CONVERGENCE OF THE HETEROGENEOUS MULTISCALE FINITE ELEMENT METHOD FOR ELLIPTIC PROBLEM WITH NONSMOOTH MICROSTRUCTURES

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ABSTRACT. We propose a condition under which the heterogeneous multiscale finite element method converges for elliptic problem with nonsmooth coefficients, and obtain the optimal convergence rate for elliptic problem with nonsymmetric periodic coefficients that allow for nonsmooth microstructures.

1. INTRODUCTION

Consider the elliptic problem

$$\begin{cases} -\operatorname{div}\left(a^\epsilon(\boldsymbol{x})\nabla u^\epsilon(\boldsymbol{x})\right) = f(\boldsymbol{x}) & \boldsymbol{x}\in D\subset\mathbb{R}^d,\\ u^\epsilon(\boldsymbol{x}) = 0 & \boldsymbol{x}\in\partial D,\end{cases} \tag{1.1}$$

where ϵ is a small parameter that signifies explicitly the multiscale nature of the coefficient a^ϵ, which is not necessarily symmetric. We assume a^ϵ belongs to a set $\mathcal{M}(\lambda,\Lambda,D)$ that is defined as

$$\mathcal{M}(\lambda,\Lambda,D)\equiv\Big\{\mathcal{B}\in[L^\infty(D)]^{d^2}\ \mid\ (\mathcal{B}(\boldsymbol{x})\boldsymbol{\xi},\boldsymbol{\xi})\ge\lambda|\boldsymbol{\xi}|^2, |\mathcal{B}(\boldsymbol{x})\boldsymbol{\xi}|\le\Lambda|\boldsymbol{\xi}| \\ \text{for any } \boldsymbol{\xi}\in\mathbb{R}^d \text{ and } a.e.\ \boldsymbol{x} \text{ in } D\Big\},$$

where D is a bounded domain in $\mathbb{R}^d$, and $(\cdot,\cdot)$ denotes the inner product on $\mathbb{R}^d$ while $|\cdot|$ the corresponding norm.

On the analytic side, the following fact is known about (1.1). In the sense of H-convergence due to MURAT AND TARTAR [27], for every $a^\epsilon\in\mathcal{M}(\lambda,\Lambda,D)$ and $f\in H^{-1}(D)$ the sequence $\{u^\epsilon\}$ the solutions of (1.1) satisfies

$$\begin{aligned} u^\epsilon &\rightharpoonup U_0 && \text{weakly in}\quad H_0^1(D),\\ a^\epsilon\nabla u^\epsilon &\rightharpoonup \mathcal{A}\nabla U_0 && \text{weakly in}\quad [L^2(D)]^d,\end{aligned}$$

where U_0 is the solution of

$$\begin{cases} -\operatorname{div}\left(\mathcal{A}(\boldsymbol{x})\nabla U_0(\boldsymbol{x})\right) = f(\boldsymbol{x}) & \boldsymbol{x}\in D,\\ U_0(\boldsymbol{x}) = 0 & \boldsymbol{x}\in\partial D,\end{cases} \tag{1.2}$$

and $\mathcal{A}\in\mathcal{M}(\lambda,\Lambda^2/\lambda,D)$. Here $H_0^1(D), L^2(D)$, and $H^{-1}(D)$ are standard Sobolev spaces [1]. We denote the $L^2(D)$ inner product by $(\cdot,\cdot)$, and the $L^2(\widetilde{D})$ inner product by $(\cdot,\cdot)_{L^2(\widetilde{D})}$ for any measurable subset $\widetilde{D}\subset D$.

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The heterogeneous multiscale method (HMM) introduced by E and Engquist [13] is a general methodology for designing sublinear scaling algorithms by exploiting scale separation and other special features of the problem. Sublinear scaling algorithm is an algorithm whose computational cost scales sublinearly with that of a brute force solver. It consists of two components: selection of a macroscopic solver and estimating the missing macroscale data by solving locally the microscale problem. The finite element method is frequently employed as the macroscopic solver in HMM (HMM-FEM for short). The convergence behavior of HMM-FEM applied to (1.1) was well-understood (see [13] and [14]). The error between U_0 and the HMM-FEM solution also consists of two parts: one is the approximation error of the macroscopic solver; the other is the error committed in estimating the effective matrix (the missing macroscale data), which is referred to as $e(\mathrm{HMM})$.

E, Ming and Zhang [14] obtained the optimal estimate of $e(\mathrm{HMM})$ when a^ϵ is a locally periodic matrix. The optimality of the estimate was confirmed by the numerical examples in [23, 12]. Their main assumptions are: 1) a^ϵ is a symmetric matrix; 2) each entry of a^ϵ is a smooth function; 3) the microscale problem is subject to the Dirichlet boundary condition. Moreover, they implicitly used a regularity assumption that was never proved; see (3.5). Although Du and Ming recently extended this result to the case when a^ϵ is a nonsymmetric matrix [12], it is still unknown under which condition $e(\mathrm{HMM})$ converges to zero if there is no further assumption on the coefficients beyond $a^\epsilon \in \mathcal{M}(\lambda, \Lambda, D)$ and to what degree the smoothness assumption on the coefficients is weakened while keeping the optimal convergence rate of $e(\mathrm{HMM})$. Besides the Dirichlet boundary condition, the microscale problem may be supplemented with other boundary conditions such as periodic and Neumann boundary conditions. They are often used in practice [26] and the Neumann microscale problem appeared in the original formulation of HMM-FEM [13]. However, the estimate of $e(\mathrm{HMM})$ has not been justified yet when the microscale problem is subject to either of the two boundary conditions.

This paper is aimed at answering the above questions and could be viewed as a follow-up of [14]. First we shall propose a condition under which $e(\mathrm{HMM})$ converges to zero. Second we shall prove the optimal convergence rate of $e(\mathrm{HMM})$ with the very weak smoothness assumption on a^ϵ that allows for nonsmooth microstructures when it is a locally periodic matrix. The proof equally applies to the three types of microscale problems. It will be found that the regularity assumption (3.5) is unnecessary, which, however, was used explicitly or implicitly in all the previous works for HMM-FEM and also for other numerical homogenization methods; see e.g. [19].

It is worth mentioning that E, Ming and Zhang [14] also estimated $e(\mathrm{HMM})$ when a^ϵ is a stationary random field. We shall not discuss this case since the proof is much more involved, but some technical results established in this paper may be helpful to improve the estimates in [14].

The remaining part of this paper is as follows. We formulate HMM-FEM in § 2 and present a condition under which $e(\mathrm{HMM})$ converges to zero. In § 3, we estimate the convergence rate of $e(\mathrm{HMM})$ when a^ϵ is a locally periodic matrix. In the last section we summarize our results and discuss certain extensions.

2. HMM-FEM Formulation and Convergence

We only present the simplest version of HMM-FEM in this section and refer to [14] for more details. The linear finite element method is employed as the macroscopic solver and the finite element space is denoted by X_H corresponding to the triangulation $\mathcal{T}_H$ with mesh size H. For each $V \in X_H$, we can see that ∇V is a piecewise constant vector. This fact will be frequently used throughout this paper.

The HMM-FEM solution $U_H \in X_H$ satisfies

$$\sum_{K\in\mathcal{T}_H} |K|\nabla V\cdot\mathcal{A}_H(\boldsymbol{x}_K)\nabla U_H = (f,V) \qquad \text{for all } V\in X_H, \tag{2.1}$$

where $\boldsymbol{x}_K$ is the barycenter of the element K.

The missing macroscale data is the effective matrix $\mathcal{A}_H(\boldsymbol{x}_K)$ which can be evaluated by

$$\mathcal{A}_H(\boldsymbol{x}_K)\left\langle\nabla v^\epsilon\right\rangle_{I_\delta} \equiv \left\langle a^\epsilon\nabla v^\epsilon\right\rangle_{I_\delta}, \tag{2.2}$$

where $I_\delta \equiv \boldsymbol{x}_K + \delta Y$ with $Y\equiv(-1/2,1/2)^d$, and δ is the size of the cell. We use $\langle\cdot\rangle_{I_\delta}$ to denote the integral mean over I_δ. Here $v^\epsilon - V\in\mathcal{V}$ satisfies

$$(a^\epsilon\nabla v^\epsilon,\nabla\varphi)_{L^2(I_\delta)} = 0 \qquad \text{for all } \varphi\in\mathcal{V}. \tag{2.3}$$

We call (2.3) the Dirichlet cell problem if $\mathcal{V} = V_{\mathrm{D}}^0 \equiv H_0^1(I_\delta)$. We call (2.3) the periodic cell problem if

$$\mathcal{V} = V_{\mathrm{P}}^0 \equiv \left\{\,\phi\in H^1_{\mathrm{per}}(I_\delta)\ \mid\ \langle\phi\rangle_{I_\delta} = 0\,\right\},$$

where $H^1_{\mathrm{per}}(I_\delta)$ is the closure of $C^\infty_{\mathrm{per}}(I_\delta)$ for the H^1 norm, and $C^\infty_{\mathrm{per}}(I_\delta)$ is the subset of $C^\infty(I_\delta)$ of $I_\delta-$periodic functions [9]. We call (2.3) the Neumann cell problem if

$$\mathcal{V} = V_{\mathrm{N}}^0 \equiv \left\{\,\phi\in H^1(I_\delta)\ \mid\ \langle\nabla\phi\rangle_{I_\delta} = \mathbf{0}\,\right\}.$$

In the numerical computation, we choose the boundary data V in the cell problem (2.3) as $\boldsymbol{e}_i\cdot\boldsymbol{x}$ where $\{\boldsymbol{e}_i\}_{i=1}^d$ are the canonical basis. Denote by ϕ_i^ϵ the solution of the corresponding cell problem. Using the definition (2.2) and the constraint (2.4), we obtain

$$\mathcal{A}_H(\boldsymbol{x}_K)\boldsymbol{e}_i = \mathcal{A}_H(\boldsymbol{x}_K)\nabla(\boldsymbol{e}_i\cdot\boldsymbol{x}) = \mathcal{A}_H(\boldsymbol{x}_K)\left\langle\nabla\phi_i^\epsilon\right\rangle_{I_\delta} = \left\langle a^\epsilon\nabla\phi_i^\epsilon\right\rangle_{I_\delta}.$$

Therefore, the effective matrix is given by

$$\mathcal{A}_H(\boldsymbol{x}_K) \equiv \left(\left\langle a^\epsilon\nabla\phi_1^\epsilon\right\rangle_{I_\delta},\dots,\left\langle a^\epsilon\nabla\phi_d^\epsilon\right\rangle_{I_\delta}\right).$$

We consider the above three types of cell problems in this paper, and refer to [29, 11, 12] for their implementation details. The discussion on the other types of cell problems can be found in [29, 17].

It may be easily checked that the solutions of the above three cell problems satisfy the following constraint:

$$\left\langle\nabla v^\epsilon\right\rangle_{I_\delta} = \nabla V. \tag{2.4}$$

Indeed, if $\mathcal{V} = V_{\mathrm{N}}^0$ then the identity is true by the definition; if $\mathcal{V} = \mathcal{V}_{\mathrm{D}}^0$ or V_{P}^0 then integrating by parts we obtain

$$\left\langle\nabla(v^\epsilon - V)\right\rangle_{I_\delta} = \frac{1}{|I_\delta|}\int_{I_\delta}\nabla(v^\epsilon - V)\,\mathrm{d}\boldsymbol{x} = \frac{1}{|I_\delta|}\int_{\partial I_\delta}(v^\epsilon - V)\mathrm{d}s = 0.$$

It follows from the definition of the cell problem (2.3) that

$$(a^\epsilon\nabla v^\epsilon,\nabla v^\epsilon)_{L^2(I_\delta)} = (a^\epsilon\nabla v^\epsilon,\nabla V)_{L^2(I_\delta)} \tag{2.5}$$

for v^ϵ the solutions of the cell problems subject to any one of the three boundary conditions. Using (2.4), we write the above relation as

$$\left\langle\nabla v^\epsilon\cdot a^\epsilon\nabla v^\epsilon\right\rangle_{I_\delta} = \nabla V\cdot\left\langle a^\epsilon\nabla v^\epsilon\right\rangle_{I_\delta} = \left\langle\nabla v^\epsilon\right\rangle_{I_\delta}\cdot\left\langle a^\epsilon\nabla v^\epsilon\right\rangle_{I_\delta},$$

which is nothing but the so-called *Hill's condition* [18]. This is actually the starting point for analyzing HMM-FEM, which has been extensively exploited in [14].

The original definition of the effective matrix in [13] and [14, (1.7)] is based on the energy, which reads as:

$$\langle \nabla w^\epsilon \rangle_{I_\delta} \cdot \mathcal{A}_H(\boldsymbol{x}_K) \langle \nabla v^\epsilon \rangle_{I_\delta} \equiv \langle \nabla w^\epsilon \cdot a^\epsilon \nabla v^\epsilon \rangle_{I_\delta},$$

where w^ϵ is defined in the same manner with v^ϵ except that the boundary value V is replaced by any $W \in X_H$. This definition is equivalent to the flux-based definition (2.2) due to the relations (2.4) and (2.5). It seems that the definition (2.2) could simplify the proof considerably.

It follows from (2.5) that

$$\|\nabla v^\epsilon\|_{L^2(I_\delta)} \le \frac{\Lambda}{\lambda} \|\nabla V\|_{L^2(I_\delta)}. \tag{2.6}$$

Using (2.4), we have

$$\|\nabla v^\epsilon\|^2_{L^2(I_\delta)} = \|\nabla V\|^2_{L^2(I_\delta)} + \|\nabla(v^\epsilon - V)\|^2_{L^2(I_\delta)}. \tag{2.7}$$

Using (2.7) and (2.6), and proceeding along the same lines of [12, Lemma 3.1], we obtain, at each $\boldsymbol{x}_K$,

$$(\mathcal{A}_H(\boldsymbol{x}_K)\boldsymbol{\xi}, \boldsymbol{\xi}) \ge \lambda |\boldsymbol{\xi}|^2 \qquad \text{and} \qquad |\mathcal{A}_H(\boldsymbol{x}_K)\boldsymbol{\xi}| \le \frac{\Lambda^2}{\lambda}|\boldsymbol{\xi}| \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^d, \tag{2.8}$$

which gives the existence and uniqueness of the HMM-FEM solution.

The following error estimate is based on the theorem of Berger, Scott and Strang [5], and can be found in [14, Theorem 1.1] except explicit constants in the estimate. We give a proof for the readers' convenience. In order to state this result, we define

$$\tilde{e}(\text{HMM}) \equiv \max_{K \in \mathcal{T}_H} \| \langle \mathcal{A} \rangle_K - \mathcal{A}_H(\boldsymbol{x}_K)\|_F,$$

where $\|\cdot\|_F$ is the Euclidean norm.

Lemma 2.1. *Let U_0 and U_H be the solutions of* (1.2) *and* (2.1)*, respectively. Then,*

$$\|\nabla(U_0 - U_H)\|_{L^2(D)} \le \frac{\Lambda}{\lambda} \inf_{V \in X_H} \|\nabla(U_0 - V)\|_{L^2(D)} + \frac{c_p}{\lambda^2}\|f\|_{H^{-1}(D)} \tilde{e}(HMM), \tag{2.9}$$

where c_p is the constant in the following discrete Poincaré's inequality

$$\|V\|_{H^1(D)} \le c_p \|\nabla V\|_{L^2(D)} \qquad \textit{for all } V \in X_H.$$

Proof. Let $\widehat{U}_0 \in X_H$ be the solution of

$$\left(\mathcal{A}\nabla \widehat{U}_0, \nabla V\right) = (\mathcal{A}\nabla U_0, \nabla V) \qquad \text{for all } V \in X_H.$$

Defining $W \equiv U_H - \widehat{U}_0$ and using the above equation we obtain

$$\begin{aligned}
(\mathcal{A}\nabla W, \nabla W) &= (\mathcal{A}\nabla U_H, \nabla W) - \left(\mathcal{A}\nabla \widehat{U}_0, \nabla W\right) = (\mathcal{A}\nabla U_H, \nabla W) - (\mathcal{A}\nabla U_0, \nabla W) \\
&= (\mathcal{A}\nabla U_H, \nabla W) - \sum_{K \in \mathcal{T}_H} |K| \nabla W \cdot \mathcal{A}_H(\boldsymbol{x}_K) \nabla U_H \\
&= \sum_{K \in \mathcal{T}_H} ((\langle \mathcal{A} \rangle_K - \mathcal{A}_H(\boldsymbol{x}_K))\nabla U_H, \nabla W)_{L^2(K)}.
\end{aligned}$$

Therefore, we obtain

$$\left\|\nabla(U_H - \widehat{U}_0)\right\|_{L^2(D)} \le \lambda^{-1} \tilde{e}(\text{HMM}) \|\nabla U_H\|_{L^2(D)}.$$

By Céa's lemma [8][1], we have

$$\left\|\nabla(U_0-\widehat{U}_0)\right\|_{L^2(D)}\le\frac{\Lambda}{\lambda}\inf_{V\in X_H}\|\nabla(U_0-V)\|_{L^2(D)}.$$

Using (2.8), we obtain

$$\|\nabla U_H\|_{L^2(D)}\le\frac{c_p}{\lambda}\|f\|_{H^{-1}(D)}.$$

Combining the above three inequalities, we come to (2.9). □

By [8, Theorem 3.2.3], we have

$$\lim_{H\to 0}\inf_{V\in X_H}\|\nabla(U_0-V)\|_{L^2(D)}=0$$

since $U_0\in H^1(D)$.

Next we prove $\tilde{e}(\mathrm{HMM})\to 0$ as $\epsilon,\delta\to 0$.

Lemma 2.2. *For any $K\in\mathcal{T}_H$, if $\boldsymbol{x}_K$ is a Lebesgue point of $\mathcal{A}$ and*

$$\max_{K\in\mathcal{T}_H}\|\langle\mathcal{A}\rangle_K-\mathcal{A}(\boldsymbol{x}_K)\|_F\to 0\qquad\text{as}\quad H\to 0,\tag{2.10}$$

then

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\tilde{e}(HMM)=0.\tag{2.11}$$

Proof. By H-convergence theory [27], the $H-$limit of Problem (2.3) satisfies

$$\begin{cases}(\mathcal{A}(\boldsymbol{x})\nabla V_0,\nabla\varphi)_{L^2(I_\delta)}=0 & \text{for all}\quad \varphi\in\mathcal{V},\\ \langle\nabla(V_0-V)\rangle_{I_\delta}=0.\end{cases}\tag{2.12}$$

By (2.6),

$$\|\nabla V_0\|_{L^2(I_\delta)}\le\liminf_{\epsilon\to 0}\|\nabla v^\epsilon\|_{L^2(I_\delta)}\le\frac{\Lambda}{\lambda}\|\nabla V\|_{L^2(I_\delta)}.$$

Using the definition of H-convergence and $(2.12)_2$, we obtain

$$\begin{aligned}(\mathcal{A}_H(\boldsymbol{x}_K)-\langle\mathcal{A}\rangle_K)\nabla V&=\langle a^\epsilon\nabla v^\epsilon\rangle_{I_\delta}-\langle\mathcal{A}\rangle_K\nabla V\\ &\xrightarrow{\epsilon\to 0}\langle\mathcal{A}(\boldsymbol{x})\nabla V_0\rangle_{I_\delta}-\langle\mathcal{A}\rangle_K\nabla V\\ &=\langle(\mathcal{A}(\boldsymbol{x})-\mathcal{A}(\boldsymbol{x}_K))\nabla V_0\rangle_{I_\delta}+(\mathcal{A}(\boldsymbol{x}_K)-\langle\mathcal{A}\rangle_K)\nabla V.\end{aligned}$$

The right hand side of the above equation is bounded by

$$\frac{\Lambda}{\lambda}\left\langle\|\mathcal{A}(\boldsymbol{x})-\mathcal{A}(\boldsymbol{x}_K)\|_F^2\right\rangle_{I_\delta}^{1/2}|\nabla V|+\|\langle\mathcal{A}\rangle_K-\mathcal{A}(\boldsymbol{x}_K)\|_F|\nabla V|.\tag{2.13}$$

Since $\boldsymbol{x}_K$ is a Lesbegue point of $\mathcal{A}$, by [15, Corollary 1 in §1.7], we have

$$\lim_{\delta\to 0}\left\langle\|\mathcal{A}(\boldsymbol{x})-\mathcal{A}(\boldsymbol{x}_K)\|_F^2\right\rangle_{I_\delta}=0.$$

Hence the first term of (2.13) tends to zero as $\delta\to 0$. The second term of (2.13) tends to zero due to the condition (2.10). This implies (2.11) by the definition of $\tilde{e}(\mathrm{HMM})$. □

If each entry of $\mathcal{A}$ is a continuous function, then (2.10) holds true. However, if any entry of $\mathcal{A}$ is discontinuous at the quadrature node $\boldsymbol{x}_K$, then (2.10) is invalid.

Lemma 2.2 is actually a reformulation of the so-called *principle of periodic localization* in H-convergence [30, (5.15)]. The above proof seems more direct.

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[1]The present form can be found in Xu and Zikatanov [28]

3. Estimate of e(HMM) with Nonsmooth Periodic Microstructute

So far we make no assumption on the form of the coefficient except that $a^\epsilon \in \mathcal{M}(\lambda, \Lambda, D)$. In order to obtain the quantitative estimate of $\tilde{e}$(HMM), we assume that a^ϵ is a locally periodic matrix, i.e., $a^\epsilon(\boldsymbol{x}) = a(\boldsymbol{x}, \boldsymbol{x}/\epsilon)$ and $a(\boldsymbol{x}, \boldsymbol{y})$ is periodic in $\boldsymbol{y}$ with period Y. By [4], the effective matrix $\mathcal{A}$ is given by

$$\mathcal{A}_{ij}(\boldsymbol{x}) = ⨍_Y \left(a_{ij} + a_{ik}\frac{\partial \chi^j}{\partial y_k}\right)(\boldsymbol{x}, \boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \qquad i, j = 1, \dots, d, \tag{3.1}$$

where $\boldsymbol{\chi}(\boldsymbol{x}, \boldsymbol{y}) = \{\chi^j(\boldsymbol{x}, \boldsymbol{y})\}_{j=1}^d$ is periodic in $\boldsymbol{y}$ with period Y and it satisfies

$$-\frac{\partial}{\partial y_i}\left(a_{ik}\frac{\partial \chi^j}{\partial y_k}\right)(\boldsymbol{x}, \boldsymbol{y}) = \left(\frac{\partial}{\partial y_i}a_{ij}\right)(\boldsymbol{x}, \boldsymbol{y}) \quad \text{in } Y, \qquad \int_Y \chi^j(\boldsymbol{x}, \boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = 0. \tag{3.2}$$

This problem is solvable and an integration by parts gives: for $j = 1, \dots, d$,

$$\|\nabla_{\boldsymbol{y}}\chi^j(\boldsymbol{x}, \boldsymbol{y})\|_{L^2(Y)} \le \frac{\Lambda}{\lambda} \quad \text{for all } \boldsymbol{x} \in D \text{ and } \boldsymbol{y} \in Y. \tag{3.3}$$

We further assume that $a(\boldsymbol{x}, \boldsymbol{y}) \in C^{0,1}(D; L^\infty(Y))$. By (3.1) and Lemma 2.2, we have if $H \to 0$ then

$$e(\text{HMM}) = \tilde{e}(\text{HMM}).$$

Therefore, we change the definition of $\tilde{e}$(HMM) to its original form [13]:

$$e(\text{HMM}) \equiv \max_{K \in \mathcal{T}_H} \|(\mathcal{A} - \mathcal{A}_H)(\boldsymbol{x}_K)\|_F.$$

When a^ϵ is a locally periodic matrix and the cell problem is subject to the Dirichlet boundary condition, E, Ming and Zhang [14] proved

$$e(\text{HMM}) \le C\left(\frac{\epsilon}{\delta} + \delta\right). \tag{3.4}$$

If the periodic boundary condition is used for the cell problem and the cell size is an integer time of the period, then

$$e(\text{HMM}) \le C\delta.$$

However, the exact period is usually unknown due to the uncertainty of the input data [10, 3]. This means that the cell problems are usually posed over a cell whose size is not necessarily an integer time of the period.

Their proof depends on the following assumptions: 1)a^ϵ is symmetric; 2)the gradient of $\boldsymbol{\chi}$ is uniformly bounded, i.e., there exists C such that for $i, j = 1, \dots, d$,

$$|\partial_{y_i}\chi^j(\boldsymbol{x}, \boldsymbol{y})| \le C \quad \text{for all } \boldsymbol{x} \in D \text{ and } \boldsymbol{y} \in Y. \tag{3.5}$$

Both assumptions are unrealistic. On the one hand, the symmetry assumption on a^ϵ is not appropriate in the conductivity problem when a magnetic field is present: Hall effect is the consequence of an anti-symmetric part in the conductivity tensor (see [21] for a discussion); on the other hand, the assumption (3.5) has not been thoroughly studied. Li and Vogelius [22] proved that the gradient of $\boldsymbol{\chi}$ is indeed uniformly bounded in the interior of the domain for the Dirichlet boundary value problems. However, less is known about the assumption (3.5) for the periodic, the Neumann, and the mixed boundary value problems which are often used in practice. Moreover, the result obtained in [22] does not apply to many important composite materials such as the fibre-reinforced composites with a touching angle for the neighboring fibres [2]; see also [7, 20] for more examples.

In this section, we shall prove (3.4) *without* the above assumptions. In order to remove the symmetric assumption on a^ϵ, we resort to the dual problem of the cell problem as in [12]; in order to remove the smoothness assumption on $\boldsymbol{\chi}$, we shall derive some new a priori estimates for the solutions of the cell problems. We shall give a unified proof for the cell problem of the Dirichlet, the periodic and the Neumann type, while the latter two were not considered in [14]. The proof is different from that in [14, 24, 12] since we employ the flux-based definition of the effective matrix.

Let $\hat{v}^\epsilon$ be the solution of the cell problem (2.3) with a^ϵ replaced by $a^\epsilon_K = a(\boldsymbol{x}_K, \boldsymbol{x}/\epsilon)$. Proceeding along the same lines that lead to (2.4) and (2.5), we have

$$\langle \nabla \hat{v}^\epsilon \rangle_{I_\delta} = \nabla V \tag{3.6}$$

and

$$(a^\epsilon_K \nabla \hat{v}^\epsilon, \nabla \hat{v}^\epsilon)_{L^2(I_\delta)} = (a^\epsilon_K \nabla \hat{v}^\epsilon, \nabla V)_{L^2(I_\delta)}\,. \tag{3.7}$$

Define

$$\widehat{V}^\epsilon \equiv V + \epsilon(\boldsymbol{\chi}_K \cdot \nabla)V, \tag{3.8}$$

where $\boldsymbol{\chi}_K = \boldsymbol{\chi}(\boldsymbol{x}_K, \boldsymbol{x}/\epsilon)$. By (3.1) and $(3.2)_2$, a straightforward calculation gives

$$\left\langle a^\epsilon_K \nabla \widehat{V}^\epsilon \right\rangle_{I_{\kappa\epsilon}} = \mathcal{A}(\boldsymbol{x}_K)\nabla V \qquad \text{and} \qquad \left\langle \nabla \widehat{V}^\epsilon \right\rangle_{I_{\kappa\epsilon}} = \nabla V, \tag{3.9}$$

where $I_{\kappa\epsilon} \equiv \boldsymbol{x}_K + \epsilon\kappa Y$ and κ is the integer part of δ/ϵ, i.e., $\kappa = \lfloor \delta/\epsilon \rfloor$. The above relation immediately implies

$$\begin{aligned}(\mathcal{A}_H - \mathcal{A})(\boldsymbol{x}_K)\nabla V &= \langle a^\epsilon \nabla v^\epsilon \rangle_{I_\delta} - \left\langle a^\epsilon_K \nabla \widehat{V}^\epsilon \right\rangle_{I_{\kappa\epsilon}} \\ &= \langle a^\epsilon \nabla v^\epsilon - a^\epsilon_K \nabla \hat{v}^\epsilon \rangle_{I_\delta} + \langle a^\epsilon_K \nabla \theta^\epsilon \rangle_{I_\delta} + \left(\left\langle a^\epsilon_K \nabla \widehat{V}^\epsilon \right\rangle_{I_\delta} - \left\langle a^\epsilon_K \nabla \widehat{V}^\epsilon \right\rangle_{I_{\kappa\epsilon}} \right),\end{aligned} \tag{3.10}$$

where $\theta^\epsilon \equiv \hat{v}^\epsilon - \widehat{V}^\epsilon$. This is the starting point to estimate $e(\text{HMM})$.

The following estimates of $\widehat{V}^\epsilon$ are old [14], while the proof is new and without using the assumption (3.5). In order to prove the estimates in (3.11), we define a cut-off function $\rho^\epsilon \in C^\infty_0(I_\delta)$ that satisfies $|\nabla \rho^\epsilon| \le C/\epsilon$, and

$$\rho^\epsilon(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \operatorname{dist}(\boldsymbol{x}, \partial I_\delta) \ge 2\epsilon, \\ 0 & \text{if } \operatorname{dist}(\boldsymbol{x}, \partial I_\delta) \le \epsilon. \end{cases}$$

Lemma 3.1. *Let $\widehat{V}^\epsilon$ be defined as* (3.8)*. Then*

$$\begin{aligned} \left\| \nabla \widehat{V}^\epsilon \right\|_{L^2(I_\delta \setminus I_{\kappa\epsilon})} &\le \sqrt{2^d - 1}\frac{\Lambda}{\lambda}\left(\frac{\epsilon}{\delta}\right)^{1/2} \|\nabla V\|_{L^2(I_\delta)}\,, \\ \left\| \nabla[(\widehat{V}^\epsilon - V)(1-\rho^\epsilon)] \right\|_{L^2(I_\delta)} &\le C\frac{\Lambda}{\lambda}\left(\frac{\epsilon}{\delta}\right)^{1/2} \|\nabla V\|_{L^2(I_\delta)}\,. \end{aligned} \tag{3.11}$$

Proof. It follows from (3.2) and the definition of $\widehat{V}^\epsilon$ that

$$\left(a^\epsilon_K \nabla \widehat{V}^\epsilon, \nabla \varphi \right)_{L^2(I_{(\kappa+1)\epsilon} \setminus I_{\kappa\epsilon})} = 0 \qquad \text{for all} \quad \varphi \in H^1_{\text{per}}(I_{(\kappa+1)\epsilon} \setminus I_{\kappa\epsilon}). \tag{3.12}$$

Taking $\varphi = \widehat{V}^\epsilon - V$ in (3.12) we obtain

$$\left(a^\epsilon_K \nabla \widehat{V}^\epsilon, \nabla \widehat{V}^\epsilon \right)_{L^2(I_{(\kappa+1)\epsilon} \setminus I_{\kappa\epsilon})} = \left(a^\epsilon_K \nabla \widehat{V}^\epsilon, \nabla V \right)_{L^2(I_{(\kappa+1)\epsilon} \setminus I_{\kappa\epsilon})},$$

which immediately implies

$$\left\| \nabla \widehat{V}^\epsilon \right\|_{L^2(I_{(\kappa+1)\epsilon} \setminus I_{\kappa\epsilon})} \le \sqrt{2^d - 1}\frac{\Lambda}{\lambda}\left(\frac{\epsilon}{\delta}\right)^{1/2} \|\nabla V\|_{L^2(I_\delta)}\,.$$

This inequality together with

$$\left\|\nabla\widehat{V}^\epsilon\right\|_{L^2(I_\delta\setminus I_{\kappa\epsilon})}\le\left\|\nabla\widehat{V}^\epsilon\right\|_{L^2(I_{(\kappa+1)\epsilon}\setminus I_{\kappa\epsilon})}$$

leads to $(3.11)_1$.

Using (3.3), we obtain

$$\begin{aligned}\left\|(\widehat{V}^\epsilon-V)\nabla(1-\rho^\epsilon)\right\|_{L^2(I_\delta)}&\le\frac{C}{\epsilon}\left\|\widehat{V}^\epsilon-V\right\|_{L^2(I_\delta\setminus I_{\kappa\epsilon})}\le C\left\|\boldsymbol{\chi}_K\right\|_{L^2(I_\delta\setminus I_{\kappa\epsilon})}|\nabla V|\\&\le C\left\|\boldsymbol{\chi}_K\right\|_{L^2(I_{(\kappa+1)\epsilon}\setminus I_{\kappa\epsilon})}|\nabla V|\\&\le C(\delta^{d-1}\epsilon)^{1/2}\left\|\boldsymbol{\chi}_K\right\|_{L^2(Y)}|\nabla V|\\&\le C\frac{\Lambda}{\lambda}\left(\frac{\epsilon}{\delta}\right)^{1/2}\left\|\nabla V\right\|_{L^2(I_\delta)}.\end{aligned}$$

It follows from $(3.11)_1$ that

$$\left\|(1-\rho^\epsilon)\nabla(\widehat{V}^\epsilon-V)\right\|_{L^2(I_\delta)}\le\left\|\nabla(\widehat{V}^\epsilon-V)\right\|_{L^2(I_\delta\setminus I_{\kappa\epsilon})}\le\sqrt{2^d-1}\left(1+\frac{\Lambda}{\lambda}\right)\left(\frac{\epsilon}{\delta}\right)^{1/2}\left\|\nabla V\right\|_{L^2(I_\delta)}.$$

Combining the above two inequalities we get $(3.11)_2$. □

Using the above estimates for $\widehat{V}^\epsilon$, we shall bound the *corrector* θ^ϵ. It can be easily checked that

$$(a_K^\epsilon\nabla\theta^\epsilon,\nabla\varphi)_{L^2(I_\delta)}=0\qquad\text{for all}\quad\varphi\in H_0^1(I_\delta).\tag{3.13}$$

The estimate of θ^ϵ arising from the Dirichlet cell problem can be found in [14, (3.9)]; see also Remark 3.3. The proof therein is an analogy of the so-called corrector estimate in the homogenization theory [30], and relies on the assumption (3.5) which is not used in our proof. We also estimate θ^ϵ arising from the periodic and Neumann cell problems. Such estimate is new and key to bound $e(\text{HMM})$. A simple analogy of the corresponding corrector estimate does not work since the cell problem (3.7) with $\mathcal{V}=V_{\text{N}}$ is not a standard Neumann boundary value problem dealt with in [30].

Lemma 3.2. *Let $\hat{v}^\epsilon$ be the solution of (3.7) and $\theta^\epsilon\equiv\hat{v}^\epsilon-\widehat{V}^\epsilon$. Then*

$$\left\|\nabla\theta^\epsilon\right\|_{L^2(I_\delta)}\le C\frac{\Lambda^2}{\lambda^2}\left(\frac{\epsilon}{\delta}\right)^{1/2}\left\|\nabla V\right\|_{L^2(I_\delta)}.\tag{3.14}$$

Proof. Define $P^\epsilon\equiv a_K^\epsilon\nabla\widehat{V}^\epsilon$ and $P^0\equiv\mathcal{A}(\boldsymbol{x}_K)\nabla V$. Using (3.7), we obtain

$$\begin{aligned}(a_K^\epsilon\nabla\hat{v}^\epsilon,\nabla\theta^\epsilon)_{L^2(I_\delta)}&=\left(a_K^\epsilon\nabla\hat{v}^\epsilon,\nabla(V-\widehat{V}^\epsilon)\right)_{L^2(I_\delta)}\\&=\left(a_K^\epsilon\nabla\theta^\epsilon,\nabla(V-\widehat{V}^\epsilon)\right)_{L^2(I_\delta)}+\left(a_K^\epsilon\nabla\widehat{V}^\epsilon,\nabla(V-\widehat{V}^\epsilon)\right)_{L^2(I_\delta)}.\end{aligned}$$

By (3.13) and (3.12), we have

$$\left(a_K^\epsilon\nabla\theta^\epsilon,\nabla[(V-\widehat{V}^\epsilon)\rho^\epsilon]\right)_{L^2(I_\delta)}=0,\quad\left(a_K^\epsilon\nabla\widehat{V}^\epsilon,\nabla(V-\widehat{V}^\epsilon)\right)_{L^2(I_{\kappa\epsilon})}=0.$$

It follows from the above three equations that

$$(a_K^\epsilon\nabla\hat{v}^\epsilon,\nabla\theta^\epsilon)_{L^2(I_\delta)}=\left(a_K^\epsilon\nabla\theta^\epsilon,\nabla[(V-\widehat{V}^\epsilon)(1-\rho^\epsilon)]\right)_{L^2(I_\delta)}+\left(a_K^\epsilon\nabla\widehat{V}^\epsilon,\nabla(V-\widehat{V}^\epsilon)\right)_{L^2(I_\delta\setminus I_{\kappa\epsilon})}.$$

Using (3.6), $(3.9)_2$, and the fact that P^0 is a constant vector over I_δ, we obtain

$$\left(P^0,\nabla\theta^\epsilon\right)_{L^2(I_\delta)}=\left(P^0,\nabla(V-\widehat{V}^\epsilon)\right)_{L^2(I_\delta)}=\left(P^0,\nabla(V-\widehat{V}^\epsilon)\right)_{L^2(I_\delta\setminus I_{\kappa\epsilon})}.$$

Combining the above two equations, we have

$$\begin{aligned}\left(a_K^\epsilon\nabla\hat{v}^\epsilon - P^0, \nabla\theta^\epsilon\right)_{L^2(I_\delta)} &= \left(a_K^\epsilon\nabla\theta^\epsilon, \nabla[(V-\widehat{V}^\epsilon)(1-\rho^\epsilon)]\right)_{L^2(I_\delta)}\\ &\quad + \left(a_K^\epsilon\nabla\widehat{V}^\epsilon - P^0, \nabla(V-\widehat{V}^\epsilon)\right)_{L^2(I_\delta\setminus I_{\kappa\epsilon})}.\end{aligned}$$

Using the estimates (3.11), we have

$$\begin{aligned}&\left|\left(a_K^\epsilon\nabla\hat{v}^\epsilon - P^0, \nabla\theta^\epsilon\right)_{L^2(I_\delta)}\right|\\ &\le \Lambda\left\|\nabla[(V-\widehat{V}^\epsilon)(1-\rho^\epsilon)]\right\|_{L^2(I_\delta)}\left\|\nabla\theta^\epsilon\right\|_{L^2(I_\delta)}\\ &\quad + \left\|\nabla(V-\widehat{V}^\epsilon)\right\|_{L^2(I_\delta\setminus I_{\kappa\epsilon})}\left(\Lambda\left\|\nabla\widehat{V}^\epsilon\right\|_{L^2(I_\delta\setminus I_{\kappa\epsilon})} + \frac{\Lambda^2}{\lambda}\left\|\nabla V\right\|_{L^2(I_\delta\setminus I_{\kappa\epsilon})}\right)\\ &\le C\frac{\Lambda^3}{\lambda^2}\frac{\epsilon}{\delta}\left\|\nabla V\right\|^2_{L^2(I_\delta)} + C\frac{\Lambda^2}{\lambda}\left(\frac{\epsilon}{\delta}\right)^{1/2}\left\|\nabla\theta^\epsilon\right\|_{L^2(I_\delta)}\left\|\nabla V\right\|_{L^2(I_\delta)}.\end{aligned}\tag{3.15}$$

Next, we define

$$(P_1^\epsilon)_i \equiv (P^0)_i + \epsilon\frac{\partial}{\partial x_j}\left(\alpha_{ij}^k\frac{\partial V}{\partial x_k}\rho^\epsilon\right),$$

where

$$\frac{\partial\alpha_{ik}^j}{\partial y_k} = g_{ij} \equiv (a_K^\epsilon)_{ij} + (a_K^\epsilon)_{ik}\frac{\partial\chi_K^j}{\partial y_k} - \mathcal{A}_{ij}(\boldsymbol{x}_K), \quad \alpha_{ik}^j = -\alpha_{ki}^j.$$

By [30, (1.12)], the tensor $\boldsymbol{\alpha}$ exists and satisfies

$$\left\|\boldsymbol{\alpha}\right\|_{L^2(Y)} + \left\|\nabla\boldsymbol{\alpha}\right\|_{L^2(Y)} \le C\left\|g\right\|_{L^2(Y)} \le C\Lambda^2/\lambda.\tag{3.16}$$

A direct calculation gives

$$(P^\epsilon)_i - (P_1^\epsilon)_i = \epsilon\frac{\partial}{\partial x_j}\left(\alpha_{ij}^k\frac{\partial V}{\partial x_k}(1-\rho^\epsilon)\right) = \frac{\partial\alpha_{ij}^k}{\partial y_j}\frac{\partial V}{\partial x_k}(1-\rho^\epsilon) - \epsilon\alpha_{ij}^k\frac{\partial V}{\partial x_k}\frac{\partial\rho^\epsilon}{\partial x_j}.$$

Using (3.16), we obtain

$$\begin{aligned}\left\|P^\epsilon - P_1^\epsilon\right\|_{L^2(I_\delta)} &\le \left\|\nabla_{\boldsymbol{y}}\boldsymbol{\alpha}\right\|_{L^2(I_\delta\setminus I_{\kappa\epsilon})}\left|\nabla V\right| + C\left\|\boldsymbol{\alpha}\right\|_{L^2(I_\delta\setminus I_{\kappa\epsilon})}\left|\nabla V\right|\\ &\le \left\|\nabla_{\boldsymbol{y}}\boldsymbol{\alpha}\right\|_{L^2(I_{(\kappa+1)\epsilon}\setminus I_{\kappa\epsilon})}\left|\nabla V\right| + C\left\|\boldsymbol{\alpha}\right\|_{L^2(I_{(\kappa+1)\epsilon}\setminus I_{\kappa\epsilon})}\left|\nabla V\right|\\ &\le C(\delta^{d-1}\epsilon)^{1/2}(\left\|\nabla\boldsymbol{\alpha}\right\|_{L^2(Y)} + \left\|\boldsymbol{\alpha}\right\|_{L^2(Y)})\left|\nabla V\right|\\ &\le C\frac{\Lambda^2}{\lambda}\left(\frac{\epsilon}{\delta}\right)^{1/2}\left\|\nabla V\right\|_{L^2(I_\delta)}.\end{aligned}\tag{3.17}$$

Integrating by parts and using the anti-symmetry of $\boldsymbol{\alpha}$, we obtain

$$\left(P_1^\epsilon - P^0, \nabla\varphi\right)_{L^2(I_\delta)} = 0 \quad \text{for all} \quad \varphi \in H^1(I_\delta),\tag{3.18}$$

which implies

$$(a_K^\epsilon\nabla\theta^\epsilon, \nabla\theta^\epsilon)_{L^2(I_\delta)} = \left(a_K^\epsilon\nabla\hat{v}^\epsilon - P^0, \nabla\theta^\epsilon\right)_{L^2(I_\delta)} + (P_1^\epsilon - P^\epsilon, \nabla\theta^\epsilon)_{L^2(I_\delta)}.$$

Using the estimates (3.15) and (3.17), we obtain

$$\begin{aligned}\left\|\nabla\theta^\epsilon\right\|^2_{L^2(I_\delta)} &\le C\frac{\Lambda^3}{\lambda^3}\frac{\epsilon}{\delta}\left\|\nabla V\right\|^2_{L^2(I_\delta)} + C\frac{\Lambda^2}{\lambda^2}\left(\frac{\epsilon}{\delta}\right)^{1/2}\left\|\nabla\theta^\epsilon\right\|_{L^2(I_\delta)}\left\|\nabla V\right\|_{L^2(I_\delta)}\\ &\le C\frac{\Lambda^3}{\lambda^3}\frac{\epsilon}{\delta}\left\|\nabla V\right\|^2_{L^2(I_\delta)} + \frac{1}{2}\left\|\nabla\theta^\epsilon\right\|^2_{L^2(I_\delta)} + C\frac{\Lambda^4}{\lambda^4}\frac{\epsilon}{\delta}\left\|\nabla V\right\|^2_{L^2(I_\delta)}\\ &\le \frac{1}{2}\left\|\nabla\theta^\epsilon\right\|^2_{L^2(I_\delta)} + C\frac{\Lambda^4}{\lambda^4}\frac{\epsilon}{\delta}\left\|\nabla V\right\|^2_{L^2(I_\delta)},\end{aligned}$$

which leads to (3.14). □

Remark 3.3. *When the cell problem is of the Dirichlet type, we may proceed as follows. Taking* $\varphi = \theta^\epsilon + (\widehat{V}^\epsilon - V)(1-\rho^\epsilon) \in H_0^1(I_\delta)$ *in* (3.13), *we have*

$$(a_K^\epsilon \nabla\theta^\epsilon, \nabla\theta^\epsilon)_{L^2(I_\delta)} = \Big(a_K^\epsilon \nabla\theta^\epsilon, \nabla[(V-\widehat{V}^\epsilon)(1-\rho^\epsilon)]\Big)_{L^2(I_\delta)},$$

which gives

$$\|\nabla\theta^\epsilon\|_{L^2(I_\delta)} \le \frac{\Lambda}{\lambda}\Big\|\nabla[(\widehat{V}^\epsilon - V)(1-\rho^\epsilon)]\Big\|_{L^2(I_\delta)}.$$

This inequality together with $(3.11)_2$ *implies* (3.14).

Using (3.10) and (3.14), we come to the main result of this paper.

Theorem 3.4. *If* $a^\epsilon = a(\boldsymbol{x}, \boldsymbol{x}/\epsilon)$ *with* $a(\boldsymbol{x},\boldsymbol{y}) \in C^{0,1}(D; L^\infty(Y))$, *and* $a(\boldsymbol{x},\boldsymbol{y})$ *is periodic in* $\boldsymbol{y}$ *with period* Y, *then*

$$e(HMM) \le C\frac{\Lambda^4}{\lambda^3}\Big(\delta + \frac{\epsilon}{\delta}\Big) \qquad \text{for all the three boundary conditions.} \tag{3.19}$$

Proof. By a standard perturbation result (see [14, Lemma 1.8]), we have

$$\|\nabla(v^\epsilon - \hat{v}^\epsilon)\|_{L^2(I_\delta)} \le C\lambda^{-1}\delta\,\|\nabla v^\epsilon\|_{L^2(I_\delta)} \le C\frac{\Lambda}{\lambda^2}\delta\,\|\nabla V\|_{L^2(I_\delta)}. \tag{3.20}$$

It follows from (2.6) and (3.20) that

$$\begin{aligned}
|\langle a^\epsilon\nabla v^\epsilon - a_K^\epsilon\nabla\hat{v}^\epsilon\rangle_{I_\delta}| &\le |\langle (a^\epsilon - a_K^\epsilon)\nabla v^\epsilon\rangle_{I_\delta}| + |\langle a_K^\epsilon \nabla(v^\epsilon - \hat{v}^\epsilon)\rangle_{I_\delta}|\\
&\le C|I_\delta|^{-1/2}\Big(\delta\,\|\nabla v^\epsilon\|_{L^2(I_\delta)} + \Lambda\,\|\nabla(v^\epsilon-\hat{v}^\epsilon)\|_{L^2(I_\delta)}\Big)\\
&\le C\frac{\Lambda^2}{\lambda^2}\delta|\nabla V|. \qquad (3.21)
\end{aligned}$$

Using $(3.11)_1$, $(3.9)_1$ and the ellipticity of $\mathcal{A}$, we obtain

$$\begin{aligned}
\Big|\Big\langle a_K^\epsilon\nabla\widehat{V}^\epsilon\Big\rangle_{I_\delta} - \Big\langle a_K^\epsilon\nabla\widehat{V}^\epsilon\Big\rangle_{I_{\kappa\epsilon}}\Big| &\le \frac{1}{|I_\delta|}\int_{I_\delta\setminus I_{\kappa\epsilon}}\Big|a_K^\epsilon\nabla\widehat{V}^\epsilon\Big|\,\mathrm{d}\boldsymbol{x} + \Big(1 - \frac{|I_{\kappa\epsilon}|}{|I_\delta|}\Big)\Big|⨍_{I_{\kappa\epsilon}} a_K^\epsilon\nabla\widehat{V}^\epsilon\,\mathrm{d}\boldsymbol{x}\Big|\\
&\le \Lambda\frac{|I_\delta\setminus I_{\kappa\epsilon}|^{1/2}}{|I_\delta|}\Big\|\nabla\widehat{V}^\epsilon\Big\|_{L^2(I_\delta\setminus I_{\kappa\epsilon})} + 3\frac{\epsilon}{\delta}\,|\,\mathcal{A}(\boldsymbol{x}_K)\nabla V\,|\\
&\le C\frac{\Lambda^2}{\lambda}\frac{\epsilon}{\delta}|\nabla V|. \qquad (3.22)
\end{aligned}$$

Let $\hat{w}^\epsilon$ be the solution of (2.3) with a^ϵ replaced by a_K^ϵ and subject to the boundary value W, and define

$$\widetilde{W}^\epsilon \equiv W + \epsilon(\widetilde{\boldsymbol{\chi}}_K\cdot\nabla)W,$$

where $\widetilde{\boldsymbol{\chi}}_K$ is defined in the same manner as $\boldsymbol{\chi}_K$ with a_K^ϵ replaced by its transpose $(a_K^\epsilon)^t$.

$$\begin{aligned}
\nabla W\cdot\langle a_K^\epsilon\nabla\theta^\epsilon\rangle_{I_\delta} &= \langle\nabla W\cdot a_K^\epsilon\nabla\theta^\epsilon\rangle_{I_\delta}\\
&= \Big\langle\nabla(W-\widetilde{W}^\epsilon)\cdot a_K^\epsilon\nabla\theta^\epsilon\Big\rangle_{I_\delta} + \Big\langle\nabla\widetilde{W}^\epsilon\cdot a_K^\epsilon\nabla\theta^\epsilon\Big\rangle_{I_\delta}\\
&= \Big\langle\nabla[(W-\widetilde{W}^\epsilon)(1-\rho^\epsilon)]\cdot a_K^\epsilon\nabla\theta^\epsilon\Big\rangle_{I_\delta} + \Big\langle\nabla\theta^\epsilon\cdot(a_K^\epsilon)^t\nabla\widetilde{W}^\epsilon\Big\rangle_{I_\delta}. \qquad (3.23)
\end{aligned}$$

Using $(3.11)_2$ and (3.14), we have

$$\begin{aligned}\left|\left\langle\nabla[(W-\widetilde{W}^\epsilon)(1-\rho^\epsilon)]\cdot a_K^\epsilon\nabla\theta^\epsilon\right\rangle_{I_\delta}\right| &\le \Lambda|I_\delta|^{-1}\left\|\nabla[(W-\widetilde{W}^\epsilon)(1-\rho^\epsilon)]\right\|_{L^2(I_\delta)}\|\nabla\theta^\epsilon\|_{L^2(I_\delta)}\\ &\le C\frac{\Lambda^4}{\lambda^3}\frac{\epsilon}{\delta}|\nabla V||\nabla W|.\end{aligned}\tag{3.24}$$

We define $\widetilde{P}^\epsilon, \widetilde{P}^0$ and $\widetilde{P}_1^\epsilon$ exactly in the same way as P^ϵ, P^0 and P_1^ϵ, respectively, provided that we replace V by W and a_K^ϵ by its transpose $(a_K^\epsilon)^t$. By (3.18), we obtain

$$\begin{aligned}\left\langle\nabla\theta^\epsilon\cdot(a_K^\epsilon)^t\nabla\widetilde{W}^\epsilon\right\rangle_{I_\delta} &= \left\langle(\widetilde{P}^\epsilon-\widetilde{P}^0)\cdot\nabla\theta^\epsilon\right\rangle_{I_\delta}+\left\langle\widetilde{P}^0\cdot\nabla\theta^\epsilon\right\rangle_{I_\delta}\\ &= \left\langle(\widetilde{P}^\epsilon-\widetilde{P}_1^\epsilon)\cdot\nabla\theta^\epsilon\right\rangle_{I_\delta}+\frac{1}{|I_\delta|}\left(\nabla(V-\widehat{V}^\epsilon),\widetilde{P}^0\right)_{L^2(I_\delta\setminus I_{\kappa\epsilon})}\end{aligned}$$

where we have used (3.6) and $(3.9)_2$.

Using (3.14) and (3.17), we obtain

$$\begin{aligned}\left|\left\langle(\widetilde{P}^\epsilon-\widetilde{P}_1^\epsilon)\cdot\nabla\theta^\epsilon\right\rangle_{I_\delta}\right| &\le |I_\delta|^{-1}\|\nabla\theta^\epsilon\|_{L^2(I_\delta)}\left\|\widetilde{P}_1^\epsilon-\widetilde{P}^\epsilon\right\|_{L^2(I_\delta)}\\ &\le C\frac{\Lambda^4}{\lambda^3}\frac{\epsilon}{\delta}|\nabla V||\nabla W|.\end{aligned}$$

Invoking $(3.11)_1$ again, we have

$$\begin{aligned}\frac{1}{|I_\delta|}\left|\left(\nabla(V-\widehat{V}^\epsilon),\widetilde{P}^0\right)_{L^2(I_\delta\setminus I_{\kappa\epsilon})}\right| &\le |I_\delta|^{-1}\frac{\Lambda^2}{\lambda}\left\|\nabla(\widehat{V}^\epsilon-V)\right\|_{L^2(I_\delta\setminus I_{\kappa\epsilon})}\|\nabla W\|_{L^2(I_\delta\setminus I_{\kappa\epsilon})}\\ &\le C\frac{\Lambda^3}{\lambda^2}\frac{\epsilon}{\delta}|\nabla V||\nabla W|.\end{aligned}$$

It follows from the above two inequalities that

$$\left|\left\langle\nabla\theta^\epsilon\cdot(a_K^\epsilon)^t\nabla\widetilde{W}^\epsilon\right\rangle_{I_\delta}\right|\le C\frac{\Lambda^4}{\lambda^3}\frac{\epsilon}{\delta}|\nabla V||\nabla W|.\tag{3.25}$$

Substituting (3.24) and (3.25) into (3.23), we obtain

$$\left|\langle a_K^\epsilon\nabla\theta^\epsilon\rangle_{I_\delta}\right|\le C\frac{\Lambda^4}{\lambda^3}\frac{\epsilon}{\delta}|\nabla V|,$$

which together with (3.21) and (3.22) leads to (3.19). □

Remark 3.5. *If the cell problem* (2.3) *is of the Dirichlet type, then we may estimate the second term of* (3.23) *as follows. Using* $\theta^\epsilon+(\widehat{V}^\epsilon-V)(1-\rho^\epsilon)\in H_0^1(I_\delta)$ *and* $(3.11)_2$, *we have*

$$\begin{aligned}\left|\left\langle\nabla\theta^\epsilon\cdot(a_K^\epsilon)^t\nabla\widetilde{W}^\epsilon\right\rangle_{I_\delta}\right| &= \left|\left\langle\nabla[(V-\widehat{V}^\epsilon)(1-\rho^\epsilon)]\cdot(a_K^\epsilon)^t\nabla\widetilde{W}^\epsilon\right\rangle_{I_\delta}\right|\\ &\le \Lambda|I_\delta|^{-1}\left\|\nabla[(V-\widehat{V}^\epsilon)(1-\rho^\epsilon)]\right\|_{L^2(I_\delta)}\left\|\nabla\widetilde{W}^\epsilon\right\|_{L^2(I_\delta\setminus I_{\kappa\epsilon})}\\ &\le C\frac{\Lambda^3}{\lambda^2}\frac{\epsilon}{\delta}|\nabla V||\nabla W|,\end{aligned}$$

which gives (3.25).

Our result, i.e., Theorem 3.4 allows for nonsmooth microstructures inside the unit cell, and suggests that the error committed in estimating data (the so-called resonance error) depends weakly on the smoothness of the microstructures. However, the corresponding estimates in [14, 12] and also [19] in the context of the multiscale finite element method require the smooth microstructures. The methods tailored to resolve the microstructures hence are in great demand to improve the overall accuracy of HMM-FEM. We refer to [16, 2, 6, 25] and the references therein for such methods.

4. Conclusion

In this paper we have clarified the condition under which $e(\mathrm{HMM})$ converges to zero, and proved the optimal convergence rate of $e(\mathrm{HMM})$ for the locally periodic coefficients under very weak and natural smoothness assumption when the cell problem is subject to the Dirichlet, the periodic, or the Neumann boundary condition.

The present work can be naturally extended to HMM for the parabolic homogenization problem; see [24, Theorem 1.2]. The most challenging issue is to prove the sharp bound for $e(\mathrm{HMM})$ under the minimal smooth assumption on the random coefficients. The analysis of the cell problem of the mixed type is also very promising and deserves a further exploration since it is easy to realize in the experiment. It is also very interesting to derive the same estimates for the multiscale finite element method and the related methods; see [19].

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