MATH 231. FINAL EXAM

Name:____

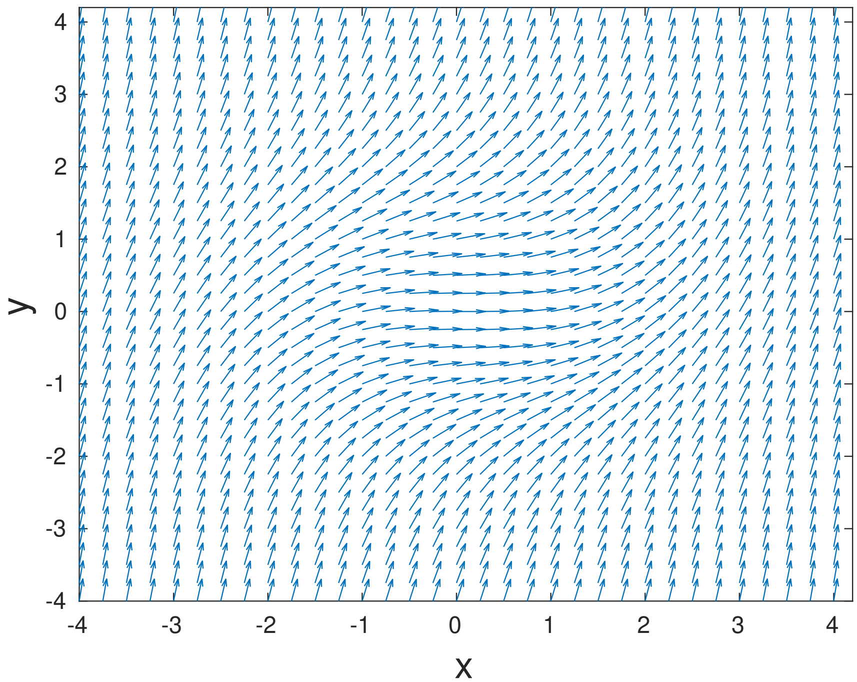
Instructions: This is a closed-book exam, and calculators can only be used to do basic arithmetic operations (not allowed for differentiation and integration). The last page contains a table for integrals and some results from textbook, which might be helpful. Read each problem carefully. You must show your work to receive credit. Partial credit will be given for any work relevant to the problem.

Problem	**Grade**
1	
2	
3	
4	
5	
6	
7	
8	
Total	

There is a total of 150 points.

Problem 1 (15 points): (a) (6 points) Trace the solution to the following IVP on the direction field.

$$\frac{dy}{dx} = \frac{x^2 + y^2}{4}, \quad y(-1) = 0.$$



(b) (9 points) Find all critical points for the system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x - 2y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y^2 - x^2 + 3. \end{cases}$$

Problem 2 (10 points): A 20L tank initially contains 5L salt water at a concentration of 1 g/L. Salt water at a concentration of 2 g/L is being pumped in at 6 L/min and the mixture is leaving the tank at 3 L/min. Write down **BUT DO NOT SOLVE** the **initial value problem** whose solution gives the amount of salt at time t.

Problem 3 (20 points): Find the recurrence relation and the first three nonzero terms in a power series expansion about $x = 0$ for the solution to the following IVP:

$$y'' - (x^2 - 2)y = 0 \quad \text{with} \quad y(0) = 4,\ y'(0) = 0.$$

Problem 4 (15 points): Solve the following ODE:

$$\frac{dy}{dx} = (x-2)(y+1)^3.$$

Problem 5 (20 points): Solve the IVP:

$$y'' + 9y = f(t) \quad \text{with} \quad y(0) = 0,\ y'(0) = -1.$$

where

$$f(t) = \begin{cases} 0, & 0 < t < 3, \\ 3, & 3 < t < 4, \\ 0, & t > 4. \end{cases}$$

Problem 6 (25 points): Consider the following equation:

$$2xy\,dx + (y^2 - 2x^2)\,dy = 0, \tag{1}$$

and let $M(x, y) = 2xy$, $N(x, y) = y^2 - 2x^2$. Show $(\partial N/\partial x - \partial M/\partial y)/M$ depends only on y (doesn't depend on x) and thus use the Theorem for Special Integrating Factors on the last page to find an integrating factor $\mu(y)$. Then use this integrating factor to solve the equation (1).

Problem 7 (20 points): Use the elimination method to solve the system with initial values:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -3x - 5y, & x(0) = 2, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x + y, & y(0) = 4. \end{cases}$$

Problem 8 (25 points): Find the general solution of

$$y'' - 2y' + 2y = \frac{e^t}{\sin(t)}.$$

Brief Table for Integrals:

$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln|x+\sqrt{x^2+a^2}|.$ $\quad$ $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x+\sqrt{x^2-a^2}|, \quad x^2 \geq a^2.$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin(\frac{x}{a}), \quad a^2 \geq x^2.$ $\quad$ $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan(\frac{x}{a}).$

$\int \tan(x)dx = -\ln|\cos x|.$ $\quad$ $\int \cot(x)dx = \ln|\sin x|.$

$\int \sec(x)dx = \ln|\sec x + \tan x|.$ $\quad$ $\int \csc(x)dx = -\ln|\csc x + \cot x|.$

Common trigonometric substitutions:

(1) For integrand involving $\sqrt{a^2-x^2}$, set $x = a\sin(\theta)$,
(2) For integrand involving $\sqrt{a^2+x^2}$, set $x = a\tan(\theta)$,
(3) For integrand involving $\sqrt{x^2-a^2}$, set $x = a\sec(\theta)$,
(4) For $\int \tan^n(x)\sec^{2m}(x)dx$, set $u = \tan(x)$,
(5) For $\int \cot^n(x)\csc^{2m}(x)dx$, set $u = \cot(x)$.

Brief Table of Laplace Transforms

$f(t)$	$\mathscr{L}[f(t)] = F(s)$	$f(t)$	$\mathscr{L}[f(t)] = F(s)$
1	$\frac{1}{s}$	e^{at}	$\frac{1}{s-a}$
$t^n,\ n=1,2,\ldots$	$\frac{n!}{s^{n+1}}$	$e^{at}t^n,\ n=1,2,\ldots$	$\frac{n!}{(s-a)^{n+1}}$
$\sin bt$	$\frac{b}{s^2+b^2}$	$\cos bt$	$\frac{s}{s^2+b^2}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$

Properties of Laplace Transforms

$$\mathscr{L}[f+g] = \mathscr{L}[f] + \mathscr{L}[g]$$

$$\mathscr{L}[cf] = c\mathscr{L}[f] \qquad \text{for any constant } c$$

$$\mathscr{L}\left[e^{at}f(t)\right](s) = \mathscr{L}[f](s-a)$$

$$\mathscr{L}[f'](s) = s\mathscr{L}[f](s) - f(0)$$

$$\mathscr{L}[f''](s) = s^2\mathscr{L}[f](s) - sf(0) - f'(0)$$

$$\mathscr{L}[t^n f(t)](s) = (-1)^n \frac{d^n}{ds^n}\left(\mathscr{L}[f](s)\right)$$

$$\mathscr{L}[f(t-a)u(t-a)](s) = e^{-as}F(s), \text{ where } F(s) = \mathscr{L}[f(t)]$$

$$\mathscr{L}^{-1}\left[e^{-as}F(s)\right] = f(t-a)u(t-a), \text{ where } f(t) = \mathscr{L}^{-1}[F(s)]$$

$$\mathscr{L}[g(t)u(t-a)](s) = e^{-as}\mathscr{L}[g(t+a)](s)$$

Theorem for Special Integrating Factors. For equation

$$M(x, y)dx + N(x, y)dy = 0, \tag{2}$$

if $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x, then

$$\mu(x) = \exp\left[\int \left(\frac{\partial M/\partial y - \partial N/\partial x}{N}\right) dx\right]$$

is an integrating factor for equation (2). If $(\partial N/\partial x - \partial M/\partial y)/M$ is continuous and depends only on y, then

$$\mu(y) = \exp\left[\int \left(\frac{\partial N/\partial x - \partial M/\partial y}{M}\right) dy\right]$$

is an integrating factor for equation (2).

Variation of Parameters Formula. Consider a 2nd order ODE

$$y'' + a_1(t)y' + a_0(t)y = f(t),$$

and assume its homogeneous version has a fundamental pair $\{Y_1(t), Y_2(t)\}$. Then a particular solution $Y_p(t)$ to the nonhomogeneous ODE is given by:

$$Y_p(t) = Y_1 \int \frac{-Y_2(t)f(t)}{W[Y_1, Y_2]}dt + Y_2 \int \frac{Y_1(t)f(t)}{W[Y_1, Y_2]}dt.$$