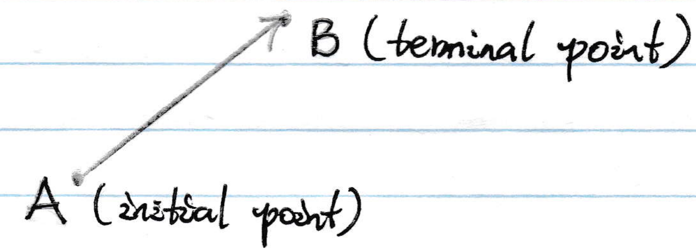
12.2 Vectors

1. Definition of a vector : magnitude + direction

Compare vectors with scalars

velocity, force ... temperature, volume, ...

Vector $\overrightarrow{AB}$:



Two vectors are equal (equivalent) if they have the same length and the same direction.

2. Component form of a vector (Don't confuse it with a point)

$$\vec{a} = \langle a_1, a_2 \rangle \text{ (2D)} \quad \text{or} \quad \vec{a} = \langle a_1, a_2, a_3 \rangle \text{ (3D)}$$

Given points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$,

★ $\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

If O is the origin, i.e. $O = (0, 0, 0)$, then

$$\overrightarrow{OA} = \langle x_1 - 0, y_1 - 0, z_1 - 0 \rangle = \langle x_1, y_1, z_1 \rangle$$

called the position vector of the point A.

Text-Ex 3 : Find the component form of $\overrightarrow{AB}$ with $A(2, -3, 4)$ and $B(-2, 1, 1)$

Ans:

$$\overrightarrow{AB} = \langle -2-2, 1-(-3), 1-4 \rangle = \langle -4, 4, -3 \rangle$$

3. Define $+$, $-$ and scalar multiple for vectors

Addition: $\underbrace{\langle a_1, a_2, a_3 \rangle}_{\vec{a}} + \underbrace{\langle b_1, b_2, b_3 \rangle}_{\vec{b}}$

$$= \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

"$-\vec{a}$": $-\vec{a} = \langle -a_1, -a_2, -a_3 \rangle$

Subtraction: $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

$$= \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

Scalar multiple: Let c be a real number, then

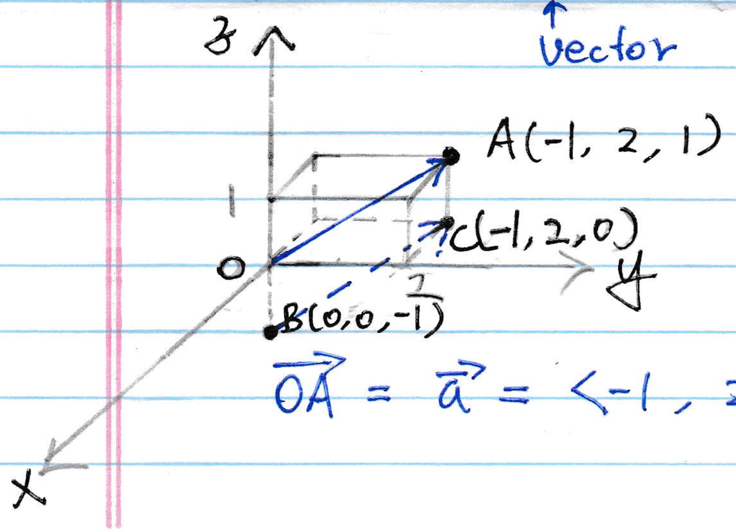
$$c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle$$

4. Visualization

(1) We may anchor a vector (associate it with two points, like $\overrightarrow{AB}$), although this is not always necessary.

Example: Let $\vec{a} = \langle -1, 2, 1 \rangle$, $A(-1, 2, 1)$.

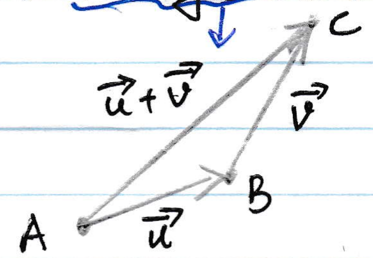
$\vec{a}$ ↑ vector; A ↑ point

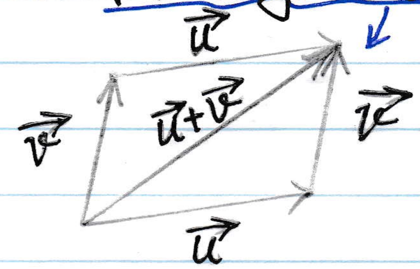


$\overrightarrow{OA} = \vec{a} = \langle -1, 2, 1 \rangle$

But we can also have $\overrightarrow{BC} = \overrightarrow{OA} = \vec{a}$.

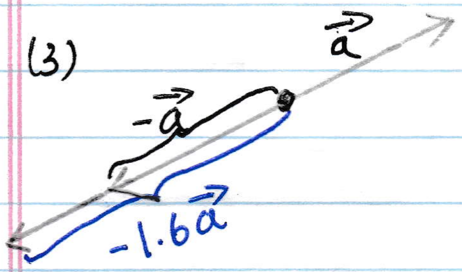
(2) Triangle Law and Parallelogram Law

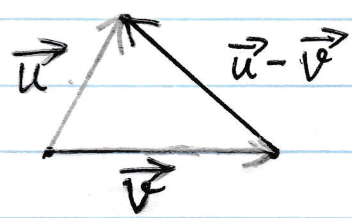




From the Triangle Law, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ always holds.

(3)





Examples: $\overrightarrow{AB} + \overrightarrow{BA} = \vec{0}$ ← zero vector $\langle 0, 0, 0 \rangle$

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AC} + \overrightarrow{CA} = \vec{0}$$

$$\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{CA} = \overrightarrow{CA} + \overrightarrow{AB} = \overrightarrow{CB}.$$

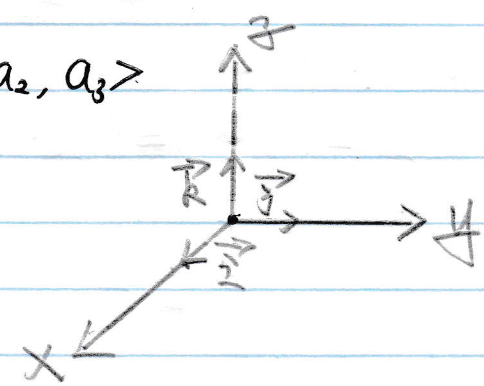
5. Definitions / Properties

(1) Standard basis vectors

$\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$

Then an alternative form of $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is

$$\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$



For example,

$$\langle 1, -2, 6 \rangle = \vec{i} - 2\vec{j} + 6\vec{k}$$

(2) For $\vec{a} = \langle a_1, a_2, a_3 \rangle$, its length (magnitude, norm) is $$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

For $\vec{a} = \langle a_1, a_2 \rangle$, $|\vec{a}| = \sqrt{a_1^2 + a_2^2}$.

(3) A unit vector is a vector with length 1.

(4) Parallel vectors are vectors which are multiples of one another.

(5) The unit vector in the direction of $\vec{a}$ (the unit vector which is parallel to $\vec{a}$) is $\dfrac{\vec{a}}{|\vec{a}|}$.

(6) Other properties (textbook p 802)

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$, $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

$\vec{a} + \vec{0} = \vec{a}$, $\vec{a} + (-\vec{a}) = \vec{0}$

$c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$, $(c+d)\vec{a} = c\vec{a} + d\vec{a}$

$(cd)\vec{a} = c(d\vec{a})$, $1\vec{a} = \vec{a}$.

Text-Ex 4: $\vec{a} = \langle 4, 0, 3 \rangle$, $\vec{b} = \langle -2, 1, 5 \rangle$, find $|\vec{a}|$ and $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$, $3\vec{b}$, $2\vec{a} + 5\vec{b}$.

Ans: $|\vec{a}| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{25} = 5$

$\vec{a} + \vec{b} = \langle 4, 0, 3 \rangle + \langle -2, 1, 5 \rangle$

$= \langle 4 + (-2), 0 + 1, 3 + 5 \rangle = \langle 2, 1, 8 \rangle$

$$\vec{a}-\vec{b} = \langle 4,0,3\rangle - \langle -2,1,5\rangle$$
$$= \langle 4-(-2),\ 0-1,\ 3-5\rangle = \langle 6,-1,-2\rangle$$

$$3\vec{b} = 3\langle -2,1,5\rangle = \langle 3(-2), 3(1), 3(5)\rangle$$
$$= \langle -6, 3, 15\rangle$$

$$2\vec{a}+5\vec{b} = 2\langle 4,0,3\rangle + 5\langle -2,1,5\rangle$$
$$= \langle 8,0,6\rangle + \langle -10,5,25\rangle = \langle -2,5,31\rangle.$$

Text-Ex 6: Find the unit vector in the direction of the vector $2\vec{i}-\vec{j}-2\vec{k}$

Ans: $2\vec{i}-\vec{j}-2\vec{k} = \langle 2,-1,-2\rangle$

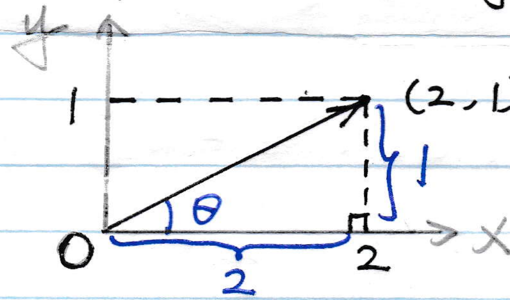
length: $|\langle 2,-1,-2\rangle| = \sqrt{2^2+(-1)^2+(-2)^2} = \sqrt{9} = 3$

By formula, the unit vector is

$$\frac{\langle 2,-1,-2\rangle}{|\langle 2,-1,-2\rangle|} = \frac{\langle 2,-1,-2\rangle}{3} = \langle \tfrac{2}{3}, -\tfrac{1}{3}, -\tfrac{2}{3}\rangle.$$

Example: What is the angle between the vector $\vec{a} = \langle 2,1\rangle$ and the positive direction of the x-axis?

Ans: Draw a picture



$$\tan(\theta) = \frac{1}{2} \Rightarrow \theta = \arctan(\tfrac{1}{2}) \approx 26.57^\circ$$

We'll discuss this kind of problem again with a better method in section 12.3.