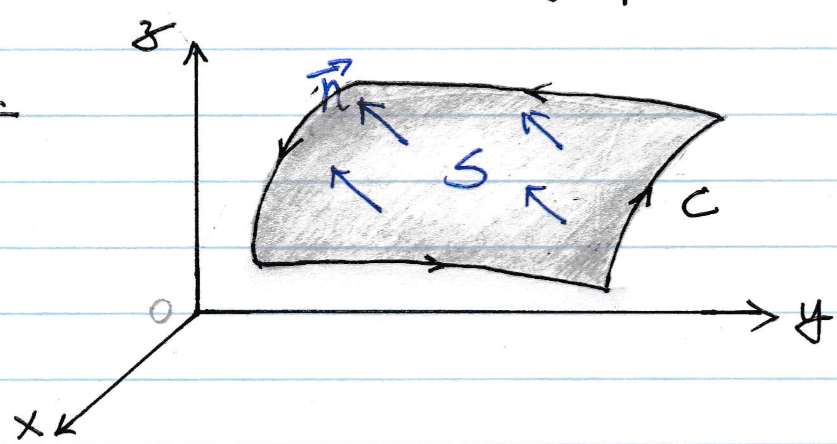
16.8 Stoke's' Theorem

1. Introduction : Stokes' Theorem can be thought of as a version of Green's Theorem which applies to surface in 3D space. In fact, it would be more accurate to say that Green's Theorem is a version of Stokes' Theorem when the surface is in the xy-plane.

2.



Induced Orientation : If C is the edge of S and then C induces an orientation on S using the right hand : if your fingers follow C's orientation then your thumb points in an orientation for S.

3. Stokes' Theorem : Suppose C is the edge of S and the orientation of S is induced by the orientation of C,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$$

Remark (1) Use stoke's Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$ through the following procedure :

We can compute the final answer from the iterated integral

$$\int_C \vec{F}\cdot d\vec{r} \xrightarrow[\text{Thm}]{\text{Stokes'}} \iint_S \xrightarrow[\text{of } S]{\text{Parametrization}} \pm\iint_D \xrightarrow[\text{inequalities}]{\text{Use}} \underline{\int_*^* \int_*^*}$$

Notice that sometimes this may not be simpler compared to using parametrization for $\int_C \vec{F}\cdot d\vec{r}$ directly.

(2) (Not Required) We use Stokes' Theorem to derive Green's Theorem. Since Green's Theorem is in 2D, we consider

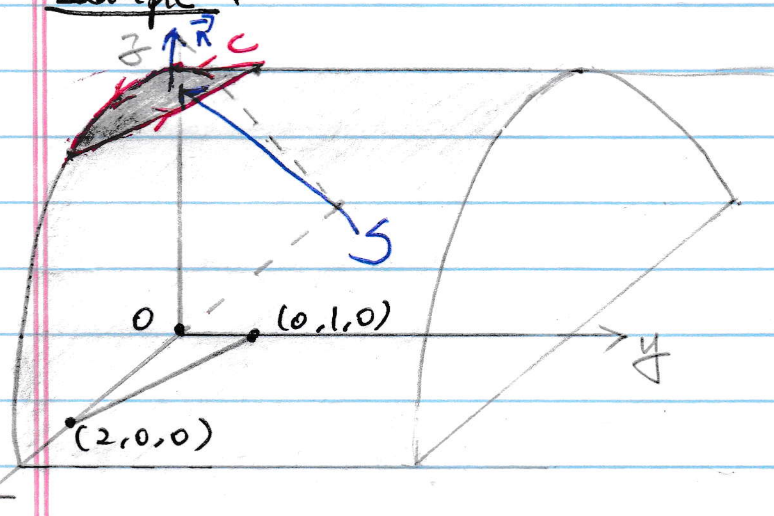
$$\vec{F}(x,y,z) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j} + 0\,\vec{k}.$$

Then $\nabla\times\vec{F} = \left\langle 0,\ 0,\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle$.

Since S is in xy-plane, we have $\vec{n} = \langle 0,0,1\rangle$, and thus

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

__Example__:



Suppose C is the curve on the surface $z = 9 - x^2$ lying above the triangle with vertices $(0,0,0)$, $(2,0,0)$, $(0,1,0)$ and having counterclockwise orientation when viewed from above. Evaluate

$$\int_C 2y\,dx + xz\,dy + z\,dz.$$

__Solution__: $\vec{F}(x,y,z) = \langle 2y,\ xz,\ z\rangle$.

$$\nabla\times\vec{F} = \langle -x,\ 0,\ z-2\rangle$$

By Stokes' Theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \langle -x, 0, z-2 \rangle \cdot \vec{n} \, dS$$

Parametrization of S :

$$\vec{r}(x, y) = x\vec{i} + y\vec{j} + (9-x^2)\vec{k}$$

$$0 \le x \le 2, \quad 0 \le y \le 1 - \frac{x}{2}$$

$$\vec{r}_x = \langle 1, 0, -2x \rangle, \quad \vec{r}_y = \langle 0, 1, 0 \rangle$$

$$\vec{r}_x \times \vec{r}_y = \langle 2x, 0, \underbrace{1}_{>0} \rangle$$

From the plot, we see that the orientation of S is upward by the right hand rule. So

$$\iint_S \langle -x, 0, z-2 \rangle \cdot \vec{n} \, dS$$

$$= + \iint_D \langle -x, 0, (9-x^2)-2 \rangle \cdot \langle 2x, 0, 1 \rangle \, dA$$

$$= \iint_D -2x^2 + 9 - x^2 - 2 \, dA$$

$$= \iint_D 7 - 3x^2 \, dA$$

$$= \int_0^2 \int_0^{1-\frac{x}{2}} 7 - 3x^2 \, dy \, dx$$

$$= \int_0^2 \left[(7-3x^2)y \right]_{y=0}^{y=1-\frac{x}{2}} dx$$

$$= \int_0^2 (7-3x^2)\left(1-\frac{x}{2}\right) dx = \cdots = 5$$

Text-Ex 1: Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = -y^2 \vec{i} + x\vec{j} + z^2 \vec{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$ and having counterclockwise orientation when viewed from above.

Solution: By right hand rule, orientation of S is upward.

Since $\nabla \times \vec{F} = \langle 0, 0, 1 + 2y \rangle$, we have

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \langle 0, 0, 1+2y \rangle \cdot \vec{n} \, dS$$

with S oriented upwards. Now consider parametrization of S:

$$\vec{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), 2 - r\sin(\theta) \rangle$$

$$0 \le \theta \le 2\pi, \quad 0 \le r \le 1.$$

$$\vec{r}_r = \langle \cos(\theta), \sin(\theta), -\sin(\theta) \rangle$$

$$\vec{r}_\theta = \langle -r\sin(\theta), r\cos(\theta), -r\cos(\theta) \rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \langle 0, r, \underbrace{r}_{\ge 0} \rangle$$

matches the orientation of S

So $\iint_S \langle 0, 0, 1+2y \rangle \cdot \vec{n} \, dS$

$$= \iint_D (1 + 2y)\vec{k} \cdot \langle 0, r, r \rangle \, dA$$

$$= \iint_D (1 + 2r\sin(\theta)) \, r \, dA$$

$$= \int_0^{2\pi} \int_0^1 r + 2r^2 \sin(\theta) \, dr \, d\theta$$

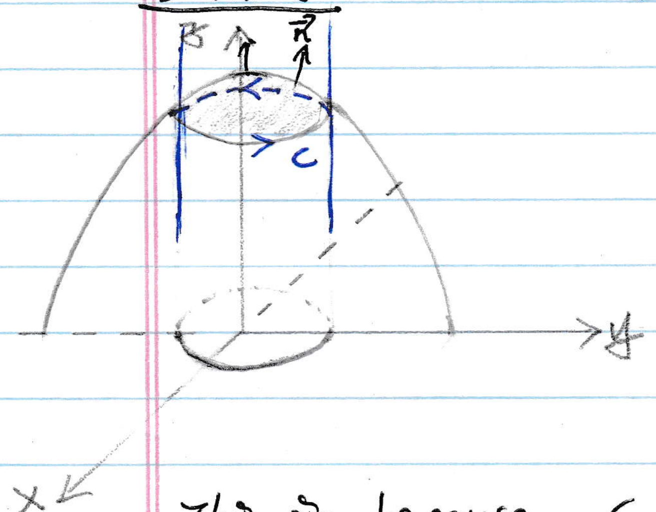
$$= \int_0^{2\pi} \left[\frac{1}{2} r^2 + \frac{2}{3} r^3 \sin(\theta) \right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} + \frac{2}{3} \sin(\theta) \, d\theta$$

$$= \left[\frac{1}{2}\theta - \frac{2}{3}\cos(\theta) \right]_0^{2\pi} = \pi .$$

Text-Ex 2 : Use Stokes' Theorem to compute the integral $\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$ where $\vec{F}(x,y,z) = \langle xz, yz, xy \rangle$ and S is the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy-plane. (S is oriented upwards)

Solution : We use Stokes' Theorem to transfer the surface integral into a line integral.



$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \int_C \vec{F} \cdot d\vec{r}$$

Parametrization of C

$$\vec{r}(t) = \cos(t)\,\vec{i} + \sin(t)\,\vec{j} + \sqrt{3}\,\vec{k}$$

$$0 \le t \le 2\pi .$$

This is because C is the intersection of

$$x^2+y^2+z^2 = 4 , \quad x^2+y^2=1 .$$

From the two equations above, we have

$$x^2+y^2 = 1 , \quad z^2 = 3 .$$

Since S is above the xy-plane, $z = \sqrt{3}$.
We also see that C should have an orientation counterclockwise when viewed from above. Combining all of these facts, we can come up with the parametrization of C.

So $$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, ds$$

$$= \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

From $\vec{F}(x, y, z) = \langle xz, yz, xy \rangle$ and

$$\vec{r}(t) = \langle \cos(t), \sin(t), \sqrt{3} \rangle,$$

we have $\vec{F}(\vec{r}(t)) = \langle \sqrt{3}\cos(t), \sqrt{3}\sin(t), \cos(t)\sin(t) \rangle$

Also $\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 0 \rangle$.

Hence $$\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} \langle \sqrt{3}\cos(t), \sqrt{3}\sin(t), \cos(t)\sin(t) \rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle\, dt$$

$$= \int_0^{2\pi} 0\, dt = 0.$$