# High-Order Differential Equations

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# 1 Generalized Jacobi Polynomials(GJPs)

- Intuition

$$\phi_k(x) = L_k(x) - L_{k+2}(x) = \frac{2k+3}{2k+2}(1-x^2)J_k^{1,1}(x).$$

$\{\phi_k\}$ are orthogonal w.r.t. $\omega^{-1,-1}(x)$

- Definition.

$$J_n^{k,l}(x) = \begin{cases} \omega^{-k,-l}(x)\, J_{n-n_0}^{-k,-l}(x), & \text{if } k,l \leqslant -1, \\ \omega^{-k,0}(x)\, J_{n-n_0}^{-k,l}(x), & \text{if } k \leqslant -1, l > -1, \\ \omega^{0,-l}(x)\, J_{n-n_0}^{k,-l}(x), & \text{if } k > -1, l \leqslant -1, \end{cases} \tag{1}$$

where $n \geqslant n_0$ with $n_0 := -(k+l), -k, -l$ for the three cases

- Simplified Notation

$$\hat{k} := \begin{cases} -k, & k \leqslant -1, \\ 0, & k > -1, \end{cases} \qquad \bar{k} := \begin{cases} -k, & k \leqslant -1, \\ k, & k > -1. \end{cases} \tag{2}$$

then

$$J_n^{k,l}(x) = \omega^{\hat{k},\hat{l}}(x) J_{n-n_0}^{\bar{k},\bar{l}}(x), \quad n \geqslant n_0 = \hat{k} + \hat{l}, \quad k,l \in \mathbb{Z}. \tag{3}$$

- Properties $(k,l \in \mathbb{Z})$
  - $\left\{J_n^{k,l}\right\}$ form a complete orthogonal system in $L^2_{\omega^{k,l}}(I)$.

$$\int_{-1}^{1} J_n^{k,l}(x) J_m^{k,l}(x) \omega^{k,l}(x)\, \mathrm{d}x = \gamma_{n-n_0}^{\bar{k},\bar{l}} \delta_{mn}. \tag{4}$$

  - Compact combination of Legendre polynomials $k,l \geqslant 1$.

$$J_n^{-k,-l}(x) = \sum_{j=n-k-l}^{n} a_j L_j(x), \quad n \geqslant k+l. \tag{5}$$

- Boundary conditions

$$\partial_x^i J_n^{-k,-l}(1) = 0, \quad i = 0, 1, ..., k-1. \quad k \geqslant 1.$$
$$\partial_x^i J_n^{-k,-l}(-1) = 0, \quad i = 0, 1, ..., l-1. \quad l \geqslant 1. \tag{6}$$

- Differentiation

$$\partial_x J_n^{k,l}(x) = C_n^k \, J_{n-1}^{k+1,l+1}(x), \tag{7}$$

$$C_n^k = \begin{cases} -2(n+k+l+1), & \text{if } k, l \leqslant -1, \\ -n, & \text{if } k \leqslant -1, l > -1, \\ -n, & \text{if } k > -1, l \leqslant -1, \\ (n+k+l+1)/2, & \text{if } k, l > -1. \end{cases} \tag{8}$$

# 2 Galerkin Methods for Even-Order Equations

## 2.1 Fourth-Order Equations

$$
\begin{aligned}
&u^{(4)} - \alpha u'' + \beta u = f, \quad x \in I = (-1, 1), \\
&u(\pm 1) = u'(\pm 1) = 0. \quad \alpha \geqslant 0, \beta > 0.
\end{aligned} \tag{9}
$$

Weak form:

$$
\begin{cases}
\text{Find } u \in H_0^2(I) \; s.t. \text{ for any } \; v \in H_0^2(I) \\
a(u, v) := (u'', v'') + \alpha(u', v') + \beta(u, v) = (f, v).
\end{cases} \tag{10}
$$

Legendre–Galerkin approximation ($I_N$: interpolation use LGL pts)

$$
\begin{cases}
\text{Find } u_N \in V_N := P_N \cap H_0^2(I) \; s.t. \\
a(u_N, v_N) = (I_N f, v_N), \quad \forall v_N \in V_N.
\end{cases} \tag{11}
$$

## Choise of basis

$$V_N = \text{span}\{\, J_k^{-2,-2} : k = 4, 5, \ldots, N \}.$$
$$\phi_k = \gamma_k \, J_{k+4}^{-2,-2}(x) \quad \text{choose } \gamma_k \text{ such that } (\phi_k'', \phi_k'') = 1.$$

Thanks to

$$\phi_k = d_k \Bigg( L_k(x) - \frac{2(2k+5)}{2k+7} L_{k+2}(x) + \frac{2k+3}{2k+7} L_{k+4}(x) \Bigg), \tag{12}$$

We have

$$\begin{aligned} a_{kj} &:= (\phi_j'', \phi_k'') = (\phi_j'''', \phi_k) = (\phi_j, \phi_k'''') = \delta_{kj}; \\ b_{kj} &:= (\phi_j, \phi_k) = 0, \quad \text{if } k \neq j, j \pm 2, j \pm 4; \\ c_{kj} &:= (\phi_j', \phi_k') = 0, \quad \text{if } k \neq j, j \pm 2. \end{aligned} \tag{13}$$

Sparse linear algebraic system (also well-conditioned)

$$(I + \beta B + \alpha\, C)\boldsymbol{u} = \boldsymbol{f}. \tag{14}$$

## 2.2 General Even-Order Equations

$$b_0\partial_x^{2m}u(x) + \sum_{k=0}^{2m-1} b_{2m-k}\partial_x^k u(x) = f(x), \quad x\in(-1,1), m\geqslant 1$$
$$\partial_x^k u(\pm 1) = 0, \qquad 0\leqslant k\leqslant m-1. \tag{15}$$

Bilinear form

$$\begin{aligned} a_m(u,v) \;:=\; & (-1)^m(\partial_x^m u, \partial_x^m(b_0 v)) + (-1)^m\big(\partial_x^{m-1}u, \partial_x^m(b_1 v)\big) \\ & +(-1)^{m-1}\big(\partial_x^{m-1}u, \partial_x^m(b_2 v)\big) + \cdots + (b_{2m}u, v) \end{aligned}$$

Assume that $\{b_j\}$ are such that

$$\begin{aligned} &|a_m(u,v)| \leqslant C_1\|u\|_m\|v\|_m, \quad \forall u,v\in H_0^m(I), \\ &a_m(u,u)\geqslant C_0\|u\|_m^2, \qquad \forall u\in H_0^m(I). \end{aligned} \tag{16}$$

Weak form

$$\begin{cases} \text{Find } u \in H_0^m(I) \text{ such that} \\ a_m(u,v) = (f,v), \quad \forall v \in H_0^m(I). \end{cases} \tag{17}$$

Legendre–Galerkin approximation ($I_N$ interpolation with LGL pts)

$$\begin{cases} \text{Find } u_N \in V_N := P_N \cap H_0^m(I) \text{ such that} \\ a_m(u_N, v_N) = (I_N f, v_N), \quad \forall v_N \in V_N. \end{cases} \tag{18}$$

Basis

$$V_N = \text{span}\{J_{2m}^{-m,-m}, J_{2m+1}^{-m,-m}, \ldots, J_N^{-m,-m}\},$$

$$\phi_k(x) := c_{k,m} J_{k+2m}^{-m,-m}(x), \qquad (\partial_x^m \phi_k, \partial_x^m \phi_k) = 1.$$

Sparse system (bandwith $2m+1$, $\text{cond}(A) \leqslant C_1/C_0$),

$$A\,\boldsymbol{u} = \boldsymbol{f},$$

where

$$a_{kj} = a_m(\phi_j, \phi_k), \;\; f_k = (I_N f, \phi_k)$$

## 3 Dual-Petrov-Galerkin Methods for Odd-Order Equations

### 3.1 Third-Order Equations

$$\begin{cases} \alpha u - \beta u_x - \gamma u_{xx} + u_{xxx} = f, \quad x \in I = (-1,1), \\ u(\pm 1) = u_x(1) = 0. \end{cases} \tag{19}$$

Let

$$\begin{aligned} V_N &= \{ u \in P_N : u(\pm 1) = u_x(1) = 0 \}, \\ V_N^* &= \{ u \in P_N : u(\pm 1) = u_x(-1) = 0 \}. \end{aligned} \tag{20}$$

The Legendre dual-Petrov-Galerkin approximation

$$\begin{cases} \text{Find } u_N \in V_N \text{ such that} \\ \alpha(u_N, v_N) - \beta(\partial_x u_N, v_N) + \gamma(\partial_x u_N, \partial_x v_N) + (\partial_x u_N, \partial_x^2 v_N) \\ \qquad = (I_N f, v_N), \quad \forall v_N \in V_N^* \end{cases} \tag{21}$$

Equivalent weigthed formulation ($v \in V_N \Longleftrightarrow \omega^{-1,1} v \in V_N^*$)

$$\begin{cases} \text{Find } u_N \in V_N \text{ such that} \\ \alpha(u_N, v)_{\omega^{-1,1}} - \beta(\partial_x u_N, v)_{\omega^{-1,1}} + \gamma(\partial_x u_N, \omega^{1,-1}\partial_x(v\,\omega^{-1,1}))_{\omega^{-1,1}} \\ \quad + (\partial_x u_N, \omega^{1,-1}\partial_x^2(v\,\omega^{-1,1}))_{\omega^{-1,1}} = (I_N f, v)_{\omega^{-1,1}}, \quad \forall v \in V_N. \end{cases} \tag{22}$$

The odd terms in (22) are now coercive !

Basis

$$\phi_k(x) = \gamma_k\, J_{k+3}^{-2,-1}(x), \quad \psi_k(x) = \gamma_k\, J_{k+3}^{-1,-2}(x), \quad (\phi_k', \psi_k'') = 1$$
$$V_N = \text{span}\{\phi_0, \phi_1, ..., \phi_{N-3}\}, \quad V_N^* = \text{span}\{\psi_0, \psi_1, ..., \psi_{N-3}\}.$$

Linear system

$$(\alpha\, M + \beta\, P + \gamma\, Q + S)\boldsymbol{u} = \boldsymbol{f}.$$

Sparsity

$$m_{ij} = 0 \quad |i - j| > 3; \qquad p_{ij} = 0 \quad |i - j| > 2;$$
$$q_{ij} = 0 \quad |i - j| > 1; \qquad s_{ij} = 0 \quad i \neq j.$$

## 3.2 General Odd-Oder Equations

$$\begin{cases} (-1)^{m+1}\partial_x^{2m+1}u(x)+\delta(-1)^m\partial_x^{2m-1}u(x)+\gamma\, u(x)=f(x), \\ \partial_x^k u(\pm 1)=\partial_x^m u(1)=0, \quad 0\leqslant k\leqslant m-1. \end{cases} \tag{23}$$

Define

$$V_N=\{v\in P_N\colon \partial_x^k u(\pm 1)=0,\ 0\leqslant k\leqslant m-1,\ \partial_x^m u(1)=0\},$$
$$V_N^*=\{v\in P_N\colon \partial_x^k u(\pm 1)=0,\ 0\leqslant k\leqslant m-1,\ \partial_x^m u(-1)=0\}.$$

Legendre dual-Petrov–Galerkin approximation

$$\begin{cases} \text{Find}\, u_N\in V_N\, s.t. \text{ for } \forall v_N\in V_N^* \\ -\big(\partial_x^{m+1}u_N,\partial_x^m v_N\big)-\delta\big(\partial_x^m u_N,\partial_x^{m-1}v_N\big)+\gamma(u_N,v_N)=(I_N f,v_N). \end{cases}$$

Sparse system ($\phi_n\colon=d_{m,n}J_{n+2m}^{-m-1,-m}$, $\psi_n\colon=d_{m,n}J_{n+2m}^{-m,-m-1}$)

$$(I+\gamma B+\delta C)\boldsymbol{u}=\boldsymbol{f} \tag{24}$$

## High Odd-Order Equations with Variable Coefficients

Legendre-Galerkin, and dual-Petrov–Galerkin methods can be extended to equations with variable coefficients.

1. Exact inner products replaced by numerical quadruatres
2. Linear algebraic system are dense
3. Iterative solvers are preferred
4. Fast matrix-vector multiplication with computational cost $O(N^2)$ or even $O(N \log N)$ can be built.
5. If the coefficients are bounded from both sides, the condition number of the resulting linear system are bounded, thus the convergence of CG type iteration solver doesn't depend on $N$.

# 4 Collocation Methods

Generalized Gauss–Lobatto quadrature ( $\{x_j\}$ zeros of $J_{N-1}^{r,l}$ )

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx \sum_{j=1}^{N-1} f(x_j)\,\omega_j + \sum_{\nu=0}^{l-1} f^{(\nu)}(-1)\omega_-^{(\nu)} + \sum_{\mu=0}^{r-1} f^{(\mu)}(1)\omega_+^{(\mu)}$$

DOF$=N+r+l-1$, DOP$=2N+r+l-3$.

Corresponding interplating

$$\left(I_N^{(G)} f\right)(x) = \sum_{j=1}^{N-1} f_j h_j(x) + \sum_{\nu=0}^{l-1} f_-^{(\nu)} h_0^{(\nu)}(x) + \sum_{\mu=0}^{r-1} f_+^{(\mu)} h_N^{(\mu)}(x).$$

$$h_j(x) = \frac{J_{N-1}^{r,l}(x)}{\partial_x J_{N-1}^{r,l}(x_j)(x-x_j)} \cdot \frac{(1+x)^l(1-x)^r}{(1+x_j)^l(1-x_j)^r}, \qquad 1 \leqslant j \leqslant N-1.$$

Reference: Huang and Sloan 1992, SINUM

## Collocation method for a fifth-order equation

$$\begin{aligned} & u^{(5)}(x) + a_1(x)u'(x) + a_0(x)u(x) = f(x) \\ & u(\pm 1) = u'(\pm 1) = u''(1) = 0. \end{aligned} \tag{25}$$

Let $X_N = \{\, u \in P_{N+3}, u(\pm 1) = u'(\pm 1) = u''(1) = 0 \,\}$, $\{x_j\}$ be zores of $J^{3,2}_{N-1}$, then the collocation methods is

$$\begin{cases} \text{Find}\, u_N \in X_N\, s.t. \text{ for } 1 \leqslant j \leqslant N-1, \\ u_N^{(5)}(x_j) + a_1(x_j)u_N'(x_j) + a_0(x_j)u_N(x_j) = f(x_j). \end{cases} \tag{26}$$

It is clear that $X_N = \text{span}\{\, h_j : 1 \leqslant j \leqslant N-1\}$, Setting

$$u_N = \sum_{j=1}^{N-1} u_j h_j(x), \quad d_{ij}^{(k)} = h_j^{(k)}(x_i), \quad A_m = \text{diag}(a_m(x_1), ..., a_m(x_{N-1}))$$

we get

$$\big(D^{(5)} + A_1 D^{(1)} + A_0\big)\boldsymbol{u} = \boldsymbol{f} \tag{27}$$

## Properties

1. Evaluation of matrices $D^{(k)} \neq D^k$
2. The matrices $\left\{ D^{(k)} \right\}$ are full with $\mathrm{cond}\left( D^{(k)} \right) \sim N^{2k}$
3. When $N$ is large, the accuracy of the nodes and of the entries of $\left\{ D^{(k)} \right\}$ is subject to severe roundoff errors.

Condition numbers of COL and GJS

| $N$ | Method | $a_0 = 0$, $a_1 = 0$ | $a_0 = 10$, $a_1 = 0$ | $a_0 = 50$, $a_1 = 1$ | $a_0 = 100x$, $a_1 = 50$ | $a_0 = 10e^{10x}$, $a_1 = \sin(10x)$ |
|---|---|---|---|---|---|---|
| 16 | COL | 3.30E+05 | 3.77E+05 | 4.46E+05 | 2.49E+05 | 4.09E+05 |
| 16 | GJS | 1.00 | 1.07 | 1.42 | 1.62 | 33.05 |
| 32 | COL | 2.70E+08 | 2.78E+08 | 3.36E+08 | 1.37E+08 | 8.22E+08 |
| 32 | GJS | 1.00 | 1.07 | 1.42 | 1.62 | 33.05 |
| 64 | COL | 2.58E+11 | 2.64E+11 | 4.43E+11 | 8.11E+10 | 1.37E+11 |
| 64 | GJS | 1.00 | 1.07 | 1.42 | 1.62 | 33.05 |
| 128 | COL | 2.05E+14 | 2.10E+14 | 2.39E+14 | 1.86E+14 | 2.64E+14 |
| 128 | GJS | 1.00 | 1.07 | 1.42 | 1.62 | 33.05 |

# 5 Error Estimates

## 5.1 Projection

Define $\pi_N^{k,l}: L^2_{\omega^{k,l}}(I) \to Q_N^{k,l} := \mathrm{span}\left\{ J_{-(k+l)}^{k,l}, \ldots, J_N^{k,l} \right\}$ as

$$\left( u - \pi_N^{k,l} u, v_N \right)_{\omega^{k,l}} = 0, \quad \forall v_N \in Q_N^{k,l} \tag{28}$$

**Theorem 1.** *For any* $k, l \in \mathbb{Z}$ *and* $u \in B^m_{k,l}(I)$, *we have that for* $0 \leqslant \mu \leqslant m \leqslant N+1$,

$$\left\| \partial_x^\mu \left( \pi_N^{k,l} u - u \right) \right\|_{\omega^{k+\mu, l+\mu}} \lesssim \sqrt{\frac{(N-m+1)!}{(N-\mu+1)!}} (N+m)^{\frac{(\mu-m)}{2}} \| \partial_x^m \|_{\omega^{k+m, l+m}}$$

**Remark 2.** $L^2_{\omega^{-m,-m}}$ projection is also a $H_0^{2m}$ projection

$$\left( \partial_x^m \left( \pi_N^{-m,-m} u - u \right), \partial_x^m v_N \right) = 0, \quad \forall v_N \in P_N \cap H_0^{2m}.$$

## 5.2 Even-Order Equations

**Theorem 3.** *Let $u$ and $u_N$ be the solution of (10) and (11), respectively. If $\alpha$, $\beta > 0$ and $u \in H_0^2(I) \cap B^m_{-2,-2}(I)$ and $f \in B^k_{-1,-1}(I)$ with $2 \leqslant m \leqslant N+1$ and $1 \leqslant k \leqslant N+1$, then*

$$\|\partial_x^\mu (u - u_N)\| \lesssim \sqrt{\frac{(N-m+1)!}{(N-\mu+1)!}}(N+m)^{\frac{\mu-m}{2}}\|\partial_x^m u\|_{\omega^{m-2,m-2}} + \sqrt{\frac{(N-k+1)!}{N!}}(N+k)^{-\frac{k+1}{2}}\|\partial_x^k f\|_{\omega^{k-1,k-1}},$$

*where $\mu = 0, 1, 2$.*

**Remark 4.** The proof is straightforward by using the error equation, and the fact that $\pi_N^{-2,-2}$ is also a projection in $H_0^2(I)$.

**Remark 5.** Similar results can be proved for the case of general even-order equations.

## 5.3 Odd-Order Equations

**Lemma 6.** *Let* $\pi_N^{-2,-1}$ *be the orthogonal projector defined in* (28). *then*

$$\left(\partial_x(u-\pi_N^{-2,-1}u),\partial_x^2 v_N\right)=0, \quad \forall u\in V, v_N\in V_N^*$$

**Theorem 7.** *Let* $u$ *and* $u_N$ *be the solution of* (19) *and* (22), *repsectively. If* $\alpha$, $\beta \geqslant 0$ *and* $-\frac{1}{3}<\gamma<\frac{1}{6}$, $u\in V\cap B^m_{-2,-1}(I)$ *and* $f\in B^m_{-1,-1}(I)$ *with* $2\leqslant m\leqslant N+1$ *and* $1\leqslant k\leqslant N+1$, *the we have*

$$\begin{aligned}\alpha\|e_N\|_{\omega^{-1,1}}&+N^{-1}\|\partial_x e_N\|_{\omega^{-1,0}}\\ &\leqslant c(1+|\gamma|N)\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{-\frac{1+m}{2}}\|\partial_x^m u\|_{\omega^{m-2,m-1}}\\ &\quad+c\sqrt{\frac{(N-k+1)!}{N!}}(N+k)^{-\frac{k+1}{2}}\|\partial_x^k f\|_{\omega^{k-1,k-1}}.\end{aligned}$$

# 6 Application

## 6.1 Cahn-Hilliard Equation

$$\begin{aligned} &u_t = -\gamma(u_{xx} - \varepsilon^{-2}(u^2-1)u)_{xx}, \quad x \in (-1,1),\ t>0, \quad \gamma>0,\\ &u(\pm 1,t) = u'(\pm 1,t) = 0, \quad t \geqslant 0,\\ &u(x,0) = u_0(x), \quad x \in [-1,1]. \end{aligned}$$

Crank–Nicolson leap-frog and Legendre–Galerkin Method

$$\begin{cases} \text{Find } u_N^{k+1} \in V_N := \{u \in P_N\colon\ u(\pm 1) = u'(\pm 1) = 0\}\ s.t. \\ \frac{1}{2\tau}\big(u_N^{k+1} - u_N^{k-1}, v_N\big) + \frac{\gamma}{2}\big(\partial_x^2(u_N^{k+1} + u_N^{k-1}), \partial_x^2 v_N\big) \\ \qquad = \frac{1}{\varepsilon^2}\big(I_N\big[(u_N^k)^3 - u_N^k\big], \partial_x^2 v_N\big), \quad \forall v_N \in V_N. \end{cases}$$

At each time step we need to solve a forth order equation.

$$\big(u_N^{k+1}, v_N\big) + \gamma\tau\big(\partial_x^2 u_N^{k+1}, \partial_x^2 v_N\big) \ = \ \mathrm{rhs}(v_N), \quad \forall v_N \in V_N.$$

# Numerical Results

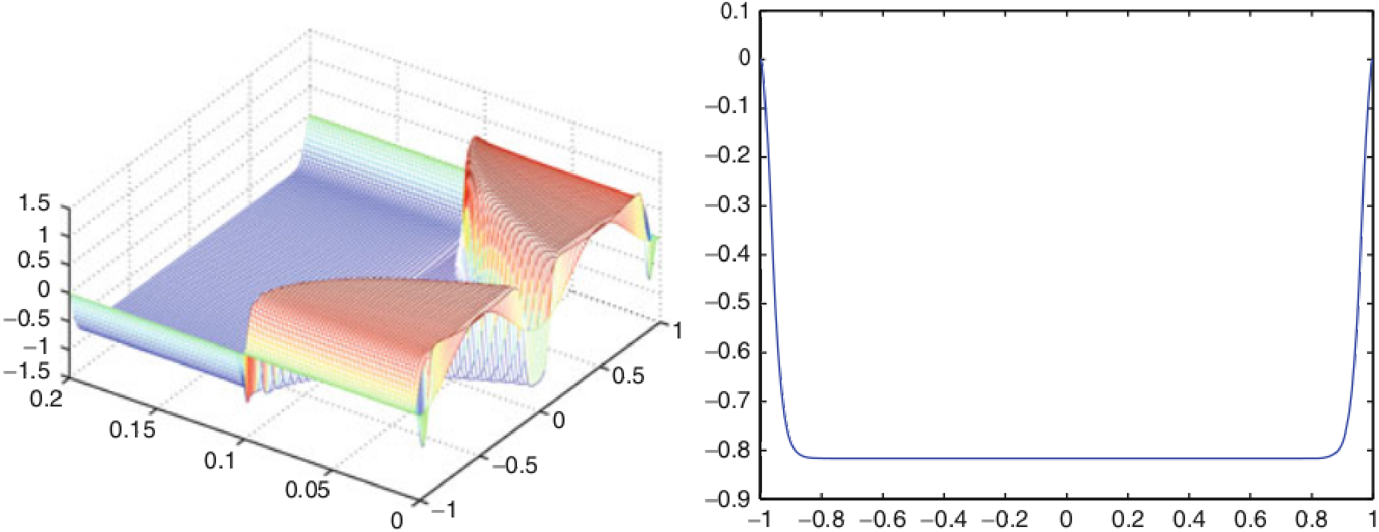



**Figure 1.** For $\varepsilon = 0.02, \gamma = 0.01$, $N = 64, \tau = 2 \times 10^{-6}$, $u_0(x) = \sin^2(\pi x)$. Left: time evolution of the numerical solution; Right: steady state solution.

## 6.2 Korteweg–de Vries (KdV) Equation

$$U_t + UU_x + U_{xxx} = 0, \quad U(x,0) = U_0(x). \tag{29}$$

Exact soliton solution

$$U(x,t) = 12\kappa^2 \text{sech}^2(\kappa(x - 4\kappa^2 t - x_0)). \tag{30}$$

$U \to 0$ exponentially as $|x| \to \infty$, domain truncation $x \in (-L, L)$.

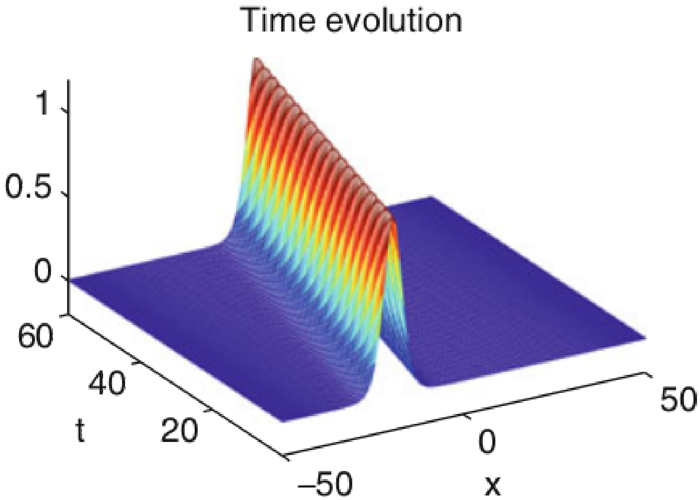
$y = x/L, \quad u(y,t) = U(x,t), \quad u_0(y) = U_0(x), \quad \alpha = 1/L^3, \quad \beta = 1/L$

$$\begin{cases} u_t + \beta u u_y + \alpha u_{yyy} = 0, & y \in (-1,1), t \in (0,T), \\ u(y,0) = u_0(y), & u(\pm 1, 0) = u'(1,t) = 0 \end{cases}$$

Crank–Nicolson leap-frog and Legendre dual-Petrov–Galerkin

$$\begin{cases} \text{Find } u_N^{k+1} \in V_N \; s.t. \text{ for } \forall v_N \in V_N^*, \\ \left(\frac{u_N^{k+1} - u_N^{k-1}}{2\tau}, v_N\right) + \alpha\left(\partial_y\left(\frac{u_N^{k+1} + u_N^{k-1}}{2}\right), \partial_y^2 v_N\right) = -\frac{\beta}{2}(\partial_y I_N(u_N^k)^2, v_N). \end{cases}$$

# Examples 1. Single soliton solution



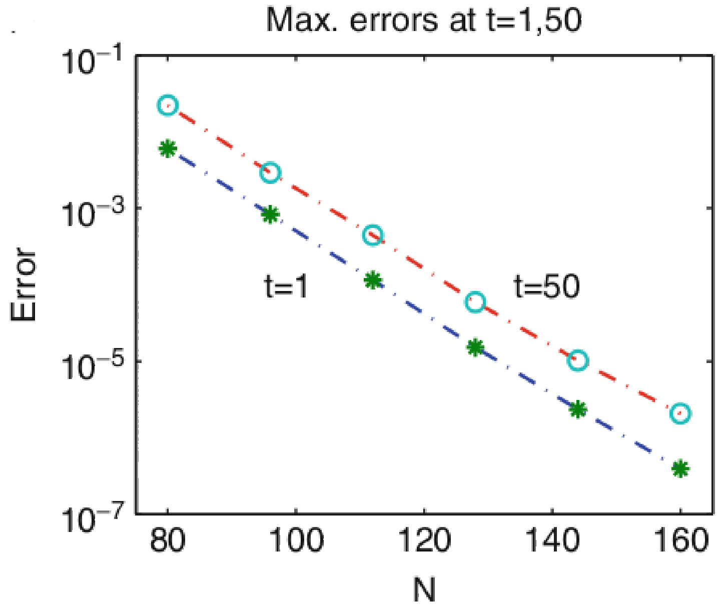




**Figure 2.** $\kappa = 0.3, x_0 = -20, L = 50$ and $\tau = 0.001,\ N \lesssim 160$.
Left: Time evolution of the numerical solution.
Right: the maximum errors at $t = 1, 50$ with different $N$.

## Example 2. Interaction of five solitons

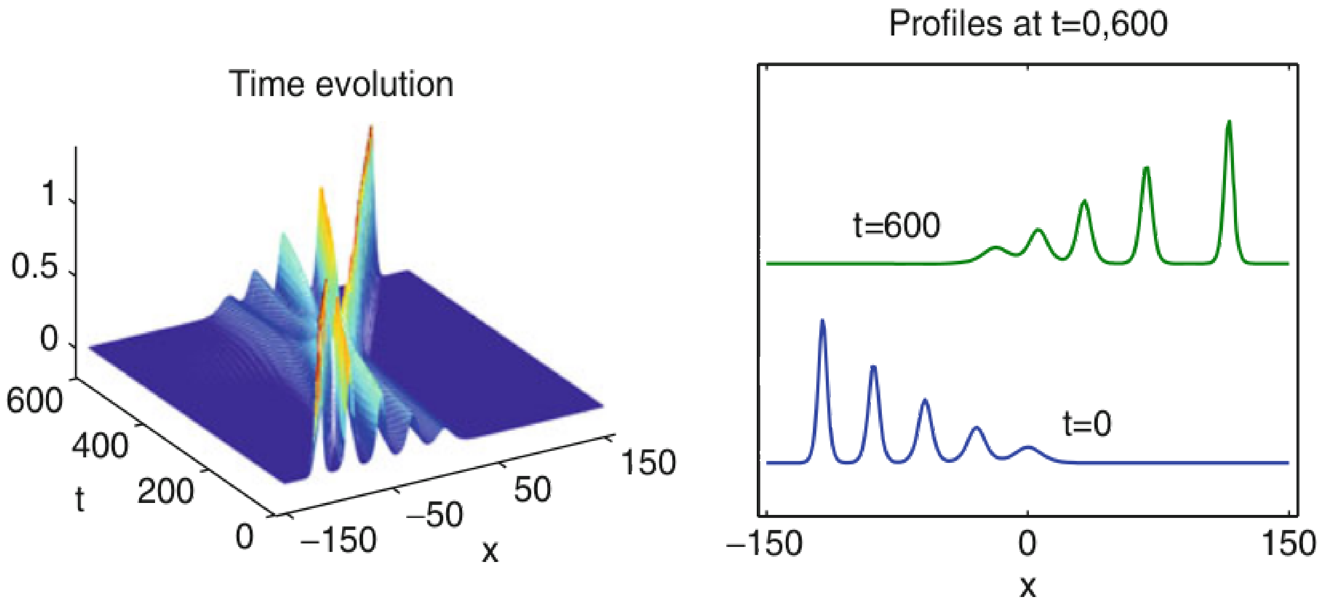



**Figure 3.** Initial value $u_0(x) = \sum_{j=1}^{5} 12\, \kappa_j^2 \mathrm{sech}^2(\kappa_j(x - x_j))$, with $x_j = 30j - 150, \kappa_j = .35 - .05j$.
Solver parameter: $L = 150, \tau = 0.02\ \ N = 256$.

# Example 3. Solitary waves generated by a Gaussian profile

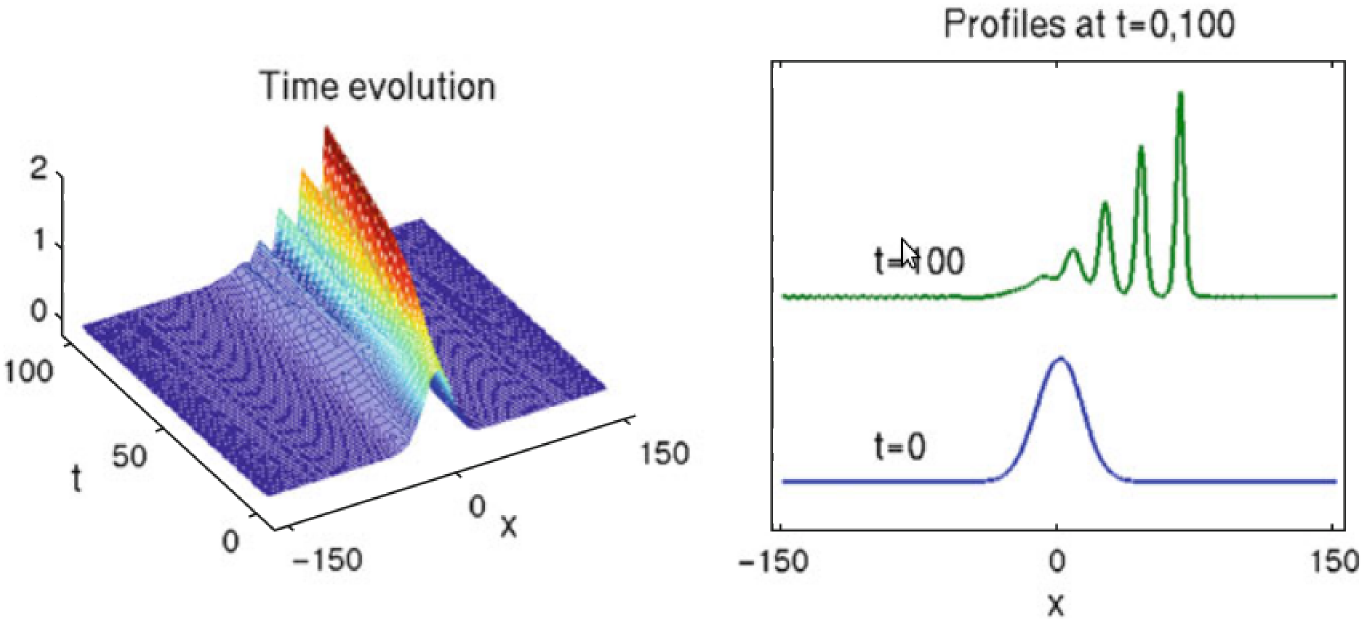



**Figure 4.** $u_0(x) = e^{-1.5(7x)^2}$.