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SHEAR LOCKING IN A PLANE ELASTICITY PROBLEM AND THE ENHANCED ASSUMED STRAIN METHOD*

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Abstract. The enhanced assumed strain (EAS) method is a popular tool for avoiding locking phenomena, e.g., a remedy for *shear locking* in plane elasticity. We consider bending-dominated problems on thin bodies which can be treated as beams and prove that the degree of approximation of the EAS method is at least as good as that of a beam model. The hypercircle method is combined with arguments of nonconforming methods.

Key words. two-dimensional-elasticity, enhanced strain method, shear locking, nonconforming element

AMS subject classification. 65N30

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1. Introduction.

It is well known that lower-order quadrilateral elements suffer from the following two drawbacks: (a) they lead to shear locking when practicable meshes are used on thin domains in the solution of bending-dominated problems; and (b) volume locking is encountered for nearly incompressible materials. Introduced by Simo and Rifai [23], the enhanced strain elements (or enhanced assumed strain [EAS] method, for short) are designed to overcome these two shortcomings. They exhibit remarkable improvements over the standard bilinear elements on rectangular grids as extensive numerical tests have shown; see, e.g., [23, 21, 22, 19]. Braess, Carstensen, and Reddy [8] have proved that the enhanced element schemes are locking-free in the incompressible limit; we also refer to [8] for a review of the earlier endeavors and to [12] for recent progress in this direction.

Standard quadrilateral elements often lead to spurious shear strain when bending-dominated problems on thin domains are treated in the framework of plane elasticity. Such phenomenon is usually called *shear locking*. Shear locking has been extensively discussed by MacNeal [14, 15] from the mechanics aspect. Pitkäranta [17] has investigated shear locking for the Turner rectangle [28].

The present paper is concerned with a different approach. The danger of shear locking is extreme on thin bodies which can be dealt with as beams. We show that the order of convergence for the EAS method is at least as good as a beam model with quadratic terms in the transverse displacement. The beam model is motivated by Morgenstern's analysis of the Kirchhoff plate by the hypercircle method [16], and the dimension reduction is justified. In this way, we obtain convergence in the thin beam limit for the bending-dominated problem. The convergence rate is independent

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of the element aspect ratio. Of course, elements with a high aspect ratio arise in engineering computations of thin bodies like composite beams or plates.

We have restricted ourselves to rectangular grids. This is in accordance with the observation that the EAS method of Simo and Rifai [23] may suffer from trapezoidal locking on general quadrilateral meshes.

The paper is organized as follows. We will introduce the linearized plane elasticity problem and the enhanced strain elements in section 2.1. The method can be reformulated as a minimization of a reduced energy [7, Chapter III, section 5]. It is shown in section 3 that the reduced energy is similar to that for the Turner rectangle as discussed by Pitkäranta [17]. An error bound consists of the model error and the constrained interpolation error that will be estimated in sections 4 and 5, respectively. In particular, the hypercircle method is combined with arguments of nonconforming methods. Finally we show in Appendix A how sensitive the analysis reacts on the choice of the *reduced energy*.

Throughout this paper, we assume that the generic constants C and c are independent of the thickness parameter t and the mesh size h.

2. Enhanced strain finite element for the linearized elasticity problem.

2.1. The linearized plane elasticity problem. We consider an isotropic, homogeneous, linearly elastic strip of length L and width d with $d \ll L$. Let $L = 1$ and the center of the strip be the origin in 2-space. The strip occupies the region $P_t := P \times I_t$, with $P = (-1/2, 1/2)$ and $I_t = (-t/2, t/2)$; see Figure 2.1. The dimensionless parameter $t = d/L$ satisfies $0 < t \ll 1$, and this is our main concern throughout the paper. We denote the union of the top and bottom surfaces of the strip by $\partial P_t^{\pm} = P \times \{-t/2, t/2\}$ and the lateral boundary by $\partial P_t^{\mathrm{L}} = \partial P \times (-t/2, t/2)$. We assume that the strip is loaded by a surface force density $\boldsymbol{g}^t$: $\partial P_t^{\pm} \to \mathbb{R}^2$, and there is no volume force. Moreover, the strip is clamped along its lateral boundary.

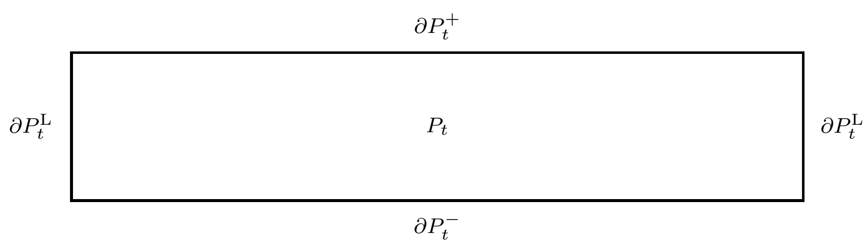

FIG. 2.1. *The plate domain P_t.*

The displacement $\boldsymbol{u} : P_t \to \mathbb{R}^2$ satisfies the following boundary-value problems:

$$\begin{cases} -\operatorname{div} \mathbb{C}\boldsymbol{\epsilon}(\boldsymbol{u}) = \boldsymbol{0} & \text{in } P_t, \\ [[\mathbb{C}\boldsymbol{\epsilon}(\boldsymbol{u})\boldsymbol{n}]] = \boldsymbol{g}^t & \text{on } \partial P_t^{\pm}, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial P_t^{\mathrm{L}}, \end{cases} \tag{2.1}$$

where the infinitesimal strain tensor $\boldsymbol{\epsilon}$ is the symmetric part of the deformation gradient

$$\boldsymbol{\epsilon}(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + [\nabla \boldsymbol{u}]^T). \tag{2.2}$$

The constitutive equation is given by the fourth-order elasticity tensor $\mathbb{C}$. Specifically,

$$\mathbb{C}\boldsymbol{\epsilon} = 2\mu\boldsymbol{\epsilon} + \lambda \operatorname{tr}(\boldsymbol{\epsilon})\mathbf{1},$$

where $\mathbf{1}$ is the identity and the Lamé constants λ, μ are related to Young's modulus E and the Poisson ratio ν by

$$\lambda = \begin{cases} \dfrac{E\nu}{1-\nu^2} & \text{plane stress,} \\ \dfrac{E\nu}{(1+\nu)(1-2\nu)} & \text{plane strain,} \end{cases} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}.$$

Usually $0 \le \nu < 1/2$. The flexural rigidity D is defined by

$$D = \begin{cases} \dfrac{E}{12} & \text{plane stress,} \\ \dfrac{E}{12(1-\nu^2)} & \text{plane strain,} \end{cases} \quad \text{and} \quad \gamma = \begin{cases} \nu & \text{plane stress,} \\ \dfrac{\nu}{1-\nu} & \text{plane strain.} \end{cases}$$

Note that $0 \le \gamma < 1$, and $\gamma \to 1$ is the incompressible limit; i.e., $\nu \to 1/2$ in the plain strain case.

By the definitions of λ, μ, D, and γ, a straightforward calculation gives the following relations:

$$\lambda + 2\mu - 24D = \lambda\gamma \quad \text{and} \quad (\lambda+2\mu)\gamma = \lambda. \tag{2.3}$$

It follows from these identities that

$$24D = 2\mu(1+\gamma). \tag{2.4}$$

As a direct consequence of (2.3) and (2.4), we have for any $\boldsymbol{v} = (u,v) \in [H^1(P_t)]^2$:

$$\begin{aligned} \int_{P_t} \mathbb{C}\boldsymbol{\epsilon}(\boldsymbol{v}) : \boldsymbol{\epsilon}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} &= 2\mu\Big(\|\partial_x u\|^2_{L^2(P_t)} + \|\partial_y v\|^2_{L^2(P_t)}\Big) \\ &\quad + \lambda\|\partial_x u + \partial_y v\|^2_{L^2(P_t)} + \mu\|\partial_y u + \partial_x v\|^2_{L^2(P_t)} \qquad (2.5) \\ &= 24D\|\partial_x u\|^2_{L^2(P_t)} + \frac{\lambda}{\gamma}\|\gamma\partial_x u + \partial_y v\|^2_{L^2(P_t)} \\ &\quad + \mu\|\partial_y u + \partial_x v\|^2_{L^2(P_t)}. \qquad (2.6) \end{aligned}$$

We consider only the bending-dominated plane elasticity problem. Therefore, we assume that the surface traction takes the following form:

$$\boldsymbol{g}^t = (g_1, g_2) \quad \text{with} \quad g_1 = 0 \ \text{ and } \ g_2(x, t/2) = g_2(x, -t/2) = tg(x).$$

The space $L^2(P_t)$ of square-integrable functions on the domain P_t is equipped with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|_{L^2(P_t)}$. Let $H^m(P_t)$ denote the standard Sobolev space and

$$\|v\|^2_{H^m(P_t)} = \sum_{k=0}^{m} |v|^2_{H^k(P_t)} \quad \text{and} \quad |v|^2_{H^k(P_t)} = \int_{P_t} \sum_{|\boldsymbol{\alpha}|=k} |D^{\boldsymbol{\alpha}} v|^2\,\mathrm{d}\boldsymbol{x}.$$

Here $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ is a multi-index whose components α_i are nonnegative integers, $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2$, and $D^{\boldsymbol{\alpha}} = \partial^{|\boldsymbol{\alpha}|}/\partial x_1^{\alpha_1} x_2^{\alpha_2}$. Define

$$\boldsymbol{V} := \left\{ \boldsymbol{v} \in [H^1(P_t)]^2 \ \mid \ \boldsymbol{v} = 0 \ \text{ on } \ \partial P_t^{\mathrm{L}} \right\}, \quad \|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{\mathbb{C}} := \langle \mathbb{C}\boldsymbol{\epsilon}(\boldsymbol{v}), \boldsymbol{\epsilon}(\boldsymbol{v})\rangle^{1/2}$$

for any $\boldsymbol{v} \in \boldsymbol{V}$.

The variational problem can be formulated as follows: find $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$\int_{P_t} \mathbb{C}\boldsymbol{\epsilon}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} = \int_{\partial P_t^{\pm}} \boldsymbol{g}^t \cdot \boldsymbol{v}\,\mathrm{d}s \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{V}. \tag{2.7}$$

2.2. Enhanced strain finite element approximation. Let $\mathcal{T}_h$ be a regular triangulation of P_t into rectangles. For any element $T \in \mathcal{T}_h$ with the parameter $h := \max_{T\in\mathcal{T}_h} \max(h_{T,x}, h_{T,y})$, we let $T := I_x \otimes I_y$ with $I_x := (x_T - h_{T,x}/2,\ x_T + h_{T,x}/2)$ and $I_y := (y_T - h_{T,y}/2,\ y_T + h_{T,y}/2)$. Define $h_x := \max_{T\in\mathcal{T}_h} h_{T,x}$ and $h_y := \max_{T\in\mathcal{T}_h} h_{T,y}$. Let ρ be the element shape parameter such that $1 \le h_{T,x}/h_{T,y} \le \rho$. It is our aim to achieve ρ-independent results.

Let $\mathcal{Q}_1(T)$ be the standard space of bilinear polynomials on T, and set

$$\boldsymbol{V}^h := \left\{ \boldsymbol{v} \in [C(\overline{P}_t)]^2 \mid \boldsymbol{v} = 0 \text{ on } \partial P_t^{\mathrm{L}},\ \boldsymbol{v}|_T \in \mathcal{Q}_1(T) \text{ for all } T \in \mathcal{T}_h \right\}. \tag{2.8}$$

The strain tensor $\boldsymbol{\epsilon}$ is regarded as an independent variable in the framework of the EAS method. In addition to the symmetric gradient of the displacement (2.2), there are enhanced strains. The finite element spaces $\widetilde{\boldsymbol{E}}^h$ for enhanced strains are subspaces of the space $\varLambda$ of symmetric 2×2 matrix-valued functions in $L^2(P_t)$, i.e., in the case of homogeneous bodies,

$$\widetilde{\boldsymbol{E}}^h \subset \varLambda := \left\{ \boldsymbol{\tau} \in L^2(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}}) \mid \int_{P_t} \mathbb{C}\boldsymbol{\tau}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \boldsymbol{0} \right\}.$$

Following [8], we formulate the enhanced strain finite element approximation:

Problem 2.1. Find $(\boldsymbol{u}_h, \boldsymbol{a}_h) \in \boldsymbol{V}^h \times \widetilde{\boldsymbol{E}}^h$ such that, for all $(\boldsymbol{v}_h, \boldsymbol{b}_h) \in \boldsymbol{V}^h \times \widetilde{\boldsymbol{E}}^h$,

$$\int_{P_t} \mathbb{C}\big(\boldsymbol{\epsilon}(\boldsymbol{u}_h) + \boldsymbol{a}_h\big) : \big(\boldsymbol{\epsilon}(\boldsymbol{v}_h) + \boldsymbol{b}_h\big)\,\mathrm{d}\boldsymbol{x} = \int_{\partial P_t^{\pm}} \boldsymbol{g} \cdot \boldsymbol{v}_h\,\mathrm{d}s. \tag{2.9}$$

There are several choices for $\widetilde{\boldsymbol{E}}^h$. We focus on the finite elements suggested by Simo and Rifai [23]. They are known to be equivalent to the nonconforming part of the element of Taylor, Beresford, and Wilson [24]:

$$\begin{aligned} \widetilde{\boldsymbol{E}}^h &:= \{ \boldsymbol{b} \in \varLambda \mid \boldsymbol{b}|_T \in \mathcal{L}(T) \quad \text{for all} \quad T \in \mathcal{T}_h \}, \\ \mathcal{L}(T) &:= \operatorname{span}\left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}, \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix} \right\}. \end{aligned} \tag{2.10}$$

Here x and y are the (local) coordinates in the reference element.

The following *strengthened Cauchy inequality* is the key to the well posedness of Problem 2.1 [6, 7]. There exists a constant $\kappa\,(0 < \kappa < 1)$ that is independent of λ and h but depends on ρ such that

$$|\langle \boldsymbol{\epsilon}(\boldsymbol{v}), \boldsymbol{b}\rangle| \le \kappa \|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{L^2(P_t)} \|\boldsymbol{b}\|_{L^2(P_t)} \quad \text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}^h,\ \boldsymbol{b} \in \widetilde{\boldsymbol{E}}^h. \tag{2.11}$$

The above inequality has been proven in [6] for $\boldsymbol{V}^h$ and $\widetilde{\boldsymbol{E}}^h$ defined in (2.8) and (2.10), respectively. A similar result may be found in [13]. By pursuing the dependence of κ on ρ, we have

$$\kappa = \frac{\rho}{\sqrt{2 + \rho^2}}. \tag{2.12}$$

By [6, Lemma A], the inequality (2.11) yields the following for all $\boldsymbol{v} \in \boldsymbol{V}^h$ and $\boldsymbol{b} \in \widetilde{\boldsymbol{E}}^h$:

$$\|\boldsymbol{\epsilon}(\boldsymbol{v}) + \boldsymbol{b}\|_{L^2(P_t)} \geq \left(\frac{1-\kappa}{2}\right)^{1/2} (\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{L^2(P_t)} + \|\boldsymbol{b}\|_{L^2(P_t)}). \tag{2.13}$$

Based on (2.13) (or equivalently on (2.11)), and under the assumption that the triangulation $\mathcal{T}_h$ is a refinement of $\mathcal{T}_{2h}$, Braess, Carstensen, and Reddy [8] proved the following result showing that the finite elements with enhanced assumed strains are free of *volumetric locking*.

THEOREM 2.2. *There exists a constant c independent of $\lambda, \mathcal{T}_h$, and $\boldsymbol{u}$ such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1(P_t)} + \|\boldsymbol{a}_h\|_{L^2(P_t)} \leq ch\|\boldsymbol{g}^t\|_{H^{3/2}(\partial P_t^{\pm})}. \tag{2.14}$$

It is clear that the right-hand side of (2.14) depends on the size of the domain P_t. A scaling argument gives that the constant c is at least $\mathcal{O}(t^{-2})$, which does not guarantee the uniform convergence of the EAS finite element method in the limit $t \to 0$. However, Theorem 5.4 below will show that the enhanced strain scheme is actually free of *shear locking*, as suggested by the extensive numerical examples in Simo and Rifai [23] and in [21, 22]. The next three sections are devoted to the proof of this result.

3. Error estimate for finite elements with enhanced strains.

Following Pitkäranta [17] we start with the characterization of a special norm that is related to the *reduced energy* which is induced by the enhanced strains. We will estimate the error with respect to this special norm.

Denote the L^2 projection onto the piecewise constant functions by Π_0. For any v, we have

$$(\Pi_0 v)|_T = \fint_T v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}.$$

Moreover, we define the L^2 projections in the x and the y direction, respectively:

$$\Pi_x v = \fint_{I_x} v(x,y)\, \mathrm{d}x, \qquad \Pi_y v = \fint_{I_y} v(x,y)\, \mathrm{d}y.$$

Recalling (2.9) we rewrite Problem 2.1 as a minimization problem:

$$(\boldsymbol{u}_h, \boldsymbol{a}_h) = \underset{(\boldsymbol{v},\boldsymbol{b}) \in \boldsymbol{V}^h \times \widetilde{\boldsymbol{E}}^h}{\operatorname{argmin}} \left\{ \frac{1}{2} \int_{P_t} \mathbb{C}(\boldsymbol{\epsilon}(\boldsymbol{v}) + \boldsymbol{b}) : (\boldsymbol{\epsilon}(\boldsymbol{v}) + \boldsymbol{b})\, \mathrm{d}\boldsymbol{x} - \int_{\partial P_t^{\pm}} \boldsymbol{g}^t \cdot \boldsymbol{v}\, \mathrm{d}s \right\}.$$

The problem above can be viewed as a two-stage minimization problem, namely, first with respect to $\boldsymbol{b}$ and then with respect to $\boldsymbol{v}$. The result of the first step is described by the following theorem.

THEOREM 3.1. *For $\boldsymbol{v} = (u, v) \in \boldsymbol{V}^h$, we have*

$$\min_{\boldsymbol{b} \in \widetilde{\boldsymbol{E}}^h} \langle \mathbb{C}(\boldsymbol{\epsilon}(\boldsymbol{v}) + \boldsymbol{b}), \boldsymbol{\epsilon}(\boldsymbol{v}) + \boldsymbol{b} \rangle = |\!|\!| \boldsymbol{v} |\!|\!|^2, \tag{3.1}$$

where

$$\begin{aligned} |\!|\!| \boldsymbol{v} |\!|\!|^2 := {} & 24D\|\partial_x u\|^2_{L^2(P_t)} + \frac{\lambda}{\gamma}\|\Pi_0(\gamma\partial_x u + \partial_y v)\|^2_{L^2(P_t)} \\ & + \mu\|\Pi_0(\partial_y u + \partial_x v)\|^2_{L^2(P_t)} + 24D\|(I - \Pi_x)\partial_y v\|^2_{L^2(P_t)} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}. \end{aligned} \tag{3.2}$$

We note that

$$\Pi_0(\partial_y v) = \Pi_x \Pi_y(\partial_y v) = \Pi_x(\partial_y v) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}^h. \tag{3.3}$$

The error bound for the EAS method will refer to the norm (3.2) and to the projector Π_x in its last term in order to cope with possible boundary layers.

Proof. Obviously, it is sufficient to prove (3.1) on each element $T \in \mathcal{T}_h$. Given $\boldsymbol{v} = (u, v) \in \boldsymbol{V}^h$, it follows from (2.6) that

$$\begin{aligned}\int_T \mathbb{C}(\boldsymbol{\epsilon}(\boldsymbol{v}) + \boldsymbol{b}) : (\boldsymbol{\epsilon}(\boldsymbol{v}) + \boldsymbol{b})\,\mathrm{d}\boldsymbol{x} &= 24D\|\partial_x u + b_{11}\|^2_{L^2(T)} \\ &+ \frac{\lambda}{\gamma}\|\gamma(\partial_x u + b_{11}) + (\partial_y v + b_{22})\|^2_{L^2(T)} + \mu\|\partial_y u + \partial_x v + 2b_{12}\|^2_{L^2(T)}.\end{aligned} \tag{3.4}$$

The functions in $\widetilde{\boldsymbol{E}}^h$ are L^2-orthogonal to piecewise constants, and we get

$$b_{12} = -\frac{1}{2}(I - \Pi_0)(\partial_y u + \partial_x v), \tag{3.5}$$

$$b_{22} = -\gamma(I - \Pi_0)\partial_x u. \tag{3.6}$$

Since the original expression (3.1) is symmetrical in x and y, we obtain by a symmetry argument from (3.6)

$$b_{11} = -\gamma(I - \Pi_0)\partial_y v. \tag{3.7}$$

The nontrivial term in (3.4) becomes

$$\begin{aligned}\gamma(\partial_x u + b_{11}) + (\partial_y v + b_{22}) &= \gamma[\partial_x u - \gamma(I - \Pi_0)\partial_y v] + \partial_y v - \gamma(I - \Pi_0)\partial_x u \\ &= \Pi_0(\gamma\partial_x u + \partial_y v) + (1 - \gamma^2)(I - \Pi_0)\partial_y v.\end{aligned}$$

The orthogonality of the terms yields

$$\begin{aligned}\|\gamma(\partial_x u + b_{11}) + (\partial_y v + b_{22})\|^2_{L^2(T)} &= \|\Pi_0(\gamma\partial_x u + \partial_y v)\|^2_{L^2(T)} \\ &\quad + (1 - \gamma^2)^2\|(I - \Pi_0)\partial_y v\|^2_{L^2(T)}.\end{aligned}$$

Similarly, $\|\partial_x u + b_{11}\|^2_{L^2(T)} = \|\partial_x u\|^2_{L^2(T)} + \gamma^2\|(I - \Pi_0)\partial_y v\|^2_{L^2(T)}$. It remains to determine the factor of the term $\|(I - \Pi_0)\partial_y v\|^2_{L^2(T)}$. It follows from (2.3) that $\frac{\lambda}{\gamma}(1 - \gamma^2) = \frac{\lambda}{\gamma} - \lambda\gamma = [\lambda + 2\mu] - [\lambda + 2\mu - 24D] = 24D$. Hence,

$$24D\gamma^2 + \frac{\lambda}{\gamma}(1 - \gamma^2)^2 = 24D\gamma^2 + 24D(1 - \gamma^2) = 24D.$$

Collecting all terms we obtain (3.2), and the proof is complete. ◻

Remark 3.2. This theorem can be viewed as a reformulation of the *static condensation* procedure in [26].

Compared to (2.6), the expression $|||\cdot|||^2$ resembles the strain energy. In what follows, we shall show that $|||\cdot|||^2$ is equivalent to the strain energy over $\boldsymbol{V}^h$.

The relations (2.3) and (2.4) yield for any $\boldsymbol{v} = (u, v) \in \boldsymbol{V}$:

$$\begin{aligned}\frac{\lambda}{\gamma}\|\Pi_0(\gamma\partial_x u + \partial_y v)\|^2_{L^2(P_t)} &= (\lambda + 2\mu - 24D)\|\Pi_0(\partial_x u)\|^2_{L^2(P_t)} \\ &\quad + 2\lambda\langle \Pi_0(\partial_x u), \Pi_0(\partial_y v)\rangle + (\lambda + 2\mu)\|\Pi_0(\partial_y v)\|^2_{L^2(P_t)} \\ &= (2\mu - 24D)\|\Pi_0(\partial_x u)\|^2_{L^2(P_t)} + 2\mu\|\Pi_0(\partial_y v)\|^2_{L^2(P_t)} \\ &\quad + \lambda\|\Pi_0(\partial_x u + \partial_y v)\|^2_{L^2(P_t)}.\end{aligned}$$

Now we rewrite $|||\,\boldsymbol{v}\,|||$ into a more symmetric form:

$$
\begin{aligned}
|||\,\boldsymbol{v}\,|||^2 = 2\mu\Big(&\|\Pi_0(\partial_x u)\|^2_{L^2(P_t)} + \|\Pi_0(\partial_y v)\|^2_{L^2(P_t)}\Big)\\
&+\lambda\|\Pi_0(\partial_x u + \partial_y v)\|^2_{L^2(P_t)} + \mu\|\Pi_0(\partial_y u + \partial_x v)\|^2_{L^2(P_t)}\\
&+24D\Big(\|(I-\Pi_0)\partial_x u\|^2_{L^2(P_t)} + \|(I-\Pi_x)\partial_y v\|^2_{L^2(P_t)}\Big).
\end{aligned} \tag{3.8}
$$

LEMMA 3.3. *For any $\boldsymbol{v} = (u, v) \in \boldsymbol{V}^h$, it holds*

$$
\sqrt{\frac{1-\gamma}{2(1+\gamma)\rho^2}}\,\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{\mathbb{C}} \le |||\,\boldsymbol{v}\,||| \le \|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{\mathbb{C}}. \tag{3.9}
$$

Proof. From (2.4) we know that $2\mu \le 24D = 2\mu(1+\gamma)$. Therefore, given $\eta \in L^2(P_t)$, we have

$$
\begin{aligned}
2\mu\,\|\eta\|^2_{L^2(P_t)} &\le 24D\|\Pi_0(\eta)\|^2_{L^2(P_t)} + 2\mu\|(I-\Pi_0)\eta\|^2_{L^2(P_t)}\\
&\le 2\mu(1+\gamma)\,\|\eta\|^2_{L^2(P_t)}.
\end{aligned} \tag{3.10}
$$

Starting from the symmetric form (3.8) and recalling (3.3), we apply the inequality above twice and drop the other two terms to obtain

$$
|||\,\boldsymbol{v}\,|||^2 \ge 2\mu\Big(\|\partial_x u\|^2_{L^2(P_t)} + \|\partial_y v\|^2_{L^2(P_t)}\Big). \tag{3.11}
$$

The components of $\boldsymbol{v} = (u, v) \in \boldsymbol{V}^h$ are bilinear functions, and a direct calculation gives

$$
\begin{aligned}
\|(I-\Pi_0)\partial_y u\|_{L^2(P_t)} &\le \rho\|(I-\Pi_0)\partial_x u\|_{L^2(P_t)},\\
\|(I-\Pi_0)\partial_x v\|_{L^2(P_t)} &\le \|(I-\Pi_x)\partial_y v\|_{L^2(P_t)},
\end{aligned} \tag{3.12}
$$

which together with the orthogonality of the different contributions implies

$$
\begin{aligned}
24D(\|(I-\Pi_0)\partial_x u\|^2_{L^2(P_t)} &+ \|(I-\Pi_x)\partial_y v\|^2_{L^2(P_t)})\\
&\ge \frac{24D}{\rho^2}\|(I-\Pi_0)(\partial_y u + \partial_x v)\|^2_{L^2(P_t)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|||\,\boldsymbol{v}\,|||^2 &\ge \mu\|\Pi_0(\partial_y u + \partial_x v)\|^2_{L^2(P_t)} + \frac{24D}{\rho^2}\|(I-\Pi_0)(\partial_y u + \partial_x v)\|^2_{L^2(P_t)}\\
&\ge \frac{\mu}{\rho^2}\,\|\partial_y u + \partial_x v\|^2_{L^2(P_t)}.
\end{aligned}
$$

A convex combination of the above inequality and (3.11) leads to

$$
|||\,\boldsymbol{v}\,|||^2 \ge \mu(\|\partial_x u\|^2_{L^2(P_t)} + \|\partial_y v\|^2_{L^2(P_t)}) + \frac{\mu}{\rho^2}\|\partial_y u + \partial_x v\|^2_{L^2(P_t)}. \tag{3.13}
$$

Now the lower bound in (3.9) follows from this inequality and

$$
\|\boldsymbol{v}\|^2_{\mathbb{C}} \le 2(\lambda+\mu)\|\boldsymbol{\epsilon}(\boldsymbol{v})\|^2_{L^2(P_t)} = (\lambda+2\mu)(1+\gamma)\|\boldsymbol{\epsilon}(\boldsymbol{v})\|^2_{L^2(P_t)}.
$$

The upper bound in the assertion is a direct consequence of the definition (3.1). ∎

The lemma asserts that $|||\cdot|||$ is equivalent to the strain energy on $\boldsymbol{V}^h$. In particular, we will make use of the upper bound that is independent of the shape parameter ρ. A similar upper bound with a slightly larger constant is obtained for elements in $\boldsymbol{V}$.

LEMMA 3.4. *For any $\boldsymbol{v}\in\boldsymbol{V}$, it holds*

$$|||\,\boldsymbol{v}\,||| \le \sqrt{1+\gamma}\,\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{\mathbb{C}}. \tag{3.14}$$

Proof. Since $\|\Pi_0(\partial_y v)\|_{L^2(P_t)} \le \|\Pi_x(\partial_y v)\|_{L^2(P_t)}$, we obtain the following from the symmetric form (3.8) by using (3.10) twice:

$$\begin{aligned}|||\,\boldsymbol{v}\,||| \le{}& 2\mu(1+\gamma)\|\partial_x u\|^2_{L^2(P_t)} + \lambda\|\partial_x u + \partial_y v\|^2_{L^2(P_t)}\\ &+ \mu\|\partial_y u + \partial_x v\|^2_{L^2(P_t)} + 2\mu(1+\gamma)\|\partial_y v\|^2_{L^2(P_t)}.\end{aligned}$$

A comparison with (2.5) yields the assertion (3.14). ∎

Remark 3.5. If we replace Π_x in the last term of the definition of $|||\,\boldsymbol{v}\,|||$ by Π_0, then the modified expression $|||\cdot|||^2$ would coincide with a modified strain energy in Belytschko and Bachrach [3]. The latter, in turn, refers to a reformulation of the classical Turner rectangle [28] as suggested by Pitkäranta [17, Theorem 5.1].

DEFINITION 3.6. *We call $\widehat{\boldsymbol{\sigma}}\in H(\mathrm{div},P_t)$ a statically admissible stress tensor, or an admissible stress tensor for short, if it satisfies*

$$\operatorname{div}\widehat{\boldsymbol{\sigma}} = \boldsymbol{0} \quad in \quad P_t, \qquad \widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{n} = \boldsymbol{g}^t \quad on \quad \partial P_t^{\pm}.$$

Let

$$\langle\,\mathbb{C}_h\boldsymbol{\epsilon}(\boldsymbol{w}),\boldsymbol{\epsilon}(\boldsymbol{w})\rangle := |||\,\boldsymbol{w}\,|||^2$$

and $\langle\,\mathbb{C}_h\boldsymbol{\epsilon}(\boldsymbol{w}),\boldsymbol{\epsilon}(\boldsymbol{v})\rangle$ be the associated bilinear form.

The following error estimate combines ideas from the theorem of Berger, Scott, and Strang [4] with the hypercircle method.

LEMMA 3.7. *Let $\widehat{\boldsymbol{\sigma}}$ be an admissible stress tensor with $\boldsymbol{g}^t=(0,tg(x))$ and $\hat{\boldsymbol{u}}\in\boldsymbol{V}$. Then*

$$\begin{aligned}|||\,\boldsymbol{u}-\boldsymbol{u}_h\,||| \le{}& \sqrt{1+\gamma}\,\|\boldsymbol{\epsilon}(\boldsymbol{u}-\hat{\boldsymbol{u}})\|_{\mathbb{C}} + \inf_{\boldsymbol{w}\in\boldsymbol{V}^h}|||\,\hat{\boldsymbol{u}}-\boldsymbol{w}\,|||\\ &+ \sup_{\boldsymbol{v}\in\boldsymbol{V}^h}\frac{|\langle\,\mathbb{C}_h\boldsymbol{\epsilon}(\hat{\boldsymbol{u}})-\widehat{\boldsymbol{\sigma}},\boldsymbol{\epsilon}(\boldsymbol{v})\rangle|}{|||\,\boldsymbol{v}\,|||}.\end{aligned} \tag{3.15}$$

Proof. Let $\tilde{\boldsymbol{u}}\in\boldsymbol{V}^h$ be the solution of the auxiliary problem

$$\langle\,\mathbb{C}_h\boldsymbol{\epsilon}(\tilde{\boldsymbol{u}}),\boldsymbol{\epsilon}(\boldsymbol{v})\rangle = \langle\,\mathbb{C}_h\boldsymbol{\epsilon}(\hat{\boldsymbol{u}}),\boldsymbol{\epsilon}(\boldsymbol{v})\rangle \quad \text{for all} \quad \boldsymbol{v}\in\boldsymbol{V}^h.$$

Setting $\boldsymbol{w}:=\tilde{\boldsymbol{u}}-\boldsymbol{u}_h$, we obtain

$$\begin{aligned}|||\,\boldsymbol{w}\,|||^2 &= \langle\,\mathbb{C}_h\boldsymbol{\epsilon}(\tilde{\boldsymbol{u}}),\boldsymbol{\epsilon}(\boldsymbol{w})\rangle - \langle\,\mathbb{C}_h\boldsymbol{\epsilon}(\boldsymbol{u}_h),\boldsymbol{\epsilon}(\boldsymbol{w})\rangle\\ &= \langle\,\mathbb{C}_h\boldsymbol{\epsilon}(\hat{\boldsymbol{u}}),\boldsymbol{\epsilon}(\boldsymbol{w})\rangle - \langle\,\mathbb{C}\boldsymbol{\epsilon}(\boldsymbol{u}),\boldsymbol{\epsilon}(\boldsymbol{w})\rangle\\ &= \langle\,\mathbb{C}_h\boldsymbol{\epsilon}(\hat{\boldsymbol{u}})-\widehat{\boldsymbol{\sigma}},\boldsymbol{\epsilon}(\boldsymbol{w})\rangle + \langle\,\widehat{\boldsymbol{\sigma}}-\mathbb{C}\boldsymbol{\epsilon}(\boldsymbol{u}),\boldsymbol{\epsilon}(\boldsymbol{w})\rangle.\end{aligned}$$

Integration by parts and the definition of $\widehat{\boldsymbol{\sigma}}$ yield

$$\begin{aligned}\langle\,\widehat{\boldsymbol{\sigma}}-\mathbb{C}\boldsymbol{\epsilon}(\boldsymbol{u}),\boldsymbol{\epsilon}(\boldsymbol{w})\rangle &= -\langle\operatorname{div}\widehat{\boldsymbol{\sigma}}-\operatorname{div}\mathbb{C}\boldsymbol{\epsilon}(\boldsymbol{u}),\boldsymbol{w}\rangle + \int_{\partial P_t}(\widehat{\boldsymbol{\sigma}}-\mathbb{C}\boldsymbol{\epsilon}(\boldsymbol{u}))\cdot\boldsymbol{n}\boldsymbol{w}\,\mathrm{d}s\\ &= 0.\end{aligned}$$

Hence,

$$\|\!|\tilde{\boldsymbol{u}} - \boldsymbol{u}_h|\!\|\!| = \sup_{\boldsymbol{v}\in\boldsymbol{V}^h} \frac{|\langle \mathbb{C}_h\boldsymbol{\epsilon}(\hat{\boldsymbol{u}}) - \widehat{\boldsymbol{\sigma}}, \boldsymbol{\epsilon}(\boldsymbol{v})\rangle|}{|\!|\!|\boldsymbol{v}|\!|\!|}. \tag{3.16}$$

From the triangle inequality and (3.14) it follows that

$$\begin{aligned}|\!|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!|\!| &\le |\!|\!|\boldsymbol{u} - \hat{\boldsymbol{u}}|\!|\!| + |\!|\!|\hat{\boldsymbol{u}} - \tilde{\boldsymbol{u}}|\!|\!| + |\!|\!|\tilde{\boldsymbol{u}} - \boldsymbol{u}_h|\!|\!|\\ &\le \sqrt{1+\gamma}\|\boldsymbol{\epsilon}(\boldsymbol{u} - \hat{\boldsymbol{u}})\|_{\mathbb{C}} + |\!|\!|\hat{\boldsymbol{u}} - \tilde{\boldsymbol{u}}|\!|\!| + |\!|\!|\tilde{\boldsymbol{u}} - \boldsymbol{u}_h|\!|\!|,\end{aligned}$$

which together with the Galerkin orthogonality $|\!|\!|\hat{\boldsymbol{u}} - \tilde{\boldsymbol{u}}|\!|\!| = \inf_{\boldsymbol{v}\in\boldsymbol{V}^h} |\!|\!|\hat{\boldsymbol{u}} - \boldsymbol{v}|\!|\!|$ and (3.16) gives (3.15). □

The lemma will be applied to $\hat{\boldsymbol{u}}$ and $\widehat{\boldsymbol{\sigma}}$ from a beam model. The first term accounts for how well the beam model approximates the solution of the bending-dominated plane elasticity problem; cf. Theorem 4.4 below. The last two terms reflect the discretization error of the beam model and will be estimated in section 5.

4. Model error estimate for beam approximation.

The main objective of this section is to estimate

$$\frac{\|\boldsymbol{\epsilon}(\boldsymbol{u} - \hat{\boldsymbol{u}})\|_{\mathbb{C}}}{\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{\mathbb{C}}},$$

which characterizes the error between the solution of the two-dimensional elasticity problem $\boldsymbol{u}$ and the $(1,2)$-model of the Euler–Bernoulli beam

$$\hat{\boldsymbol{u}} := \left(-y\omega'(x),\, \omega(x) + \frac{\gamma}{2}y^2\psi(x)\right). \tag{4.1}$$

Specifically, ω is the main term of the transverse displacement augmented by a correction of the second order in y. In contrast to the Timoshenko beam [25] with a more general ansatz for the rotation θ: $\boldsymbol{u} := (y\theta(x),\, \omega(x) + \frac{\gamma}{2}y^2\psi(x))$, the rotation is fixed here in the spirit of the Kirchhoff hypothesis by $\theta = -\omega'$.

Here ω is the solution of the boundary-value problem:

$$\begin{cases} Dt^2\omega^{(4)}(x) = g(x) \qquad \text{in } P,\\ \omega(-1/2) = \omega(1/2) = \omega'(-1/2) = \omega'(1/2) = 0.\end{cases} \tag{4.2}$$

And ψ is the solution of

$$\begin{cases} -t^2\psi''(x) + A^2\psi(x) = A^2\omega''(x) \qquad \text{in } P,\\ \psi(-1/2) = \psi(1/2) = 0,\end{cases} \tag{4.3}$$

with $A := (20/(1-\gamma))^{1/2}\mu$. To this end we shall exploit the following special form of the *hypercircle* theorem (see Prager and Synge [18] or [7, p. 148]) and recall Definition 3.6.

THEOREM 4.1. *Let $\widehat{\boldsymbol{\sigma}}$ be an admissible stress tensor and $\bar{\boldsymbol{u}} \in \boldsymbol{V}$. Then for the solution $\boldsymbol{u}$ of Problem* 2.1,

$$\langle \mathbb{C}\boldsymbol{\epsilon}(\boldsymbol{u} - \bar{\boldsymbol{u}}), \boldsymbol{\epsilon}(\boldsymbol{u} - \bar{\boldsymbol{u}})\rangle + \langle \mathbb{C}^{-1}(\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}), \boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}\rangle = \langle \mathbb{C}^{-1}\big(\widehat{\boldsymbol{\sigma}} - \mathbb{C}\boldsymbol{\epsilon}(\bar{\boldsymbol{u}})\big), \widehat{\boldsymbol{\sigma}} - \mathbb{C}\boldsymbol{\epsilon}(\bar{\boldsymbol{u}})\rangle.$$

To exploit Prager and Synge's theorem, we construct an admissible stress tensor. Morgensten's treatment of the $(1,1,2)$-model of the Kirchhoff–Love plate [16] (see also [2, 9, 11, 20]) motivates the following choice of the admissible stress tensor and the decision to combine it with the Euler–Bernoulli beam model (4.1). Set

$$\widehat{\boldsymbol{\sigma}} := \begin{pmatrix} -24Dy\omega'' & symm. \\ 24D(y^2/2 - t^2/8)\omega^{(3)}(x) & 24\,(1/8 - y^2/(6t^2))yg(x) \end{pmatrix}. \tag{4.4}$$

A priori estimates of ω and ψ will be derived by the inequality (4.5) below, which can be regarded as a special case of the general Gagliardo–Nirenberg inequality in [1]. We include an elementary proof for the reader's convenience.

LEMMA 4.2. *Let (a,b) be a finite interval and $\phi \in H^1(a,b)$. Then*

$$\|\phi\|_{L_\infty} := \max_{a\le x\le b} |\phi(x)| \le \left(\frac{1}{b-a} + 2\right)^{1/2} \|\phi\|^{1/2}_{L^2(a,b)} \|\phi\|^{1/2}_{H^1(a,b)}. \tag{4.5}$$

Moreover, if ϕ has a zero in $[a,b]$, then

$$\|\phi\|_{L_\infty} \le \sqrt{2}\, \|\phi\|^{1/2}_{L^2(a,b)} \|\phi'\|^{1/2}_{L^2(a,b)}. \tag{4.6}$$

Proof. For any $x, y \in [a,b]$ we have

$$\begin{aligned} \phi^2(y) &= \phi^2(x) + 2\int_x^y \phi(s)\phi\,'(s)\,\mathrm{d}s \;\le\; \phi^2(x) + 2\int_a^b |\phi(s)|\;|\phi\,'(s)|\,\mathrm{d}s \\ &\le \phi^2(x) + 2\|\phi\|_{L^2(a,b)}\|\phi\,'\|_{L^2(a,b)}. \end{aligned} \tag{4.7}$$

Integrating the above inequality with respect to x, we get

$$\begin{aligned} \phi^2(y) &\le \frac{1}{b-a}\|\phi\|^2_{L^2(a,b)} + 2\|\phi\|_{L^2(a,b)}\|\phi\,'\|_{L^2(a,b)} \\ &= \|\phi\|_{L^2(a,b)} \left(\frac{1}{b-a}\|\phi\|_{L^2(a,b)} + 2\|\phi\,'\|_{L^2(a,b)}\right) \\ &\le \left(\frac{1}{b-a} + 2\right) \|\phi\|_{L^2(a,b)}\|\phi\|_{H^1(a,b)}. \end{aligned}$$

This gives (4.5). If ϕ has a zero x, we apply (4.7) to obtain (4.6). ∎

We are ready to derive the following a priori estimates for ψ. It can be understood as an analogue of Lemma 5 in [2].

LEMMA 4.3. *Let ψ be solution of* (4.3) *and $t \le A$. Then*

$$t^2\|\psi\,'\|^2_{L^2(P)} + A^2\|\psi - \omega\,''\|^2_{L^2(P)} \le 6At\|\omega\,''\|^2_{H^1(P)}. \tag{4.8}$$

Proof. Set $w := \omega\,''$. Multiplying (4.3) by $-\psi\,''$ and integrating by parts, we obtain

$$\begin{aligned} &t^2\|\psi\,''\|^2_{L^2(P)} + A^2\|\psi\,'\|_{L^2(P)} \\ &\qquad = A^2\int_P w\,'\psi\,'\,\mathrm{d}x - A^2[w(1/2)\psi\,'(1/2) - w(-1/2)\psi\,'(-1/2)] \\ &\qquad \le A^2\|\psi\,'\|_{L^2(P)}\|w\,'\|_{L^2(P)} + 2A^2\|\psi\,'\|_{L_\infty}\|w\|_{L_\infty}. \end{aligned} \tag{4.9}$$

By Rolle's theorem, $\psi\,'$ has a zero in $(-1/2, +1/2)$, and it follows from Lemma 4.2 that

$$\|\psi\,'\|^2_{L_\infty} \le 2\,\|\psi\,'\|_{L^2(P)}\|\psi\,''\|_{L^2(P)}. \tag{4.10}$$

Young's inequality yields $\|\psi'\|_{L^2(P)}\|w'\|_{L^2(P)} \le \frac{1}{4}\|\psi'\|^2_{L^2(P)} + \|w'\|^2_{L^2(P)}$ and with (4.10) also

$$\begin{aligned} A^2\|\psi'\|_{L_\infty}\|w\|_{L_\infty} &\le \frac{At}{4}\|\psi'\|^2_{L_\infty} + \frac{A^3}{t}\|w\|^2_{L_\infty} \\ &\le \frac{1}{2}At\|\psi'\|_{L^2(P)}\|\psi''\|_{L^2(P)} + \frac{A^3}{t}\|w\|^2_{L_\infty} \\ &\le \frac{1}{4}\left(t^2\|\psi''\|^2_{L^2(P)} + A^2\|\psi'\|^2_{L^2(P)}\right) + \frac{A^3}{t}\|w\|^2_{L_\infty}\,. \end{aligned}$$

We insert the last two inequalities into (4.9) and note that the terms with the function ψ can be absorbed by the left-hand side:

$$\frac{1}{2}t^2\|\psi''\|^2_{L^2(P)} + \frac{1}{2}A^2\|\psi'\|_{L^2(P)} \le A^2\|w'\|^2_{L^2(P)} + \frac{A^3}{t}\|w\|^2_{L_\infty}\,.$$

Multiplying the above inequality by t^2/A^2 and using the first equation in (4.3), we obtain

$$\frac{A^2}{2}\|\psi - w\|^2_{L^2(P)} + \frac{t^2}{2}\|\psi'\|^2_{L^2(P)} \le t^2\|w''\|^2_{L^2(P)} + At\|w\|^2_{L_\infty}. \tag{4.11}$$

Since $\|w'\|_{L^2(P)} \le \|w\|_{H^1(P)}$, the Gagliardo–Nirenberg inequality (4.5) yields the bound $\|w\|_{L_\infty} \le \sqrt{3}\,\|w\|_{H^1(P)}$, and the right-hand side of (4.11) is smaller than $(1+\sqrt{3})At\|w\|_{H^1(P)}$. This completes the proof of (4.8). ∎

As for (4.2), we have the a priori estimate

$$\|\omega''\|_{H^1(P)} \le Ct^{-2}\|g\|_{H^{-1}(P)}, \qquad \|\omega^{(4)}\|_{L^2(P)} \le D^{-1}t^{-2}\|g\|_{L^2(P)}. \tag{4.12}$$

THEOREM 4.4. *Let $\hat{\boldsymbol{u}}$ be the beam mode defined in* (4.1). *There exists a constant $c = c[g]$ such that for sufficiently small t*

$$\frac{\|\boldsymbol{\epsilon}(\boldsymbol{u}-\hat{\boldsymbol{u}})\|_{\mathbb{C}}}{\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{\mathbb{C}}} \le c\,t^{1/2}. \tag{4.13}$$

Proof. For any $\boldsymbol{\tau} \in L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})$,

$$\begin{aligned} \langle\,\widehat{\boldsymbol{\sigma}} - \mathbb{C}\boldsymbol{\epsilon}(\hat{\boldsymbol{u}}),\boldsymbol{\tau}\rangle &= \langle\,\widehat{\sigma}_{11} - 24D\partial_x\hat{u}_1,\tau_{11}\rangle + \langle\,\widehat{\sigma}_{22},\tau_{22}\rangle \\ &\quad - \frac{\lambda}{\gamma}\langle\,\gamma\partial_x\hat{u}_1 + \partial_y\hat{u}_2, \gamma\tau_{11} + \tau_{22}\rangle + 2\langle\,\widehat{\sigma}_{12} - 2\mu\boldsymbol{\epsilon}_{12}(\hat{\boldsymbol{u}}),\tau_{12}\rangle. \end{aligned}$$

A direct calculation yields $\widehat{\sigma}_{11} - 24D\partial_x\hat{u}_1 = 0$. Therefore, we obtain

$$\begin{aligned} \langle\,\widehat{\boldsymbol{\sigma}} - \mathbb{C}\boldsymbol{\epsilon}(\hat{\boldsymbol{u}}),\boldsymbol{\tau}\rangle &= \langle\,\widehat{\sigma}_{22},\tau_{22}\rangle - \frac{\lambda}{\gamma}\langle\,\gamma\partial_x\hat{u}_1 + \partial_y\hat{u}_2,\gamma\tau_{11}+\tau_{22}\rangle \\ &\quad + 2\langle\,\widehat{\sigma}_{12} - 2\mu\boldsymbol{\epsilon}_{12}(\hat{\boldsymbol{u}}),\tau_{12}\rangle. \end{aligned} \tag{4.14}$$

By the definitions of $\widehat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{u}}$ in (4.4) and (4.1), respectively, we have

$$\begin{aligned} &\|\widehat{\sigma}_{22}\|^2_{L^2(P_t)} = \frac{t^3}{3}\|g\|^2_{L^2(P)}, \qquad \|\gamma\partial_x\hat{u}_1 + \partial_y\hat{u}_2\|^2_{L^2(P)} = \frac{t^3\gamma^2}{12}\|\psi - \omega''\|^2_{L^2(P)}, \\ &\|\widehat{\sigma}_{12} - 2\mu\boldsymbol{\epsilon}_{12}(\hat{\boldsymbol{u}})\|^2_{L^2(P_t)} \le \frac{\mu^2\gamma^2}{60}t^5\|\psi'\|^2_{L^2(P)} + 2\mu^2(1+\gamma)^2t^5\|\omega^{(3)}\|^2_{L^2(P)}. \end{aligned}$$

Substituting the above equations into (4.14) and using (4.8) and (4.12), we obtain

$$
\begin{aligned}
|\langle \widehat{\boldsymbol{\sigma}} - \mathbb{C}\boldsymbol{\epsilon}(\hat{\boldsymbol{u}}), \boldsymbol{\tau}\rangle| &\le C(\mu)\Big[t^{3/2}(t\|\psi'\|_{L^2(P)} + A\|\psi - \omega''\|_{L^2(P)}) \\
&\qquad + t^{5/2}\|\omega^{(3)}\|_{L^2(P)} + t^{3/2}\|g\|_{L^2(P)}\Big]\|\boldsymbol{\tau}\|_{\mathbb{C}} \\
&\le C(\mu)\Big(\|g\|_{H^{-1}(P)} + t^{3/2}\|g\|_{L^2(P)}\Big)\|\boldsymbol{\tau}\|_{\mathbb{C}}. \qquad (4.15)
\end{aligned}
$$

It follows from Theorem 4.1 that

$$
\|\boldsymbol{\epsilon}(\boldsymbol{u} - \hat{\boldsymbol{u}})\|_{\mathbb{C}} \le C(\mu)\left(\|g\|_{H^{-1}(P)} + t^{3/2}\|g\|_{L^2(P)}\right). \qquad (4.16)
$$

Recalling (2.6), we evaluate the denominator in (4.13),

$$
\begin{aligned}
\|\boldsymbol{\epsilon}(\hat{\boldsymbol{u}})\|_{\mathbb{C}}^2 &= 24D\|y\omega''(x)\|_{L^2(P_t)}^2 + \lambda\gamma\|y(\psi - \omega'')(x)\|_{L^2(P_t)}^2 + \mu\gamma^2\|\phi(y)\psi'(x)\|_{L^2(P_t)}^2 \\
&= 2Dt^3\|\omega''\|_{L^2(P)}^2 + \frac{\lambda\gamma t^3}{12}\|\psi - \omega''\|_{L^2(P)}^2 + \frac{\mu\gamma^2 t^5}{320}\|\psi'\|_{L^2(P)}^2.
\end{aligned}
$$

Using Lemma 4.3, we conclude that the second and third terms on the right-hand side of the above identity are bounded by a constant C uniformly with respect to t. A simple scaling argument shows that

$$
\|\omega''\|_{L^2(P)} = ct^{-2}, \qquad (4.17)
$$

where the nonzero constant c depends on $\|g\|_{H^{-1}(P)}$. Therefore,

$$
\|\boldsymbol{\epsilon}(\hat{\boldsymbol{u}})\|_{\mathbb{C}} \ge ct^{-1/2}
$$

holds with a nonzero constant c that depends on $\|g\|_{H^{-1}(P)}$. From (4.16) and the triangle inequality it follows that

$$
\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{\mathbb{C}} \ge \|\boldsymbol{\epsilon}(\hat{\boldsymbol{u}})\|_{\mathbb{C}} - \|\boldsymbol{\epsilon}(\boldsymbol{u} - \hat{\boldsymbol{u}})\|_{\mathbb{C}} \ge \frac{1}{2}ct^{-1/2} \qquad (4.18)
$$

holds for t sufficiently small. By combining this inequality with (4.16), we complete the proof of (4.13). $\square$

We note that the constant in (4.13) depends on the quotient $\|\omega''\|_{H^1}/\|\omega''\|_{L_2}$.

5. Discretization error of the beam model. In this section we estimate the last two terms on the right-hand side of (3.15) and start with a construction of a finite element approximation $\boldsymbol{w} \in \boldsymbol{V}^h$. Let $\theta(x) := \omega'(x)$, $s(x) := 1/4 - x^2$, $\phi(y) := y^2/2$, and

$$
\boldsymbol{w} = (w_1, w_2) := \big(-y\theta_h(x),\, \omega_h(x) + \gamma\phi_h(y)\psi_h(x)\big), \qquad (5.1)
$$

where

$$
\theta_h := \mathit{\Pi}_1\theta + \alpha\mathit{\Pi}_1 s \quad \text{with} \quad \alpha := \frac{\int_P(\theta - \mathit{\Pi}_1\theta)(x)\,\mathrm{d}x}{\int_P \mathit{\Pi}_1 s(x)\,\mathrm{d}x},
$$

$$
\phi_h := \mathit{\Pi}_1\phi, \quad \psi_h := \mathit{\Pi}\psi, \quad \omega_h(x) := \int_{-1/2}^{x} \mathit{\Pi}_0\theta_h(x')\,\mathrm{d}x'.
$$

Here $\mathit{\Pi}_1$ is the standard linear interpolation operator, and $\mathit{\Pi}$ is a Clément interpolation operator (see [10]).

By definition, $\omega_h(-1/2) = 0$, and we have

$$\omega_h(1/2) = \int_P \Pi_0\theta_h = \int_P \theta_h = \int_P \Pi_1\theta + \alpha\int_P \Pi_1 s = \int_P \theta$$
$$= \omega(1/2) - \omega(-1/2) = 0.$$

By construction, we get $\theta_h(\pm 1/2) = 0$. This shows that $\boldsymbol{w} \in \boldsymbol{V}^h$.

A standard finite element interpolation estimate yields

$$\|\theta' - \theta_h'\|_{L^2(P)} \le Ch_x\|\theta''\|_{L^2(P)} = Ch_x\|\omega^{(3)}\|_{L^2(P)}. \tag{5.2}$$

From $\phi(y) \le t^2/8$ it follows that also $\phi_h(y) \le t^2/8$ and

$$\|\phi_h\|_{L^2(I_t)} \le \frac{1}{8}t^{5/2}. \tag{5.3}$$

The following lemma is the key to an estimate of the interpolation error.

LEMMA 5.1. *Let ψ_h be defined by* (5.1). *It holds that*

$$\|\psi - \psi_h\|_{L^2(P)} \le C\big(\|\psi - \omega''\|_{L^2(P)} + h_x\|\omega^{(3)}\|_{L^2(P)}\big), \tag{5.4}$$
$$\|\psi - \Pi_x\psi\|_{L^2(P)} \le \|\psi - \omega''\|_{L^2(P)} + Ch_x\|\omega^{(3)}\|_{L^2(P)}. \tag{5.5}$$

Proof. We write

$$\psi - \psi_h = (I - \Pi)(\psi - \omega'') + (I - \Pi)\omega''.$$

By the following properties of the Clément operator [10]

$$\|\Pi\psi\|_{L^2(P)} \le C\|\psi\|_{L^2(P)}, \qquad \|(I - \Pi)\omega''\|_{L^2(P)} \le Ch_x\|\omega^{(3)}\|_{L^2(P)},$$

we get the first estimate (5.4).

Proceeding along the same line and using the above estimate with Π replaced by Π_x, we obtain (5.5). ∎

It is our main task to show that locking phenomena are eliminated. Note that the shear term of the beam model

$$\boldsymbol{\epsilon}_{12}(\hat{\boldsymbol{u}}) = \frac{\gamma}{2}y^2\psi'(x)$$

is small. It is crucial that the *reduced* shear term of the finite element interpolant $\Pi_0\boldsymbol{\epsilon}_{12}(\boldsymbol{w})$ is also small.

LEMMA 5.2. *Let $\boldsymbol{w}$ be defined by* (5.1). *Then*

$$|||\, \hat{\boldsymbol{u}} - \boldsymbol{w} \,||| \le C(1 + h_x/t^{1/2})\|g\|_{H^{-1}(P)}. \tag{5.6}$$

Proof. From

$$\phi_h'(y) = \fint_{I_y} \phi_h'(\tilde{y})d\tilde{y} = \fint_{I_y} \phi'(\tilde{y})d\tilde{y} = \Pi_y\phi' \tag{5.7}$$

it follows that $\Pi_0[\phi_h'(y)\psi_h(x)] = \Pi_0[\phi'(y)\psi_h(x)]$ and

$$\|\Pi_0[\gamma\partial_x(\hat{u}_1 - w_1) + \partial_y(\hat{u}_2 - w_2)]\|_{L^2(P_t)}$$
$$\le \gamma\|\partial_x(\hat{u}_1 - w_1)\|_{L^2(P_t)} + \gamma\|\phi'(y)\Pi_x(\psi - \psi_h)(x)\|_{L^2(P_t)}.$$

Invoking (5.7) once more, we get

$$(I-\Pi_x)\partial_y(\hat u_2-w_2)=(I-\Pi_y)\phi'(I-\Pi_x)\psi+\Pi_y\phi'(I-\Pi_x)(\psi-\psi_h).$$

This leads to

$$\|(I-\Pi_x)\partial_y(\hat u_2-w_2)\|_{L^2(P_t)}\le\|\phi'(I-\Pi_x)\psi\|_{L^2(P_t)}+\|\phi'(\psi-\psi_h)\|_{L^2(P_t)}.$$

Obviously,

$$\|\Pi_0[\partial_y(\hat u_1-w_1)+\partial_x(\hat u_2-w_2)]\|_{L^2(P_t)}\le\|\boldsymbol{\epsilon}_{12}(\hat{\boldsymbol{u}})\|_{L^2(P_t)}+\|\Pi_0\boldsymbol{\epsilon}_{12}(\boldsymbol{w})\|_{L^2(P_t)}.$$

Combining the above three inequalities with (3.2) and (2.3), we have

$$\begin{aligned}\|\!|\, \hat{\boldsymbol{u}}-\boldsymbol{w}\,|\!\|\le 2\sqrt{2\lambda/\gamma}\Big[&\|\partial_x(\hat u_1-w_1)\|_{L^2(P_t)}+\|\boldsymbol{\epsilon}_{12}(\hat{\boldsymbol{u}})\|_{L^2(P_t)}+\|\Pi_0\boldsymbol{\epsilon}_{12}(\boldsymbol{w})\|_{L^2(P_t)}\\ &+\|\phi'(\psi-\psi_h)\|_{L^2(P_t)}+\|\phi'(I-\Pi_x)\psi\|_{L^2(P_t)}\Big].\end{aligned}\tag{5.8}$$

By (5.2),

$$\|\partial_x(\hat u_1-w_1)\|_{L^2(P_t)}=\|y(\theta'-\theta_h')\|_{L^2(P_t)}\le Ch_xt^{3/2}\|\omega^{(3)}\|_{L^2(P)}.$$

A direct calculation leads to

$$\|\boldsymbol{\epsilon}_{12}(\hat{\boldsymbol{u}})\|_{L^2(P_t)}=\frac{\gamma}{2}\|\phi\|_{L^2(I_t)}\|\psi'\|_{L^2(P)}\le Ct^{5/2}\|\psi'\|_{L^2(P)},$$

and

$$\Pi_0\boldsymbol{\epsilon}_{12}(\boldsymbol{w})=\frac{\gamma}{2}\Pi_0[\phi_h(y)\psi_h'(x)]+\frac12\Pi_0(\theta_h-\Pi_0\theta_h)=\frac{\gamma}{2}\Pi_0[\phi_h(y)\psi_h'(x)].$$

It follows from (5.3) and $\|\psi_h'\|_{L^2(P)}\le C\|\psi'\|_{L^2(P)}$ that

$$\|\Pi_0\boldsymbol{\epsilon}_{12}(\boldsymbol{w})\|_{L^2(P_t)}\le\frac{\gamma}{2}\|\phi_h(y)\psi_h'(x)\|_{L^2(P_t)}\le Ct^{5/2}\|\psi'\|_{L^2(P)}.$$

By (5.4) and (5.5), the last two terms in (5.8) are bounded by

$$\|\phi'(\psi-\psi_h)\|_{L^2(P_t)}+\|\phi'(I-\Pi_x)\psi\|_{L^2(P_t)}\le Ct^{3/2}(\|\psi-\omega''\|_{L^2(P)}+h_x\|\omega^{(3)}\|_{L^2(P)}).$$

Summing up the above estimates for the terms on the right-hand side of (5.8) and using the a priori estimate (4.8), we obtain

$$\begin{aligned}\|\!|\, \hat{\boldsymbol{u}}-\boldsymbol{w}\,|\!\|&\le Ct^{3/2}\big(\|\psi-\omega''\|_{L^2(P)}+t\|\psi'\|_{L^2(P)}+h_x\|\omega^{(3)}\|_{L^2(P)}\big)\\&\le C(1+h_x/t^{1/2})\|g\|_{H^{-1}(P)}.\qquad\square\end{aligned}$$

In Theorem 4.4, we have shown that the quantity

$$\frac{\langle\mathbb{C}^{-1}(\widehat{\boldsymbol{\sigma}}-\mathbb{C}\boldsymbol{\epsilon}(\hat{\boldsymbol{u}})),\widehat{\boldsymbol{\sigma}}-\mathbb{C}\boldsymbol{\epsilon}(\hat{\boldsymbol{u}})\rangle^{1/2}}{\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{\mathbb{C}}}$$

is small. Next we show that it remains small with $\mathbb{C}$ replaced by $\mathbb{C}_h$.

LEMMA 5.3. *Let $\widehat{\boldsymbol{\sigma}}$ and $\hat{\mathbf{u}}$ be defined as in* (4.4) *and* (4.1), *respectively. Then*

$$\sup_{\boldsymbol{w}\in\boldsymbol{V}^h}\frac{|\langle\,\widehat{\boldsymbol{\sigma}}-\mathbb{C}_h\boldsymbol{\epsilon}(\hat{\boldsymbol{u}}),\boldsymbol{\epsilon}(\boldsymbol{w})\rangle|}{\|\!|\,\boldsymbol{w}\,|\!\|}\le C\big((1+h_x/t^{1/2})\|g\|_{H^{-1}(P)}+t^{3/2}\|g\|_{L^2(P)}\big).\tag{5.9}$$

Proof. Similar to (4.14), we have the following expansion:

$$\begin{aligned}\langle \widehat{\boldsymbol{\sigma}} - \mathbb{C}_h \boldsymbol{\epsilon}(\hat{\boldsymbol{u}}), \boldsymbol{\epsilon}(\boldsymbol{w})\rangle &= \langle \widehat{\sigma}_{22} - 24D(I-\Pi_x)\partial_y \hat{u}_2, \partial_y w_2\rangle \\ &\quad - \frac{\lambda}{\gamma}\langle \Pi_0[\gamma\partial_x \hat{u}_1 + \partial_y \hat{u}_2], \gamma\partial_x w_1 + \partial_y w_2\rangle + 2\langle \widehat{\sigma}_{12} - 2\mu\Pi_0\boldsymbol{\epsilon}_{12}(\hat{\boldsymbol{u}}), \boldsymbol{\epsilon}_{12}(\boldsymbol{w})\rangle,\end{aligned}$$

which may be rewritten as

$$\begin{aligned}\langle \widehat{\boldsymbol{\sigma}} - \mathbb{C}_h \boldsymbol{\epsilon}(\hat{\boldsymbol{u}}), \boldsymbol{\epsilon}(\boldsymbol{w})\rangle &= \langle \widehat{\sigma}_{22}, \Pi_0 \partial_y w_2\rangle - \frac{\lambda}{\gamma}\langle \gamma\partial_x \hat{u}_1 + \partial_y \hat{u}_2, \Pi_0[\gamma\partial_x w_1 + \partial_y w_2]\rangle \\ &\quad + 2\langle \widehat{\sigma}_{12} - 2\mu\boldsymbol{\epsilon}_{12}(\hat{\boldsymbol{u}}), \Pi_0\boldsymbol{\epsilon}_{12}(\boldsymbol{w})\rangle + 2\langle (I-\Pi_0)\widehat{\sigma}_{12}, \boldsymbol{\epsilon}_{12}(\boldsymbol{w})\rangle \\ &\quad + \langle \widehat{\sigma}_{22}, (I-\Pi_0)\partial_y w_2\rangle - 24D\langle (I-\Pi_x)\partial_y \hat{u}_2, \partial_y w_2\rangle \\ &= \big[\langle \widehat{\boldsymbol{\sigma}} - \mathbb{C}\boldsymbol{\epsilon}(\hat{\boldsymbol{u}}), \Pi_0\boldsymbol{\epsilon}(\boldsymbol{w})\rangle + \langle \widehat{\sigma}_{22}, (I-\Pi_x)\partial_y w_2\rangle\big] \\ &\quad + 2\langle \widehat{\sigma}_{12}, (I-\Pi_0)\boldsymbol{\epsilon}_{12}(\boldsymbol{w})\rangle - 24D\langle (I-\Pi_x)\partial_y \hat{u}_2, \partial_y w_2\rangle. \end{aligned} \tag{5.10}$$

It is clear to see that

$$2\langle \widehat{\sigma}_{12}, (I-\Pi_0)\boldsymbol{\epsilon}_{12}(\boldsymbol{w})\rangle = \langle \widehat{\sigma}_{12}, (I-\Pi_x)\partial_y w_1\rangle + \langle \widehat{\sigma}_{12}, (I-\Pi_0)\partial_x w_2\rangle.$$

We use $(I-\Pi_x)\partial_y w_1 = \partial_y(I-\Pi_x)w_1$ and integrate by parts (note that $(I-\Pi_x)w_1$ is continuous in y and $\widehat{\sigma}_{12}$ vanishes on $\partial P_t^{\pm}$) to obtain

$$\begin{aligned}\langle \widehat{\sigma}_{12}, \partial_y(I-\Pi_x)w_1\rangle &= -\langle \partial_y\widehat{\sigma}_{12}, (I-\Pi_x)w_1\rangle + \int_{\partial P_t^{\pm}} \widehat{\sigma}_{12} n_y (I-\Pi_x)w_1 \,\mathrm{d}x \\ &= -\langle \partial_y\widehat{\sigma}_{12}, (I-\Pi_x)w_1\rangle.\end{aligned}$$

Here $\boldsymbol{n} = (n_x, n_y)$ is the outward unit normal to $\partial P_t^{\pm}$. We proceed by applying the Cauchy–Schwarz inequality

$$\begin{aligned}|\langle \widehat{\sigma}_{12}, \partial_y(I-\Pi_x)w_1\rangle| &\le \|\partial_y\widehat{\sigma}_{12}\|_{L^2(P_t)}\|(I-\Pi_x)w_1\|_{L^2(P_t)} \\ &\le C(24D)h_x t^{3/2}\|\omega^{(3)}\|_{L^2(P)}\|\partial_x w_1\|_{L^2(P_t)} \\ &\le C h_x t^{-1/2}\|g\|_{H^{-1}(P)}|\!|\!|\, \boldsymbol{w} \,|\!|\!|.\end{aligned}$$

Moreover, it follows from (4.12) that $\|\widehat{\sigma}_{12}\|_{L^2(P_t)} \le Ct^{1/2}\|g\|_{H^{-1}(P)}$, which together with the second equation in (3.12) leads to

$$\begin{aligned}|\langle \widehat{\sigma}_{12}, (I-\Pi_0)\partial_x w_2\rangle| &\le \|\widehat{\sigma}_{12}\|_{L^2(P_t)}\|(I-\Pi_0)\partial_x w_2\|_{L^2(P_t)} \\ &\le Ct^{1/2}\|g\|_{H^{-1}(P)}\|(I-\Pi_x)\partial_y w_2\|_{L^2(P_t)} \\ &\le Ct^{1/2}\|g\|_{H^{-1}(P)}|\!|\!|\, \boldsymbol{w} \,|\!|\!|.\end{aligned}$$

Combining the above two estimates, we obtain

$$|2\langle \widehat{\sigma}_{12}, (I-\Pi_0)\boldsymbol{\epsilon}_{12}(\boldsymbol{w})\rangle| \le C(h_x/t^{1/2} + t^{1/2})\|g\|_{H^{-1}(P)}|\!|\!|\, \boldsymbol{w} \,|\!|\!|. \tag{5.11}$$

It follows from (5.5) that

$$\begin{aligned}|\langle (I-\Pi_x)\partial_y \hat{u}_2, \partial_y w_2\rangle| &\le Ct^{3/2}\|\psi - \Pi_x\psi\|_{L^2(P)}\|\partial_y w_2\|_{L^2(P_t)} \\ &\le Ct^{3/2}(\|\psi - \omega''\|_{L^2(P)} + Ch_x\|\omega^{(3)}\|_{L^2(P)})|\!|\!|\, \boldsymbol{w} \,|\!|\!| \\ &\le C(1 + h_x/t^{1/2})\|g\|_{H^{-1}(P)}|\!|\!|\, \boldsymbol{w} \,|\!|\!|. \end{aligned} \tag{5.12}$$

Using (3.12), (4.15), and $\|\Pi_0\boldsymbol{\epsilon}(\boldsymbol{w})\|_{\mathbb{C}} \le |||\,\boldsymbol{w}\,|||$, we bound the terms in the square bracket of (5.10) as

$$\begin{aligned}
&|\langle\,\widehat{\boldsymbol{\sigma}} - \mathbb{C}\boldsymbol{\epsilon}(\hat{\boldsymbol{u}}), \Pi_0\boldsymbol{\epsilon}(\boldsymbol{w})\rangle + \langle\,\widehat{\sigma}_{22}, (I-\Pi_x)\partial_y w_2\rangle|\\
&\qquad\le Ct^{3/2}\|g\|_{L^2(P)}\|(I-\Pi_x)\partial_y w_2\|_{L^2(P_t)} + C(\|g\|_{H^{-1}(P)} + t^{3/2}\|g\|_{L^2(P)})|||\,\boldsymbol{w}\,|||\\
&\qquad\le C\big(\|g\|_{H^{-1}(P)} + t^{3/2}\|g\|_{L^2(P)}\big)|||\,\boldsymbol{w}\,|||.
\end{aligned}$$

By inserting the above estimate, (5.12), and (5.11) into (5.10), we complete the proof of the lemma. $\square$

Now we are ready to establish the main result.

THEOREM 5.4. *There exists $c = c[g]$ independent of t and ρ such that for sufficiently small t*

$$\frac{|||\,\boldsymbol{u} - \boldsymbol{u}_h\,|||}{\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{\mathbb{C}}} \le c(t^{1/2} + h_x). \tag{5.13}$$

Proof. The substitution of (5.6) and (5.9) into (3.15) leads to

$$|||\,\boldsymbol{u} - \boldsymbol{u}_h\,||| \le C(1 + h_x/t^{1/2})\|g\|_{H^{-1}(P)} + Ct^{3/2}\|g\|_{L^2(P)} + \|\boldsymbol{\epsilon}(\boldsymbol{u} - \hat{\boldsymbol{u}})\|_{\mathbb{C}}\,.$$

Using (4.13) to estimate the last term on the right-hand side of the above inequality and recalling (4.18), we get (5.13). $\square$

Proceeding along the same lines that led to (5.13) and using (3.14), (4.17), and (4.16), we get

$$\begin{aligned}
|||\,\boldsymbol{u}\,||| &\ge |||\,\hat{\boldsymbol{u}}\,||| - |||\,\boldsymbol{u} - \hat{\boldsymbol{u}}\,|||\\
&\ge \sqrt{24D}\,\|\partial_x\hat{u}_1\|_{L^2(P_t)} - \sqrt{1+\gamma}\,\|\boldsymbol{\epsilon}(\boldsymbol{u} - \hat{\boldsymbol{u}})\|_{\mathbb{C}}\\
&\ge ct^{-1/2}
\end{aligned}$$

for a nonzero constant c that depends on $\|g\|_{H^{-1}(P)}$. This immediately implies the following.

COROLLARY 5.5. *There exists $c = c[g]$ independent of t and ρ such that for sufficiently small t*

$$\frac{|||\,\boldsymbol{u} - \boldsymbol{u}_h\,|||}{|||\,\boldsymbol{u}\,|||} \le c(t^{1/2} + h_x). \tag{5.14}$$

In view of the equivalence between the enhanced strain finite element and the method of incompatible modes [23] over the rectangular mesh, we have actually proved that the Wilson nonconforming elements [27, 24] are also free of shear locking in the thin beam limit.

6. Conclusion and perspective. In this paper, we have investigated the bending-dominated plane elasticity problem and proved that the EAS method yields finite element solutions with an order of approximation that is at least as good as a beam model with dimension reduction. The convergence result holds also for meshes with a high element aspect ratio, which is common when solving problems with thin domains. In this way we exclude *shear locking*, although a generic constant in the error estimate depends on the smoothness of the load. The main result, Theorem 5.4, may also apply to other EAS methods as discussed in [5].

Appendix A. A negative result for the triple norm. The error bound in our main result heavily depends on the right choice for the reduced energy. If we replace Π_x in the last term of $|||\cdot|||$ by Π_0 and denote the modified norm by $|||\cdot|||_*$, then we have the following lower bound for the interpolation error in this norm, which plays the same role as the lower bound in [6, Theorem 3] for the volumetric locking.

LEMMA A.1. *Let $\hat{\boldsymbol{u}}$ be the beam model defined in* (4.1)*. Then*

$$\inf_{\boldsymbol{w}\in\boldsymbol{V}^h} |||\,\hat{\boldsymbol{u}}-\boldsymbol{w}\,|||_* \geq C h_y t^{-3/2}. \tag{A.1}$$

Proof. For any $\boldsymbol{w}\in\boldsymbol{V}^h$ we have

$$|||\,\hat{\boldsymbol{u}}-\boldsymbol{w}\,|||_*^2 \geq |||\,\hat{\boldsymbol{u}}-\boldsymbol{w}\,|||_*^2 - |||\,\hat{\boldsymbol{u}}-\boldsymbol{w}\,|||^2 = 24D\|(\Pi_x-\Pi_0)\partial_y(\hat{u}_2-w_2)\|^2_{L^2(P_t)}.$$

Noting that $(\Pi_x-\Pi_0)\partial_y w_2=0$, we get

$$\inf_{\boldsymbol{w}\in\boldsymbol{V}^h} |||\,\hat{\boldsymbol{u}}-\boldsymbol{w}\,|||_* \geq \sqrt{24D}\|(\Pi_x-\Pi_0)\partial_y\hat{u}_2\|_{L^2(P_t)}. \tag{A.2}$$

A straightforward calculation gives

$$\begin{aligned}24D\|(\Pi_x-\Pi_0)\partial_y\hat{u}_2\|^2_{L^2(P_t)} &= 24D\|(I-\Pi_y)\phi'\|^2_{L^2(I_t)}\|\Pi_x\psi\|^2_{L^2(P)}\\ &= 2Dh_y^2 t\|\Pi_x\psi\|^2_{L^2(P)}.\end{aligned}$$

Proceeding along the same line as that leading to (5.5) and using the a priori estimate (4.8) and (4.17), we obtain for sufficiently small t and h_x

$$\begin{aligned}\|\Pi_x\psi\|_{L^2(P)} &\geq \|\omega''\|_{L^2(P)} - \|(I-\Pi_x)\omega''\|_{L^2(P)} - \|\Pi_x(\psi-\omega'')\|_{L^2(P)}\\ &\geq ct^{-2} - ch_x\|\omega^{(3)}\|_{L^2(P)} - \|\psi-\omega''\|_{L^2(P)}\\ &\geq ct^{-2}(1-h_x-t^{1/2})\\ &\geq (c/2)t^{-2}.\end{aligned}$$

The substitution of the above two inequalities in (A.2) yields (A.1). ◻

Based on the above lemma, we shall obtain a lower bound for the relative error in the norm $|||\cdot|||_*$. For any $\boldsymbol{w},\boldsymbol{v}\in\boldsymbol{V}$, let

$$\langle\,\tilde{\mathbb{C}}_h\boldsymbol{\epsilon}(\boldsymbol{w}),\boldsymbol{\epsilon}(\boldsymbol{w})\rangle := |||\,\boldsymbol{w}\,|||_*^2$$

and $\langle\,\tilde{\mathbb{C}}_h\boldsymbol{\epsilon}(\boldsymbol{w}),\boldsymbol{\epsilon}(\boldsymbol{v})\rangle$ be the associated bilinear form. Proceeding the same way as for the derivation of (3.14), we obtain

$$|||\,\boldsymbol{v}\,|||_* \leq \sqrt{1+\gamma}\,\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{\mathbb{C}} \quad \text{for any} \quad \boldsymbol{v}\in\boldsymbol{V}. \tag{A.3}$$

LEMMA A.2. *There exists a constant c that depends on $\|g\|_{H^{-1}(P)}$ but is independent of t,ρ, and h such that*

$$\frac{|||\,\boldsymbol{u}-\boldsymbol{u}_h\,|||_*}{\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{\mathbb{C}}} \geq c. \tag{A.4}$$

Proof. Let $\tilde{\boldsymbol{u}}\in\boldsymbol{V}^h$ be the solution of

$$\langle\,\tilde{\mathbb{C}}_h\boldsymbol{\epsilon}(\tilde{\boldsymbol{u}}),\boldsymbol{\epsilon}(\boldsymbol{w})\rangle = \langle\,\tilde{\mathbb{C}}_h\boldsymbol{\epsilon}(\hat{\boldsymbol{u}}),\boldsymbol{\epsilon}(\boldsymbol{w})\rangle \quad \text{for all} \quad \boldsymbol{w}\in\boldsymbol{V}^h.$$

By the Galerkin orthogonality, we obtain

$$|||\,\hat{\boldsymbol{u}}-\boldsymbol{u}_h\,|||_*^2 = |||\,\hat{\boldsymbol{u}}-\tilde{\boldsymbol{u}}\,|||_*^2 + |||\,\tilde{\boldsymbol{u}}-\boldsymbol{u}_h\,|||_*^2 = \inf_{\boldsymbol{w}\in\boldsymbol{V}^h} |||\,\hat{\boldsymbol{u}}-\boldsymbol{w}\,|||_*^2 + |||\,\tilde{\boldsymbol{u}}-\boldsymbol{u}_h\,|||_*^2,$$

which immediately implies

$$|||\,\hat{\boldsymbol{u}}-\boldsymbol{u}_h\,|||_* \geq \inf_{\boldsymbol{w}\in\boldsymbol{V}^h} |||\,\hat{\boldsymbol{u}}-\boldsymbol{w}\,|||_*\,.$$

This together with the triangle inequality, the inequality (A.3), Lemma A.1, and Theorem 4.4 implies for sufficiently small t:

$$\begin{aligned}|||\,\boldsymbol{u}-\boldsymbol{u}_h\,|||_* &\geq |||\,\hat{\boldsymbol{u}}-\boldsymbol{u}_h\,|||_* - |||\,\boldsymbol{u}-\hat{\boldsymbol{u}}\,|||_*\\ &\geq \inf_{\boldsymbol{w}\in\boldsymbol{V}^h} |||\,\hat{\boldsymbol{u}}-\boldsymbol{w}\,|||_* - \sqrt{1+\gamma}\|\boldsymbol{\epsilon}(\boldsymbol{u}-\hat{\boldsymbol{u}})\|_{\mathbb{C}}\\ &\geq ch_y/t^{3/2}.\end{aligned}$$

Recalling (4.18), we get (A.4). ∎

Remark A.3. In view of Lemma A.1 and Lemma 5.2, we have

$$\inf_{\boldsymbol{w}\in\boldsymbol{V}^h}\frac{|||\,\hat{\boldsymbol{u}}-\boldsymbol{w}\,|||}{\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{\mathbb{C}}} \leq C(t^{1/2}+h_x) < C\frac{h_y}{t} \leq \inf_{\boldsymbol{w}\in\boldsymbol{V}^h}\frac{|||\,\hat{\boldsymbol{u}}-\boldsymbol{w}\,|||_*}{\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{\mathbb{C}}},$$

which shows that the alternate triple norm $|||\cdot|||_*$ does not soften the strain energy sufficiently enough such that the approximation result holds, while the triple norm $|||\cdot|||$, indeed, reduces the strain energy to the desired degree.

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