MATH 231. EXAM 1

Name:____

Instructions: This is a closed-book exam, and calculators can only be used to do basic arithmetic operations (not allowed for differentiation and integration). The last page contains a table for integrals and some results from textbook, which might be helpful. Read each problem carefully. You must show your work to receive credit. Partial credit will be given for any work relevant to the problem.

Problem	**Grade**
1	
2	
3	
4	
5	
Total	

There is a total of 100 points.

Problem 1 (15 points): Are the following statements true or false? For each one of them provide a short explanation.

(a) (5 points) $x = \cos(2t) - 2\sin(2t)$ is a solution to the differential equation $x'' + 4x = 0$.

(b) (5 points) The equation

$$(xy^3 - 4y^3)\frac{dy}{dx} - 2x^2 = 0$$

is separable.

(c) (5 points) The equation

$$(xe^{xy} - 2y)dy + (ye^{xy} + 2x)dx = 0$$

is exact.

Problem 2 (20 points): Solve the following initial value problem:

$$\frac{y}{x}(y^2 - x^2) - (x^2 + y^2)\frac{dy}{dx} = 0, \qquad y(e) = e.$$

Problem 3 (20 points): Show that $\mu(x, y) = xe^y$ is an integrating factor for the following equation and then solve it.

$$(3x + 2y)\, dx + \left(x^2 + xy + x + \frac{2y}{x} e^{-y} \right) dy = 0.$$

Problem 4 (20 points): Solve the following ODE:

$$(y^2 - y)e^x \, \frac{dy}{dx} = y(y+1)^3.$$

Problem 5 (25 points): A 20 L tank initially is full of salt water at a concentration of 1 g/L. Salt water at a concentration of 6 g/L is being pumped in at 2 L/min and the mixture is leaving the tank at 4 L/min. Compute the amount of salt in the tank when the tank is exactly half full.

Brief Table for Integrals:

$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln|x + \sqrt{x^2+a^2}|$.

$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x + \sqrt{x^2-a^2}|, \quad x^2 \geq a^2$.

$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin(\frac{x}{a}), \quad a^2 \geq x^2$.

$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan(\frac{x}{a})$.

Common trigonometric substitutions:

(1) For integrand involving $\sqrt{a^2-x^2}$, set $x = a\sin(\theta)$,
(2) For integrand involving $\sqrt{a^2+x^2}$, set $x = a\tan(\theta)$,
(3) For integrand involving $\sqrt{x^2-a^2}$, set $x = a\sec(\theta)$,
(4) For $\int \tan^n(x)\sec^{2m}(x)dx$, set $u = \tan(x)$,
(5) For $\int \cot^n(x)\csc^{2m}(x)dx$, set $u = \cot(x)$.

Theorem for Special Integrating Factors. For equation

$$M(x,y)dx + N(x,y)dy = 0, \tag{1}$$

if $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x, then

$$\mu(x) = \exp\left[\int \left(\frac{\partial M/\partial y - \partial N/\partial x}{N}\right) dx\right]$$

is an integrating factor for equation (1). If $(\partial N/\partial x - \partial M/\partial y)/M$ is continuous and depends only on y, then

$$\mu(y) = \exp\left[\int \left(\frac{\partial N/\partial x - \partial M/\partial y}{M}\right) dy\right]$$

is an integrating factor for equation (1).