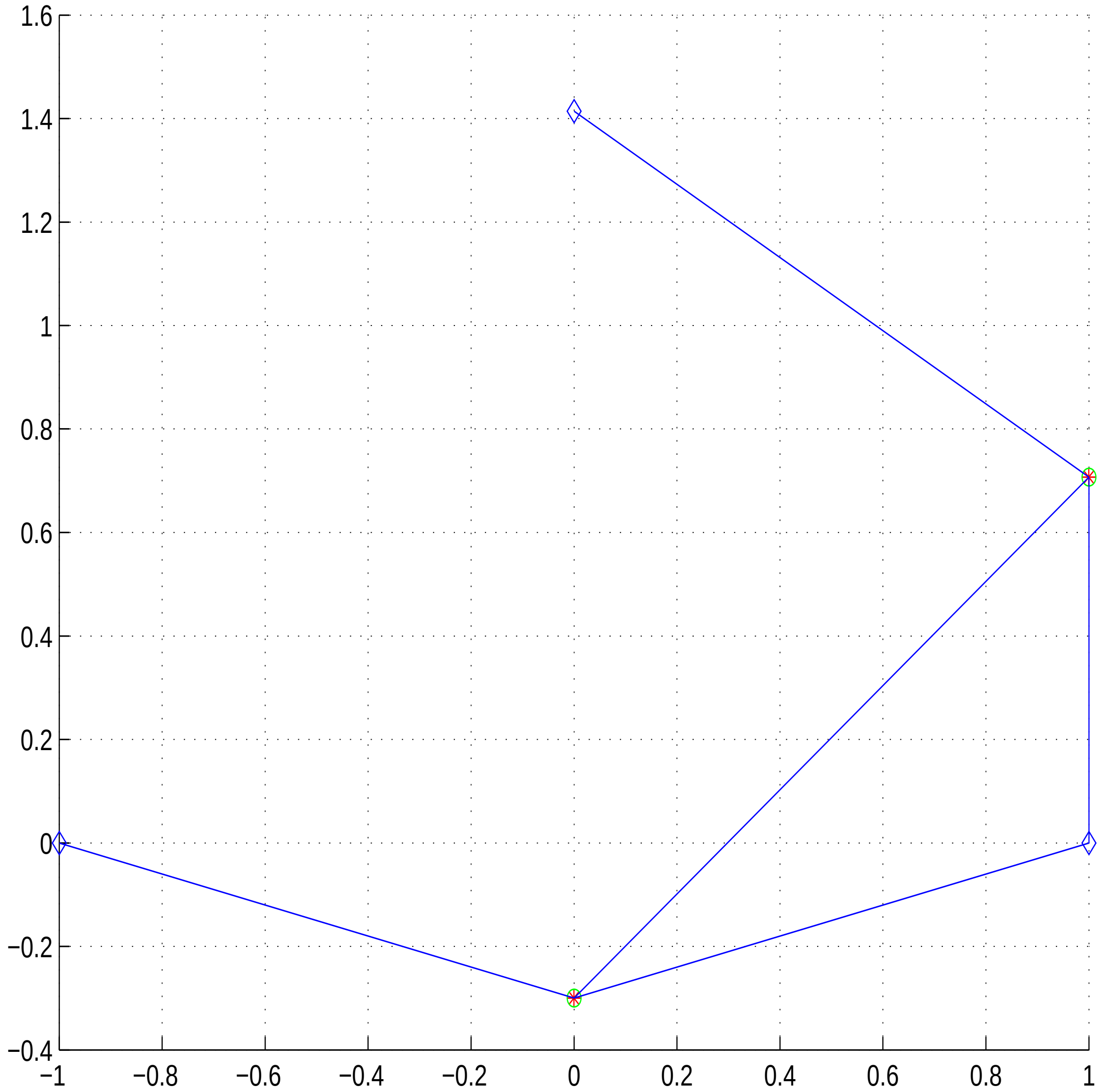


Figure 7: Two sensor-Three anchors: Stronly Localizable



Proof of Theorem

If the primal feasible matrix generated from the interior-point algorithm has rank 2, that is, $\bar{Y} = \bar{X}^T\bar{X}$ or the trace of $\bar{Y} = \bar{X}^T\bar{X}$ equal 0, then the feasible solution for the original problem is unique.

Proof continued

First, every feasible matrix has rank at least 2 since $Y \succeq X^T X$.

Second, since the matrix solution computed from the interior-point algorithm has the maximal rank and it is 2, we conclude that every feasible matrix has rank exact 2.

Suppose that the system has two rank-2 feasible matrices:

$$Z_1 = \begin{pmatrix} I & X_1 \\ X_1^T & X_1^T X_1 \end{pmatrix} \quad \text{and} \quad Z_2 = \begin{pmatrix} I & X_2 \\ X_2^T & X_2^T X_2 \end{pmatrix}$$

Consider $Z = \alpha Z_1 + \beta Z_2$, where $\alpha + \beta = 1$ and $\alpha, \beta > 0$. Then Z is a feasible solution and its rank must be 2.

$$Z = \begin{pmatrix} I & \alpha X_1 + \beta X_2 \\ \alpha X_1^T + \beta X_2^T & \alpha X_1^T X_1 + \beta X_2^T X_2 \end{pmatrix} =$$

$$\begin{pmatrix} I & \alpha X_1 + \beta X_2 \\ \alpha X_1^T + \beta X_2^T & (\alpha X_1 + \beta X_2)^T (\alpha X_1 + \beta X_2) \end{pmatrix}$$

Thus,

$$0 = \alpha X_1^T X_1 + \beta X_2^T X_2 - (\alpha X_1 + \beta X_2)^T (\alpha X_1 + \beta X_2) =$$

$$\alpha\beta (X_1 - X_2)^T (X_1 - X_2)$$

or

$$\|X_1 - X_2\| = 0.$$

Localize All Localizable Points

Theorem 4 *If a problem (graph) contains a subproblem (subgraph) that is localizable, then the submatrix solution corresponding to the subproblem in the SDP solution has rank* 2. *That is, the SDP relaxation computes a solution that localize all possibly localizable unknown sensor points.*

Implication: Trace,

$$\text{Trace}(\bar{Y} - \bar{X}^T \bar{X}) = \sum_{j=1}^{n} (\bar{Y}_{jj} - \|\bar{x}_j\|^2)$$

$\bar{Y}_{jj} - \|\bar{x}_j\|^2$ can be used as a measure to see whether jth sensor's estimated position is reliable or not. In particular, the individual diagonal entry of $\bar{Y} - \bar{X}^T \bar{X}$ tells whether or not the corresponding sensor is correctly localized.

SDP Relaxation with Noisy Data

Find a symmetric matrix $Z \in \mathbf{R}^{(2+n)\times(2+n)}$ and α_{ij} and α_{kj} such that

$$\begin{array}{ll} \text{minimize} & \sum_{(i,j)\in N_x} |\alpha_{ij}| + \sum_{(k,j)\in N_a} |\alpha_{kj}| \\ \text{subject to} & Z_{1:2,1:2} = I, \\ & (\mathbf{0}; (e_i - e_j))(\mathbf{0}; (e_i - e_j))^T \bullet Z + \alpha_{ij} = d_{ij}^2, \ \forall\, (i,j) \in N_x, \\ & (a_k; -e_j)(a_k; -e_j)^T \bullet Z + \alpha_{kj} = d_{kj}^2, \ \forall\, (k,j) \in N_a, \\ & Z \succeq 0. \end{array}$$

Project continued: Develop a Matlab code, with Sedumi or DSDP as subroutines, to solve the localization problem. Turn in a report to summarize your implementations and findings.

NSDP Decomposition: a Further Relaxation

Replace

$$(C1): \quad \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \succeq 0;$$

by

$$(C2): \quad \begin{pmatrix} I & x_i & x_{i_1} & \dots & x_{i_{d(i)}} \\ x_i^T & Y_{ii} & Y_{ii_1} & \dots & Y_{ii_{d(i)}} \\ x_{i_1}^T & Y_{i_1 i} & Y_{i_1 i_1} & \dots & Y_{i_1 i_{d(i)}} \\ \dots & \dots & \dots & \dots & \dots \\ x_{i_{d(i)}}^T & Y_{i_{d(i)} i} & Y_{i_{d(i)} i_1} & & Y_{i_{d(i)} i_{d(i)}} \end{pmatrix} \succeq 0, \quad \forall i,$$

where $(i, i_{ik}) \in N_x$ and $d(i)$ is the degree of sensor node i.

Node-based Decomposition

ESDP Decomposition: a Further Relaxation

Replace

$$(C1): \quad \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \succeq 0;$$

by

$$(C2): \quad \begin{pmatrix} I & x_i & x_j \\ x_i^T & Y_{ii} & Y_{ij} \\ x_j^T & Y_{ji} & Y_{jj} \end{pmatrix} \succeq 0, \quad \forall (i,j) \in N_x.$$

Edge-Based Decomposition

Analyses

A undirected graph is a chordal graph if every cycle of length greater than three has a chord.

A square matrix is called to be partial symmetric if it is symmetric to the extent of its specified entries, i.e., if the (i, j) entry of the matrix is specified, then so is the (j, i) entry and the two are equal. A partial semi-definite matrix is a partial symmetric matrix and every fully specified principal submatrix is positive semi-definite.

Lemma 3 *(Hogben 2001) Every partial positive semi-definite matrix with undirected graph G has positive semi-definite completion if and only if G is chordal.*

The Equivalence Theorem for NSDP

Theorem 5 *(Wang, Zheng, Boyd and Ye [2006]) Suppose the undirected graph of sensor nodes with edge set N_x is chordal, then SDP and NSDP relaxations are equivalent.*

The Trace Theorem for ESDP

Homework 14:

Theorem 6 *(Wang, Zheng, Boyd and Ye [2006]) Let*

$$Z = \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix}$$

be a max-rank solution of ESDP. If the diagonal entry

$$(Y - X^T X)_{\bar{i}\bar{i}} = 0$$

then the $\bar{i}$*th column of* X*,* $x_{\bar{i}}$*, must be the true location of the* $\bar{i}$*th sensor, and* $x_{\bar{i}}$ *is invariant over all solutions* Z *for ESDP.*

The Kissing Problem

- Given a unit center sphere, the maximum number of unit spheres, in d dimensions, can touch or kiss the center sphere?
- General Solutions does not exist.
- Delsarte Method uses linear programming to provide an upper bound on the number of spheres.
- K(1)=2, K(2)=6, K(3)= 12, K(8) = 240, K(24) = 196650.
- K(4) = 24: proved using Delsarte Method by Oleg Musin only 3 years ago.
- For other dimensions, lower bounds have been provided by constructing a lattice structure. There also exists a bound using the Riemann zeta function, but is non-constructive.

The Kissing Problem as Localization

- Can be formulated as a SDP feasibility problem; but SDP solution may not provide proper rank.

$$\begin{aligned} (e_i - e_j)^T Y (e_i - e_j) &\geq 4, \ \forall i \neq j, \\ e_i^T Y e_i &= 4, \ \forall i, \\ Y &\succeq 0. \end{aligned}$$

- Construct a nonzero SDP objective function to reduce the rank of a solution.

$$\begin{array}{lrl} \min & C \bullet Y, & \\ \text{s.t.} & (e_i - e_j)^T Y (e_i - e_j) & \geq 4, \ \forall i \neq j, \\ & e_i^T Y e_i & = 4, \ \forall i, \\ & Y & \succeq 0. \end{array}$$

The Kissing Problem and Coding

Given a number of points, find the largest radius r such that

$$\begin{aligned} (e_i - e_j)^T Y (e_i - e_j) &\geq (1+r)^2, \ \forall i \neq j, \\ e_i^T Y e_i &= (1+r)^2, \ \forall i, \\ Y &\succeq 0, \end{aligned}$$

has a rank 3 matrix solution Y.

Schoenberg's theorem on the Gegenbauer polynomial may be used to strenghten the above SDP formulation.

Schoenberg's theorem on the Gegenbauer polynomial matrix

The Gegenbauer polynomial:

$$G_0^{(r)}(t) = 1,\ G_1^{(r)}(t) = t,\ \ldots,$$

$$G_k^{(r)}(t) = \frac{(2k+r-4)tG_{k-1}^{(r)}(t) - (k-1)G_{k-2}^{(r)}(t)}{k+r-3}.$$

Given symmetric matrix $Y \succeq 0$ with rank r and all its diagonals equal 1.

Theorem 7 *The Gegenbauer polynomial matrix, $G_k^{(r)}[y_{ij}]$, remains positive semidefinite for $k = 0, \ldots$, where symmetric matrix $G_k^{(r)}[y_{ij}]$ has the same dimension of Y and its corresponding component equals $G_k^{(r)}(y_{ij})$.*