Hence, using Lemma 4.9 again yields

$$|u-u_N|_1+k\|u-u_N\| \le c\big(|e_N|_1+k\|e_N\|+|\tilde{e}_N|_1+k\|\tilde{e}_N\|\big)$$
$$\le c\big(1+k^2N^{-1}+kN^{-1/2}\big)\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{(1-m)/2}\|(r-r^2)^{(m-1)/2}\partial_r^m u\|.$$

This ends the proof. □

Problems

4.1. Show that under the assumption (4.3), the bilinear form $\mathscr{B}(\cdot,\cdot)$ defined by (4.9) is continuous and coercive in $H^1_\diamond(I)\times H^1_\diamond(I)$.

4.2. Let $\{h_j\}_{j=0}^N$ be the Lagrange basis polynomials relative to the Jacobi-Gauss-Radau points $\{x_j\}_{j=0}^N$ with $x_0=-1$ (see Theorem 3.26). Let $\widetilde{D}=(d_{kj}:=h_j'(x_k))_{1\le k,j\le N}$ be the differentiation matrix corresponding to the interior collocation point (see (3.163)). Write down the matrix form of the Jacobi-Gauss-Radau collocation method for

$$u'(x)=f(x),\quad x\in(-1,1);\quad u(-1)=c_-,$$

where $f\in C[-1,1]$ and c_- is a given value. Use the uniqueness of the approximate solution to show that the matrix $\tilde{D}$ is nonsingular.

4.3. Prove Lemma 4.8.

4.4. Consider the Burgers' equation:

$$\frac{\partial u}{\partial t}=\varepsilon\frac{\partial^2 u}{\partial x^2}-u\frac{\partial u}{\partial x},\quad \varepsilon>0. \tag{4.120}$$

(i) Verify that it has the soliton solution

$$u(x,t)=\kappa\Big[1-\tanh\Big(\frac{\kappa(x-\kappa t-x_c)}{2\varepsilon}\Big)\Big], \tag{4.121}$$

where the parameter $\kappa>0$ and the center $x_c\in\mathbb{R}$.

(ii) Take $\varepsilon=0.1,\kappa=0.5,x_c=-3,x\in[-5,5]$, and impose the initial value $u(x,0)$ and the boundary conditions $u(\pm5,t)$ by using the exact solution. Use the Crank-Nicolson leap-frog scheme to in time (see (1.2)–(1.3)), and the Chebyshev collocation method in space to solve the equation. Output the discrete maximum errors for $\tau=10^{-k}$ (time step size) with $k=2,3,4$ and $N=32,64,128$ at $t=12$. Refer to Table 1 in Wu et al. (2003) for the behavior of the errors (obtained by other means).

(iii) Replace the Chebyshev-collocation method in (ii) by the Chebyshev-Galerkin method. Do the same test and compare two methods. Refer to Sect. 3.4.3 for the

Chebyshev differentiation process using FFT and to Trefethen (2000) for a handy `MATLAB` code for this process.

(iv) Consider the Burgers' equation (4.120) in $(-1,1)$ with the given data

$$u(\pm 1,t) = 0, \quad u(x,0) = -\sin(\pi x), \;\; x \in [-1,1]. \tag{4.122}$$

Solve this problem by the methods in (ii) and (iii) by taking $\varepsilon = 0.02, \tau = 10^{-4}$ and $N = 128$ and plot the numerical solution at $t = 1$. Refer to Shen and Wang (2007b) for some profiles of the numerical solution (obtained by other means).

4.5. Consider the Fisher equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u). \tag{4.123}$$

(i) Verify that it has the traveling solution

$$u(x,t) = \Big[1 + \exp\Big(\frac{x}{\sqrt{6}} - \frac{5}{6}t\Big)\Big]^{-2}. \tag{4.124}$$

(ii) Since $u(x,t) \to 0$ (resp. 1) as $x \to +\infty$ (resp. $-\infty$), we can approximate (4.123) in $(-L,L)$, where L is large enough so that the wave front does not reach the boundary $x = L$, by imposing the boundary conditions

$$u(-L,t) = 1, \quad u(L,t) = 0,$$

and taking the initial value as $u(x,0)$. Use the second-order splitting scheme (D.30) with $Au = \partial_x^2 u$ and $Bu = u(1-u)$ in time, and the Legendre-Galerkin method in space to solve this problem with $\tau = 10^{-3}, N = 128, L = 100$ up to $t = 6$. Output the discrete maximum errors between the exact and approximate solutions at $t = 1, 2, \ldots, 6$. An advantage of the splitting scheme is that the sub-problem (a Bernoulli's equation for t):

$$\frac{\partial u}{\partial t} = u(1-u)$$

can be solved exactly, so it suffices to solve a linear equation in each step. Refer to Wang and Shen (2005) for this numerical study by a mapping technique.