Aug 22, 2019

0. Introduction of the course (Math 231 - Differential Equations)

+ 1. Background
   - Welcome
   - Why differential equations?

It appears ubiquitously in the mathematical models developed to aid in the understanding of physical phenomena.

- Example : Radioactive decay

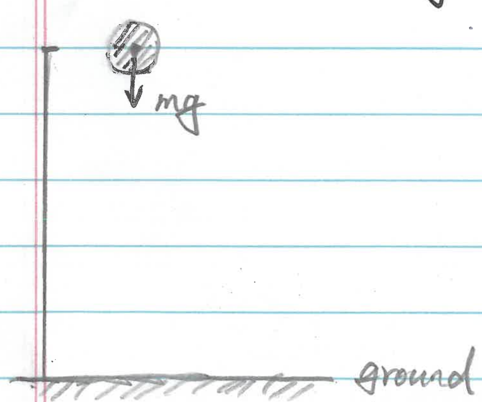
Assume that the rate of decay[1] of a substance is proportional[2] to the amount of substance present[3].

$$\frac{dA}{dt} = -kA, \quad (1)$$

where $A(t)$ is the unknown amount of radioactive substance at time $t$, $k$ is a positive constant.

Equation (1) involves the first derivative of $A$, so it is a DE.

- Example : Falling body




An object of mass $m$ falls under the action of gravity.

$-mg$ (negative sign due to direction)

Newton's second law : $F = ma$,

$a$ (acceleration) = rate of change of speed

$= \cancel{\frac{dx}{dt}}$ derivative of speed $= \frac{dv}{dt}$

Speed = derivative of position (height) $= \frac{dh}{dt}$

So : $a = \frac{dv}{dt} = \frac{d^2h}{dt^2}$, and thus $F = ma$ implies

$$-mg = m\frac{d^2h}{dt^2} \Rightarrow \frac{d^2h}{dt^2} = -g \quad (2)$$

Equation (2) involves the second derivative of $h$, so it is also a DE.

• Going through the syllabus + Questions

• What is a differential equation and what does it mean to solve one?

(a) The definition of a differential equation (DE) is that it's an equation involving some or all of the following: An unknown function of one or more variables such as $y(t)$, derivatives of that function such as $y'$, $y''$, and so on, and other functions of the same variable(s) such as $\sin(t)$ and $t^2$.

DE must involve this, other terms mentioned here are optional.

Examples:

$$\frac{dA}{dt} = -kA \quad ; \quad A \text{ is unknown function of } t$$

$$\frac{d^2h}{dt^2} = -g \quad ; \quad h \text{ is unknown function of } t$$

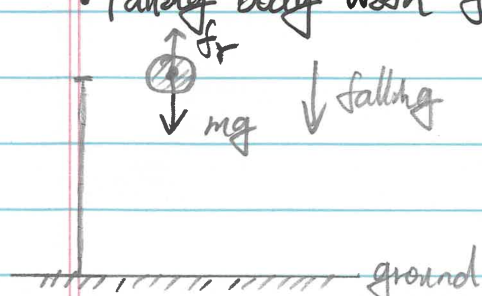
$$y'' + 3y' - ty = 6 \quad ; \quad y \text{ is unknown function of } t$$

$$2019\frac{dy}{dx} - x\frac{d^2y}{dx^2} = xy\sin(y) \quad ; \quad y \text{ is unknown function of } x$$

$$\partial_x u + \sin(x)\,\partial_y u = y^3\,\partial^2_{xy} u \quad ; \quad u \text{ is unknown function of } x, y$$

Examples from physical phenomena:

• Falling body with friction




compared to ~~gravity force~~ velocity

$f_r$ : friction, opposite direction, assumed to be proportional to the velocity here

$f_r = -kv$ (b/c $v$ is downward, then $f_r$ is upward)

From $F = ma$, we get

$$-k\frac{dh}{dt} - mg = -kv - mg = m\frac{d^2h}{dt^2} \Longrightarrow \frac{d^2h}{dt^2} = -g - \frac{k}{m}\frac{dh}{dt} \quad (3)$$

In (3), $h$ is unknown function of $t$.

- Vibration of a string


Vertical displacement at $x$ is $u(x)$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad (4)$$

$t$ is time

In (4), $u$ is unknown function of $t$ and $x$.

- Heat equation : let $u$ be the temperature at position $(x, y, z)$ in a room, then :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \quad (5)$$

In (5), $u$ is unknown function of $t, x, y, z$.

- Dependent variables and independent variables :
If an equation involves one variable w.r.t. another variable, then the former is <u>dependent</u>, the latter is <u>independent</u>.

(6) Solving a DE means finding a function which makes the DE true when you plug that in

<u>Examples</u> : $f(t) = e^t$ is a solution to the DE $f(t) - f'(t) = 0$.
$f(t) = t + e^{2t}$ is a solution to the DE $f''(t) + 4t = 4f(t)$
$f(x) = x^2$ is not a solution to the DE $x f'(x) = f(x)$

Just as regular equations can have more than one solution ($x^2 - 9 = 0$ has two solutions) so can a DE. In fact DE will have infinitely many solutions.

$f(t) = 231e^t$ is a solution to $f(t) - f'(t)$ ; in fact $f(t) = Ce^t$

with any constant $C$ is a solution to that DE.

• Associated definitions:

(a) A DE is called ordinary (so an ODE) if the unknown function is just a function of one variable, i.e. there is only one independent variable.

Otherwise it's partial (so a PDE). In this course we focus on ODE, so when we say DE we usually mean ODE.

coefficients

Examples: $\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0$ ODE; $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = x - 2y$ PDE.

(b) The order of a DE is the highest derivative that appears in it. We say things like first-order, second-order and so on.

Examples: $x^7 f'(x) + \cos(x) f(x) + x = e^x$ ← first-order

$t f(t) + e^t f''(t) = 1 - f'(t)$ ← second-order

(c) A DE is linear if it can be written as a sum of some or all of:

(i) An unknown function $y$ multiplied by a coefficient.

(ii) Derivatives of the unknown $y$ multiplied by coefficients

(iii) Coefficients: by coefficients we mean they can be other functions of the independent variables, including just constants and $0$.

So a linear DE can always be rewritten as:

$$a_n(x) y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \cdots + a_1(x) y'(x) + a_0(x) y = f(x)$$

Otherwise, a DE is nonlinear.

Examples :

$5t\, f(t) + \ln(t) f'(t) = 5$ : linear

$(\tan t)\, y(t) - t^3 y'(t) + 7y''(t) = 1$ : linear

$f(x)\sqrt{x} + (1-x) f'''(x) = f'(x)$ : linear

$f(t)^2 + f'(t) = 7$ : nonlinear b/c $f(t)^2$ is not permitted

$\sin(y') + y' - y = x$ : nonlinear b/c $\sin(y')$ is not permitted

$y'y + y = xy$ : nonlinear b/c $y'y$ is not permitted

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(d) A system of DEs is just a collection of more than one DE.

Examples : $\begin{cases} \dfrac{dx}{dt} = x - 2y \\ \dfrac{dy}{dt} = 2y - x \end{cases}$ ; $\begin{cases} \dfrac{du}{dt} = v \\ \dfrac{dv}{dt} = u - u^3 \end{cases}$

1.2 Solutions and initial value problems (IVP)

• Explicit solution : suppose our dependent variable is $y$ and independent variable is $x$. Then $\phi(x)$ is an explicit solution if substituting $y = \phi(x)$ makes the DE true for $x$ in some interval.

Ex 1 $\phi(x) = x^2 - x^{-1}$ is an explicit solution of $x^2 y'' = 2y$ on interval $(0, \infty)$.

Check : $\phi(x) = x^2 - x^{-1}$ defined on $(0,\infty)$

$\phi'(x) = 2x + \dfrac{1}{x^2}$ on $(0,\infty)$

$\phi''(x) = 2 - \dfrac{2}{x^3}$ on $(0,\infty)$

So on $(0,\infty)$, we have

$$\underbrace{x^2\phi''(x)}_{= x^2 y''} = x^2\left(2 - \frac{2}{x^3}\right) = 2x^2 - \frac{2}{x} = 2(x^2 - x^{-1}) = \underbrace{2\phi(x)}_{= 2y}$$

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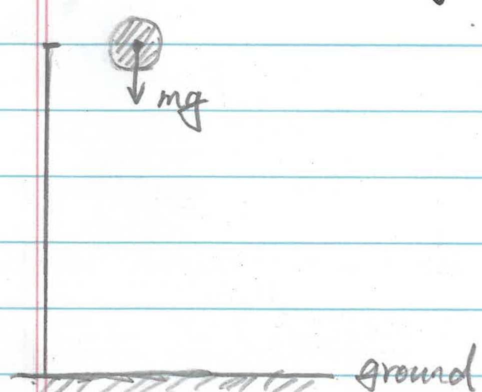


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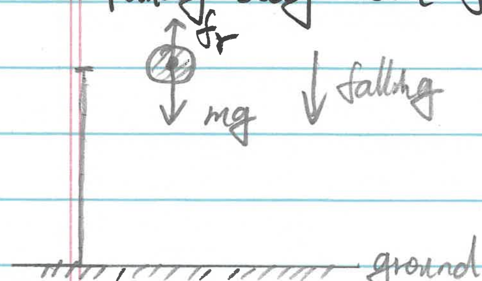


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(Remark: We say either $y = \phi(x)$ is an explicit solution or $\phi(x)$ is an explicit solution)

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Ex 1 $\phi(x) = c_1 \sin(x) + c_2 \cos(x)$ is a solution to $y'' + y = 0$ for any choice of $c_1, c_2$.

Check: $\phi'(x) = c_1 \cos(x) - c_2 \sin(x)$, $\phi''(x) = -c_1 \sin(x) - c_2 \cos(x)$

So $y'' + y = c_1 \sin(x) + c_2 \cos(x) + (-c_1 \sin(x) - c_2 \cos(x))$

$$= (c_1 - c_1)\sin(x) + (c_2 - c_2)\cos(x) = 0.$$

Discuss this part after IVP

• Implicit solution: suppose our dependent variable is $y$ and independent variable is $x$. Then a relation $\boxed{G(x,y)=0}$ is an implicit solution if it defines one or more explicit solution on $I$.

Q: How can tell for example a relation

$$G(x,y) = x + y + e^{xy} = 0 \qquad (6)$$

defines an explicit solution or not?

Ans: Using Implicit ~~Fun~~ Function Theorem!

(Review)

Exercise: Use implicit function theorem to find out $\frac{dy}{dx}$ for $0 = G(x,y) = x + y + e^{xy}$ at $y(0) = -1$.

Solution: Just take derivative $\frac{d}{dx}$ for $G(x,y) = 0$:

$$\frac{d}{dx}(x + y + e^{xy}) = \frac{d}{dx}(0) = 0$$

We use Product rule for $xy$ here

Since $\frac{d}{dx}(x+y+e^{xy}) = 1 + \frac{dy}{dx} + e^{xy}\left(\left(\frac{d}{dx}(x)\right)y + x\frac{dy}{dx}\right)$

$$= 1 + \frac{dy}{dx} + e^{xy}\left(y + x\frac{dy}{dx}\right)$$

$$= (1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0,$$

So at $y(0) = -1$ ( $x=0$, $y=-1$ ) we have

$$\left(1 + 0 \cdot e^{0\cdot(-1)}\right)\frac{dy}{dx} + 1 + (-1)e^{0\cdot(-1)} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} + 1 + (-1) = 0 \quad \Rightarrow \quad \frac{dy}{dx} = 0 \text{ at } (0,-1).$$

**Ex** Show that $x + y + e^{xy} = 0$ is an implicit solution to the equation $\left(1 + x e^{xy}\right)\frac{dy}{dx} + 1 + y e^{xy} = 0$.

Check: like the exercise above, take derivative $\frac{d}{dx}$ on both sides of the equation $x + y + e^{xy} = 0$ and get

$$\frac{d}{dx}\left(x + y + e^{xy}\right) = 0$$

$$\Rightarrow \quad 1 + \frac{dy}{dx} + e^{xy}\left(y + x\frac{dy}{dx}\right) = 0$$

Rearrange the terms and get

$$\left(1 + x e^{xy}\right)\frac{dy}{dx} + 1 + y e^{xy} = 0$$

Therefore $x + y + e^{xy} = 0$ is an implicit solution to the given ODE.

Remark: You may wonder why we want to talk about implicit solutions. In fact, it will naturally arrive in sec 2.2 (Separable equations), and sometimes we cannot write it as an explicit solution.

like in the example above, we do not know how to write an expression $y = \phi(x)$ from the relation $x + y + e^{xy} = 0$.

Both explicit solutions and implicit solutions are called solutions.

- Initial Value (IV), Initial Value Problem (IVP)

We often solve ODEs together with IVs. For an ODE with independent variable $x$ and dependent variable $y$, if the ODE is first-order, then IV is like

$y(x_0) = y_0$ ( $x_0, y_0$ are given, for example $y(0)=7$ );

if the ODE is second-order, then IV is like

$$\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y_1 \end{aligned}$$ ( $x_0, y_0, y_1$ are give, for example $\begin{aligned} y(1)&=2 \\ y'(1)&=-1 \end{aligned}$ );

and similarly for higher oder.

The differential equation (DE) and the IV together form an IVP.

- Example: $(7)\begin{cases} \frac{dy}{dx} = 3y \\ y(0) = 7 \end{cases}$ is an IVP (first-order)

$(8)\begin{cases} y''(x) - y'(x) - 2y(x) = 0 \\ y(0) = 2, \quad y'(0) = -3 \end{cases}$ is an IVP (second-order).

- More on the examples above

Exercise: One can check that $y = Ce^{3x}$ is a solution to $\frac{dy}{dx} = 3y$ for any $C$. Find the value of $C$ that makes the IV in (7) also satisfied.

Ans: $y(x) = Ce^{3x}$, plug this into the IV $y(0) = 7$, we get $7 = Ce^{3\cdot 0} \Rightarrow 7 = C$ so $C = 7$.

- General solution: a general solution to a DE is a solution involving constants and for which different constants will give all solutions.

Comment: usually for an ODE with order $n$, there are $n$ different constants in its general solution.

<u>Examples</u>: $y = ce^{3x}$ is a general solution to $\frac{dy}{dx} = 3y$;

$\phi(x) = c_1 \sin(x) + c_2 \cos(x)$ is a general solution to $y'' + y = 0$.

• Specific (Particular) Solution: a specific solution or a particular solution is a solution in which a specific choice of constant(s) has been made.

<u>Example</u>. $y = 7e^{3x}$ is a specific solution to $\frac{dy}{dx} = 3y$.

• Solving an IVP: to solve an IVP, we <u>first solve the DE to get the general solution</u> and then use the IV to get the specific solution.

We will learn this later, just take the general solution for granted now.

<u>Example</u>: Recall the example on top of page 6, we know $\phi(x) = c_1 \sin(x) + c_2 \cos(x)$ is a general solution to $y'' + y = 0$.

Solve the IVP $\begin{cases} y'' + y = 0 \\ y(\frac{\pi}{4}) = 2, \quad y'(\frac{\pi}{4}) = 0 \end{cases}$

solution: $\phi(x) = c_1 \sin(x) + c_2 \cos(x)$, $\phi'(x) = c_1 \cos(x) - c_2 \sin(x)$.

Plugging this into the IVP we get

$$\begin{cases} c_1 \sin(\frac{\pi}{4}) + c_2 \cos(\frac{\pi}{4}) = 2 \\ c_1 \cos(\frac{\pi}{4}) - c_2 \sin(\frac{\pi}{4}) = 0 \end{cases} \Rightarrow \begin{cases} \frac{\sqrt{2}}{2} c_1 + \frac{\sqrt{2}}{2} c_2 = 2 \\ \frac{\sqrt{2}}{2} c_1 - \frac{\sqrt{2}}{2} c_2 = 0 \end{cases}$$

Solving equations gives us $c_1 = c_2 = \sqrt{2}$. Therefore the particular solution of the IVP is $\phi(x) = \sqrt{2} \sin(x) + \sqrt{2} \cos(x)$.

- Theorem on Existence and Uniqueness

For an IVP (first-order) of the following form

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0,$$

if both functions $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are continuous around $(x_0, y_0)$ then the problem has a unique solution $y(x)$ in some interval around $x_0$.

Rigorously, our textbook says there is a rectangle $R$ s.t. $f$ and $\frac{\partial f}{\partial y}$ are continuous in $R$.

Never mind if you can't understand the theorem, let's look at examples.

Ex 1 $\quad y' - ty = \sin^2 t, \quad y(\pi) = 5$

To use the theorem, first write the ODE into the form $\frac{dy}{dx} = f(x, y)$.

In this example: (1) $\quad y' = \underbrace{\sin^2 t + ty}_{f(t, y)}$

Notice here $t$ is our independent variable, not $x$.

Second step is to take partial derivative w.r.t. $y$ (dependent variable):

(2) $\quad \frac{\partial f}{\partial y} = t$

Last step, from IV, find the point to plug into $f$ and $\frac{\partial f}{\partial y}$, if the answer is well-defined and not $\infty$, then the theorem can be applied and there exists a unique solution. Otherwise we are not sure whether the solution exists or is unique.

(3) From IV, $y(\pi) = 5$, so plug $(\pi, 5)$ into $f$ and $\frac{\partial f}{\partial y}$, both are finite numbers. So we can apply the theorem to conclude there is a unique solution.

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**Ex 1** $\quad \dfrac{dy}{dx} = 3y^{2/3}, \quad y(2) = 0$

Now repeat the procedure :

① $\dfrac{dy}{dx} = 3y^{2/3} := f(x,y)$, we are lucky we don't need to do anything

② $\dfrac{\partial f}{\partial y} = 3 \cdot \frac{2}{3} y^{2/3 - 1} = 2y^{-\frac{1}{3}}$

③ From the IV $y(2) = 0$, we need to plug $(x,y) = (2,0)$ into $f$ and $\dfrac{\partial f}{\partial y}$. For $f(x,y)$, we get $f(2,0) = 3 \cdot 0^{2/3} = 0$, which is a finite number ; however for $\dfrac{\partial f}{\partial y}$,

$$\frac{\partial f}{\partial y}(2,0) = 2 \cdot 0^{-\frac{1}{3}} \quad \text{is not defined}$$

Since $-\frac{1}{3}$ is negative, this term is like $\frac{1}{0}$, so it is not well-defined.

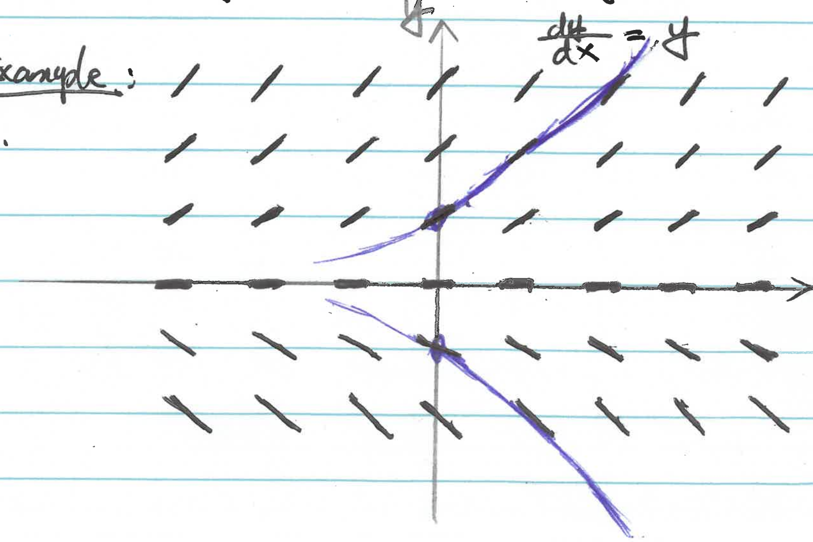
Therefore we cannot apply the theorem .

You will see in the homework problem 1.2 : 29 that the solution to the IVP is not unique in this example .

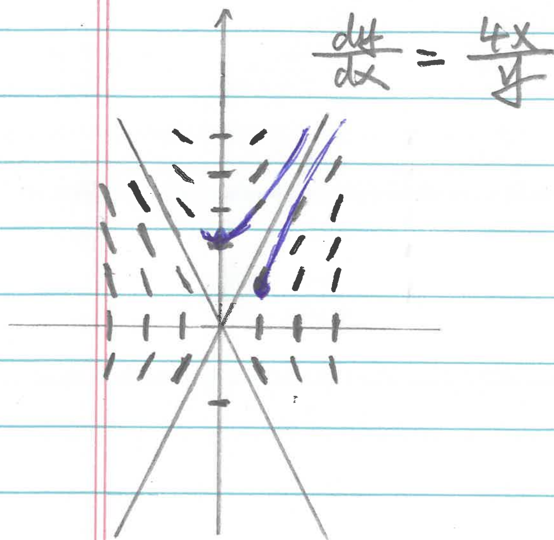
## 1.3 Direction Fields

$\dfrac{dy}{dx} = f(x,y)$ is telling us the slope of the solution at every point. Plotting this information on lots of points gives us a direction field or slope field .

Example :




Given the field, we can trace the solution which follows the field and draw conclusions .




Example : $\frac{dy}{dx} = \frac{4x}{y}$

Sketch the solution curve for $y(0) = 2$ and $y(2) = 1$.

Observe as $x \to +\infty$, $y \to +\infty$.