5.2 Elimination Method for A System of ODEs

A System of ODEs: A collection of more than one ODE.

Examples: (1) $\begin{cases} x'(t) = y(t) + 1 \\ y'(t) = x(t) - t \end{cases}$

(2) $\begin{cases} x'' - x + 5y = e^t \\ 2x + y'' + 2y = 0 \end{cases}$.

Rewrite ODE with differential operator $\underline{D}$:

$$y'' + 4y' + 3y = 0$$

$$\Longleftrightarrow D^2 y + 4Dy + 3y = 0$$

$$\Longleftrightarrow (D^2 + 4D + 3)y = 0$$

Examples: (1) $\begin{cases} x'(t) = y(t) + 1 \\ y'(t) = x(t) - t \end{cases}$

$$\Longleftrightarrow \begin{cases} Dx - y = 1 \\ Dy - x = -t \end{cases}.$$

(2) $x'' + 2x' + 1 = 0$

$$\Longleftrightarrow (D^2 + 2D + 1)x = 0$$

$$\Longleftrightarrow (D+1)^2 x = 0$$

(3) $D(D+1)y = 0$

$$\Longleftrightarrow (D^2 + D)y = 0$$

$$\Longleftrightarrow y'' + y' = 0$$

Remark: The reason for the calculations above is explained as follows:

Example: Check $(D+1)(D+3) = D^2 + 4D + 3$.

Test with y: $(D+1)(D+3)[y] = (D+1)(Dy + 3y)$

$= (D+1)(y' + 3y) = D(y'+3y) + (y'+3y)$

$= y'' + 3y' + y' + 3y = y'' + 4y' + 3y$.

However when the calculation ~~is~~ involving with $\boxed{\text{non-constant}}$ functions, it may not be as simple as we expect.

Example: $t \cdot D \neq D \cdot t$

Test with y: $t \cdot D[y] = t \cdot (Dy) = ty'$

$D \cdot t[y] = D[ty] = \frac{d}{dt}(ty)$

$= ty' + y$.

So $t \cdot D \neq D \cdot t$.

End of the remark above. Fortunately, we won't need to deal with something like $t \cdot D$ as an operator.

Idea of Solving System: Elimination (sometimes just substitution)

Similar to solving a linear system.

Example: $\begin{cases} x'(t) = y(t) + 1 \\ y'(t) = x(t) - t \end{cases}$

Solution: First rewrite the system and put the terms involving dependent variables (x, y here) on the left:

$$\begin{cases} Dx - 1\cdot y = 1 & \text{①} \\ -1\cdot x + D\cdot y = -t & \text{②} \end{cases}$$

$\underbrace{D\cdot ①}$ + ② : $D(Dx - 1\cdot y) + (\overset{-1\cdot x}{\cancel{\quad}} + Dy) = \underbrace{D\cdot 1}_{=0} + (-t)$

Operator D times the first line, might be weird, but this is the way how it works!

$D^2x - Dy - \overset{1\cdot x}{\cancel{\quad}} + Dy = -t$

$\underline{D^2x - x = -t}$

Eliminates the terms with y.

$$D^2x - x = -t \iff x'' - x = -t$$

We can solve this ODE as we discussed earlier.

CE: $r^2 - 1 = 0 \Rightarrow r = 1, -1 \Rightarrow$ fundamental pair $\{e^t, e^{-t}\}$

For $f(t) = -t$, use the form

$$X_p(t) = t^0\cdot(A_1t + A_0)\cdot e^{0\cdot t} = A_1t + A_0$$

$$X_p'(t) = A_1, \quad X_p''(t) = 0$$

$$X_p'' - X_p = 0 - (A_1t + A_0) = -A_1t - A_0 = -t$$

$$\Rightarrow \begin{cases} -A_1 = -1 \\ -A_0 = 0 \end{cases} \Rightarrow \begin{cases} A_1 = 1 \\ A_0 = 0 \end{cases} \Rightarrow X_p(t) = t$$

So $x(t) = t + C_1e^t + C_2e^{-t}$

How do we find $y(t)$?

Like solving linear system, find one line (① or ②) to plug in the expression of $x(t)$ we found.

We use ① here: $D(t + C_1e^t + C_2e^{-t}) - y = 1$

$$
\begin{aligned}
y &= D(t + c_1 e^{t} + c_2 e^{-t}) - 1 \\
&= \frac{d}{dt}(t + c_1 e^{t} + c_2 e^{-t}) - 1 \\
&= 1 + c_1 e^{t} + (-c_2) e^{-t} - 1 \\
&= c_1 e^{t} - c_2 e^{-t}
\end{aligned}
$$

In conclusion, we find solution of the system:

$$
\begin{cases}
x(t) = t + c_1 e^{t} + c_2 e^{-t} \\
y(t) = c_1 e^{t} - c_2 e^{-t}
\end{cases}
$$

solving

<u>Remark</u>: An alternative way of finding $x(t)$ in the example above.

↳ like substitution

From ①: $y = Dx - 1$, plug in ②:

$$
\begin{aligned}
-x + D(Dx - 1) &= -t \\
-x + D^2 x - 0 &= -t \\
x'' - x &= -t
\end{aligned}
$$

The remaining part / procedure would be the same.

<u>Example</u>:
$$
\begin{cases}
x'(t) = 3x(t) - 4y(t) + 1 \\
y'(t) = 4x(t) - 7y(t) + 10t
\end{cases}
$$

<u>Solution</u>: Rewrite the system using "D" and put x, y on the left hand side.

You also need to combine terms of x, y together, i.e instead of $D[x] - 3x$ we should write $(D-3)[x]$

$$
\begin{cases}
\underbrace{(D-3)x}_{x' - 3x} + 4y = 1 & ① \\
-4x + \underbrace{(D+7)y}_{y' + 7y} = 10t & ②
\end{cases}
$$

Now try to eliminate x or y by summing / taking difference eq. ① times something, eq ② times something else.

Say we want to eliminate x.

4. ① + $(D-3)\cdot$ ② :

$$\begin{aligned} &4(D-3)x + 4\cdot 4y && = 4\cdot 1 \\ &+(D-3)(-4x) + (D-3)(D+7)y && + (D-3)(10t) \end{aligned}$$

$$\begin{aligned} LHS &= 4 + \frac{d}{dt}(10t) - 3\cdot 10t \\ LHS &= 4 + 10 - 30t \\ LHS &= 14 - 30t \end{aligned}$$

$$LHS = 4(D-3)x + 16y - 4(D-3)x + \underbrace{(D^2+4D-21)}_{=(D-3)(D+7)}y$$

$$\begin{aligned} &= 16y + (D^2+4D-21)y \\ &= (D^2+4D-5)y \\ &= y'' + 4y' - 5y. \end{aligned}$$

Therefore we get $y'' + 4y' - 5y = 14 - 30t$

Solve as before: CE $r^2+4r-5=0$

$(r+5)(r-1)=0$

$\Rightarrow$ Two roots: $r=-5, 1$ fundamental pair $\{e^{-5t}, e^{t}\}$.

For $f(t) = 14 - 30t$, use the form

$$Y_p(t) = t^0\cdot(A_1t + A_0)\cdot e^{0t} = A_1t + A_0$$

$$Y_p' = A_1, \quad Y_p'' = 0$$

$$Y_p'' + 4Y_p' - 5Y_p = 0 + 4A_1 - 5(A_1t + A_0) = -5A_1t + (4A_1 - 5A_0) = 14 - 30t$$

$$\Rightarrow \begin{cases} -5A_1 = -30 \\ 4A_1 - 5A_0 = 14 \end{cases} \Rightarrow \begin{cases} A_1 = 6 \\ A_0 = 2 \end{cases}$$

$y_p(t) = 6t + 2$, and thus

$$y(t) = 6t + 2 + c_1 e^{-5t} + c_2 e^{t}.$$

Now to find $x(t)$, plug in $y(t)$ into ②:

$$-4x + (D+7)(6t + 2 + c_1 e^{-5t} + c_2 e^{t}) = 10t$$

$$-4x + \frac{d}{dt}(6t + 2 + c_1 e^{-5t} + c_2 e^{t}) + 7(6t + 2 + c_1 e^{-5t} + c_2 e^{t}) = 10t$$

$$-4x + (6 + (-5c_1)e^{-5t} + c_2 e^{t}) + 42t + 14 + 7c_1 e^{-5t} + 7c_2 e^{t} = 10t$$

$$32t + 20 + 2c_1 e^{-5t} + 8c_2 e^{t} = 4x$$

$$\Rightarrow x = \frac{1}{2}c_1 e^{-5t} + 2c_2 e^{t} + 8t + 5$$

So
$$\begin{cases} x(t) = \frac{1}{2}c_1 e^{-5t} + 2c_2 e^{t} + 8t + 5 \\ y(t) = c_1 e^{-5t} + c_2 e^{t} + 6t + 2. \end{cases}$$

General Procedure:

~~[illegible]~~ dependent variables

1. Rewrite system using "D", put terms involving x, y on the left
2. Eliminate x or y by doing "$+$ or $-$, $\times$" (add, minus, times)
3. Solve the ODE for the remaining variable
4. Plug in back to find another variable (Try to avoid solving another ODE here)
5. Remove extra constants if needed.

Solve

Example: $\begin{cases} x''(t) + y'(t) - x(t) + y(t) = -1 \\ x'(t) + 2y(t) - x(t) = t \end{cases}$

Solution: Rewrite system using "D", x, y on the left

$$\begin{cases} (D^2-1)(x) + (D+1)(y) = -1 & ① \\ (D-1)(x) + 2y = t & ② \end{cases}$$

$(D-1)(D+1)$

Notice $(D^2-1) = (D-1)(D+1)$, we calculate

$(D+1)\cdot ② - ①$:

$$\begin{aligned} &(D+1)(D-1)(x) + (D+1)2y = (D+1)t \\ -&\big((D^2-1)(x) + (D+1)y\big) \quad\quad +1 \end{aligned}$$

$$(D+1)y = t+2$$

$$y' + y = t+2.$$

linear ODE: $a(t) = 1$, $A(t) = t$

$$y(t) = e^{-t}\int e^{t}(t+2)\,dt$$

Integration by parts

$$= e^{-t}\left(e^{t}(t+2) - \int e^{t}\cdot 1\,dt\right)$$

$$= e^{-t}\left(e^{t}(t+2) - e^{t} + c_1\right)$$

$$= e^{-t}\cdot\left(e^{t}(t+1) + c_1\right) = t+1+c_1e^{-t}$$

Plug into ② : $(D-1)x + 2(t+1) + 2c_1e^{-t} = t$

$$x' - x = -t-2-2c_1e^{-t}$$

linear ODE:

$$x(t) = e^{t}\left(\int e^{-t}(-t-2-2c_1e^{-t})\,dt\right)$$

$$= e^{t}\left(\int -2c_1e^{-2t}\,dt - \int e^{-t}(t+2)\,dt\right)$$

$$= e^{t}\left(c_1e^{-2t} - \int e^{-t}(t+2)\,dt\right)$$

Integration by parts

$$= e^{t}\left(c_1e^{-2t} + e^{-t}(t+2) - \int e^{-t}\,dt\right)$$

$$= e^{t}\left(c_1e^{-2t} + e^{-t}(t+2) + e^{-t} + c_2\right)$$

$$= t+3+c_1e^{-t}+c_2e^{t}$$

$$\Rightarrow \begin{cases} x(t) = t+3+c_1e^{-t}+c_2e^{t} \\ y(t) = t+1+c_1e^{-t} \end{cases}$$

Example:
$$\begin{cases} (D-1)[x] + 2D[y] = 1 & (1) \\ D[x] + (D+1)[y] = \cos(t)+\sin(t) & (2) \end{cases}$$

Solution: It is already in the form with "D" and x, y on the left. So let's try to eliminate x now.

$D\cdot(1) - (D-1)\cdot(2)$:

$$D(D-1)[x] + 2D^2[y] = 0 - (D-1)[\cos(t)+\sin(t)]$$
$$-\left((D-1)D[x] + (D-1)(D+1)[y]\right)$$

$$2D^2[y] - (D^2-1)[y] = (\cos(t)+\sin(t)) - \frac{d}{dt}(\cos(t)+\sin(t))$$

$$(D^2+1)[y] = \cos(t)+\sin(t) - (-\sin(t)+\cos(t))$$

$$y'' + y = 2\sin(t) \qquad ③$$

Solve ③ as before: CE $r^2+1=0 \Rightarrow r=\pm i$

fundamental pair $\{\cos(t), \sin(t)\}$

Since $f(t) = 2\sin(t)$, use the form

b/c $\pm i$ is a root for the CE.

$$Y_p(t) = t A_0 \cos(t) + t B_0 \sin(t)$$

$$Y_p'(t) = A_0(\cos(t) - t\sin(t)) + B_0(\sin(t) + t\cos(t))$$

$$Y_p''(t) = A_0(-2\sin(t) - t\cos(t)) + B_0(2\cos(t) - t\sin(t))$$

$$Y_p'' + Y_p = -2A_0\sin(t) + 2B_0\cos(t) = 2\sin(t)$$

$$\Rightarrow A_0 = -1,\ B_0 = 0 \Rightarrow Y_p(t) = -t\cos(t)$$

So $y(t) = -t\cos(t) + C_1\cos(t) + C_2\sin(t)$

How to find $x(t)$?

(complicated) Method 1: Plug the expression $y(t)$ into ②

$$D[x] + (D+1)[-t\cos(t) + C_1\cos(t) + C_2\sin(t)] = \cos(t) + \sin(t)$$

Since $(D+1)(-t\cos(t) + C_1\cos(t) + C_2\sin(t))$

$$= (-\cos(t) + t\sin(t) - C_1\sin(t) + C_2\cos(t))$$
$$+ (-t\cos(t) + C_1\cos(t) + C_2\sin(t))$$

$$\Rightarrow x' = \cos(t) + \sin(t) - (-\cos(t) + t\sin(t) - C_1\sin(t) + C_2\cos(t))$$
$$- (-t\cos(t) + C_1\cos(t) + C_2\sin(t))$$

$$= (2 - C_2 - C_1 + t)\cos(t) + (1 + C_1 - C_2 - t)\sin(t)$$

So:

$$x(t) = \int (2 - c_2 - c_1 + t)\cos(t) + (1 + c_1 - c_2 - t)\sin(t)\, dt$$

$$= (2 - c_2 - c_1)\sin(t) - (1 + c_1 - c_2)\cos(t)$$

$$+ \int t\cos(t) - t\sin(t)\, dt$$

Integration by parts

$$= (2 - c_2 - c_1)\sin(t) - (1 + c_1 - c_2)\cos(t)$$

$$+ t\sin(t) + t\cos(t) - \int \sin(t) + \cos(t)\, dt$$

$$= (2 - c_2 - c_1)\sin(t) - (1 + c_1 - c_2)\cos(t)$$

$$+ t\sin(t) + t\cos(t) + \cos(t) - \sin(t) + c_3$$

$$= (2 - c_2 - c_1 + t - 1)\sin(t) + (-1 - c_1 + c_2 + t + 1)\cos(t) + c_3$$

$$= (1 - c_2 - c_1 + t)\sin(t) + (-c_1 + c_2 + t)\cos(t) + c_3$$

Why do we have three constants c_1, c_2, c_3?

b/c we haven't finished yet. We need to ~~remove~~ check or remove constants by plugging into ① too.

$$(D-1)[x] + 2D[y] = 1$$

$$(D-1)\Big((1 - c_2 - c_1 + t)\sin(t) + (-c_1 + c_2 + t)\cos(t) + c_3\Big)$$

$$+ 2D\Big(-t\cos(t) + c_1\cos(t) + c_2\sin(t)\Big) = 1$$

If you do the messy calculation above, you should get:

$$-(1-c_2-c_1+t)\sin(t) - (-c_1+c_2+t)\cos(t) - c_3$$
$$+ (1-c_2-c_1+t)\cos(t) + \cancel{\cos(t)}\ \sin(t)$$
$$+ (-c_1+c_2+t)(-\sin(t)) + \cos(t)$$
$$+ 2\left(-\cos(t) + t\sin(t) - c_1\sin(t) + c_2\cos(t)\right) = 1$$

$$\Rightarrow \quad -c_3 = 1 \quad \Rightarrow \quad c_3 = -1$$

So
$$\begin{cases} x(t) = (1-c_2-c_1+t)\sin(t) + (-c_1+c_2+t)\cos(t) - 1 \\ y(t) = -t\cos(t) + c_1\cos(t) + c_2\sin(t) \end{cases}$$

(Simpler) Method 2: Observe if we do ② − ①, we get:

$$x - (D-1)[y] = \cos(t) + \sin(t) - 1$$

$$\Rightarrow x(t) = (D-1)(y) + \cos(t) + \sin(t) - 1$$
$$= (D-1)\left(-t\cos(t) + c_1\cos(t) + c_2\sin(t)\right) + \cos(t) + \sin(t) - 1$$
$$= (1-c_2-c_1+t)\sin(t) + (-c_1+c_2+t)\cos(t) - 1.$$

The trick here is trying to do some combination

$$\tilde{L}_1 \cdot ① + \tilde{L}_2 \cdot ②$$ ($\tilde{L}_1, \tilde{L}_2$ may contain "D" or just constant numbers)

to get rid of derivatives of x when knowing y and wanting to find x (after solving y already)

5.4 Introduction to the Phase Plane

(My notes are different with textbook, ignore things in the textbook which are not here.)

$$\begin{cases} \dfrac{dx}{dt} = f(x, y) \\ \dfrac{dy}{dt} = g(x, y) \end{cases}$$

← Autonomous because the right hand side f, g do not depend on independent variable t.

↑ Autonomous system of two first-order ODEs.

Part I: How to find critical point, or equilibrium point?

Def: A point (x_0, y_0) is called a critical point if $f(x_0, y_0) = 0$, $g(x_0, y_0) = 0$.

For a given critical point, there is an equilibrium solution:

$$\begin{cases} x(t) = x_0 \\ y(t) = y_0 \end{cases}$$

→ Notice here the solution is constant, in the sense it doesn't change along the time.

Example: Find critical point for $\begin{cases} \dfrac{dx}{dt} = -x \\ \dfrac{dy}{dt} = -2y \end{cases}$

Solutions: $\begin{cases} -x = 0 \\ -2y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$ ← Critical point.

Example: Find critical point for $\begin{cases} \dfrac{dx}{dt} = 5x - 3y - 2 \\ \dfrac{dy}{dt} = 4x - 3y - 1 \end{cases}$

Solutions: $\begin{cases} 5x - 3y - 2 = 0 \\ 4x - 3y - 1 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}$ ← Critical point.

Example: Find the critical points for $\begin{cases} \frac{dx}{dt} = x - y \\ \frac{dy}{dt} = \sin(x+y) \end{cases}$

Solutions: $\begin{cases} x - y = 0 \\ \sin(x+y) = 0 \end{cases} \Rightarrow x = y, \quad \sin(x+y) = \sin(2x) = 0$

$2x = k\pi$ (k: integer), $x = \frac{k}{2}\pi$.

So critical points are: $\begin{cases} x = \frac{k}{2}\pi \\ y = \frac{k}{2}\pi \end{cases}$ (k is integer).

Part II: How to find eigenvalues of a 2×2 matrix A?

Method: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. solve λ from

$$0 = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$$

$$= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}).$$

Examples: Find eigenvalues of $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

Solutions $\begin{vmatrix} 1-\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 0 = (1-\lambda)^2 = 0$

$\Rightarrow \lambda = 1$ (multiplicity $= 2$).

Examples: Find eigenvalues of $\begin{pmatrix} -5 & 4 \\ 8 & -1 \end{pmatrix}$.

Solutions: $\begin{vmatrix} -5-\lambda & 4 \\ 8 & -1-\lambda \end{vmatrix} = (-5-\lambda)(-1-\lambda) - 4\cdot 8$

$$= (\lambda+5)(\lambda+1) - 32$$
$$= \lambda^2 + 6\lambda + 5 - 32$$
$$= \lambda^2 + 6\lambda - 27$$
$$= (\lambda+9)(\lambda-3) = 0 \Rightarrow \lambda_1 = -9,\ \lambda_2 = 3.$$

Example: Find eigenvalues of $\begin{pmatrix} 1 & 2 \\ -4 & 5 \end{pmatrix}$.

Solutions: $\begin{vmatrix} 1-\lambda & 2 \\ -4 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda) - 2\cdot(-4)$

$$= (\lambda^2 - 6\lambda + 5) + 8$$
$$= \lambda^2 - 6\lambda + 13 = 0$$

$$\Rightarrow \lambda = \frac{6 \pm \sqrt{6^2 - 4\cdot 13}}{2} = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i.$$

Part III: Classification of critical points for a linear system:

$$(*)\begin{cases} \frac{dx}{dt} = a_{11}x + a_{12}y \\ \frac{dy}{dt} = a_{21}x + a_{22}y \end{cases}$$

Can also be written as $\longrightarrow$ $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$

where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

$(*)$ always has a critical point $(0,0)$.

Remark: See the end of this section in this note if you are interested in why we can use the system $(*)$ to approximate the behavior of general system $\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases}$.

<u>Direction Field</u>: On xy-plane, plot a vector $c \cdot \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$ at points (x, y).

There is a constant c b/c we only care about the directions at (x, y).

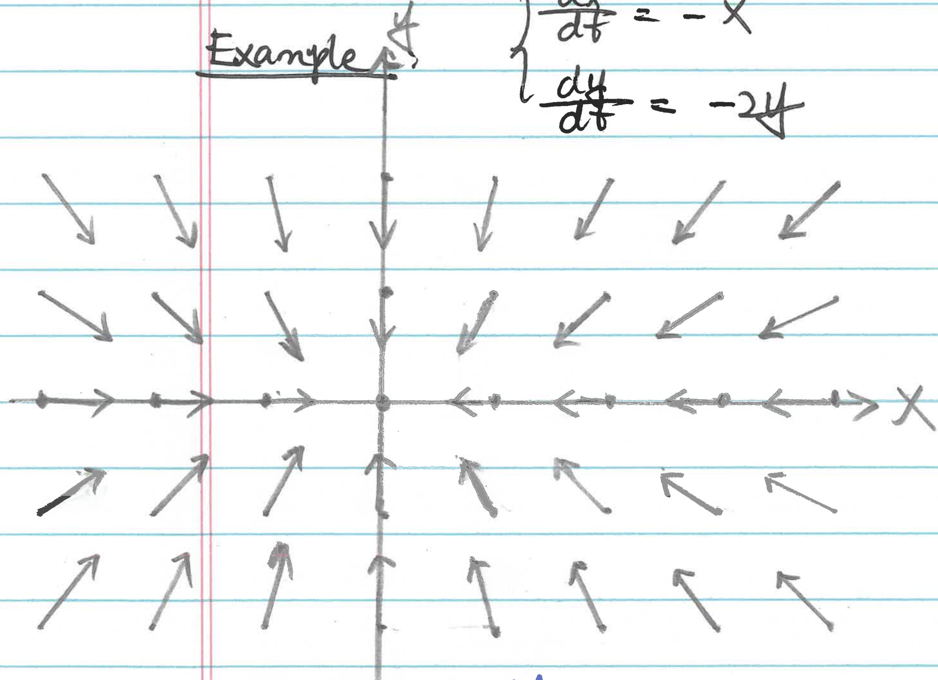
<u>Trajectory</u>: A curve $x = x(t)$, $y = y(t)$ with arrows indicating its direction with increasing t.

<u>Phase portrait</u>: A representative set of trajectories in the xy-plane.

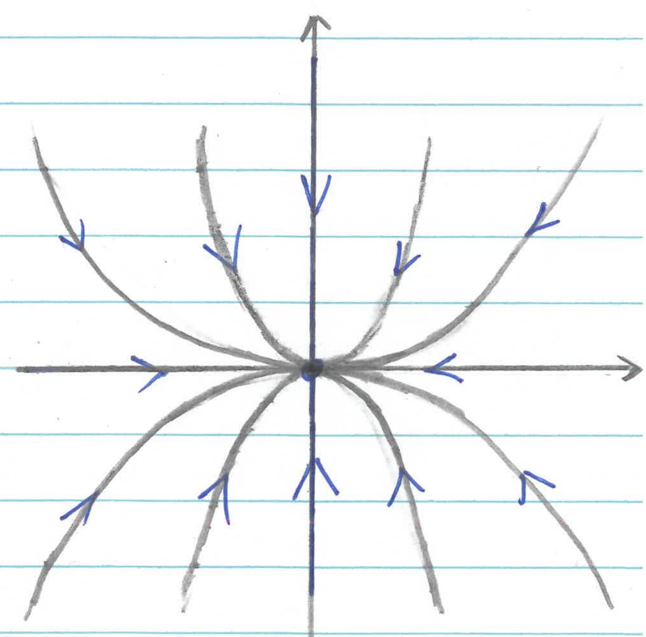
Called phase plane

<u>Example</u>

$$\begin{cases} \dfrac{dx}{dt} = -x \\ \dfrac{dy}{dt} = -2y \end{cases}$$



Direction Field



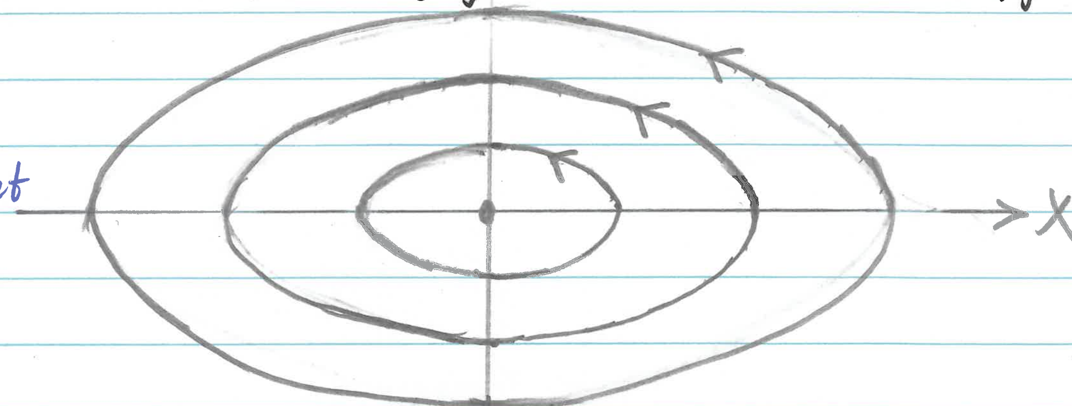
Phase Portrait

Notice that the critical point (equilibrium point) is $(0, 0)$. And in this case if we start from somewhere near $(0, 0)$ at $t = 0$, the trajectory will "flow into" the critical point. This critical point is called (asymptotically) <u>stable</u>.

Don't worry, you will never be asked to plot the phase portrait in this class. It will be given in quizzes and exams if needed.

Example: $A = \begin{pmatrix} 0 & -12 \\ 3 & 0 \end{pmatrix}$, $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$.

In this case, the critical point is stable.



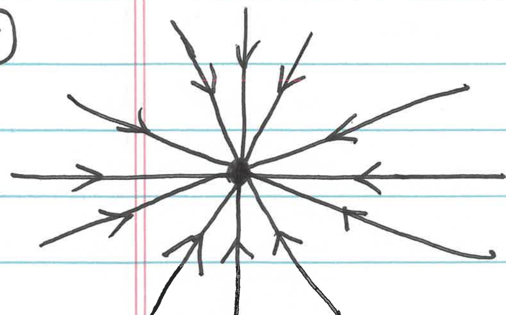
Phase Portrait

(Assume 0 is not an eigenvalue of A)

Classification of critical points (stable or unstable, what phase portrait does it have)

(1) Compute eigenvalues of matrix A

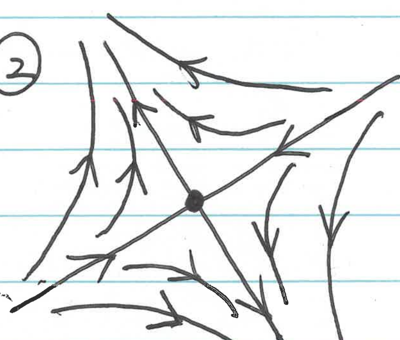
(2) Find the corresponding ~~type~~ type in the following (textbook p267)

① 

Nodal Sink

"Asymptotically" stable

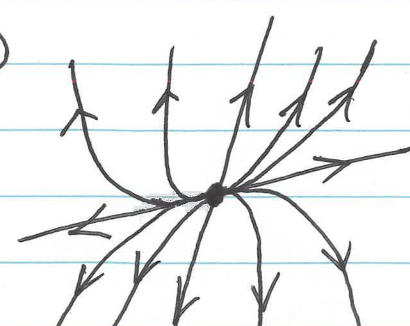
$\lambda_1, \lambda_2 < 0$.

② 

Saddle Point

Unstable

$\lambda_1 < 0, \lambda_2 > 0$

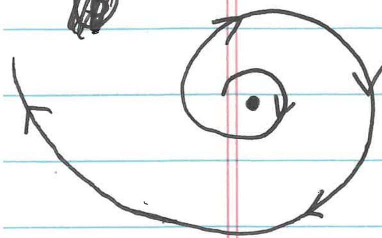
③ 

Nodal Source

Unstable

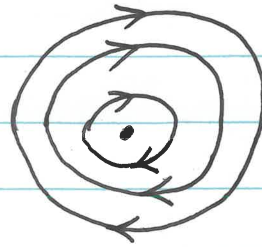
$\lambda_1, \lambda_2 > 0$

④



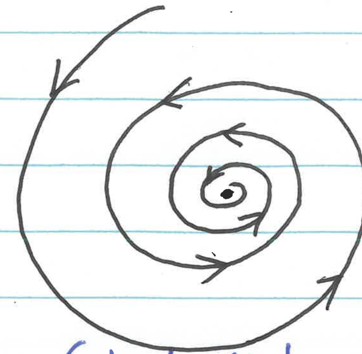
Spiral Source
Unstable
$\lambda = r \pm si \ (r>0)$

⑤



Center
Stable
$\lambda = 0 \pm si$.

⑥



Spiral Sink
"Asymptotically" stable
$\lambda = r \pm si \ (r<0)$

Summary: Critical points are ~~stable~~ $\boxed{\text{unstable}}$ if at least one eigenvalue is >0 or has real part >0.

Example: Determine whether the critical points of the following systems are stable or not:

(1) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(2) $\begin{cases} \frac{dx}{dt} = 2x - 3y \\ \frac{dy}{dt} = 3x + 2y \end{cases}$

Solutions: (1) $A = \begin{pmatrix} 5 & 4 \\ 2 & 7 \end{pmatrix}$. Solve eigenvalues of A.

$$0 = |A - \lambda I| = \begin{vmatrix} 5-\lambda & 4 \\ 2 & 7-\lambda \end{vmatrix} = (5-\lambda)(7-\lambda) - 4 \cdot 2$$

$$= (\lambda - 5)(\lambda - 7) - 8 = \lambda^2 - 12\lambda + 35 - 8$$

$$= \lambda^2 - 12\lambda + 27 = (\lambda - 3)(\lambda - 9)$$

$\Rightarrow \lambda_1 = 3, \ \lambda_2 = 9$. Both eigenvalues are real and >0,

so the critical point is a nodal source and unstable.

↑ Not required to memorize for quiz and exam.

(2) $A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$. Solve eigenvalues of A.

$$0 = \begin{vmatrix} 2-\lambda & -3 \\ 3 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - (-3)\cdot 3$$

$$= (\lambda-2)^2 + 9 = \lambda^2 - 4\lambda + 13$$

$$\Rightarrow \lambda = \frac{4 \pm \sqrt{4^2 - 4\cdot 13}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i.$$

Both are complex, and real part is $2 > 0$.
So the critical point is a spiral source and thus unstable.

Not required to memorize

(3) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(4) $\begin{cases} \frac{dx}{dt} = -2x + 2y \\ \frac{dy}{dt} = -4x + 2y \end{cases}$

Solutions: (3) $A = \begin{pmatrix} -3 & 4 \\ 0 & 2 \end{pmatrix}$. Solve eigenvalues to get

$$0 = \begin{vmatrix} -3-\lambda & 4 \\ 0 & +2-\lambda \end{vmatrix} = (\lambda+3)(\lambda-2) \Rightarrow \lambda_1 = -3,\ \lambda_2 = 2$$

$\Rightarrow$ Critical point is a saddle point and thus unstable.

Not required to memorize

(4) $A = \begin{pmatrix} -2 & 2 \\ -4 & 2 \end{pmatrix}$. Solve eigenvalues

$$0 = \begin{vmatrix} -2-\lambda & 2 \\ -4 & 2-\lambda \end{vmatrix} = (-2-\lambda)(2-\lambda) - 2\cdot(-4) = \lambda^2 + 4$$

$\Rightarrow \lambda = \pm 2i \Rightarrow$ Critical point is center and thus stable.

Not required to memorize.

(Not required)

Remark: Explanations on why we only need to consider $(*)$ and can use it as an approximation for the general system

$$\begin{cases} \dfrac{dx}{dt} = f(x, y) \\ \dfrac{dy}{dt} = g(x, y) \end{cases} \quad (\Delta)$$

(1) First we can always assume $(0,0)$ is the critical point we are interested. If not, for critical point (x_0, y_0), consider substitution $\hat{x} = x - x_0$, $\hat{y} = y - y_0$, then (Δ) becomes

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{d\hat{x}}{dt} = f(\hat{x} + x_0, \hat{y} + y_0) \\ \dfrac{dy}{dt} = \dfrac{d\hat{y}}{dt} = g(\hat{x} + x_0, \hat{y} + y_0) \end{cases} \quad (\spadesuit)$$

Now $(0,0)$ is a critical point for $\spadesuit$.

(2) Suppose $(0,0)$ is a critical point, which meas $f(0,0) = g(0,0) = 0$. Then using Taylor's expansion to approximate $f(x,y)$, $g(x,y)$:

$$f(x,y) \approx \underbrace{f(0,0)}_{= 0} + \underbrace{\left(\frac{\partial f}{\partial x}(0,0)\right)}_{= a_{11}} x + \underbrace{\left(\frac{\partial f}{\partial y}(0,0)\right)}_{= a_{12}} y$$

$$g(x,y) \approx \underbrace{g(0,0)}_{= 0} + \underbrace{\left(\frac{\partial g}{\partial x}(0,0)\right)}_{= a_{21}} x + \underbrace{\left(\frac{\partial g}{\partial y}(0,0)\right)}_{= a_{22}} y$$

Let a_{ij} be the corresponding partial derivatives at $(0,0)$. then we can approximate (Δ) with a new system:

$$\begin{cases} \dfrac{dx}{dt} = a_{11} x + a_{12} \\ \dfrac{dy}{dt} = a_{21} x + a_{22} \end{cases} \quad (*).$$

This usually approximates the original system very well if we only consider the behavior near the critical point $(0,0)$.