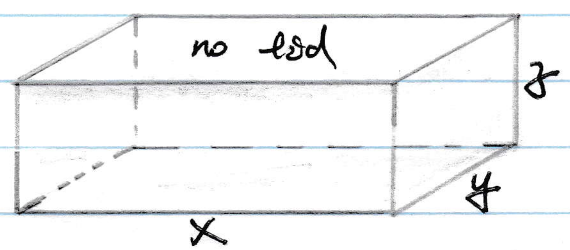
14.8 Lagrange Multipliers

the objective function

1. Goal: To find the max and/or min of $f(x, y)$ subject to $g(x, y) = k$ (the constraint equation)

Example: A rectangular box without a lid is to be made from $12\,m^3$ of cardboard.

If we want to maximize the volume of such a box, then



$$\max \quad V = xyz$$

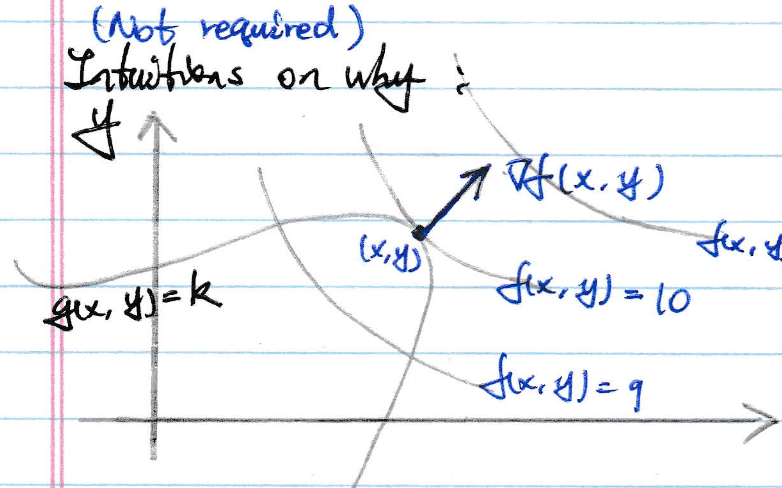
subject to $\quad g(x, y, z) = \underbrace{2xz + 2yz + xy}_{\text{surface area without lid}} = 12$

2. Fact: If $f(x, y)$ attains such a max and/or min, then the max/min occur where $\nabla f = \lambda \nabla g$

λ is called a Lagrange multiplier

(Not required)

Intuitions on why:



When max/min is attained,

$\nabla f \perp g(x, y) = k$ level curve

$\Rightarrow \quad \nabla f \parallel \nabla g \quad \Rightarrow \quad \nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$

3. Method / Procedure

(1) Identify objective $f(x, y)$

Identify constraint $g(x, y) = k$

(2) Solve for all (x, y) satisfying the system
(or (x, y, z) if we are in 3D)

Lagrange System

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ (f_z = \lambda g_z) \\ g(x, y) = k \end{cases}$$

$\longleftarrow$ If we are in 3D

Note: You may also find λ while doing this. That's fine, but in the end we only need (x, y) or (x, y, z).

(3) Plug each resulting (x, y) into f and identify the largest and/or smallest.

Example: Find max and min of $f(x, y) = x + 3y$ under the constraint $x^2 + y^2 = 9$.

Ans: (1) $f(x, y) = x + 3y$
$g(x, y) = x^2 + y^2$
} Identify f and g

(2) $\nabla f = \langle 1, 3 \rangle$, $\nabla g = \langle 2x, 2y \rangle$

Solve

$$\begin{cases} 1 = \lambda (2x) & ① \\ 3 = \lambda (2y) & ② \\ x^2 + y^2 = 9 & ③ \end{cases}$$

To solve these equations, notice from ① and ②:

$$x = \frac{1}{2\lambda}, \quad y = \frac{3}{2\lambda}.$$

Plug the above equations into ③:

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 9$$

$$\Rightarrow \quad \frac{1}{4\lambda^2} + \frac{9}{4\lambda^2} = 9$$

$$\Rightarrow \quad \frac{10}{4\lambda^2} = 9 \Rightarrow \lambda^2 = \frac{5}{18} \Rightarrow \lambda = \pm\sqrt{\frac{5}{18}}$$

If $\lambda = \sqrt{\frac{5}{18}}$, then

$$x = \frac{1}{2\lambda} = \frac{3}{\sqrt{10}}, \quad y = \frac{3}{2\lambda} = \frac{9}{\sqrt{10}}.$$

If $\lambda = -\sqrt{\frac{5}{18}}$, then

$$x = \frac{1}{2\lambda} = \frac{-3}{\sqrt{10}}, \quad y = \frac{3}{2\lambda} = \frac{-9}{\sqrt{10}}$$

So we have two points $\left(\frac{3}{\sqrt{10}}, \frac{9}{\sqrt{10}}\right)$, $\left(\frac{-3}{\sqrt{10}}, \frac{-9}{\sqrt{10}}\right)$.

(3) Plug each (x, y) into f.

$$f\left(\frac{3}{\sqrt{10}}, \frac{9}{\sqrt{10}}\right) = \frac{3}{\sqrt{10}} + 3\left(\frac{9}{\sqrt{10}}\right) = \frac{30}{\sqrt{10}} = 3\sqrt{10}$$

$$f\left(\frac{-3}{\sqrt{10}}, \frac{-9}{\sqrt{10}}\right) = \frac{-3}{\sqrt{10}} + 3\left(\frac{-9}{\sqrt{10}}\right) = -3\sqrt{10}$$

Therefore the max value is $\boxed{f\left(\frac{3}{\sqrt{10}}, \frac{9}{\sqrt{10}}\right) = 3\sqrt{10}}$

the min value is $\boxed{f\left(\frac{-3}{\sqrt{10}}, \frac{-9}{\sqrt{10}}\right) = -3\sqrt{10}}$

Remark: Another way of solving $\begin{cases} 1 = \lambda(2x) & ① \\ 3 = \lambda(2y) & ② \\ x^2 + y^2 = 9 & ③ \end{cases}$.

From ① : $2\lambda x = 1 \Rightarrow \lambda \neq 0, \; x \neq 0$

From ② : $2\lambda y = 3 \Rightarrow \lambda \neq 0, \; y \neq 0$.

Use $\frac{①}{②}$: $\frac{x}{y} = \frac{1}{3} \Rightarrow y = 3x$.

Plug $y = 3x$ into ③ :

$$x^2 + (3x)^2 = 9 \Rightarrow 10x^2 = 9 \Rightarrow x^2 = \frac{9}{10}$$

$$\Rightarrow x = \frac{3}{\sqrt{10}} \text{ or } -\frac{3}{\sqrt{10}}$$

Since $y = 3x$, we have

$$(x, y) = \left(\frac{3}{\sqrt{10}}, \frac{9}{\sqrt{10}}\right) \text{ or } \left(-\frac{3}{\sqrt{10}}, \frac{-9}{\sqrt{10}}\right)$$

Example: Find minimum value of $f(x, y) = xy$ with $y = 2x + 4$.

Ans: (1) $f(x, y) = xy$, $g(x, y) = y - 2x - 4$

We write $y = 2x + 4$ as $y - 2x - 4 = 0$.

(2) $\nabla f = \langle y, x \rangle$, $\nabla g = \langle -2, 1 \rangle$

Solve $\begin{cases} y = -2\lambda & ① \\ x = \lambda & ② \\ y - 2x - 4 = 0. & ③ \end{cases}$

From ① and ③ $y = -2\lambda = -2x$.

Plug this into ③ :

$$(-2x) - 2x - 4 = 0 \Rightarrow -4x - 4 = 0$$

$$\Rightarrow 4x = -4 \Rightarrow x = -1 .$$

So $y = -2x = 2$. We have $(x, y) = (-1, 2)$.

(3) Only one point, so the minimum value is

$$\boxed{f(-1, 2) = (-1)(2) = -2}$$

Remark : 1. Only one point, where is the max ?
Ans : No maximum value . Or the max value $= +\infty$
because as $x \to +\infty$ we have $y \to +\infty$ and thus $xy \to +\infty$.

2. You can also plug $y = 2x + 4$ into $f(x, y)$ and have

$$f(x, y) = x(2x+4) = 2x^2 + 4x .$$

In this way you can find the minimum value easily.
We just use this as an example for Lagrange multipliers.

Example : Find $\underbrace{\text{extreme values}}_{}$ of $f(x, y) = x + xy$ under the constraint $x^2 + y^2 = 1$.

(extreme values = max and min)

Ans : $f(x, y) = x + xy$, $g(x, y) = x^2 + y^2$

$\nabla f = \langle 1+y, x \rangle$, $\nabla g = \langle 2x, 2y \rangle$

Solve $\begin{cases} 1+y = \lambda(2x) & ① \\ x = \lambda(2y) & ② \\ x^2+y^2 = 1 & ③ \end{cases}$

Plug ② into ① : $1+y = 2\lambda(2\lambda y) = 4\lambda^2 y$

$\Rightarrow 1 = y(4\lambda^2-1) \Rightarrow y = \dfrac{1}{4\lambda^2-1}$

Hence $x \underset{\uparrow}{=} 2\lambda y = \dfrac{2\lambda}{4\lambda^2-1}$

from ②

Now plug $x = \dfrac{2\lambda}{4\lambda^2-1}$ and $y = \dfrac{1}{4\lambda^2-1}$ into ③

$$\left(\frac{2\lambda}{4\lambda^2-1}\right)^2 + \left(\frac{1}{4\lambda^2-1}\right)^2 = 1$$

$$\frac{4\lambda^2}{(4\lambda^2-1)^2} + \frac{1}{(4\lambda^2-1)^2} = 1$$

$$4\lambda^2+1 = (4\lambda^2-1)^2$$

$$4\lambda^2+1 = 16\lambda^4 - 8\lambda^2 + 1$$

$$0 = 16\lambda^4 - 12\lambda^2$$

$$0 = 4\lambda^2(4\lambda^2-3)$$

$\Rightarrow \lambda^2 = 0$ or $\lambda^2 = \dfrac{3}{4}$

$\Rightarrow \lambda = 0$ or $\dfrac{\sqrt{3}}{2}$ or $\dfrac{-\sqrt{3}}{2}$.

When $\lambda = 0$, we have

$$x = \frac{2\lambda}{4\lambda^2-1} = 0, \quad y = \frac{1}{4\lambda^2-1} = -1.$$

When $\lambda = \frac{\sqrt{3}}{2}$, we have

$$x = \frac{2\lambda}{4\lambda^2-1} = \frac{\sqrt{3}}{2}, \quad y = \frac{1}{4\lambda^2-1} = \frac{1}{2}.$$

When $\lambda = \frac{-\sqrt{3}}{2}$, we have

$$x = \frac{2\lambda}{4\lambda^2-1} = \frac{-\sqrt{3}}{2}, \quad y = \frac{1}{4\lambda^2-1} = \frac{1}{2}.$$

Therefore we have three points

$$(0, -1), \quad \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \quad \left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right).$$

Evaluate $f(x, y) = x + xy$ at these points

$$f(0, -1) = 0 + 0(-1) = 0$$

$$f\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\frac{1}{2} = \frac{3\sqrt{3}}{4}$$

$$f\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{-\sqrt{3}}{2} + \left(\frac{-\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{-3\sqrt{3}}{4}.$$

So max value is $\boxed{f\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{3\sqrt{3}}{4}}$

min value is $\boxed{f\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{-3\sqrt{3}}{4}}$