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H^2-Korn's Inequality and the Nonconforming Elements for The Strain Gradient Elastic Model

Hongliang Li[1] · Pingbing Ming[2,3] · Huiyu Wang[4]

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Abstract
We establish a new H^2-Korn's inequality and its discrete analog, which greatly simplify the construction of nonconforming elements for a linear strain gradient elastic model. The Specht triangle (Specht in Int J Numer Methods Eng 28:705–715, 1988) and the NZT tetrahedron (Wang et al. in Numer Math 106:335–347, 2007) are analyzed as two typical representatives for robust nonconforming elements in the sense that the rate of convergence is independent of the small material parameter. We construct the regularized interpolation operators and the enriching operators for both elements, and prove the error estimates under minimal smoothness assumption on the solution. Numerical results for the smooth solution, and the solution with boundary layer are consistent with the corresponding theoretical prediction.

Keywords Strain gradient elasticity · H^2-Korn's inequality · Robust finite elements

1 Introduction

Let u be the solution of the following boundary value problem

$$\begin{cases} (\iota^2\triangle - I)\,(\mu\triangle u + (\lambda+\mu)\nabla\nabla\cdot u) = f & \text{in } \Omega, \\ u = \partial_n u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

✉ Pingbing Ming
mpb@lsec.cc.ac.cn

Hongliang Li
lhl@sicnu.edu.cn

Huiyu Wang
why@lsec.cc.ac.cn

1 School of Mathematics and Science, Sichuan Normal University, Chengdu 610066, China

2 LSEC, Institute of Computational Mathematics and Scientific/Engineering Computing, AMSS Chinese Academy of Sciences, No. 55, East Road Zhong-Guan-Cun, Beijing 100190, China

3 School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

4 Beijing 101 Middle School, Beijing, China

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where λ and μ are the Lamé constants, and ι is the material parameter satisfying $0 < \iota \le 1$. In particular, we are interested in the regime when ι is close to zero. This boundary value problem arises from a linear strain gradient elastic model proposed by Aifantis et al [4,44], and the unknown u is the displacement. This model may be regarded as a simplification of the more general strain gradient elasticity models in [39] because it contains only one extra material parameter ι besides the Lamé constants λ and μ. This strain gradient model successfully eliminated the strain singularity of the brittle crack tip field [22], and we refer to [21] and [24] for other strain gradient models.

Problem (1) is essentially a singularly perturbed elliptic system of fourth order due to the strain gradient $\nabla\epsilon(u)$. C^1-conforming finite element such as Argyris triangle [5] seems a natural choice for approximation (1). The performance of Argyris triangle and several other C^1-conforming finite elements has been carefully studied in [23] for a nonlinear strain gradient elastic model. A drawback of the C^1-conforming elements is that the number of the degrees of freedom (dofs) is large and high order polynomial has to be used in the shape functions, which is more pronounced for three dimensional problems; See, e.g., the finite element for a three-dimensional strain gradient model proposed in [43] locally has 192 dofs. We aim to develop some simple and robust nonconforming elements for (1), where the robustness is understood in the sense that the elements converge uniformly in the energy norm with respect to parameter ι.

To this end, we firstly prove a new H^2-Korn's inequality and its discrete analog in any dimension. This H^2-Korn's inequality may be viewed as a quantitative version of the so-called *vector version of J.L. Lions lemma* [19, Theorem 6.19-1] (cf. the statement (13)), while our proof is constructive and may be adapted to prove a Korn's inequality for the piecewise H^2 vector fields (broken H^2-Korn's inequality for short), which may be viewed as a higher-order counterpart of BRENNER'S seminal Korn's inequality [13] for the piecewise H^1 vector fields. Compared to the broken H^2-inequality proved in [32], the jump term associated with the gradient tensor of the piecewise vector field may be dropped. Therefore, the degrees of freedom associated with the gradient tensor along each face or edge may be dropped, which simplify the construction of the elements. Based on this observation, all H^1 conforming but H^2 nonconforming elements are suitable candidates to approximate (1). We choose the Specht triangle [49] and the NZT tetrahedron [53] as two typical representatives. The Specht triangle is simpler than those in [32], because the elements therein locally belong to a 21 dimensional subspace of quintic polynomials, while the tensor products of the Specht triangle locally belong to an 18 dimensional subspace of quartic polynomials. It is worth mentioning that the broken H^2-Korn's inequality may also be exploited to develop C^0 interior penalty method [16,20] for the strain gradient elastic model.

To prove the robustness of both elements, we construct a regularized interpolation operator and an enriching operator. The regularized interpolation operator may be viewed as a combination of the interpolation operator defined in [28] and the enriching operator defined in [41]. The enriching operator satisfies certain interpolation estimates and a kind of Petrov-Galerkin orthogonality, the latter differs from the standard enriching operator (cf. [11]), while its is ubiquitous for deriving error estimate for rough solution.

The remaining part of the paper is organized as follows. We prove the continuous and the broken H^2-Korn's inequalities in Sect. 2. The Specht triangle and the NTZ tetrahedron are introduced in Sect. 3 and the corresponding regularized interpolant are constructed and analyzed therein. We introduce enriching operators for both elements in Sect. 4, and derive the error bounds uniformly with respect to ι in the same part. The numerical tests of both elements are reported in the last section, which confirm the theoretical prediction in Sect. 4.

Throughout this paper, the constant C may differ from line to line, while it is independent of the mesh size h and the materials parameter ι.

2 H^2-Korn's Inequality and the Broken H^2-Korn's Inequality

In this part we prove the H^2-Korn's inequalities and the broken H^2-Korn's inequalities. Let us fix some notations firstly.

2.1 Notations

Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be a bounded convex polytope. We shall use the standard notations for Sobolev spaces, norms and semi-norms [2]. The function space $L^2(\Omega)$ consists of functions that are square integrable over Ω, which is equipped with norm $\| \cdot \|_{L^2(\Omega)}$ and the inner product $(\cdot,\cdot)$. Let $H^m(\Omega)$ be the Sobolev space of square integrable functions whose weak derivatives up to order m are also square integrable, the corresponding norm $\| v \|^2_{H^m(\Omega)} := \sum_{k=0}^m |v|^2_{H^k(\Omega)}$ with the semi-norm $|v|^2_{H^k(\Omega)} := \sum_{|\alpha|=k} \| \partial^\alpha v \|^2_{L^2(\Omega)}$ for all $v \in H^m(\Omega)$. For a positive number s that is not an integer, $H^s(\Omega)$ is the fractional order Sobolev space. Let $m = \lfloor s \rfloor$ be the largest integer less than s and $\varrho = s - m$. The sem-inorm $|v|_{H^s(\Omega)}$ and the norm $\| v \|_{H^s(\Omega)}$ are respectively given by

$$|v|^2_{H^s(\Omega)} = \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|(\partial^\alpha)v(x) - (\partial^\alpha)v(y)|^2}{|x-y|^{2+2\varrho}} \,\mathrm{d}x\ \mathrm{d}y,$$

$$\| v \|^2_{H^s(\Omega)} = \| v \|^2_{H^m(\Omega)} + |v|^2_{H^s(\Omega)}.$$

By [2, §7], the above definition for the fractional order Sobolev space $H^s(\Omega)$ is equivalent to the one obtained by interpolation, i.e.,

$$H^s(\Omega) = \left[H^{m+1}(\Omega), H^m(\Omega)\right]_\theta \qquad \text{with} \quad \theta = m + 1 - s.$$

In particular, there exists C that depends on Ω and s such that

$$\| v \|_{H^s(\Omega)} \le C \| v \|^{1-\theta}_{H^{m+1}(\Omega)} \| v \|^{\theta}_{H^m(\Omega)}. \tag{2}$$

For $s \ge 0$, $H^s_0(\Omega)$ is the closure in $H^s(\Omega)$ of the space of $C^\infty(\Omega)$ functions with compact supports in Ω.

For any vector-valued function v, its gradient ∇v is a matrix-valued function given by $(\nabla v)_{ij} = \partial_i v_j$ for $i, j = 1, \cdots, d$. The strain tensor $\epsilon(v)$ is given by $\epsilon(v) = \frac{1}{2}(\nabla v + [\nabla v]^T)$ with $\epsilon_{ij} = \frac{1}{2}\big(\partial_i v_j + \partial_j v_i\big)$. The divergence operator is defined by $\nabla \cdot v = \sum_{i=1}^d \partial_i v_i$. The vector-valued spaces are given by $[H^m(\Omega)]^d$, $[H^m_0(\Omega)]^d$ and $[L^2(\Omega)]^d$. Without abuse of notation, we employ $|\cdot|$ to denote the abstract value of a scalar, the ℓ_2 norm of a vector, and the Euclidean norm of a matrix. Throughout this paper, we may drop the subscript Ω whenever no confusion occurs.

Let $\mathcal{T}_h$ be a simplicial triangulation of Ω with maximum mesh size h. We assume all elements in $\mathcal{T}_h$ are shape-regular in the sense of Ciarlet and Raviart [18], i.e., there exists a constant γ such that $h_K/\rho_K \le \gamma$, where h_K is the diameter of the element K, and ρ_K is the diameter of the largest ball inscribed into K, and γ is the so-called chunkiness parameter [17]. We denote by $\mathcal{F}_h$, $\mathcal{E}_h$ and $\mathcal{V}_h$ the sets of $(d-1)$-dimensional faces, edges and vertices, respectively. Let $\mathcal{F}_h^B = \{\, f \in \mathcal{F}_h \mid f \subset \partial\Omega \,\}$ be the set of boundary faces. We denote by

$\mathcal{F}_h^I = \mathcal{F}_h \setminus \mathcal{F}_h^B$ the set of interior faces. Similar notations apply to $\mathcal{E}_h$ and $\mathcal{V}_h$. We denote by $\mathcal{V}_h(K)$ (resp. $\mathcal{F}_h(K)$, resp. $\mathcal{E}_h(K)$) the four vertices of K (resp. four faces, resp. six edges). Define by $\omega(a)$ (resp. $\omega(e)$) the set of elements that have a (resp. e) as a common vetex (resp. edge), and $\omega(K) = \cup_{a\in\mathcal{V}_h(K)}\omega(a)$ is the local element star of K.

Following [41], we classify the boundary vertices as follows. We say that a node $a \in \mathcal{V}_h^B$ is a *flat node* if the normal vectors of all the faces in $\mathcal{F}_h^B(a)$ are parallel. Otherwise, such vertex a is a *sharp node*. We let $\mathcal{V}_h^B = \mathcal{V}_h^\flat \cup \mathcal{V}_h^\#$, where $\mathcal{V}_h^\flat$ and $\mathcal{V}_h^\#$ denote the sets of the flat node and sharp node, respectively. By [41, Remark 3], for any $a \in \mathcal{V}_h^\#$, and let $\{t_{1,i}\}_{i=1}^{d-1}$ and $\{t_{2,i}\}_{i=1}^{d-1}$ span the tangential space of some $f_1, f_2 \in \mathcal{F}_h^B(a)$ with non-parallel unit normal vectors, then there exists j such that $\{t_{1,j}, t_{2,1}, \cdots, t_{2,i-1}\}$ forms a basis of $\mathbb{R}^d$.

2.2 H^2-Korn's Inequality

We write the boundary value problem (1) into the following variational problem: Find $u \in [H_0^2(\Omega)]^d$ such that

$$a(u, v) = (f, v) \quad \text{for all} \quad v \in [H_0^2(\Omega)]^d, \tag{3}$$

where the bilinear form a is defined for any $v, w \in [H_0^2(\Omega)]^d$ as

$$a(v, w) := (\mathbb{C}\epsilon(v), \epsilon(w)) + \iota^2(\mathbb{D}\nabla\epsilon(v), \nabla\epsilon(w)),$$

and the fourth-order tensors $\mathbb{C}$ and the sixth-order tensor $\mathbb{D}$ are defined by

$$\mathbb{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + 2\mu\delta_{ik}\delta_{jl} \quad \text{and} \quad \mathbb{D}_{ijklmn} = \lambda\delta_{il}\delta_{jk}\delta_{mn} + 2\mu\delta_{il}\delta_{jm}\delta_{kn},$$

respectively. Here δ_{ij} is the Kronecker delta function. The strain gradient $\nabla\epsilon(v)$ is a third-order tensor defined by $(\nabla\epsilon(v))_{ijk} = \epsilon_{jk,i}$.

The wellposedness of problem (3) depends on the coercivity of the bilinear form a over $[H_0^2(\Omega)]^d$, which is a direct consequence of the following H^2-Korn's inequality

$$\| \epsilon(v) \|_{L^2}^2 + \| \nabla\epsilon(v) \|_{L^2}^2 \geq C(\Omega)\| \nabla v \|_{H^1}^2 \qquad \text{for all} \quad v \in [H_0^2(\Omega)]^d. \tag{4}$$

This inequality was proved in [32, Theorem 1] for $d = 2$ with $C(\Omega) = 1/2$ by exploiting the community property of the strain operator ϵ and the partial derivative operator ∂. The proof therein easily carries over to $d > 2$. Such idea has been implicitly used in [1][1] to prove an inequality similar to (4) with an unknown constant $C(\Omega)$.

In Theorem 1, we shall prove that (4) remains valid for a vector field v belonging to $[H^2(\Omega) \cap H_0^1(\Omega)]^d$ with $C(\Omega) = 1 - 1/\sqrt{2}$. The proof relies on the fact that the strain gradient field fully controls the Hessian of the displacement algebraically (cf. (6)). This fact will be further exploited to prove a discrete analog of (5) for a piecewise vector field (cf. Theorem 2).

Theorem 1 *For any $v \in [H_0^1(\Omega)]^d$ and $\nabla\epsilon(v) \in [L^2(\Omega)]^{d\times d\times d}$, there holds $\nabla v \in [H^1(\Omega)]^{d\times d}$ and*

$$\| \epsilon(v) \|_{L^2}^2 + \| \nabla\epsilon(v) \|_{L^2}^2 \geq (1 - 1/\sqrt{2}) \left(\| \nabla v \|_{L^2}^2 + \| \nabla^2 v \|_{L^2}^2\right). \tag{5}$$

Proof The core of the proof is the following algebraic inequality

$$|\nabla\epsilon(v)|^2 \geq (1 - 1/\sqrt{2})|\nabla^2 v|^2. \tag{6}$$

[1] The inequality below (3.15), which is exactly (7) for a vector filed satisfying periodic boundary condition over a thin domain.

Integrating (6) over Ω, we obtain

$$\| \nabla\epsilon(v) \|_{L^2}^2 \geq (1 - 1/\sqrt{2})\| \nabla^2 v \|_{L^2}^2, \tag{7}$$

which together with the first Korn's inequality [30,31]

$$2\| \epsilon(v) \|_{L^2}^2 \geq \| \nabla v \|_{L^2}^2 \qquad \text{for all} \quad v \in [H_0^1(\Omega)]^d \tag{8}$$

implies (5).

To prove (6), we start with the identity

$$\begin{aligned} |\nabla\epsilon(v)|^2 &= \sum_{1\le i,j,k\le d} |\partial_i \epsilon_{jk}|^2 \\ &= \sum_{i=1}^d |\partial_i\epsilon_{ii}|^2 + \sum_{1\le i<j\le d} \left(|\partial_i\epsilon_{jj}|^2 + 2|\partial_j\epsilon_{ij}|^2\right) + \left(|\partial_j\epsilon_{ii}|^2 + 2|\partial_i\epsilon_{ij}|^2\right) \\ &\quad + \sum_{i\neq,j\neq k,i\neq k} |\partial_i\epsilon_{jk}|^2 =: I_1 + I_2 + I_3, \end{aligned} \tag{9}$$

where I_3 vanishes for $d = 2$.

Employing the elementary algebraic inequality

$$a^2 + \frac{1}{2}(a+b)^2 \geq (1 - 1/\sqrt{2})(a^2 + b^2), \qquad a, b \in \mathbb{R},$$

we obtain

$$\begin{cases} |\partial_j\epsilon_{ii}|^2 + 2|\partial_i\epsilon_{ij}|^2 \geq (1 - 1/\sqrt{2}) \left(|\partial_{ij} v_i|^2 + |\partial_{ii} v_j|^2\right), \\ |\partial_i\epsilon_{jj}|^2 + 2|\partial_j\epsilon_{ij}|^2 \geq (1 - 1/\sqrt{2}) \left(|\partial_{ij} v_j|^2 + |\partial_{jj} v_i|^2\right). \end{cases} \tag{10}$$

A direct calculation gives

$$\begin{aligned} &\sum_{i\neq j,j\neq k,i\neq k} |\partial_i\epsilon_{jk}|^2 \\ &= \frac{1}{2} \sum_{\substack{1\le i<j\le d \\ 1\le k\le d,k\neq i,k\neq j}} |\partial_{ij} v_k|^2 + \frac{1}{2} \left(\sum_{\substack{1\le i<j\le d \\ 1\le k\le d,k\neq i,k\neq j}} \partial_{ij} v_k \right)^2. \end{aligned} \tag{11}$$

Combining (9), (10) and (11), we obtain (6) immediately. □

A direct consequence of Theorem 1 is the following full H^2-Korn's inequality.

Corollary 1 *Let $\Omega \subset \mathbb{R}^d$ be a domain such that the following Korn's inequality is valid for any vector field $v \in [L^2(\Omega)]^d$ and $\epsilon(v) \in [L^2(\Omega)]^{d\times d}$,*

$$\| v \|_{L^2} + \| \epsilon(v) \|_{L^2} \geq C(\Omega)\| v \|_{H^1}.$$

If $v \in [L^2(\Omega)]^d$, $\epsilon(v) \in [L^2(\Omega)]^{d\times d}$ and $\nabla\epsilon(v) \in [L^2(\Omega)]^{d\times d\times d}$, then $v \in [H^2(\Omega)]^d$ and

$$\| v \|_{L^2} + \| \epsilon(v) \|_{L^2} + \| \nabla\epsilon(v) \|_{L^2} \geq \min\left(C(\Omega), \sqrt{1 - 1/\sqrt{2}}\right) \| v \|_{H^2}. \tag{12}$$

The following vector version of *J.L. Lions Lemma* is proved in [19, Theorem 6.19-1]: For any domain D in $\mathbb{R}^d$ and $m \in \mathbb{Z}$, then

$$v \in [H^m(D)]^d \quad \text{and} \quad \epsilon(v) \in [H^m(D)]^{d\times d} \quad \text{implies} \quad v \in [H^{m+1}(D)]^d. \tag{13}$$

The full H^2-Korn's inequality (12) may be viewed as a quantitative version of (13) with $m = 1$, while the proof in [19, Theorem 6.19-1] is nonconstructive and does not seem easy to be extended to prove the broken H^2-Korn's inequality for a piecewise vector filed.

The regularity of problem (3) is essential to prove a uniform error estimate. Unfortunately, it does not seem easy to identify such estimates in the literature, and we give a proof for the readers' convenience. We firstly assume the following regularity estimate.

Hypothesis 1 Let u be the solution of

$$\begin{cases} \triangle\big(\mathcal{L}u\big) = f & \text{in } \Omega, \\ u = \partial_n u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathcal{L}u{:} = \mu\triangle u + (\lambda+\mu)\nabla\nabla\cdot u$. Then for any $f \in H^{-1}(\Omega)$, there holds

$$\| u \|_{H^3} \le C\| f \|_{H^{-1}}. \tag{14}$$

If Ω is a smooth domain, then the regularity property (14) is standard; See e.g., [3]. While it is unclear whether (14) is true for a convex polytope. Nevertheless, if $\mathcal{L}$ is replaced by the Laplacian operator $\triangle$, then (14) was proved in [38, Chapter 4, Theorem 4.3.10].

Lemma 1 *Assume Hypothesis 1 is valid and let u be the solution of (3), then there exists C that may depend on Ω but independent of ι such that*

$$\| \nabla^k(u-u_0) \|_{L^2} \le C\iota^{3/2-k}\| f \|_{L^2} \quad \textit{for} \quad k = 1, 2, \tag{15}$$

where $u_0 \in [H_0^1(\Omega)]^d$ satisfies

$$(\mathbb{C}\epsilon(u_0), \epsilon(v)) = (f, v) \qquad \textit{for all} \quad v \in [H_0^1(\Omega)]^d. \tag{16}$$

Moreover, we have

$$\| u \|_{H^{3/2}} \le C\| f \|_{L^2}, \tag{17}$$

and

$$\| u \|_{H^{5/2}} \le C\iota^{-1}\| f \|_{L^2}. \tag{18}$$

Under Hypothesis 1, we may prove the above estimates by following essentially the same line that leads to [42, Lemma 5.1].

Proof Denoting $\phi = u - u_0$, using (1) and (16), we have

$$\triangle\mathcal{L}(u) = \iota^{-2}\mathcal{L}(u-u_0) = \iota^{-2}\mathcal{L}(\phi).$$

Using the regularity hypothesis (14), we obtain

$$\| u \|_{H^3} \le C\iota^{-2}\| \mathcal{L}(\phi) \|_{-1} \le C\iota^{-2}\big(\mathbb{C}\epsilon(\phi), \epsilon(\phi)\big)^{\frac{1}{2}}. \tag{19}$$

By the regularity estimate for (16) [38, Theorem 4.3.3], there exists C depends on Ω, λ and μ such that

$$\| u_0 \|_{H^2} \le C\| f \|_{L^2}. \tag{20}$$

Integration by parts, we have

$$a(\phi,\phi) = -\iota^2\big(\mathbb{D}\nabla\epsilon(u_0),\nabla\epsilon(\phi)\big) + \iota^2\int_{\partial\Omega} M_{nn}(u)\partial_n u_0\,\mathrm{d}\sigma(x),$$

where $M_{nn}(u) = n^{\mathrm{T}}\cdot\mathbb{D}\nabla\epsilon(u)\cdot n$.

Using (20), we bound there exists C such that

$$\begin{aligned}\iota^2|\big(\mathbb{D}\nabla\epsilon(u_0),\nabla\epsilon(\phi)\big)| &\le \frac{\iota^2}{2}\big(\mathbb{D}\nabla\epsilon(\phi),\nabla\epsilon(\phi)\big) + \frac{\iota^2}{2}\big(\mathbb{D}\nabla\epsilon(u_0),\nabla\epsilon(u_0)\big)\\ &\le \frac{\iota^2}{2}\big(\mathbb{D}\nabla\epsilon(\phi),\nabla\epsilon(\phi)\big) + (\mu+d\lambda/2)\iota^2\|\nabla\epsilon(u_0)\|_{L^2}\\ &\le \frac{\iota^2}{2}\big(\mathbb{D}\nabla\epsilon(\phi),\nabla\epsilon(\phi)\big) + C\iota^2\|f\|_{L^2}^2,\end{aligned}$$

where C depends on Ω, λ and μ.

Using the trace inequality (25), we obtain, for $\delta>0$ to be chosen later,

$$\begin{aligned}\iota^2\left|\int_{\partial\Omega} M_{nn}(u)\partial_n u_0\mathrm{d}\sigma(x)\right| &\le \iota^3\delta\|M_{nn}(u)\|_{L^2(\partial\Omega)}^2 + \frac{\iota}{4\delta}\|\partial_n u_0\|_{L^2(\partial\Omega)}^2\\ &\le C\delta\left(\iota^4\|\nabla^2 u\|_{H^1}^2 + \iota^2\|\nabla^2 u\|_{L^2}^2\right) + C\frac{\iota}{\delta}\|u_0\|_{H^2}^2.\end{aligned}$$

Using (7), we obtain

$$\|\nabla^2 u\|_{L^2}^2 \le 2\|\nabla^2\phi\|_{L^2}^2 + 2\|\nabla^2 u_0\|_{L^2}^2 \le \frac{4}{\mu}\big(\mathbb{D}\nabla\epsilon(\phi),\nabla\epsilon(\phi)\big) + 2\|\nabla^2 u_0\|_{L^2}^2.$$

Using the regularity estimates (19) and (20), we bound the right-hand side of the above inequality as

$$\iota^2\left|\int_{\partial\Omega} M_{nn}(u)\partial_n u_0\mathrm{d}\sigma(x)\right| \le C\delta a(\phi,\phi) + C\iota\left(\delta\iota+\frac{1}{\delta}\right)\|f\|_{L^2}^2.$$

Combining the above inequalities, we obtain

$$a(\phi,\phi) \le C\delta a(\phi,\phi) + \frac{\iota^2}{2}\big(\mathbb{D}\nabla\epsilon(\phi),\nabla\epsilon(\phi)\big) + C\iota^2(1+\delta)\|f\|_{L^2}^2 + \frac{C\iota}{\delta}\|f\|_{L^2}^2,$$

which immediately implies

$$\frac{1}{2}a(\phi,\phi) \le C\delta a(\phi,\phi) + C\iota^2(1+\delta)\|f\|_{L^2}^2 + \frac{C\iota}{\delta}\|f\|_{L^2}^2.$$

Choosing δ properly and using (5), we obtain (15).

Using (15) and the Poincaré inequality, and noting that $\iota<1$, we obtain

$$\|u-u_0\|_{H^2} \le C\left(\iota^{1/2}+\iota^{-1/2}\right)\|f\|_{L^2} \le C\iota^{-1/2}\|f\|_{L^2}, \tag{21}$$

and

$$\|u\|_{H^2} \le \|u-u_0\|_{H^2} + \|u_0\|_{H^2} \le C\iota^{-1/2}\|f\|_{L^2}. \tag{22}$$

Interpolating (21) and (15) with $k=1$, and using the interpolation inequality (2), we obtain

$$\|u-u_0\|_{H^{3/2}} \le C\|f\|_{L^2}.$$

Invoking the interpolation inequality (2) again, we obtain

$$\|u_0\|_{H^{3/2}} \le C\|u_0\|_{H^1}^{1/2}\|u_0\|_{H^2}^{1/2} \le C\|f\|_{L^2}.$$

A combination of the above two inequalities yields (17).

Combining (19) and (15), we obtain

$$\| u \|_{H^3} \leq C\iota^{-2} \| \nabla(u - u_0) \|_{L^2} \leq C\iota^{-3/2} \| f \|_{L^2}.$$

Interpolating the above inequality and (22), we obtain (18). □

2.3 The Broken H^2-Korn's Inequality

For any $m \in \mathbb{N}$, the space of piecewise vector fields is defined by

$$[H^m(\Omega, \mathcal{T}_h)]^d := \{ v \in [L^2(\Omega)]^d \mid v|_K \in [H^m(K)]^d \quad \text{for all} \quad K \in \mathcal{T}_h \},$$

which is equipped with the broken norm

$$\| v \|_{H_h^k} := \| v \|_{L^2} + \sum_{k=1}^{m} \| \nabla_h^k v \|_{L^2},$$

where $\| \nabla_h^k v \|_{L^2}^2 = \sum_{K \in \mathcal{T}_h} \| \nabla^k v \|_{L^2(K)}^2$ with $(\nabla_h^k v)|_K = \nabla^k (v|_K)$. Moreover, $\epsilon_h(v) = (\nabla_h v + [\nabla_h v]^T)/2$. For any $v \in H^m(\Omega, \mathcal{T}_h)$, we denote by $[\![v]\!]$ and $\{\{v\}\}$ the jump and the average of v across the face or the edge, respectively; See, e.g., [6] for the definitions.

The main result of this part is the following broken H^2-Korn's inequality.

Theorem 2 *For any $v \in [H^2(\Omega, \mathcal{T}_h)]^d$, there exits C that depends on Ω and γ but independent of h such that*

$$\begin{aligned} \| v \|_{H_h^2}^2 \leq C\Big(&\| \nabla_h \epsilon_h(v) \|_{L^2}^2 + \| \epsilon_h(v) \|_{L^2}^2 + \| v \|_{L^2}^2 \\ &+ \sum_{f \in \mathcal{F}_h} h_f^{-1} \| [\![\Pi_f v]\!] \|_{L^2(f)}^2 \Big), \end{aligned} \tag{23}$$

where $\Pi_f : [L^2(f)]^d \mapsto [P_{1,-}(f)]^d$ is the L^2 projection and

$$[P_{1,-}(f)]^d := \{ v \in [P_1(f)]^d \mid v_t \in RM(f) \},$$

where $v_t = v - (v \cdot n)n$ is the tangential components of v, and n is the normal vector of the face f (or edge for $d = 2$), and $RM(f)$ is the infinitesimal rigid motion on f.

For a piecewise vector field v, the inequality (23) improves the one proved in [32, Theorem 2] by removing the jump term

$$\sum_{i=1}^{2} \sum_{f \in \mathcal{F}_h} h_f^{-1} \| [\![\Pi_f (v_{,i})]\!] \|_{L^2(f)}^2.$$

This term stands for the jump of ∇v across the element boundary. This would simplify the construction of the strain gradient elements as shown in below.

Proof of Theorem 2 Integrating (6) over element $K \in \mathcal{T}_h$, we obtain,

$$\| \nabla \epsilon(v) \|_{L^2(K)}^2 \geq (1 - 1/\sqrt{2}) \| \nabla^2 v \|_{L^2(K)}^2.$$

Summing up all $K \in \mathcal{T}_h$, we get

$$\| \nabla_h \epsilon_h(v) \|_{L^2} \geq (1 - 1/\sqrt{2}) \| \nabla_h^2 v \|_{L^2}^2, \tag{24}$$

which together with the following Korn's inequality for a piecewise H^1 vector field proved by Mardal and Winther [37]

$$\| v \|_{H_h^1}^2 \le C \left(\|\epsilon_h(v)\|_{L^2}^2 + \| v \|_{L^2}^2 + \sum_{f \in \mathcal{F}_h} h_f^{-1} \| [\![\Pi_f v]\!] \|_{L^2(f)}^2 \right)$$

implies (23). □

We shall frequently use the following trace inequalities.

Lemma 2 *For any Lipschitz domain D, there exists C depending on D such that*

$$\| v \|_{L^2(\partial D)} \le C \| v \|_{L^2(D)}^{1/2} \| v \|_{H^1(D)}^{1/2}. \tag{25}$$

For an element K, there exists C independent of h_K, but depends on γ such that

$$\| v \|_{L^2(\partial K)} \le C \left(h_K^{-1/2} \| v \|_{L^2(K)} + h_K^{1/2} \| \nabla v \|_{L^2(K)} \right). \tag{26}$$

If $v \in \mathbb{P}_m(K)$, then there exists C independent of v, but depends on γ and m such that

$$\| v \|_{L^2(\partial K)} \le C h_K^{-1/2} \| v \|_{L^2(K)}. \tag{27}$$

The multiplicative type trace inequality (25) may be found in [27, Theorem 1.5.1.10], while (26) is a direct consequence of (25). The inequality (27) is a combination of (26) and the inverse inequality for any polynomial $v \in \mathbb{P}_m(K)$.

3 Interpolation for Nonsmooth Data

Motivated by the broken H^2-Korn's inequality (23), we conclude that the H^1-conforming but H^2-nonconforming finite elements are natural choices to approximate (1). A family of rectangular elements in this vein may be found in [35], and two nonconforming tetrahedron elements were constructed and analyzed in [52]. Note that the tensor product of certain finite elements for the singular perturbation problem of fourth order may also be used to approximate (1), we refer to [28,42,46,47,50] and references therein for such elements. In what follows, we select the Specht triangle [49] and the NZT tetrahedron [53] as the representatives. The Specht triangle is a successful plate bending element, *which passes all the patch tests and performs excellently, and is one of the best thin plate triangles with* 9 *degrees of freedom that currently available* [57, Quatation in p. 345]. The NZT tetrahedron may be regarded as a three-dimensional extension of the Specht triangle. One may wonder whether the tensor product of certain non H^1-conforming elements for the singular perturbation problem of fourth order can be used to approximate (1). This is indeed the case for the Morley triangle with a modified strain energy; See, e.g., [33] for details. More non H^1-conforming elements [54] have to be carefully studied to approximate (1).

The Specht triangle and the NZT tetrahedron may be defined by the finite element triple (K, P_K, Σ_K) [18] in a unifying way as follows. Let K be a simplex, and

$$\begin{cases} P_K = Z_K + b_K \mathbb{P}_1(K), \\ \Sigma_K = \{p(a_i), (e_{ij} \cdot \nabla p)(a_i), 1 \le i \ne j \le d+1\} \end{cases}$$

with extra constraints

$$\frac{1}{|f_i|} \int_{f_i} \partial_n p = \frac{1}{d} \sum_{1 \le k \le d+1, k \ne i} \partial_n p(a_k), \quad i = 1, \cdots, d+1, \tag{28}$$

where f_i is a $(d-1)$-simplex opposite to the vertex a_i, and e_{ij} is the edge vector from a_i to a_j. Here Z_K is the Zienkiewicz space defined by

$$Z_K = \mathbb{P}_2(K) + \operatorname{Span}\{\lambda_i^2\lambda_j - \lambda_i\lambda_j^2 \mid 1 \le i \ne j \le d+1\},$$

where λ_i is the barycentric coordinate with respect to the vertex a_i.

The finite element space is defined by

$$X_h := \{ v \in H^1(\Omega) \mid v|_K \in P_K, K \in \mathcal{T}_h; v(a), \nabla v(a) \text{ are continuous for } a \in \mathcal{V}_h \}.$$

The corresponding homogenous finite element space is given by

$$X_h^0 := \{ v \in X_h \mid v(a), \nabla v(a) \text{ vanish for } a \in \mathcal{V}_h^B \}.$$

It is clear that $X_h^0 \subset H_0^1(\Omega)$. We denote $V_h = [X_h^0]^d$, and the approximating problem reads as: Find $u_h \in V_h$ such that

$$a_h(u_h, v) = (f, v) \quad \text{for all} \quad v \in V_h, \tag{29}$$

where a_h is defined for any $v, w \in V_h$ as

$$a_h(v, w) := (\mathbb{C}\epsilon(v), \epsilon(w)) + \iota^2(\mathbb{D}\nabla_h\epsilon(v), \nabla_h\epsilon(w))$$

with

$$(\mathbb{D}\nabla_h\epsilon(v), \nabla_h\epsilon(w)) := \sum_{K\in\mathcal{T}_h} \int_K \mathbb{D}\nabla\epsilon(v)\nabla\epsilon(w)\,\mathrm{d}x.$$

The energy norm is defined by $\|\!|v|\!\|_{\iota,h} := \left(\| v \|_{H^1}^2 + \iota^2 \| \nabla_h^2 v \|_{L^2}^2\right)^{1/2}$. The bilinear form is coercive in this energy norm as shown in the next lemma.

Lemma 3 *For any $v \in V_h$,*

$$a_h(v, v) \ge C_\flat \|\!|v|\!\|_{\iota,h}^2, \tag{30}$$

where $C_\flat = \mu/(2 + 2C_p^2)$ with C_p appears in the Poincaré inequality

$$\| v \|_{L^2} \le C_p \| \nabla v \|_{L^2}, \qquad \textit{for all} \quad v \in [H_0^1(\Omega)]^d. \tag{31}$$

The estimate (30) immediately implies the wellposedness of problem (29) for any fixed ι.

Proof For any $v \in V_h$, there holds

$$a_h(v, v) \ge 2\mu \left(\| \epsilon(v) \|_{L^2}^2 + \iota^2 \| \nabla_h\epsilon(v) \|_{L^2}^2\right).$$

Using the first Korn's inequality (8) and the estimate (24), we obtain

$$a_h(v, v) \ge \frac{\mu}{2} \left(\| \nabla v \|_{L^2}^2 + \iota^2 \| \nabla_h^2 v \|_{L^2}^2\right),$$

which together with the Poincare inequality (31) implies (30) . □

The degrees of freedom of X_h involve the first order derivatives at the vertices and the mean of the normal derivatives at each face (resp. edge when $d = 2$), the associated canonical interpolant is not well-defined on H^1. In what follows we construct a regularized interpolant that is H^1 bounded. The construction is a combination of a regularized interpolant in [28] and an enriching operator in [41]. The regularized interpolant $I_h = \Pi_h \circ \Pi_C : H_0^1(\Omega) \to X_h$. For any $v \in H_0^2(\Omega)$, we define $I_h^0 : H_0^2(\Omega) \to X_h^0$ with

$$
\begin{aligned}
&I_h^0 v(a) = I_h v(a),\ \nabla I_h^0 v(a) = \nabla I_h v(a) \qquad && \text{for all } a \in \mathcal{V}_h^I,\\
&I_h^0 v(a) = 0,\ \nabla I_h^0 v(a) = 0 && \text{for all } a \in \mathcal{V}_h^B.
\end{aligned}
$$

Here we denote by $\Pi_C : H_0^1(\Omega) \to L_h$ the Scott-Zhang interpolant [45] , where L_h is the quadratic Lagrangian finite element space with vanishing trace. The auxiliary operator $\Pi_h : L_h \to X_h$ is locally defined as follows.

1. If $a \in \mathcal{V}_h^I$ is an interior vertex, then we fix an element K' from $\omega(a)$,

$$\Pi_h w(a) := w(a) \quad \text{and} \quad \nabla \Pi_h w(a) := \nabla w_{K'}(a).$$

2. If $a \in \mathcal{V}_h^\flat$ is a flat node, then we fix an element K' from $\omega(a)$,

$$\Pi_h w(a) := 0 \quad \text{and} \quad \nabla \Pi_h w(a) := \nabla w_{K'}(a).$$

3. If $a \in \mathcal{V}_h^\#$ is a sharp node, then

$$\Pi_h w(a) := 0 \quad \text{and} \quad \nabla \Pi_h w(a) := 0.$$

The properties for the regularized interpolants I_h and I_h^0 are as follows.

Theorem 3 *There exists an operator $I_h : H_0^1(\Omega) \to X_h$ such that for any $v \in H^m(\Omega)$ with $1 \le m \le 3$, there holds*

$$\| v - I_h v \|_{H_h^k} \le C h^{m-k} |v|_{H^m}, \quad 0 \le k \le m. \tag{32}$$

Moreover, there exists $I_h^0 : H_0^2(\Omega) \to X_h^0$ such that for any $v \in H^m(\Omega) \cap H_0^2(\Omega)$ with $1 \le m \le 3$, there holds

$$\| v - I_h^0 v \|_{H_h^k} \le C h^{m-k} |v|_{H^m}, \quad 0 \le k \le m. \tag{33}$$

The interpolant $I_h^0 v$ is enough to our ends, while $I_h v$ is a useful tool for the strain gradient elastic model with other boundary conditions (cf. [7]).

Proof For any $\phi \in P_K$, a standard scaling argument yields that

$$\| \phi \|_{L^2(K)}^2 \le C h_K^d \sum_{a \in \mathcal{V}_h(K)} \left(|\phi(a)|^2 + h_K^2 |\nabla \phi(a)|^2 \right).$$

Let $\phi = w - \Pi_h w$ with $w = \Pi_C v$. Noting $\phi(a) = 0$, we obtain

$$\| \phi \|_{L^2(K)}^2 \le C h_K^{d+2} \sum_{a \in \mathcal{V}_h(K)} | \nabla \phi(a) |^2. \tag{34}$$

If $a \in \mathcal{V}_h^I$ or $a \in \mathcal{V}_h^\flat$, then $\nabla \Pi_h w(a) = \nabla w_K(a)$. We may select a sequence of elements $\{K_1, \cdots, K_J\} \subset \omega(a)$ such that $K_1 = K$, $K_J = K'$, and $f_j = \partial K_j \cap \partial K_{j+1}$ is a common face of K_j and K_{j+1}. We write the right-hand side of (34) as the telescopic sum

$$|\nabla \phi(a)|^2 \le \sum_{j=1}^{J-1} \left| \nabla w_{K_j}(a) - \nabla w_{K_{j+1}}(a) \right|^2.$$

It follows from the inverse inequality that

$$
\begin{aligned}
|\nabla \phi(a)|^2 &\leq \sum_{j=1}^{J-1} \| \nabla w_{K_j} - \nabla w_{K_{j+1}} \|_{L^\infty(f_i)}^2 \leq C \sum_{j=1}^{J-1} h_{f_i}^{1-d} \| \nabla w_{K_i} - \nabla w_{k_{i+1}} \|_{L^2(f_i)}^2 \\
&\leq C \sum_{j=1}^{J-1} h_{f_i}^{1-d} \| [\![\nabla w]\!] \|_{L^2(f_i)}^2,
\end{aligned}
$$

where the jump of ∇w across f_j is defined as $[\![\nabla w]\!]|_{f_j} := \nabla w_{K_j} \,|_{f_j} - \nabla w_{K_{j+1}} \,|_{f_j}$. Note that w is continuous across f_j, we rewrite the above inequality as

$$
|\nabla \phi(a)|^2 \leq C \sum_{\substack{K, K' \in \omega(a) \\ \partial K \cap \partial K' = f \neq \emptyset}} h_f^{1-d} \| [\![\partial_{n_f} w]\!] \|_{L^2(f)}^2. \tag{35}
$$

At the sharp node $a \in \mathcal{V}_h^\#$, we write $\nabla \phi(a) = \nabla w_K(a)$. Since a is a sharp node, there exists two simplexes $K_1, K_2 \in \omega(a)$, and boundary faces $f_1 \in \partial K_1 \cap \partial \Omega$ and $f_2 \in \partial K_2 \cap \partial \Omega$, and f_1, f_2 do not have a common normal vector. Hence, there exists a tangential vector $t_{1,i}$ of f_1 and $d-1$ tangential vector $\{t_{2,1}, \cdots, t_{2,d-1}\}$ of f_2, such that these d vectors form a basis of $\mathbb{R}^d$. Proceeding along the same line that leads to (35) and using the fact that the tangential derivatives of v vanishes on $\partial \Omega$, we have

$$
\begin{aligned}
\left| \frac{\partial w_K}{\partial t_{1,i}}(a) \right|^2 &\leq C \left(\sum_{\substack{K, K' \in \omega(a) \\ \partial K \cap \partial K' = f \neq \emptyset}} h_f^{1-d} \| [\![\partial_{t_{1,i}} w]\!] \|_{L^2(f)}^2 + h_{f_1}^{-2} \| \partial_{t_{1,i}} w_{K_1} \|_{L^2(f)}^2 \right) \\
&\leq C \sum_{\substack{K, K' \in \omega(a) \\ \partial K \cap \partial K' = f \neq \emptyset}} h_f^{1-d} \| [\![\partial_{n_f} w]\!] \|_{L^2(f)}^2.
\end{aligned}
$$

Proceeding along the same line that leads to the above inequality, we obtain, for $j = 1, \cdots, d-1$,

$$
\left| \frac{\partial w_K}{\partial t_{2,j}}(a) \right|^2 \leq C \sum_{\substack{K, K' \in \omega(a) \\ \partial K \cap \partial K' = f \neq \emptyset}} h_f^{1-d} \| [\![\partial_{n_f} w]\!] \|_{L^2(f)}^2.
$$

Therefore, for $a \in \mathcal{V}_h^\#$,

$$
|\nabla \phi(a)|^2 \leq C \sum_{\substack{K, K' \in \omega(a) \\ \partial K \cap \partial K' = f \neq \emptyset}} h_f^{1-d} \| [\![\partial_{n_f} w]\!] \|_{L^2(f)}^2. \tag{36}
$$

Substituting (35) and (36) into (34), we obtain

$$
\| \phi \|_{L^2(K)} \leq C h_K^{3/2} \sum_{\substack{K, K' \in \omega(a) \\ \partial K \cap \partial K' = f \neq \emptyset}} \| [\![\partial_n w]\!] \|_{L^2(f)},
$$

where we have used the fact that $h_K \simeq h_f$ because $\mathcal{T}_h$ is locally quasi-uniform. Using the above inequality, the estimate for the Scott-Zhang interpolant, the inverse inequality and the trace inequality (26), we obtain

$$\begin{aligned}\| \Pi_C v - \Pi_h \Pi_C v \|_{H^k(K)} &\le C h_K^{-k} \| \Pi_C v - \Pi_h \Pi_C v \|_{L^2(K)} \\ &\le C h_K^{3/2-k} \sum_{\substack{K, K' \in \omega(a) \\ \partial K \cap \partial K' = f \neq \emptyset}} \| [\![\partial_n \Pi_C v]\!] \|_{L^2(f)} \\ &\le C h_K^{m-k} \| v \|_{H^m(\omega(K))}.\end{aligned}$$

Finally, using the triangle inequality, we obtain

$$\begin{aligned}\| v - I_h v \|_{H^k(K)} &\le \| v - \Pi_C v \|_{H^k(K)} + \| \Pi_C v - \Pi_h \Pi_C v \|_{H^k(K)} \\ &\le C h_K^{m-k} \| v \|_{H^m(\omega(K))}.\end{aligned}$$

Summing up the above inequalities for $K \in \mathcal{T}_h$, we obtain (32).

For any $v \in H_0^2(\Omega) \cap H^m(\Omega)$, the estimate (33) may be proceeded along the same line that leads to (32), we omit the details. □

4 Error Estimate for Less Smooth Solution

The standard error estimate argument holds true if $u \in H^s(\Omega)$ with $s \ge 3$, while such assumption is usually invalid for the point load or nonconvex domain [9], even for the biharmonic problems. In this part we shall exploit an enriching operator to derive a new error estimate for problem (1) with less smooth solution. The enriching operator measures the distance between V_h and $H^2(\Omega)$, which was firstly introduced by Brenner [11,12] to analyze nonconforming elements in the context of fast solvers. It also plays an important role in deriving a priori and a posteriori error estimates for the fourth order problems [14, 25,26,34,51]. A recent application of the enriching operator to Hamilton-Jacobi-Bellman equation may be found in [41]. The construction of these enriching operators are mainly based on the averaging of the degrees of freedom. Two enriching operators constructed with different ways have appeared in [15] and [48] recently. Besides the standard interpolation error estimates, the enriching operator should satisfy a kind of Petrov-Galerkin orthogonality, which is key to derive the error estimates for rough solution as demonstrated in [34, Lemma 4.1], in which the authors have constructed an enriching operator for the quadratic Specht triangle and have obtained optimal error estimate for approximating the biharmonc problems with rough solution. The construction and the proof therein equally applies to the Specht triangle.

We shall construct an enriching operator for the NZT tetrahedron with the aid of the tenth polynomial C^1-conforming element introduced by Zhang [56]. The enriching operator $E_h : X_h^0 \to H_0^2(\Omega)$ is defined as follows.

1. For any $a \in \mathcal{V}_h^I$, we fix an element K_a from the element star $\omega(a)$,

$$\big(\nabla^\alpha E_h v\big)(a) := \big(\nabla^\alpha v|_{K_a}\big)(a) \quad |\alpha| \le 4.$$

2. For any edge $e \in \mathcal{E}_h^I$, we choose two unit vectors s_1, s_2 orthogonal to the edge e. We fix an element K_e from the element star $\omega(e)$.

(a) For a the middle point of the edge e,

$$E_h v(a) := v|_{K_e}(a).$$

(b) For a the 2 equally-distributed interior points of the edge e,

$$\partial_{s_i} E_h v(a) := \big(\partial_{s_i} v|_{K_e}\big)(a) \quad i = 1, 2. \tag{37}$$

(c) For a the 3 equally-distributed interior points of the edge e,

$$\frac{\partial^2 E_h v}{\partial s_i \partial s_j}(a) := \frac{\partial^2 v|_{K_e}}{\partial s_i \partial s_j}(a) \quad i, j = 1, 2. \tag{38}$$

3. For any $f \in \mathcal{F}_h^I$, and for any $w \in \mathbb{P}_1(f)$,

$$\int_f E_h v w \,\mathrm{d}\sigma(x) := \int_f v w \,\mathrm{d}\sigma(x), \tag{39}$$

and for any $w \in \mathbb{P}_3(f)$,

$$\int_f \partial_n \big(E_h v\big) w \,\mathrm{d}\sigma(x) := \int_f \{\!\{\partial_n v\}\!\} w \,\mathrm{d}\sigma(x). \tag{40}$$

4. For any $K \in \mathcal{T}_h$, and for $w \in \mathbb{P}_2(K)$,

$$\int_K E_h v w \,\mathrm{d}x := \int_K v w \,\mathrm{d}x.$$

5. All the degrees of freedom of $E_h v$ vanish on $\partial\Omega$.

We summarize the properties of E_h in the following lemma.

Lemma 4 *The enriching operator E_h has the following properties:*

1. Petrov-Galerkin orthogonality:

$$a_h(v - E_h v, w) = 0 \qquad \textit{for all} \quad v \in V_h, w \in W_h, \tag{41}$$

where $W_h = [L_h]^3$ is the tensorized quadratic Lagrangian finite element space with vanishing trace.

2. E_h is stable in the sense that

$$|\!|\!| E_h v |\!|\!|_{\iota,h} \le (1 + \beta) |\!|\!| v |\!|\!|_{\iota,h} \qquad \textit{for all} \quad v \in V_h. \tag{42}$$

3. For any $v \in V_h$, we have

$$\| \nabla_h^k (v - E_h v) \|_{L^2} \le \beta h^{j-k} \| \nabla_h^j v \|_{L^2}, \quad 0 \le k \le j \le 2. \tag{43}$$

The stability estimate (42) is a direct consequence of (43). We only prove (41) and (43).

Proof For any $v \in V_h$ and $w \in W_h$, integration by parts, we obtain

$$a_h(v - E_h v, w) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \big(\iota^2 \partial_n (v - E_h v) \cdot M_{nn}(w) + (v - E_h v) \cdot \sigma_n(w)\big) \mathrm{d}\sigma(x),$$

where $\sigma_n(w) = n^{\mathrm{T}} \cdot \big(\mathbb{C}\epsilon(w)\big)$. Using the magic formula [6], we write

$$
\begin{aligned}
&a_h(v - E_h v, w) \\
&\quad = \iota^2 \sum_{f \in \mathcal{F}_h^I} \int_f \big([\![\partial_n (v - E_h v)]\!] \cdot \{\!\{M_{nn}(w)\}\!\} + \{\!\{\partial_n (v - E_h v)\}\!\} \cdot [\![M_{nn}(w)]\!] \big) \mathrm{d}\sigma(x) \\
&\qquad + \sum_{f \in \mathcal{F}_h^I} \int_f \big([\![v - E_h v]\!] \cdot \{\!\{\sigma_n(w)\}\!\} + \{\!\{v - E_h v\}\!\} \cdot [\![\sigma_n(w)]\!] \big) \mathrm{d}\sigma(x),
\end{aligned}
$$

Using the facts that $M_{nn}(w) \in [\mathbb{P}_0(K)]^3$ and $\sigma_n(w) \in [\mathbb{P}_1(K)]^3$,

$$
\begin{aligned}
[\![\partial_n (v - E_h v)]\!] &= [\![\partial_n v]\!] \qquad \text{and} \quad \{\!\{\partial_n (v - E_h v)\}\!\} = \{\!\{\partial_n v\}\!\} - \partial_n E_h v, \\
[\![v - E_h v]\!] &= 0 \qquad\qquad \text{and} \quad \{\!\{v - E_h v\}\!\} = v - E_h v,
\end{aligned}
$$

and using (39) and (40), we obtain the identity (41).

For any $K \in \mathcal{T}_h$, we let $\mathcal{N}(K)$, $\mathcal{E}(K)$, $\mathcal{F}(K)$ and $\mathcal{V}(K)$ be the set of the nodal variables N, the set of the edge variables E, the set of the face variable F, and the set of the volume variables V of $\mathbb{P}_{10}$ conforming element, respectively. For any $v \in V_h$, $v - E_h v \in \mathbb{P}_{10}$, and it follows from the scaling argument that

$$
\begin{aligned}
\| v - E_h v \|_{L^2(K)}^2 \le C &\sum_{N \in \mathcal{N}(K)} h_K^{3+2\mathrm{order}(N)} \left(N(v - E_h v)\right)^2 \\
&+ C \sum_{E \in \mathcal{E}(K)} h_K^{3+2\mathrm{order}(E)} \left(E(v - E_h v)\right)^2 \\
&+ C \sum_{F \in \mathcal{F}(K)} h_K^{3+2\mathrm{order}(F)} \left(F(v - E_h v)\right)^2 \\
&+ C \sum_{V \in \mathcal{V}(K)} h_K^{3+2\mathrm{order}(V)} \left(V(v - E_h v)\right)^2 \\
=: &I_1 + \cdots + I_4,
\end{aligned}
$$

where $\mathrm{order}(N)$ is the order of the differentiation in the definition of N, and the same rule applies to $\mathrm{order}(E)$, $\mathrm{order}(F)$ and $\mathrm{order}(V)$. It is clear that $I_4 = 0$ because $V(v) = V(E_h v)$.

Note that $N(v - E_h v) = 0$ when $\mathrm{order}(N) = 0, 1$. To estimate the terms with $\mathrm{order}(N) = 2, 3, 4$. Using the inverse inequality, we obtain

$$
\begin{aligned}
I_1 &\le C \sum_{l=2}^{4} \sum_{a \in \mathcal{V}_h(K)} h_{K'}^{3+2l} \sum_{K' \in \omega(a)} \| \nabla^l v \|_{L^\infty(K')}^2 \\
&\le C \sum_{l=2}^{4} \sum_{a \in \mathcal{V}_h(K)} h_{K'}^{3+2l} \sum_{K' \in \omega(a)} h_{K'}^{1-2l} \| \nabla^2 v \|_{L^2(K')}^2 \\
&\le C \sum_{K' \in \omega(K)} h_{K'}^4 \| \nabla_h^2 v \|_{L^2(K')}^2,
\end{aligned}
$$

Next, we note $E(v - E_h v) = 0$ when $\mathrm{order}(E) = 0$. We denote I_2^1 (resp. I_2^2) for the term in I_2 with $\mathrm{order}(E) = 1$ (resp. $\mathrm{order}(E) = 2$). For $\mathrm{order}(E) = 1$ and for any $a \in e$ with $e \in \mathcal{E}_h(K)$, using (37), we have

$$
|E(v - E_h v)|^2 \le |\big(\nabla v|_K\big)(a) - \big(\nabla v|_{K_e}\big)(a)|^2.
$$

Proceeding along the same line that leads to (35), we obtain

$$\left| (\nabla v|_K)(a) - (\nabla v|_{K_e})(a) \right|^2 \leq C \sum_{\substack{K, K' \in \omega(e) \\ \partial K \cap \partial K' = f \neq \emptyset}} |f|^{-1} \| [\![\partial_n v]\!] \|_{L^2(f)}^2.$$

Note that $\int_f [\![\partial_n v]\!] \, \mathrm{d}\sigma(x) = 0$, it follows from the Poincaré inequality and the trace inequality (27) that

$$\begin{aligned} I_2^1 &\leq C \sum_{e \in \mathcal{E}_h(K)} \sum_{\substack{K, K' \in \omega(e) \\ \partial K \cap \partial K' = f \neq \emptyset}} h_K^5 |f|^{-1} \| [\![\partial_n v]\!] \|_{L^2(f)}^2 \\ &\leq C \sum_{e \in \mathcal{E}_h(K)} \sum_{\substack{K, K' \in \omega(e) \\ \partial K \cap \partial K' = f \neq \emptyset}} h_K^5 |f|^{-1} h_f^2 \| \nabla [\![\partial_n v]\!] \|_{L^2(f)}^2 \\ &\leq C \sum_{K' \in \omega(K)} h_K^4 \| \nabla^2 v \|_{L^2(K')}^2. \end{aligned}$$

Using (38) and the inverse inequality, we obtain

$$I_2^2 \leq C \sum_{e \in \mathcal{E}_h(K)} \sum_{K' \in \omega(e)} h_{K'}^7 \| \nabla^2 v \|_{L^\infty(K')}^2 \leq C \sum_{K' \in \omega(K)} h_{K'}^4 \| \nabla^2 v \|_{L^2(K')}^2.$$

Lastly, we have $F(v - E_h v) = 0$ when $\mathrm{order}(F) = 0$. It follows from a standard scaling argument and (40) that

$$I_3 \leq C \sum_{f \in \mathcal{F}_h(K)} h_F^5 \left| \fint_f [\![\partial_n v]\!] \mathrm{d}\sigma(x) \right|^2 \leq C \sum_{f \in \mathcal{F}_h(K)} h_f^5 |f|^{-1} \| [\![\partial_n v]\!] \|_{L^2(f)}^2.$$

Note that $\int_f [\![\partial_n v]\!] \mathrm{d}\sigma(x) = 0$ for any face $f \in \mathcal{F}_h$, using the Poincaré inequality and the trace inequality again, we obtain

$$I_3 \leq C \sum_{K' \in \omega(K)} h_{K'}^4 \| \nabla^2 v \|_{L^2(K')}^2.$$

Summing up the estimates for $I_1, \cdots, I_4$, we obtain, for $j = 0, 1, 2$,

$$\| v - E_h v \|_{L^2(K)}^2 \leq C \sum_{K' \in \omega(K)} h_{K'}^4 \| \nabla^2 v \|_{L^2(K')}^2 \leq C \sum_{K' \in \omega(K)} h_{K'}^{2j} \| \nabla^j v \|_{L^2(K')}^2.$$

For $0 \leq k \leq j$, it follows from the inverse inequality that

$$\| \nabla^k (v - E_h v) \|_{L^2(K)}^2 \leq C h_K^{-2k} \| v - E_h v \|_{L^2(K)}^2 \leq C \sum_{K' \in \omega(K)} h_{K'}^{2(j-k)} \| \nabla^j v \|_{L^2(K')}^2.$$

Summing up all the elements $K \in \mathcal{T}_h$, we obtain (43). $\square$

Based on Lemmas 3 and 4, we derive the error estimate without the regularity assumption on the solution u.

Theorem 4 *Let u and u_h be the solutions of problem (3) and problem (29), respectively. Then there exists C depends on β*

$$
\begin{aligned}
|\!|\!| u-u_h |\!|\!|_{\iota,h} \le \left(1+\frac{2(2\mu+d\lambda)\beta}{C_\flat}\right)\inf_{v\in V_h}|\!|\!| u-v|\!|\!|_{\iota,h} \\
+\frac{(2\mu+d\lambda)\beta}{C_\flat}\inf_{w\in W_h}|\!|\!| u-w|\!|\!|_{\iota,h}+\frac{\beta}{C_\flat}\mathrm{Osc}(f),
\end{aligned}
\tag{44}
$$

where the oscillation of f is defined as

$$
\mathrm{Osc}(f) := \left(\sum_{K\in\mathcal{T}_h} h_K^2 \inf_{\overline{f}\in\mathbb{P}_2(K)} \| f-\overline{f}\|^2_{L^2(K)}\right)^{1/2}.
$$

Remark 1 The estimate (44) may be viewed as the generation of Céa lemma for approximating the strain gradient elastic model. We refer to [36] for the similar estimate for second order elliptic problem and [29] and [34] for similar estimates for the fourth order problems.

It is worthwhile to mention that the smoothness assumption on f may be further weakened if we change the definition of $\mathrm{Osc}(f)$ to

$$
\mathrm{Osc}(f) := \left(\sum_{K\in\mathcal{T}_h} \inf_{\overline{f}\in\mathbb{P}_2(K)} \| f-\overline{f}\|^2_{H^{-1}(K)}\right)^{1/2}.
$$

Such estimate may be useful for dealing with the cracked problem (cf. [4,34]).

Proof For any $v\in V_h$, we denote $w=v-u_h$ and $E_hw=(E_hw_1,\cdots,E_hw_d)$. By the Petrov-Galerkin orthogonality (41) of the enriching operator, we obtain, for any $z\in W_h$,

$$
\begin{aligned}
a_h(w,w) &= a_h(v,w)-a_h(u_h,w) = a_h(v,w-E_hw)+a_h(v,E_hw)-(f,w)\\
&= a_h(v-z,w-E_hw)+a_h(v-u,E_hw)+(f,E_hw-w)\\
&= a_h(v-z,w-E_hw)+a_h(v-u,E_hw)+(f-\overline{f},E_hw-w),
\end{aligned}
\tag{45}
$$

where we have used $(40)_3$ in the last step for any $\overline{f}|_K\in\mathbb{P}_2(K)$. The energy estimate (44) follows from (42) and (43) with $k=0$, $j=1$ and the triangle inequality and the estimate

$$
|a_h(v-z,w-E_hw)| \le (2+\beta)(2\mu+d\lambda)\big(|\!|\!| u-v|\!|\!|_{\iota,h}+|\!|\!| u-z|\!|\!|_{\iota,h}\big)|\!|\!| w|\!|\!|_{\iota,h}.
$$

□

We are ready to derive the rates of convergence for the Specht triangle and the NZT tetrahedron.

Theorem 5 *Let u and u_h be the solutions of problem (3) and problem (29), if the hypothesis 1 is true, then*

$$
|\!|\!| u-u_h|\!|\!|_{\iota,h} \le Ch^{1/2}\| f\|_{L^2}. \tag{46}
$$

If $u\in H^3(\Omega)$, then

$$
|\!|\!| u-u_h|\!|\!|_{\iota,h} \le C(h^2+\iota h)\| u\|_{H^3}. \tag{47}
$$

Proof Using (33), the regularity estimates (17) and (18), we obtain

$$
\inf_{v\in V_h}|\!|\!| u-v|\!|\!|_{\iota,h} \le \iota\|\nabla_h^2(u-I_h^0u)\|_{L^2}+\|\nabla(u-I_h^0u)\|_{L^2} \le Ch^{1/2}\| f\|_{L^2}.
$$

Using the interpolation estimate of SCOTT- ZHANG interpolant [45], we obtain

$$\inf_{v\in W_h} \|\!|u - v|\!\|_{\iota,h} \leq \iota\| \nabla_h^2(u - \Pi_C u) \|_{L^2} + \| \nabla(u - \Pi_C u) \|_{L^2} \leq Ch^{1/2}\| f \|_{L^2}.$$

It is clear that

$$\text{Osc}(f) \leq Ch\| f \|_{L^2}.$$

Substituting all the above inequalities into (44), we obtain (46).

The estimate (47) may be proved in a standard manner as that leads to [32, Theorem 4]. By the theorem of Berger, Scott and Strang [8], we have

$$\|\!|u - u_h|\!\|_{\iota,h} \leq C \left(\inf_{v\in V_h} \|\!|u - v|\!\|_{\iota,h} + \sup_{w\in V_h} \frac{E_h(u, w)}{\|\!|w|\!\|_{\iota,h}} \right),$$

where $E_h(u, w) = a_h(u, w) - (f, w)$.

Using the interpolation estimate (33), we obtain

$$\inf_{v\in V_h} \|\!|u - v|\!\|_{\iota,h} \leq \|\!|u - I_h^0 u|\!\|_{\iota,h} \leq C(h^2 + \iota h)\| u \|_{H^3}.$$

Integration by parts and using the continuity of w, we write E_h as

$$E_h(u, w) = \iota^2 \sum_{f\in\mathcal{F}_h} \int_f M_{nn}(u)[\![\partial_n w]\!]\mathrm{d}\sigma(x).$$

Employing the trace inequalities (26) and (27), we obtain

$$|E_h(u, w)| \leq C\iota^2 h\| u \|_{H^3}\| \nabla_h^2 w \|_{L^2} \leq C\iota h\| u \|_{H^3}\|\!|w|\!\|_{\iota,h}.$$

Combining all the above estimates, we obtain (47). □

5 Numerical Experiments

In this part, we test the accuracy of the Specht triangle and the NZT tetrahedron for a smooth solution and the numerical pollution effect for a solution with strong boundary layers. In all the examples, we let $\Omega = (0, 1)^d$ and set $\lambda = 10$, $\mu = 1$. For $d = 2$, the initial unstructured mesh consists of 220 triangles and 127 vertices, and the maximum mesh size is $h = 1/8$; See Fig. 1_a. For $d = 3$, we construct an initial mesh by splitting origin cube into 512 small cubes, and each small cube is divided into 6 tetrahedrons; See Fig. 1_b.

Throughout the simulation, we employ the analytical basis functions for the Specht triangle [49,57] and the NZT tetrahedron [53]. The computation for the NZT tetrahedron is performed in a parallel hierarchical grid platform (PHG) [55].[2] For all the tests, we measure the rates of convergence in the relative energy norm $\|\!|u - u_h|\!\|_{\iota,h}/\|\!|u|\!\|_{\iota,h}$ for different ι and h.

[2] http://lsec.cc.ac.cn/phg.

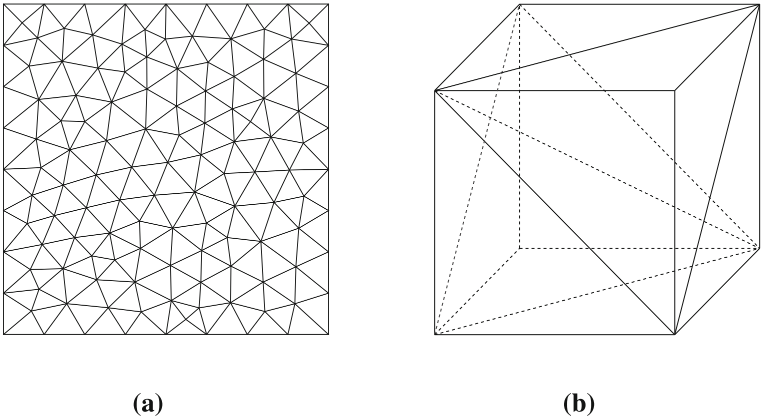

Fig. 1 Plots of meshes: **a** $d = 2$; **b** $d = 3$

Table 1 Rate of convergence of the Specht triangle

$\iota \backslash h$	1/8	1/16	1/32	1/64	1/128	1/256
1e+0	1.99e−01	9.87e−02	4.80e−02	2.36e−02	1.17e−02	5.85e−03
rate		1.01	1.04	1.02	1.01	1.00
1e−2	3.16e−02	1.21e−02	5.30e−03	2.53e−03	1.25e−03	6.21e−04
rate		1.39	1.19	1.07	1.02	1.01
1e−4	2.20e−02	5.57e−03	1.39e−03	3.48e−04	8.75e−05	2.26e−05
rate		1.98	2.00	2.00	1.99	1.95
1e−6	2.20e−02	5.57e−03	1.39e−03	3.47e−04	8.65e−05	2.16e−05
rate		1.98	2.00	2.00	2.00	2.00

5.1 Example for Smooth Solution

This example is to test the accuracy of the elements for the smooth solution, which is given by $u = (u_1, u_2, u_3)$ with

$$u_1 = \prod_{i=1}^{d} \big(\exp(\cos 2\pi x_i) - \exp(1)\big),\ u_2 = \prod_{i=1}^{d} (\cos 2\pi x_i - 1),$$
$$u_3 = \prod_{i=1}^{d} x_i^2 (x_i - 1)^2.$$

The source term f is computed by $(1)_1$. For $d = 2$, we drop the third component u_3. We report the rates of convergence for the Specht triangle and the NZT tetrahedorn in Tables 1 and 2, respectively. We observe that the rates of convergence appear to be linear when ι is large, while it turns out to be quadratic when ι is close to zero, which is consistent with the theoretical predication (47).

Table 2 Rates of convergence of the NZT tetrahedron

$\iota\backslash h$	1/8	1/16	1/32	1/64	1/128
1e+0	1.02e−01	7.54e−01	5.05e−01	2.84e−01	1.47e−01
rate		0.43	0.58	0.83	0.95
1e−2	5.12e−01	2.28e−01	8.99e−02	3.57e−02	1.55e−02
rate		1.17	1.34	1.33	1.20
1e−4	3.03e−01	7.14e−02	1.79e−02	4.66e−03	1.27e−03
rate		2.08	1.99	1.94	1.89
1e−6	3.01e−01	6.98e−02	1.71e−02	4.26e−03	1.07e−03
rate		2.11	2.03	2.00	1.99

Table 3 Rates of convergence for $\iota = 10^{-6}$

h	1/8	1/16	1/32	1/64	1/128	1/256
NZT	2.74e−01	1.73e−02	1.18e−01	8.24e−02	5.80e−02	4.10e−2
rate		0.66	0.56	0.52	0.51	0.50
Specht	1.57e−01	1.10e−01	7.70e−02	5.42e−02	3.82e−02	2.70e−02
rate		0.51	0.51	0.51	0.50	0.50

5.2 Example with Boundary Layer

In this example, we test the performance of both elements for a solution with a boundary layer, such boundary layer is one of the main difficulties for the strain gradient elastic model, and we refer to [20] for a one dimensional example with analytical expression. Based on this example, we construct a displacement field $u = (u_1, u_2, u_3)$ as

$$u_1 = \prod_{i=1}^{d}(\exp(\sin \pi x_i) - 1 - \varphi(x_i)), \quad u_2 = \prod_{i=1}^{d}(\sin \pi x_i - \varphi(x_i)),$$
$$u_3 = \prod_{i=1}^{d}\bigl(\pi x_i(1 - x_i) - \varphi(x_i)\bigr)$$

with

$$\varphi(x) = \pi\iota \frac{\cosh[1/2\iota] - \cosh[(2x-1)/2\iota]}{\sinh[1/2\iota]}.$$

A direct calculation gives

$$\lim_{\iota\to 0} u = u_0 = \left(\prod_{i=1}^{d}\exp(\sin \pi x_i) - 1, \prod_{i=1}^{d}\sin \pi x_i, \prod_{i=1}^{d}\pi x_i(1 - x_i)\right),$$

with $u_0|_{\partial\Omega} = 0$ and $\partial_n u_0|_{\partial\Omega} \neq 0$. It is clear that $\partial_n u$ has boundary layers. The source term f is also computed from $(1)_1$. We report the rates of convergence for both elements in Table 3. The half order rates of convergence are observed for both elements, which is consistent with the theoretical prediction (46).

6 Conclusion

We prove a new H^2-Korn's inequality and a new broken H^2-Korn's inequality. The former is crucial for the well-posedness of a strain gradient elasticity model, while the latter motivates us to construct robust nonconforming elements for this model, and the elements are simpler than the known elements in the literature; See, e.g., [32]. With the aid of the new regularized interpolant and the enriching operator, we proved that the tensor product of the Specht triangle and the NZT tetrahedron converges uniformly with respect to the small materials parameter under the minimal smoothness assumption on the solution. Moreover, the technicalities may also be used to derive shaper error bounds for the elements in [28,42,50]. Guided by the broken H^2-Korn's inequality, we can design robust elements for the nonlinear strain gradient elastic models, thin beam and thin plate with strain gradient effect in [20,23] by combining the tricks in [10,40] and the machinery developed in the present work, which will be left for further pursuit.

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