Linear and Semidefinite Programming

Yinyu Ye
Department of Management Science and Engineering
Stanford University

http://www.stanford.edu/~yyye

Introduction to Optimization

Any scenario in which you are trying to make certain decisions and reach the best possible outcome. Optimization is the common goal of Management Science and Engineering.

Optimization is concerned with the study of maximization and minimization of mathematical functions. Very often the arguments of (i.e., variables or unknowns in) these functions are subject to side conditions or constraints. By virtue of its great utility in such diverse areas as applied science, engineering, economics, finance, medicine, and statistics, optimization holds an important place in the practical world and the scientific world. Indeed, as far back as the Eighteenth Century, the famous Swiss mathematician and physicist Leonhard Euler (1707-1783) proclaimed[a] that . . . nothing at all takes place in the Universe in which some rule of maximum or minimum does not appear.

[a]See Leonhardo Eulero, *Methodus Inviendi Lineas Curvas Maximi Minimive Proprietate Gaudentes*, Lausanne & Geneva, 1744, p. 245.

Where do Optimization Problems come from?

- Economics: Consumer theory / supplier theory
- Finance: Optimal hedging / pricing
- Science / Engineering: Aerospace, product design, data mining
- Other Business decisions: scheduling, production, organizational decisions
- Government: Military applications, fund allocation, etc
- Other Personal decisions: Sports, on-field decisions, player acquisition, marketing

Quantitative or Mathematical Models

The class of optimization problems considered in this course can all be expressed in the form

$$\begin{array}{lll} \text{(P)} & \text{minimize} & f(\mathbf{x}) \\ & \text{subject to} & \mathbf{x} \in \mathcal{X} \end{array}$$

where $\mathcal{X}$ usually specified by constraints:

$$\begin{array}{rcll} c_i(\mathbf{x}) & = & 0 & i \in \mathcal{E} \\ c_i(\mathbf{x}) & \leq & 0 & i \in \mathcal{I}. \end{array}$$

Production Management

The Wyndor Glass Co. is a producer of high-quality glass products. It has three plans. Aluminum frames and hardware are made in Plant 1, wood frames are made in Plant 2, and Plant 3 is used to produce glass and assemble the products.

Wyndor produces to products which require the resources of the 2 plants as follows:

Plant	Aluminum	Wood	Resources
1	1	0	4
2	0	2	12
3	3	2	18
Unit Profit	$\$3000$	$\$5000$	

Mathematical Formulation

$$
\begin{array}{lrrl}
\text{maximize} & 3x_1 + & 5x_2 & \\
\text{subject to} & x_1 & \leq 4, & \\
 & & 2x_2 & \leq 12, \\
 & 3x_1 + & 2x_2 & \leq 18, \\
 & x_1, & x_2 & \geq 0.
\end{array}
$$

Objective; Decision Variables; Constraints; Data Parameters.

Linear Programming

$$
\begin{array}{rl}
\text{min(or max)imize} & c_1x_1 + c_2x_2 + ... + c_nx_n \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \; \{\leq, =, \geq\} \; b_1, \\
& a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \; \{\leq, =, \geq\} \; b_2, \\
& \quad\quad ..., \\
& a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \; \{\leq, =, \geq\} \; b_m, \\
& x_j \; \{\geq, \leq\} \; u_j, \quad j = 1, ..., n,
\end{array}
$$

LP in Matrix Form

- $$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

- $$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}.$$

-

$$\begin{array}{rl} \text{min(or max)imize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x}\ \{\leq, =, \geq\}\ \mathbf{b}, \\ & \mathbf{x}\ \{\geq, \leq\}\ \mathbf{0}. \end{array}$$

Important Terms

- decision variable/activity, data/parameter
- objective/goal/target
- constraint/limitation/requirement
- equality/inequality constraint
- constraint function/the right-hand side
- direction of inequality
- coefficient vector/coefficient matrix
- nonnegativity constraint
- integrality constraint
- satisfied/violated

Why Quantitative?

- Some management decisions inevitably need quantitative models and can significantly benefit from using quantitative models
- Allow us to make rankings and use the power of computers
- Model building involves a great deal of experience, intuition, art and imagination as well as technical know-how.

The Optimization Process

- Formulate real life problems into mathematical models
  - Study the environment and clearly understand the problem
  - Formulate the problem using verbal description
  - Define notations for parameters and decision variables
  - Construct a model using mathematical expressions
  - Collect necessary data; Transform the raw data to parameter values
- Implement the model and solution algorithms using a computer: analyze the models and develop efficient procedures to obtain best solutions
- interpret computer solutions and perform sensitivity analysis
- Implementation: put the knowledge gained from the solution to work
- Monitor the validity and effectiveness of the model and update it when necessary

What do You Learn?

- Models –the art: How we choose to represent real problems
- Theory – the science: What we know about different classes of models; e.g. necessary and sufficient conditions for optimality
- Algorithms – the tools: How we apply the theory to robustly and efficiently solve powerful models

The Art of Modeling

Objective to distill the real-world as accurately and succinctly as possible into a quantitative model

- Dont want models to be too generalized might not draw much real world value from your results. Ex. Analyzing traffic flows assuming every person has the same characteristics
- Dont want models to be too specific might lose the ability to solve problems or gain insights. Ex. Trying to analyze traffic flows by modeling every single individual using different assumptions

Formulation of Optimization Models: Four-Step Rule

- Sort out data and parameters from the verbal description
- Define the set of decision variables
- Formulate the objective function of data and decision variables
- Set up equality and/or inequality constraints

Formulation 1: Air Traffic Control I

Air plane $j, \quad j = 1, ..., n$ arrives at the airport within the time interval $[a_j, b_j]$ in the order of $1, 2, ..., n$. The airport wants to find the arrival time for each air plane such that the narrowest metering time (inter-arrival time between two consecutive airplanes) is the greatest.

Let: t_j be the arrival time of plane j. Then

$$\begin{array}{ll} \text{maximize} & \min_{j=1,...,n-1}(t_{j+1} - t_j) \\ \text{subject to} & a_j \le t_j \le b_j, \quad j = 1, 2, ..., n. \end{array}$$

Air Traffic Control II

Equivalent smooth formulation:

$$
\begin{array}{ll}
\text{maximize} & \Delta \\
\text{subject to} & t_2 - t_1 - \Delta \geq 0, \\
& t_3 - t_2 - \Delta \geq 0, \\
& ..., \\
& t_n - t_{n-1} - \Delta \geq 0, \\
& a_j \leq t_j \leq b_j, \quad j = 1, 2, ..., n.
\end{array}
$$

Formulation 2: Data Fitting I

Given data points $\mathbf{a}_j$, $j = 1, ..., n$, and the observation value c_j at data point $\mathbf{a}_j$, the least squares problem is to find $\mathbf{y}$ such that

$$\sum_j (\mathbf{a}_j^T \mathbf{y} - c_j)^2$$

is minimized.

Sometime, it is desired to minimize the p norm, where $p = 1$ or $p = \infty$,

$$\sum_j |\mathbf{a}_j^T \mathbf{y} - c_j|| \quad \text{or} \quad \max_j |\mathbf{a}_j^T \mathbf{y} - c_j||$$

Homework 1: Rewrite the problems as linear programs.

Data Fitting II

Constrained data fitting–Fingerprint Matching: c_j is the measured signal strength from base-station j at a location, and $\mathbf{a}_j$ contains base-station j's signal strengths for all known individual locations.

$$\begin{array}{rl} \text{minimize} & \sum_{j=1}^{n} |\mathbf{a}_j^T\mathbf{y} - c_j| \\ \text{subject to} & e^T\mathbf{y} = 1,\ y_i \in \{0,1\}. \end{array}$$

LP relaxation:

$$\begin{array}{rl} \text{minimize} & \sum_{j=1}^{n} |\mathbf{a}_j^T\mathbf{y} - c_j| \\ \text{subject to} & e^T\mathbf{y} = 1,\ \mathbf{y} \geq \mathbf{0}. \end{array}$$

Formulation 3: Minimize max-TSP-tour I

Given two-dimension sensor points $\mathbf{a}_j$, $j = 1, ..., n$, and the vehicle locations $\mathbf{b}_i$, $i = 1, ..., m$; find the best m clusters assigned to each vehicle such that the maximum of the TSP tour length is minimized.

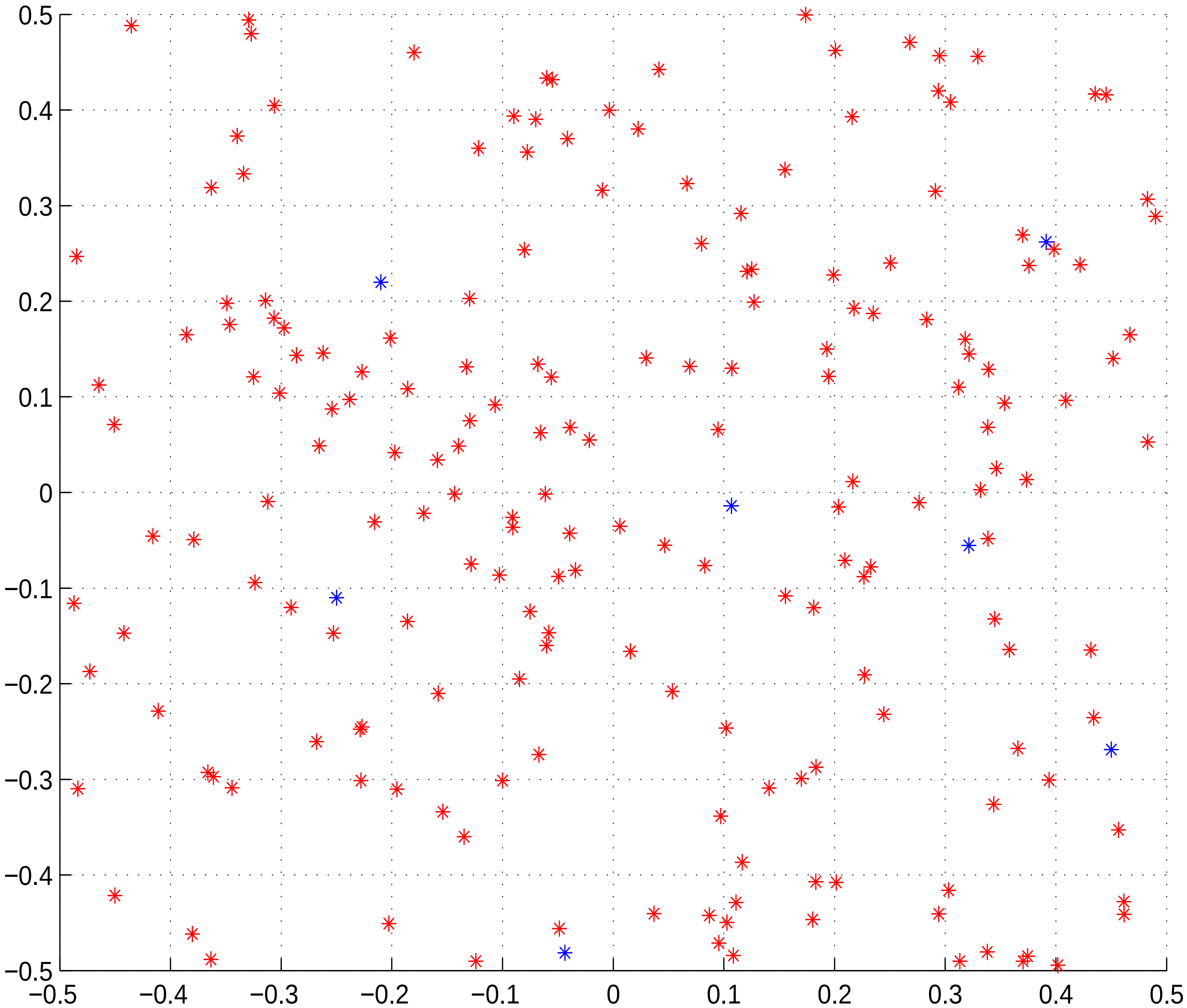


Figure 1: Base-Station Location

Minimize max-TSP-tour II

Let: x_{ij} be the binary decision variable to assign sensor point j to vehicle i.

Then

$$\begin{array}{rl} \text{minimize} & \sum_{i,j} \|\mathbf{a}_j - \mathbf{b}_i\| x_{ij} \\ \text{subject to} & \sum_j x_{ij} = \frac{n}{m}, \ \forall i, \\ & \sum_i x_{ij} = 1, \ \forall j, \\ & x_{ij} \in \{0, 1\}. \end{array}$$

Minimize max-TSP-tour III

Let: x_{ij} be the continuous variable to assign sensor point j to vehicle i. Then

$$
\begin{array}{ll}
\text{minimize} & \sum_{i,j} \|\mathbf{a}_j - \mathbf{b}_i\| x_{ij} \\
\text{subject to} & \sum_j x_{ij} = \frac{n}{m}, \ \forall i, \\
& \sum_i x_{ij} = 1, \ \forall j, \\
& x_{ij} \geq 0.
\end{array}
$$

There is no difference between the binary and continuous models. LP will generate an optimal binary solution!

Formulation 4: Supporting Vector Machine I

Suppose we have two-class discrimination data. We assign the first class with 1 and the second with -1. A powerful discrimination method is the Supporting Vector Machine (SVM).

Let the data point i be given by $\mathbf{a}_i \in R^d$, $i = 1, ..., n$. With this data set, we have some $\bar{y}_i = 1$ (in the first class) and the rest $\bar{y}_i = -1$ (in the second class).

We wish to find a hyper-plane in R^d to separate $\mathbf{a}_i$s with $\mathbf{y}_i = 1$ from $\mathbf{a}_j$s with $\mathbf{y}_j = -1$. Mathematically, we wish to find $\omega \in R^d$ and $\beta \in R$ such that

$$\mathbf{a}_i^T \omega + \beta > 0 \, \forall \{i : \bar{y}_i = 1\}$$

and

$$\mathbf{a}_i^T \omega + \beta < 0 \, \forall \{i : \bar{y}_i = -1\}.$$

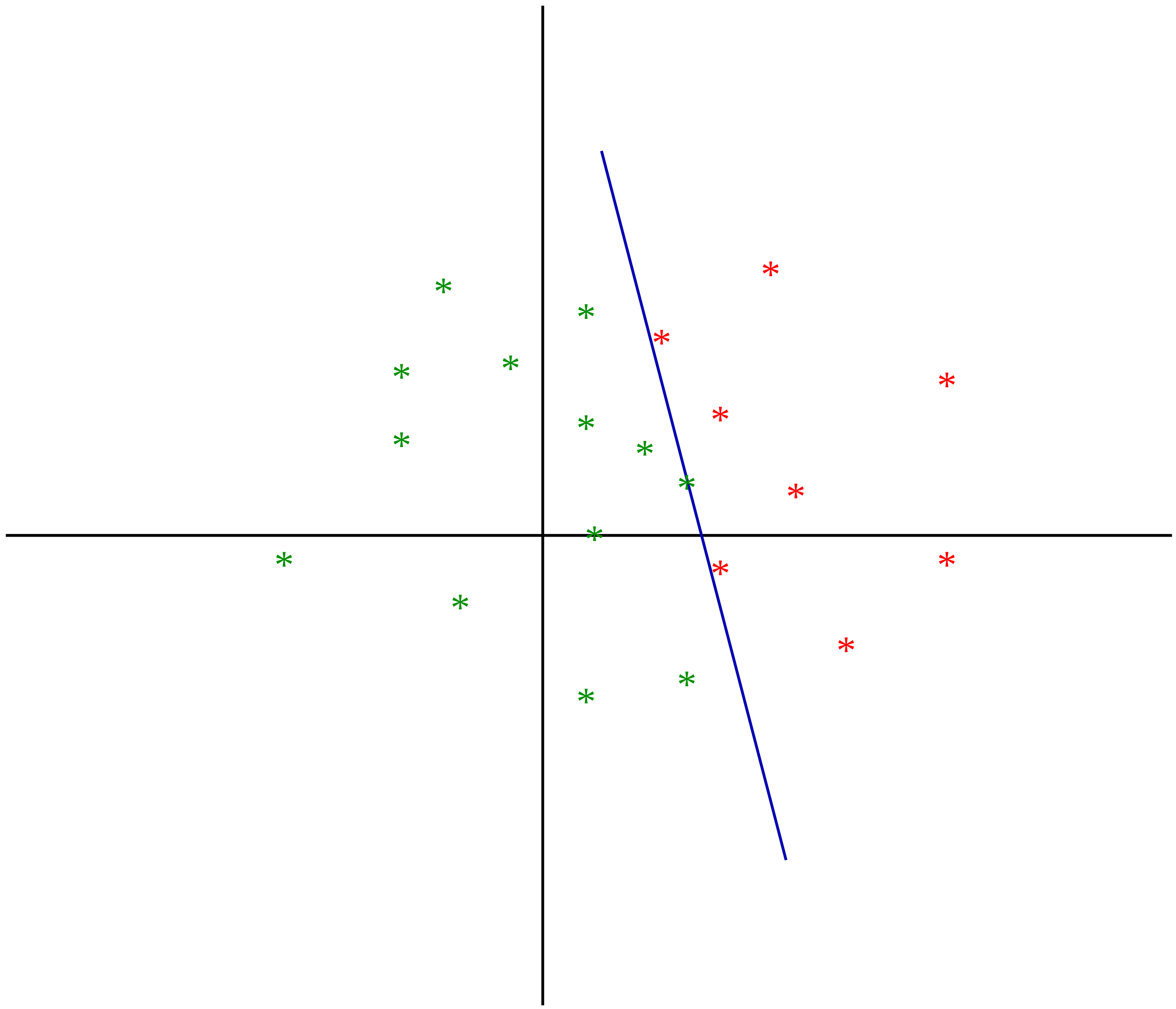


Figure 2: Linear Support Vector Machine

Supporting Vector Machine II

We wish to find a hyper-plane in R^d to separate $\mathbf{a}_i$s with red from $\mathbf{a}_j$s with green. Mathematically, we wish to find $\omega \in R^d$ and $\beta \in R$ such that

$$\mathbf{a}_i^T \omega + \beta > 1 \, \forall \{i : \bar{y}_i = 1\}$$

and

$$\mathbf{a}_i^T \omega + \beta < -1 \, \forall \{i : \bar{y}_i = -1\},$$

that is,

$$\bar{y}_i(\mathbf{a}_i^T \omega + \beta) > 1 \, \forall i.$$

The hyperp-lane would be

$$\{\mathbf{x} : \; \mathbf{x}^T \omega + \beta = 0\}.$$

Supporting Vector Machine III

If a clean separation is possible, we can formulate the problem as a maximization problem:

$$\begin{array}{rl} \text{minimize} & \|\omega\|_p \\ \text{subject to} & \bar{y}_i(\mathbf{a}_i^T\omega + \beta) > 1, \ \forall i. \end{array}$$

Supporting Vector Machine IV

A clean separation may not be possible for noisy data. Another formulation of the problem is a minimization problem:

$$\begin{array}{ll} \text{minimize} & \|\omega\|_p + \gamma \cdot \sum_{i=1}^n |e_i| \\ \text{subject to} & \bar{y}_i(\mathbf{a}_i^T \omega + \beta) > 1 + e_i, \ \forall i. \end{array}$$

Formulation 5: Combinatorial Auction I

Given the m different states that are mutually exclusive and exactly one of them will be true at the maturity.

A contract on a state is a paper agreement so that on maturity it is worth a notional \$$w$ if it is on the winning state and worth \$$0$ if is not on the winning state. There are n orders betting on one or a combination of states, with a price limit and a quantity limit.

Combinatorial Auction II: an order

The jth order is given as $(\mathbf{a}_j \in R^m_+,\ \pi_j \in R_+,\ q_j \in R_+)$: $\mathbf{a}_j$ is the combination betting vector where each component is either 1 or 0

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \ldots \\ a_{mj} \end{pmatrix},$$

where 1 is betting state and 0 is non-betting state; π_j is the price limit for one such a contract, and q_j is the maximum number of contracts the better like to buy.

Combinatorial Auction III: qualified orders

Let x_j be the number of contracts awarded to the jth order. Then, the jth bidder will pay the amount $\pi_j \cdot x_j$ and the total amount paid is $\pi^T \mathbf{x}$.

If the ith state is the winning state, then the auction organizer need to pay back

$$w \cdot \left(\sum_{j=1}^{n} a_{ij} x_j \right) = w \cdot \mathbf{a}_{i.}\mathbf{x}$$

The question is, how to decide $\mathbf{x} \in R^n$, that is, how to fill the orders.

Combinatorial Auction Pricing IV: Robust model

$$
\begin{array}{lrl}
\max & \pi^T \mathbf{x} - w \cdot \max_i \{\mathbf{a}_i.\mathbf{x}\} & \\
\text{s.t.} & \mathbf{x} & \leq \mathbf{q}, \\
 & \mathbf{x} & \geq \mathbf{0}.
\end{array}
$$

$\pi^T \mathbf{x}$: the amount can be collected.

$$
\begin{array}{lrl}
\max & \pi^T \mathbf{x} - w \cdot \max(A\mathbf{x}) & \\
\text{s.t.} & \mathbf{x} & \leq \mathbf{q}, \\
 & \mathbf{x} & \geq \mathbf{0}.
\end{array}
$$

Combinatorial Auction Pricing V: linear model

$$
\begin{array}{lrl}
\max & \pi^T \mathbf{x} - w \cdot s & \\
\text{s.t.} & A\mathbf{x} - \mathbf{e} \cdot s & \leq \mathbf{0}, \\
& \mathbf{x} & \leq \mathbf{q}, \\
& \mathbf{x} & \geq \mathbf{0}.
\end{array}
$$

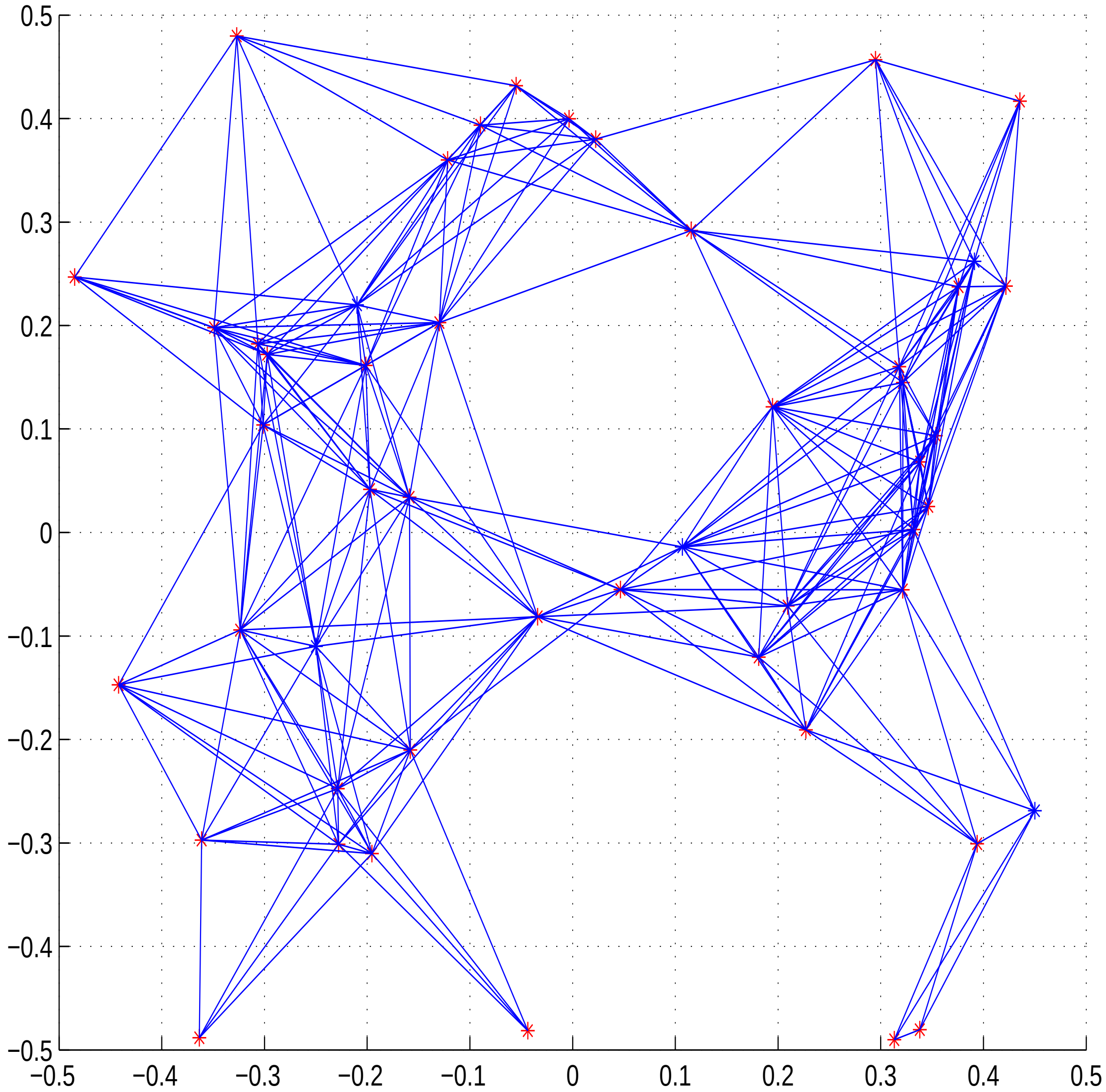
$\pi^T \mathbf{x}$: the potentiall revenue can be collected.

$w \cdot s$: the worst-case amount need to pay out (cost).

Formulation 6: Sensor Localization I

- **Input** m known points (anchors) $a_k \in \mathbf{R}^2$, $k = 1, ..., m$, and n unknown points (sensors or targets) $x_j \in \mathbf{R}^2$, $j = 1, ..., n$. For some pair of two points, we have a Euclidean distance measure $\hat{d}_{kj}$ between a_k and x_j, or distance measure $\hat{d}_{ij}$ between x_i and x_j.
- **Output** Position estimation for all unknown points, and confidence measures on reliability of each position estimation.
- **Objective** Robust, fast and accurate.

Figure 3: 50-Sensor Network with Radio Range .3



Sensor Localization II: Distance Geometry Model

System of nonlinear equations for x_i's:

$$\|x_i - x_j\| = d_{ij},\ \forall\,(i,j) \in N_x,\ i < j,$$

$$\|a_k - x_j\| = d_{kj},\ \forall\,(k,j) \in N_a,$$

Sensor Localization III: Convex Relaxation

System of nonlinear inequalities for x_i's:

$$\|x_i - x_j\| \le d_{ij},\ \forall\,(i,j) \in N_x,\ i < j,$$
$$\|a_k - x_j\| \le d_{kj},\ \forall\,(k,j) \in N_a,$$

Homework 2: Prove the solution set of $x_i,\ i = 1, ..., n$, is convex.