12.3 The Dot Product

1. Define the dot product : $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Text-Ex 1 : $\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2(3) + 4(-1) = 2$

$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\frac{1}{2} \rangle = (-1)6 + 7(2) + 4(-\frac{1}{2}) = 6$

$\underbrace{(\vec{i} + 2\vec{j} - 3\vec{k})}_{= \langle 1, 2, -3 \rangle} \cdot \underbrace{(2\vec{j} - \vec{k})}_{= \langle 0, 2, -1 \rangle} = 1(0) + 2(2) + (-3)(-1) = 7$.

↳ b/c there is no "$\vec{i}$" part

2. Basic properties (textbook p 807) :

(1) $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ (2) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

(3) $\vec{a} \cdot (\vec{b} \pm \vec{c}) = \vec{a} \cdot \vec{b} \pm \vec{a} \cdot \vec{c}$

(4) $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$ (5) $\vec{0} \cdot \vec{a} = 0$.

3. Advanced properties :

(1) $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos(\theta)$ where θ is the angle between $\vec{a}$ and $\vec{b}$. This follows from the Law of Cosines.

(2) $\cos(\theta) = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$ (we usually use this formula to compute θ)

(3) $\vec{a}$ and $\vec{b}$ are called perpendicular or orthogonal (notation: $\vec{a} \perp \vec{b}$) if $\theta = \frac{\pi}{2}$. So

$\vec{a} \perp \vec{b}$ iff (if and only if) $\vec{a} \cdot \vec{b} = 0$

Text-Ex 2: If $|\vec{a}| = 4$, $|\vec{b}| = 6$ and the angle between them is $\frac{\pi}{3}$, find $\vec{a}\cdot\vec{b}$.

Ans: Using $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos(\theta)$, we have

$$\vec{a}\cdot\vec{b} = 4\cdot 6\cdot\cos\left(\frac{\pi}{3}\right) = 12.$$

Text-Ex 3: Find the angle between $\vec{a} = \langle 2,2,-1\rangle$ and $\vec{b} = \langle 5,-3,2\rangle$.

Ans: To use the formula

$$\cos(\theta) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|},$$

we compute

$$\vec{a}\cdot\vec{b} = 2(5) + 2(-3) + (-1)(2) = 2,$$

$$|\vec{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3$$

$$|\vec{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38},$$

and then:

$$\cos(\theta) = \frac{2}{3\sqrt{38}}$$

So

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \quad (\approx 84^\circ)$$

No need to do this part in the quiz or exam.

Text-Ex 4: Show $2\vec{i} + 2\vec{j} - \vec{k}$ is perpendicular to $5\vec{i} - 4\vec{j} + 2\vec{k}$.

Ans:

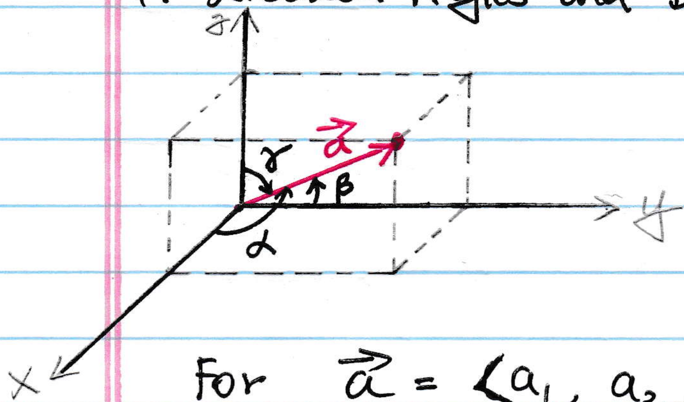
To determine perpendicular or not, check the dot product.

$$(2\vec{i} + 2\vec{j} - \vec{k})\cdot(5\vec{i} - 4\vec{j} + 2\vec{k})$$

$$= 2(5) + 2(-4) + (-1)(2) = 0 \Rightarrow$$

These vectors are perpendicular.

4. Direction Angles and Direction Cosines



Direction angles α, β, γ of $\vec{a}$: angles between $\vec{a}$ and positive x-, y-, z-axes respectively ($\vec{i}, \vec{j}, \vec{k}$).

For $\vec{a} = \langle a_1, a_2, a_3 \rangle$

direction cosines

$$\cos(\alpha) = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}|\,|\vec{i}|} = \frac{a_1}{|\vec{a}|}, \qquad \alpha = \cos^{-1}\left(\frac{a_1}{|\vec{a}|}\right)$$

$$\cos(\beta) = \frac{\vec{a}\cdot\vec{j}}{|\vec{a}|\,|\vec{j}|} = \frac{a_2}{|\vec{a}|}, \qquad \beta = \cos^{-1}\left(\frac{a_2}{|\vec{a}|}\right)$$

$$\cos(\gamma) = \frac{\vec{a}\cdot\vec{k}}{|\vec{a}|\,|\vec{k}|} = \frac{a_3}{|\vec{a}|}, \qquad \gamma = \cos^{-1}\left(\frac{a_3}{|\vec{a}|}\right)$$

Properties:

$$\underbrace{\frac{\vec{a}}{|\vec{a}|}}_{\text{unit vector in the direction of } \vec{a}} = \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$$

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1$$

Text-Ex 5: Find the direction angles of $\vec{a} = \langle 1, 2, 3 \rangle$

Ans:

$$|\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}.$$

By the formula above,

$$\cos(\alpha) = \frac{1}{\sqrt{14}}, \quad \cos(\beta) = \frac{2}{\sqrt{14}}, \quad \cos(\gamma) = \frac{3}{\sqrt{14}}$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right), \quad \beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right), \quad \gamma = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right)$$

5. Projections

Scalar projection of $\vec{b}$ onto $\vec{a}$: $\text{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

Vector projection of $\vec{b}$ onto $\vec{a}$:

$$\text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a}$$

Text-Ex 6 : Find the scalar and vector projections of $\vec{b} = \langle 1, 1, 2\rangle$ onto $\vec{a} = \langle -2, 3, 1\rangle$.

Ans : Just use the formulas.

$$|\vec{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$$

$$\vec{a}\cdot\vec{b} = (-2)(1) + 3\,(1) + 1\,(2) = 3$$

$$\Rightarrow \text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{3}{\sqrt{14}}$$

$$\text{proj}_{\vec{a}}\vec{b} = \frac{3}{\sqrt{14}}\cdot\frac{\langle -2, 3, 1\rangle}{\sqrt{14}} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right\rangle.$$

(Not important) 6. Application in Physics : Work

$$\underset{\text{Work}}{W} = \underset{\text{Force}}{\vec{F}} \cdot \underset{\text{Displacement}}{\vec{D}}$$

See Text-Ex 7 and Text-Ex 8.