16.7 Surface Integrals

1. Surface integral of function f over the surface S.

$$\iint_S f(x, y, z)\, dS$$ ← Notation

Meaning:

(1) If $f(x, y, z) = 1$, we get the surface area of S. This is the case discussed at the end of section 16.7.

(2) If $f(x, y, z)$ is the density (mass or electrical charge or whatever) at point (x, y, z), then we get the total (mass or charge or whatever).

2. Evaluation of $\iint_S f(x, y, z)\, dS$

Suppose S has a parametrization

$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$$

for $(u, v) \in D$ ← 2D region in the uv-plane

then

$$\iint_S f(x, y, z)\, dS = \iint_D \underbrace{f(x(u,v), y(u,v), z(u,v))}_{\uparrow} \left|\vec{r}_u \times \vec{r}_v\right| dA$$

Write function f in terms of u and v: $f(\vec{r}(u,v))$

Text-Ex 1: Compute $\iint_S x^2\, dS$ where S is the unit sphere

$$x^2 + y^2 + z^2 = 1.$$

Solution: Use spherical coordinates (ρ, ϕ, θ), we see the equation of S is given by

$$\rho = 1 .$$

Therefore we have a parametrization of S

$$\vec{r}(\phi, \theta) = \sin(\phi)\cos(\theta)\,\vec{i} + \sin(\phi)\sin(\theta)\,\vec{j} + \cos(\phi)\,\vec{k}$$

$$\underline{0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi} .$$

↳ These inequalities define D.

By calculation,

$$\vec{r}_\phi = \langle \cos(\phi)\cos(\theta), \cos(\phi)\sin(\theta), -\sin(\phi) \rangle,$$

$$\vec{r}_\theta = \langle -\sin(\phi)\sin(\theta), \sin(\phi)\cos(\theta), 0 \rangle,$$

$$\vec{r}_\phi \times \vec{r}_\theta = \langle \sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \sin(\phi)\cos(\phi) \rangle,$$

$$|\vec{r}_\phi \times \vec{r}_\theta| = \sqrt{(\sin^2(\phi)\cos(\theta))^2 + (\sin^2(\phi)\sin(\theta))^2 + (\sin(\phi)\cos(\phi))^2}$$

$$= \sqrt{\sin^4(\phi)\cos^2(\phi) + \sin^4(\phi)\sin^2(\theta) + \sin^2(\phi)\cos^2(\phi)}$$

$$= \sqrt{\sin^2(\phi)} = \sin(\phi) .$$

So by the formula

$$\iint_S x^2\, dS = \iint_D (\sin(\phi)\cos(\theta))^2\, |\vec{r}_\phi \times r_\theta|\, dA$$

$$= \iint_D \sin^2(\phi)\cos^2(\theta)\sin(\phi)\, dA$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^3(\phi)\cos^2(\theta)\, d\phi\, d\theta$$

$$= \cdots = \frac{4\pi}{3}.$$

<u>Text-Ex 2</u> Evaluate $\iint_S y\, dS$ where S is the part of $z = x + y^2$ with $0 \le x \le 1$, $0 \le y \le 2$.

<u>Solution</u>: We use x, y as parameters in the parametrization

$$\vec{r}(x, y) = x\,\vec{i} + y\,\vec{j} + (x + y^2)\,\vec{k}$$

$$0 \le x \le 1, \quad 0 \le y \le 2.$$

We compute

$$\vec{r}_x = \langle 1, 0, 1 \rangle$$

$$\vec{r}_y = \langle 0, 1, 2y \rangle$$

$$\vec{r}_x \times \vec{r}_y = \langle -1, -2y, 1 \rangle$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{(-1)^2 + (-2y)^2 + 1^2} = \sqrt{2 + 4y^2}.$$

By the formula,

$$\iint_S y\, dS = \iint_D y\sqrt{2 + 4y^2}\, dA$$

$$= \int_0^1 \int_0^2 y \sqrt{2+4y^2} \, dy \, dx$$

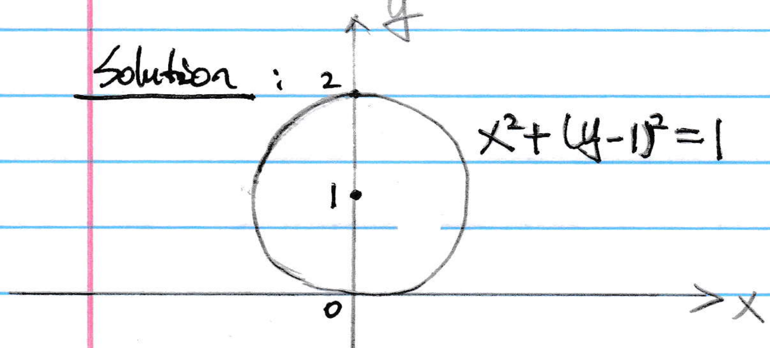
$$= \int_0^1 \left[\frac{1}{12} \left(2+4y^2\right)^{\frac{3}{2}} \right]_{y=0}^{y=2} dx$$

$$= \int_0^1 \left[\frac{9\sqrt{2}}{2} - \frac{\sqrt{2}}{6} \right] dx$$

$$= \frac{13\sqrt{2}}{3}.$$

Example Compute $\iint_S f(x,y,z)\,dS$ for $f(x,y,z) = x^2 z$ and S the part of the part of $z = 7 - x$ inside $x^2 + (y-1)^2 = 1$.

Solution:



This region D can be described easily in polar

$$\underbrace{0 \leq \theta \leq \pi, \quad 0 \leq r \leq 2\sin(\theta)}_{\text{region } D}.$$

So we choose r, θ as our parameters and write the parametrization of S:

$$\vec{r}(r,\theta) = r\cos(\theta)\,\vec{i} + r\sin(\theta)\,\vec{j} + \left(7 - r\cos(\theta)\right)\vec{k}$$

$$0 \leq \theta \leq \pi, \quad 0 \leq r \leq 2\sin(\theta).$$

We compute

$$\vec{r}_r(r,\theta) = \langle \cos(\theta), \sin(\theta), -\cos(\theta) \rangle$$

$$\vec{r}_\theta(r,\theta) = \langle -r\sin(\theta), r\cos(\theta), r\sin(\theta) \rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \langle r, 0, r \rangle$$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{r^2 + 0^2 + r^2} = \sqrt{2r^2} = \sqrt{2}\, r.$$

So
$$\iint_S x^2 z \, dS = \iint_D (r\cos(\theta))^2 (7 - r\cos(\theta)) \sqrt{2}\, r \, dA$$

$$= \int_0^\pi \int_0^{2\sin(\theta)} 7\sqrt{2}\, r^3 \cos^2(\theta) - \sqrt{2}\, r^4 \cos^3(\theta) \, dr \, d\theta$$

$$= \int_0^\pi \left[\frac{7\sqrt{2}}{4} r^4 \cos^2(\theta) - \frac{\sqrt{2}}{5} r^5 \cos^3(\theta) \right]_{r=0}^{r=2\sin(\theta)} d\theta$$

$$= \int_0^\pi 28\sqrt{2} \sin^4(\theta)\cos^2(\theta) - \frac{32\sqrt{2}}{5} \sin^5(\theta)\cos^3(\theta) \, d\theta$$

$$= 28\sqrt{2} \int_0^\pi \sin^4(\theta)\cos^2(\theta)\, d\theta - \frac{32\sqrt{2}}{5} \underbrace{\int_0^\pi \sin^5(\theta)\cos^3(\theta)\, d\theta}_{=0}$$

$$= 28\sqrt{2} \int_0^\pi \left(\frac{1-\cos(2\theta)}{2} \right) \left(\frac{\sin(2\theta)}{2} \right)^2 d\theta$$

$$= \frac{7\sqrt{2}}{2} \int_0^\pi \sin^2(2\theta) - \sin^2(2\theta)\cos(2\theta) \, d\theta$$

$$= \frac{7\sqrt{2}}{2} \int_0^\pi \sin^2(2\theta)\, d\theta = \frac{7\sqrt{2}}{2} \left[\frac{\theta}{2} - \frac{\sin(4\theta)}{8} \right]_{\theta=0}^{\theta=\pi} = \frac{7\pi\sqrt{2}}{4}$$

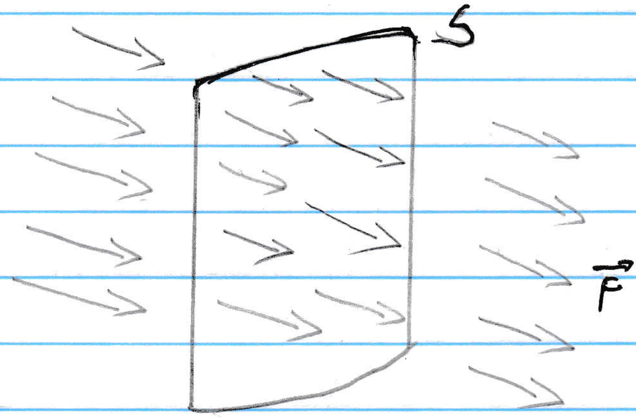
3. Surface Integrals of vector field $\vec{F}$ over S

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, dS$$

$\leftarrow$ Two Acceptable Notations

$\vec{n}$: unit normal vector of S

Meaning:



Suppose $\vec{F}$ is the fluid flow telling you the velocity at each point, then $\iint_S \vec{F} \cdot \vec{n} \, dS$ gives you the rate at which $\vec{F}$ flows through S.

Orientation matters! We must know the direction of $\vec{n}$ in order to compute the integral.

4. Evaluation of $\iint_S \vec{F} \cdot \vec{n} \, dS$

Suppose S has a parametrization

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

$$(u,v) \in D$$

then

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \pm \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dA$$

Here "$\pm$" is determined as follows :

- Use "$+$" if the vector $\vec{r}_u \times \vec{r}_v$ matches the direction of the orientation of Σ
- Use "$-$" if they are opposite .

<u>Example</u> : Evaluate $\iint_S \vec{F} \cdot \vec{n}\, dS$ for $\vec{F}(x,y,z) = y\vec{i} + x\vec{j} + z\vec{k}$ and S is part of $z = 1 - x^2 - y^2$ for $x^2 + y^2 \leq 1$ oriented upwards .

<u>Solution</u> : Since D is determined by $x^2 + y^2 \leq 1$, we use polar r, θ as parameters . (Another method is to use x, y at first and then change to r, θ when doing integrals . See Textbook example 5 for details .)

Then the parametrization of D is

$$\vec{r}(r, \theta) = r\cos(\theta)\,\vec{i} + r\sin(\theta)\,\vec{j} + \underbrace{(1 - r^2)}_{\text{obtained from } 1 - x^2 - y^2}\vec{k}$$

$0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$.

We compute

$$\vec{r}_r = \langle \cos(\theta) , \sin(\theta) , -2r \rangle$$

$$\vec{r}_\theta = \langle -r\sin(\theta) , r\cos(\theta) , 0 \rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \langle 2r^2\cos(\theta) , 2r^2\sin(\theta) , \underbrace{r}_{\geq 0} \rangle$$

"Oriented upwards" means the orientation of S has positive z-component. So we see that $\vec{r}_r \times \vec{r}_\theta$ is in the same direction of the orientation of S.

Use "+" in the formula:

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \vec{F}(\vec{r}(r,\theta)) \cdot \langle 2r^2\cos(\theta), 2r^2\sin(\theta), r \rangle\, dA$$

$$= \iint_D \underline{\langle r\sin(\theta), r\cos(\theta), 1-r^2 \rangle} \cdot \langle 2r^2\cos(\theta), 2r^2\sin(\theta), r \rangle\, dA$$

← This is $\langle y, x, z \rangle$. See also our parametrization.

$$= \iint_D 2r^3\sin(\theta)\cos(\theta) + 2r^3\sin(\theta)\cos(\theta) + r - r^3\, dA$$

$$= \int_0^{2\pi}\int_0^1 4r^3\sin(\theta)\cos(\theta) + r - r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[r^4\sin(\theta)\cos(\theta) + \frac{1}{2}r - \frac{1}{4}r^4 \right]_{r=0}^{r=1} d\theta$$

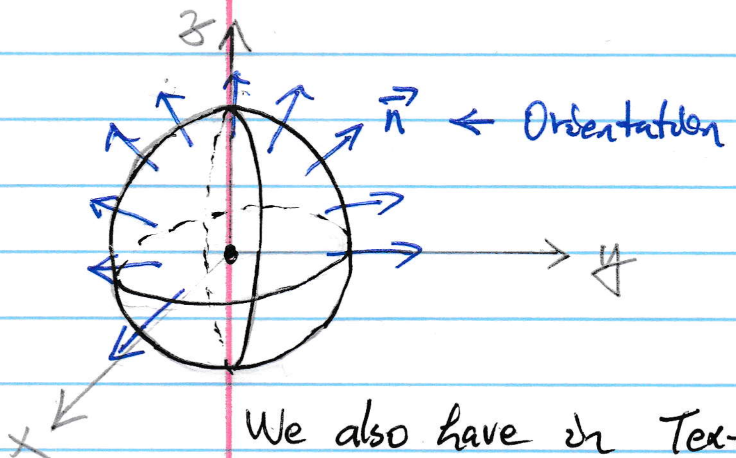
$$= \int_0^{2\pi} \sin(\theta)\cos(\theta) + \frac{1}{2} - \frac{1}{4}\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\sin(2\theta) + \frac{1}{4}\, d\theta$$

$$= \left[-\frac{1}{4}\cos(2\theta) + \frac{1}{4}\theta \right]_0^{2\pi} = \frac{\pi}{2}$$

Text-EX 4 Compute $\iint_S \vec{F}(x,y,z)\cdot\vec{n}\,dS$ where $\vec{F}(x,y,z) = \langle z, y, x\rangle$ and S is sphere $x^2+y^2+z^2=1$ oriented outwards.

Solution: As we did in Text-Ex 1, the parametrization is



$$\vec{r}(\phi,\theta) = \sin(\phi)\cos(\theta)\,\vec{i} + \sin(\phi)\sin(\theta)\,\vec{j} + \cos(\phi)\,\vec{k}$$

$$0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi.$$

We also have in Text-Ex 1 that

$$\vec{r}_\phi(\phi,\theta) \times \vec{r}_\theta(\phi,\theta) = \langle \sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \sin(\phi)\cos(\phi)\rangle.$$

To see whether this has the same direction as the orientation of S, we consider $\phi = \frac{\pi}{2}$, $\theta = 0$. This corresponds to the point $\vec{r}(\frac{\pi}{2}, 0) = \langle 1, 0, 0\rangle$, where the unit normal outward vector is $\vec{n}(0,0,1) = \langle 1, 0, 0\rangle$.

$$\vec{r}_\phi(\tfrac{\pi}{2},0)\times\vec{r}_\theta(\tfrac{\pi}{2},0) = \langle 1, 0, 0\rangle.$$

So they have the same direction. Use "+":

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_D \langle \cos(\phi), \sin(\phi)\sin(\theta), \sin(\phi)\cos(\theta)\rangle \cdot \langle \sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \sin(\phi)\cos(\phi)\rangle\,dA$$

$$= \iint_D 2\sin^2(\phi)\cos(\phi)\cos(\theta) + \sin^3(\phi)\sin^2(\theta)\, dA$$

$$= \int_0^{2\pi}\int_0^{\pi} 2\sin^2(\phi)\cos(\phi)\cos(\theta) + \sin^3(\phi)\sin^2(\theta)\, d\phi\, d\theta$$

$$= \cdots = \frac{4\pi}{3}.$$

A few more comments. The vector $\langle \cos(\phi), \sin(\phi)\sin(\theta), \sin(\phi)\cos(\theta) \rangle$ is obtained by using substitution

$$\begin{cases} x = \sin(\phi)\cos(\theta) \\ y = \sin(\phi)\sin(\theta) \\ z = \cos(\phi) \end{cases} \leftarrow \text{From the parametrization of } S$$

to rewrite $\vec{F} = \langle z, y, x \rangle$.

Also, another way to determine whether $\vec{r_\phi} \times \vec{r_\theta}$ is in the same direction of the orientation or not is to write

$$\vec{r_\phi} \times \vec{r_\theta} = \sin(\phi) \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi) \rangle$$

$$= \underbrace{\sin(\phi)}_{>0} \langle x, y, z \rangle.$$

So this points outwards and is in the same direction of the orientation.