Lecture 2: Fourier Spectral Method

This part is based on Chapter 2 of ShenTangWang's book. We will first introduce the basic ingredient of Fourier methods in section 1, then we will do some analysis on the Fourier approximation, in the last section, we will apply the Fourier method to solve some PDEs.

1 Introduction to Fourier Transforms and FFT

1.1 Fourier Transform of a function on $\mathbb{R}$

The **Fourier transform** of a function $u(x)$, $x\in\mathbb{R}$, is the function $\hat{u}(k)$ defined by

$$\hat{u}(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)\,e^{-ikx}\,\mathrm{d}x,\quad k\in\mathbb{R}. \tag{1}$$

we can construct u from $\hat{u}$ by the **inverse Fourier transform**

$$u(x)=\int_{-\infty}^{\infty}\hat{u}(k)\,e^{ikx}\,\mathrm{d}k,\quad x\in\mathbb{R}. \tag{2}$$

x: physical variable, k: Fourier variable or wavenumber.

Now we consider x ranging over $h\mathbb{Z}$ rather than $\mathbb{R}$. Since the spatial domain is discrete, the wavenumber k will no longer range over all of $\mathbb{R}$, since e^{-ik_1x} and e^{-ik_2x} will be equal on $h\mathbb{Z}$ if $k_1-k_2=2\pi/h$. So, we consider let $k\in[-\pi/h,\pi/h]$.

$$\begin{array}{lcccl} \text{Physical space} & : & \text{discrete,} & \text{unbounded} & : \; x\in h\,\mathbb{Z} \\ & & \updownarrow & \updownarrow & \\ \text{Fourier space} & : & \text{bounded,} & \text{continuous} & : \; k\in[-\pi/h,\pi/h] \end{array}$$

For a function v defined on $h\mathbb{Z}$ with value v_j at x_j, the **semidiscrete Fourier transform** is defined by

$$\hat{v}(k)=\frac{h}{2\pi}\sum_{j=-\infty}^{\infty} v_j\,e^{-ikx_j},\quad k\in[-\pi/h,\pi/h], \tag{3}$$

and the inverse semidiscrete Fourier transform is

$$v_j=\int_{-\pi/h}^{\pi/h}\hat{v}(k)e^{ikx_j}\,\mathrm{d}k,\quad j\in\mathbb{Z}. \tag{4}$$

1.1.1 Differentiation on infinite grid $h\mathbb{Z}$

For spectral differentiation, we need an interpolant. Give $v(x_j)$, $x_j\in h\mathbb{Z}$, we define the **banded-limited** interpolant of v by

$$p(x)=\int_{-\pi/h}^{\pi/h}\hat{v}(k)e^{ikx}\,\mathrm{d}k,\quad x\in\mathbb{R}, \tag{5}$$

where $\hat{v}(k)$ are given by equation (3). There are two procedure to spectral differentiation. One is the description in phsyical space:

1. Given v, determine its band-limited interpolant p by (5), then
2. Set $w_j=p'(x_j)$.

Another approach is doing the differentiation in Fourier space:

1. Given v, compute its semidiscrete Fourier transform $\hat{v}$ by (3).
2. Define $\hat{w}(k)=ik\;\hat{v}(k)$.
3. Compute w from $\hat{w}$ by inverse semidiscrete Fourier transform (4).

To make the formulation looks nicer, to let's first define the **differentiation matrix**. Denote

$$\delta_j\equiv\delta(x_j):=\begin{cases}1, & j=0,\\ 0, & \text{otherwise.}\end{cases}$$

It is very easy to calculate the semidiscrete Fourier transfrom is $\hat{\delta}(k)=h/2\pi$, and the banded-limit interpolant of δ is

$$\frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx}\,\mathrm{d}k=\frac{\sin(\pi x/h)}{\pi x/h}:=S_h(x),$$

This is the famous **sinc** function.

Since a general grid function v can by written as

$$v_j=\sum_{m=-\infty}^{\infty} v_m\delta_{j-m}.$$

its interpolant can be written as

$$p(x)=\sum_{m=-\infty}^{\infty} v_m\,S_h(x-x_m).$$

The derivative is accordingly

$$w_j=p'(x_j)=\sum_{m=-\infty}^{\infty} v_m S_h'(x_j-x_m)$$

If we take $S_h'(x_j-x_m)$ as components of a matrix D, then $S_h'(x_j)=S_h'(x_j-x_0)$ is the components of column $m=0$ of D. the other columns are obtained by shitting this column. It is easy to verify that

$$S_h'(x_j)=\begin{cases} 0, & j=0,\\ \dfrac{(-1)^j}{jh}, & j\neq 0.\end{cases}$$

Similarly we can calculate the high order differentiation matrix by

$$p''(x_j)=\sum_{m=-\infty}^{\infty} v_m S_h''(x_j-x_m),$$

and

$$S_h''(x_j)=\begin{cases} -\dfrac{\pi^2}{3h^2}, & j=0,\\ 2\dfrac{(-1)^{j+1}}{j^2h^2}, & j\neq 0.\end{cases}$$

1.2 Fourier transform on periodic grid

1.2.1 Continuous Fourier series

If the function considered is 2π periodic, then its spectrum must be integer, since e^{ikx} is 2π periodic iff k is an integer. For any complex-valued function $u\in L^2(0,2\pi)$, its **Fourier series** is defined by

$$\mathcal{F}(u)(x):=\sum_{k=-\infty}^{\infty}\hat{u}_k\,e^{ikx}, \tag{6}$$

where the Fourier coefficients are determined by

$$\hat{u}_k=\frac{1}{2\pi}\int_0^{2\pi} u(x)\,e^{-ikx}\,\mathrm{d}x. \tag{7}$$

Actually, $\{E_k=e^{ikx}\colon k\in\mathbb{Z}\}$ forms a set of complete orthogonal bases in the complex Hilbert space $L^2(0,2\pi)$, equipped with the inner product and the norm

$$(u,v):=\frac{1}{2\pi}\int_0^{2\pi} u(x)\bar{v}(x)\,\mathrm{d}x,\quad \|u\|=\sqrt{(u,u)},$$

where $\bar{v}$ is the complex conjugate of v. The orthogonality of $\{E_k=e^{ikx}\colon k\in\mathbb{Z}\}$ reads

$$(E_k,E_m)=\frac{1}{2\pi}\int_0^{2\pi} e^{i(k-m)x}\mathrm{d}x=\delta_{km}.$$

The basic properties of the Fourier series:

1. if u is a real-valued function, then $\hat{u}_{-k} = \bar{\hat{u}_k}$, $\forall k \in \mathbb{Z}$.
2. For any $u \in L^2(0, 2\pi)$, its truncated Fourier series $\mathcal{F}_N(u) := \sum_{|k| \leqslant N} \hat{u}_k e^{ikx}$ converges to u in the L^2-sense, and there holds the Parseval's identity:
$$\|u\|^2 = \sum_{k=-\infty}^{\infty} |\hat{u}_k|^2.$$
3. If u is continuous, periodic and of bounded variation on $[0, 2\pi]$, then $\boldsymbol{\mathcal{F}}_N(u)$ uniformly converges to u.
4. $\mathcal{F}_N(u)(x) = (\mathcal{D}_N * u)(x) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{D}_N(x-t)u(t)\,\mathrm{d}t$, where $\mathcal{D}_N = \sum_{k=-N}^{N} e^{ikx} = \frac{\sin((N+1/2)x)}{\sin(x/2)}$.

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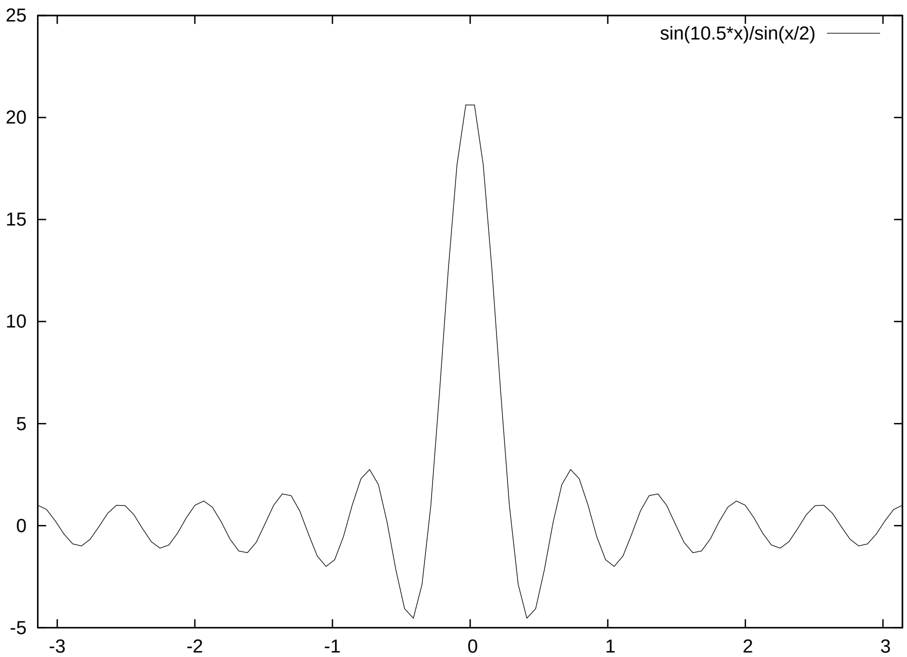


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Sometimes, it is convenient to express the Fourier series in terms of the trigonometric polynomials

$$u(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx)), \tag{8}$$

where

$$a_k = \frac{1}{\pi}\int_0^{2\pi} u(x)\cos(kx)\,\mathrm{d}x, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} u(x)\sin(kx)\,\mathrm{d}x.$$

The coefficients of the two different representation of (6) and (8) are related by

$$\hat{u}_0 = \frac{a_0}{2}, \quad \hat{u}_k = \begin{cases} \dfrac{a_k - i\,b_k}{2}, & \text{if } k \geqslant 1, \\ \dfrac{a_{-k} + i\,b_{-k}}{2}, & \text{if } k \leqslant -1. \end{cases} \tag{9}$$

In particular, if u is a real-valued function(in which we call (8) **real Fourier series**), then

$$a_0 = 2\,\hat{u}_0, \quad a_k = 2\,\mathrm{Re}(\hat{u}_k), \quad b_k = -2\,\mathrm{Im}(\hat{u}_k), \quad k \geqslant 1.$$

1.2.2 Discrete Fourier Series

Given a positive integer N, let $x_j = j\,h = j\frac{2\pi}{N}$, $0 \leqslant j \leqslant N-1$ be the N-equispaced grids in $[0, 2\pi)$, which are refered to as Fourier collocation points. Since $e^{ikx_j} = e^{i\,(k+N)x_j}$, so we will again require $k \in [-\pi/h, \pi/h]$. In this lecture we will assume N is an even number, such that $\frac{\pi}{h} = \frac{N}{2}$ is an integer.

$$\begin{array}{llllll} \text{Physical space} & : & \text{discrete,} & \text{bounded (periodic)} & : & u(x_j), \quad x_j = j\dfrac{2\pi}{N},\ j \in \mathbb{Z}_N = \{0, 1, \ldots, N-1\} \\ & & \updownarrow & \updownarrow & & \\ \text{Fourier space} & : & \text{bounded,} & \text{discrete} & : & E_k, \quad k \in \{-N/2+1, -N/2+2, \ldots, N/2\} \end{array}$$

We define the discrete inner product by

$$\langle u, v\rangle_N := \frac{1}{N}\sum_{j=0}^{N-1} u(x_j)\bar{v}\,(x_j).$$

Lemma 1. *Let $E_k(x) = e^{ikx}$. For any integer $N \geqslant 1$, we have*

$$\langle E_k, E_m\rangle_N = \begin{cases} 1, & \text{if } k - m = l\,N,\ \forall l \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. *If $k - m$ is not divisible by N, then*

$$\begin{aligned} \langle E_k, E_m\rangle_N &= \frac{1}{N}\sum_{j=0}^{N-1} e^{i(k-m)x_j} = \frac{1}{N}\sum_{j=0}^{N-1} \left(e^{i(k-m)2\pi/N}\right)^j \\ &= \frac{1}{N}\frac{e^{2\pi i(k-m)-1}}{e^{2\pi i(k-m)/N}-1} = 0. \end{aligned}$$

If $k - m$ is divisible by N, we have $e^{2\pi i(k-m)/N} = 1$, so the summation above equals to 1. □

It is easy to verify that use the rectangular quadrature formular on discrete grids

$$\frac{1}{2\pi}\int_0^{2\pi} v(x)\,\mathrm{d}x \approx \frac{1}{N}\sum_{j=0}^{N-1} v(x_j), \quad \forall v \in C[0, 2\pi). \tag{10}$$

is exact for all $v \in \operatorname{span}\{e^{ikx}: 0 \leqslant |k| \leqslant N-1\}$, and $v = \sin(\pm Nx)$, but not for $v = \cos(\pm Nx)$.

Since $E_{-N/2} = E_{N/2}$, it will be problematic if we including only $E_{N/2}$. For example, to approximate $2\cos(N\,x/2)$ on $\{x_j,\ j \in \mathbb{Z}_N\}$, we get a sawtooth wave on the grid. Using $\{E_k, \ -N/2 < k \leqslant N/2\}$ to approximate it, we will get $e^{iNx/2}$, its derivative will be $i\,N/2\,e^{iNx/2}$ instead of 0 on the grid points. So we will use the approximation space

$$\mathcal{T}_N = \{u = \sum_{k=-N/2}^{N/2} \tilde{u}_k\,e^{ikx}: \quad \tilde{u}_{-N/2} = \tilde{u}_{N/2}\}.$$

Define the the mapping $I_N: C[0, 2\pi) \to \mathcal{T}_N$ by

$$(I_N u)(x) = \sum_{k=-N/2}^{N/2} \tilde{u}_k\,e^{ikx}, \tag{11}$$

where $\{\tilde{u}_k\}$ are given by the **Discrete Fourier Transfrom** (DFT)

$$\tilde{u}_k = \frac{1}{c_k} \times \frac{1}{N}\sum_{j=0}^{N-1} u(x_j)\,e^{-ikx_j}, \quad k = 0, \pm 1, \ldots, \pm N/2. \tag{12}$$

Here

$$c_k = \begin{cases} 2, & k = \pm N/2, \\ 1, & \text{otherwise.} \end{cases}$$

The following lemma shows that I_N is the interpolation operator from $C[0,2\pi)$ to $\mathcal{T}_N$, i.e.

$$(I_N u)(x_j) = u(x_j), \quad x_j = \frac{2\pi j}{N}, \quad 0 \leqslant j \leqslant N-1. \tag{13}$$

Lemma 2. *For any* $u \in C[0,2\pi)$,

$$(I_N u)(x) = \sum_{j=0}^{N-1} u(x_j) h_j(x), \tag{14}$$

where

$$h_j(x) = \frac{1}{N}\sin\Big[N\frac{x-x_j}{2}\Big]\cot\Big[\frac{x-x_j}{2}\Big] \quad \in \mathcal{T}_N$$

satisfying

$$h_j(x_k) = \delta_{jk}, \quad \forall j,k = 0,1,\ldots,N-1.$$

Proof. *By* (11) *and* (12),

$$\begin{aligned}(I_N u)(x) &= \sum_{k=-N/2}^{N/2}\left(\frac{1}{N c_k}\sum_{j=0}^{N-1} u(x_j)\, e^{-ikx_j}\right)e^{ikx}\\ &= \sum_{j=0}^{N-1}\left(\frac{1}{N}\sum_{k=-N/2}^{N/2}\frac{1}{c_k}e^{ik(x-x_j)}\right)u(x_j)\end{aligned}$$

Comparing this with (14), *we get*

$$\begin{aligned}h_j(x) &= \frac{1}{N}\sum_{k=-N/2}^{N/2}\frac{1}{c_k}e^{ik(x-x_j)}\\ &= \frac{1}{N}\Big(\mathcal{D}_{N/2-1}(x-x_j) + \cos\Big[N\frac{x-x_j}{2}\Big]\Big)\\ &= \frac{1}{N}\left(\frac{\sin\big[(N-1)\frac{x-x_j}{2}\big]}{\sin\frac{x-x_j}{2}} + \cos\Big[N\frac{x-x_j}{2}\Big]\right)\\ &= \frac{1}{N}\sin\Big[N\frac{x-x_j}{2}\Big]\cot\Big[\frac{x-x_j}{2}\Big].\end{aligned}$$

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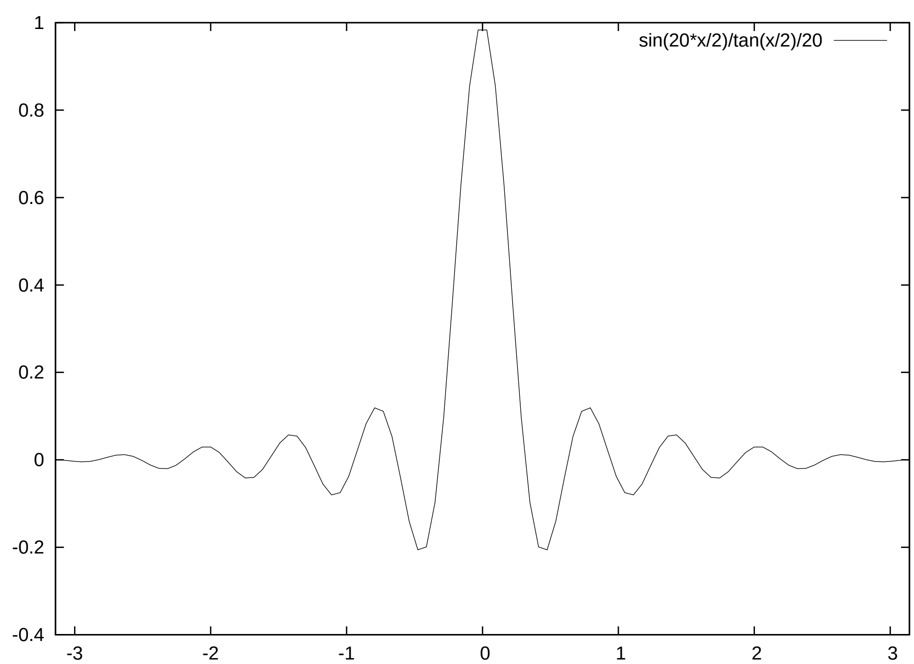

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□

Taking $x=x_j$ in (11) leads to the **inverse Discrete Fourier Transfrom**

$$u(x_j)=\sum_{k=-N/2}^{N/2}\tilde{u}_k\, e^{ikx_j},\quad j=0,1,\ldots,N-1. \tag{15}$$

Remark 3. It is obvious that the discrete Fourier transform (12) and its inverse (15) can be carried out through matrix-vector multiplication with $O(N^2)$ operations. However, thanks to the fast Fourier trasnform duce to Cooley and Tukey (1965), such processes can be accomplished with $O(N\log_2 N)$ operations.

1.2.3 FFT and iFFT in MATLAB

Given the data $\{\, v(j)=u(x_{j-1})\,\}_{j=1}^{N}$ sampled at $\{\, x_j=2\pi j/N\}_{j=0}^{N-1}$, the command "$\tilde{v}=\mathrm{fft}(v)$" returns the vector $\{\bar{v}\,(k)\}_{k=1}^{N}$, defined by

$$\bar{v}\,(k)=\sum_{j=1}^{N} v(j)e^{-2\pi i(j-1)(k-1)/N},\quad 1\leqslant k\leqslant N,$$

while the inverse FFT can be computed with the command "v=ifft($\tilde{\mathrm{v}}$)" which returns the physical values $\{\, v(j)\}_{j=1}^{N}$ via

$$v(j)=\frac{1}{N}\sum_{k=1}^{N}\bar{v}\,(k)e^{2\pi i(j-1)(k-1)/N},\quad 1\leqslant j\leqslant N.$$

For $u(x_j)=v(j+1)$, $x_j=\frac{2\pi j}{N}$, $0\leqslant j\leqslant N-1$, let $\tilde{u}_k$ be the FFT defined by (12), then

$$\tilde{u}_k=\begin{cases}\dfrac{1}{N}\bar{v}\,(k+1), & 0\leqslant k\leqslant\dfrac{N}{2}-1,\\ \dfrac{1}{N}\bar{v}\,(k+N+1), & -\dfrac{N}{2}+1\leqslant k\leqslant -1,\\ \dfrac{1}{2N}\bar{v}\,(N/2+1), & k=\pm\dfrac{N}{2}.\end{cases}$$

1.2.4 Differentiation in the Physical Space

By using the nodal basis representation (14), we can easily represent the derivatives of $u(x)$ in terms of $u(x_j)$:

$$u_N^{(m)}(x)=(I_N u)^{(m)}(x)=\sum_{j=0}^{N-1}u(x_j)h_j^{(m)}(x).$$

This precess can be formulated as a matrix-vector multiplication

$$\boldsymbol{u}_N^{(m)}=D^{(m)}\boldsymbol{u}_N,\quad m\geqslant 0, \tag{16}$$

where

$$D^{(m)}=\Big(d_{kj}^{(m)}:=h_j^{(m)}(x_k)\Big)_{k,j=0,\ldots,N-1},$$

$$\boldsymbol{u}_N=(u(x_0),u(x_1),\ldots,u(x_{N-1}))^T,$$

$$\boldsymbol{u}_N^{(m)}=\Big(u_N^{(m)}(x_0),u_N^{(m)}(x_1),\ldots,u_N^{(m)}(x_{N-1})\Big)^T.$$

Lemma 4. *The entries of the first-order Fourier differentiation matrix D are given by*

$$d_{kj}^{(1)}=h_j'(x_k)=\begin{cases}\dfrac{(-1)^{k+j}}{2}\cot\left[\dfrac{(k-j)\pi}{N}\right], & \text{if } k\neq j,\\ 0, & \text{if } k=j.\end{cases} \tag{17}$$

Proof. *Let* $\theta = (x - x_j)/2$*, then*

$$h_j'(x) = \frac{1}{2}\cos(N\theta)\cot(\theta) - \frac{1}{2N}\sin(N\theta)\csc^2(\theta). \tag{18}$$

If $x = x_k \neq x_j$*, then* $\theta = (k - j)\pi/N$*, the second term is 0, the first term can be simplified into the desired expression in* (17)*. If* $k = j$*, then* $\theta=0$*, it can be showed that* $h_j'(x_j) = 0$ *by the Taylor expansion of* (18)*.* □

In general

$$h_j^{(m)}(x_i) = \frac{1}{N}\sum_{k=-N/2}^{N/2} \frac{(i\,k)^m}{c_k}\, e^{2\pi i k(i-j)/N}. \tag{19}$$

In particular, the entries of the second-order differentiation matrix $D^{(2)}$ are given by

$$d_{kj}^{(2)} = h_j''(x_k) = \begin{cases} -\dfrac{(-1)^{k+j}}{2}\sin^{-2}\left[\dfrac{(k-j)\pi}{N}\right], & \text{if } k \neq j, \\ -\dfrac{N^2}{12} - \dfrac{1}{6}, & \text{if } k = j. \end{cases} \tag{20}$$

Remark 5. It is worthwhile to point out that $D^{(2)} \neq D^2$. Consider $u(x) = \cos(Nx/2)$, then $u(x_j) = (-1)^j$, we have $D\boldsymbol{u} = 0$, but $D^{(2)}\boldsymbol{u} = -N^2\boldsymbol{u}/4$.

1.2.5 Differentiation in the Frequency Space.

It is clear that the differentiation procedure using (16) requires $O(N^2)$ operations. Now we demonstrate how to perform the differentiation in Fourier space with $O(N\log_2 N)$ operations.

1. Given $u(x_j)$, $j \in \mathbb{Z}_N$ using FFT to calculate $\tilde{u}_k$, $k = 0, 1, \ldots, \pm N/2$. Note that $\tilde{u}_{N/2} = \tilde{u}_{-N/2}$.
2. Compute the coefficients of the expansion of the derivative:

$$\tilde{w}_k = \begin{cases} i\,k\,\tilde{u_k}, & k = 0, 1, \ldots, \pm N/2 - 1, \\ 0, & k = \pm N/2. \end{cases}$$

3. Compute the derivative $w_j = (I_N u)'(x_j)$ by using IFFT on $\tilde{w}_k$, $k = 0, 1, \ldots, \pm N/2$.

2 Approximation Properties in Sobolev Space

2.1 Inverse Inequalities

Since all norms of a finite dimensional space are equivalent, we can bound a strong norm by a weaker one with bounding constants depending on the dimension of the spce. This type of inequality is called inverse inequality. Our aim of this section is to find the optimal constants in such inequalities.

Define

$$X_N := \text{span}\{\, e^{i\,k\,x} \colon -N \leqslant k \leqslant N \,\}.$$

Lemma 6. *(Nikolski's inequality) For any* $u \in X_N$ *and* $1 \leqslant p \leqslant q \leqslant \infty$*,*

$$\|u\|_{L^q} \leqslant \left(\frac{N p_0 + 1}{2\,\pi}\right)^{\frac{1}{p} - \frac{1}{q}} \|u\|_{L^p},$$

where p_0 *is the least even integer* $\geqslant p$*.*

Lemma 7. *(Bernstein inequality) For any* $u \in X_N$ *and* $1 \leqslant p \leqslant \infty$*,*

$$\|\partial_x^m u\|_{L^p} \lesssim N^m \|u\|_{L^p}, \quad m \geqslant 1.$$

In particular, for $p = 2$*,*

$$\|\partial_x^m u\| \lesssim N^m \|u\|. \tag{21}$$

The proofs of these inverse inequalities can be found in Butzer and Nessel (1971).

2.2 Orthogonal Projection

Let $P_N: L^2(I) \to X_N$, $I := (0, 2\pi)$ be the L^2-orthogonal projection, defined by

$$(P_N u - u, v) = 0, \quad \forall v \in X_N.$$

It is obvious that $P_N u$ is the truncated Fourier series, namely,

$$(P_N u)(x) = \sum_{k=-N}^{N} \hat{u}_k e^{ikx},$$

where $\{\hat{u}_k\}$ are given by (7).

The norm and seminorm of $H_p^m(I)$ can be characterized in the frequency space by

$$\|u\|_m = \left(\sum_{k=-\infty}^{\infty} (1+k^2)^m |\hat{u}_k|^2 \right)^{1/2}, \quad |u|_m = \left(\sum_{k=-\infty}^{\infty} |k|^{2m} |\hat{u}_k|^2 \right)^{1/2}.$$

It is easy to verify that

$$\partial_x^l (P_N u) = P_N (\partial_x^l u), \quad 0 \leqslant l \leqslant m.$$

Theorem 8. *For any $u \in H_p^m(I)$ and $0 \leqslant \mu \leqslant m$,*

$$\|P_N u - u\|_\mu \lesssim N^{\mu - m} |u|_m \tag{22}$$

Proof.

$$\begin{aligned}
\|P_N u - u\|_\mu^2 &= \sum_{|k|>N} (1+k^2)^\mu |\hat{u}_k|^2 \\
&\lesssim N^{2\mu - 2m} \sum_{|k|>N} |k|^{2m-2\mu} (1+k^2)^\mu |\hat{u}_k|^2 \\
&\lesssim N^{2\mu - 2m} \sum_{|k|>N} |k|^{2m} |\hat{u}_k|^2 \\
&\lesssim N^{2\mu - 2m} |u|_m^2.
\end{aligned}$$

□

Theorem 9. *For any $u \in H_p^m(I)$ with $m > 1/2$, (Sobolev embedding $H_p^{1/2}(I) \subseteq L^\infty(I)$)*

$$\max_{x \in [0, 2\pi]} |(P_N u - u)(x)| \leqslant \sqrt{\frac{1}{2m-1}} N^{1/2 - m} |u|_m.$$

Proof. *By the Cauchy–Schwarz inequality,*

$$\begin{aligned}
|(P_N u - u)(x)| \leqslant \sum_{|k|>N} |\hat{u_k}| &\leqslant \left(\sum_{|k|>N} |k|^{-2m} \right)^{1/2} \left(\sum_{|k|>N} |k|^{2m} |\hat{u}_k|^2 \right)^{1/2} \\
&\leqslant \sqrt{\frac{1}{2m-1}} N^{1/2 - m} |u|_m.
\end{aligned}$$

□

2.2.1 Interpolation

We consider the Fourier interpolation on $2N$ collocation points $\{\, x_j = \pi j / N\}_{j=0}^{2N-1}$, but still denote the interpolation operator by I_N, that is

$$(I_N u)(x) = \sum_{k=-N}^{N} \tilde{u}_k e^{ikx},$$

with $\tilde{u}_N = \tilde{u}_{-N}$ and

$$\tilde{u}_k = \frac{1}{2N c_k} \sum_{j=0}^{2N-1} u(x_j) e^{-ikx_j}, \quad -N \leqslant k \leqslant N.$$

Theorem 10. *For any $u \in H_p^m(I)$ with $m > 1/2$,*

$$\|\partial_x^l (I_N u - u)\| \lesssim N^{l-m} |u|_m, \quad 0 \leqslant l \leqslant m.$$

Proof. *First, the asliasing formula*

$$c_k \tilde{u}_k = \hat{u}_k + \sum_{|p|>0}^{\infty} \hat{u}_{k+2pN}. \tag{23}$$

By using this formula, a direct calculation leads to

$$\begin{aligned}
\|P_N u - I_N u\|^2 &= \sum_{|k| \leqslant N} |\hat{u}_k - \tilde{u}_k|^2 \\
&= \sum_{|k|<N} |\hat{u}_k - \tilde{u}_k|^2 + \frac{1}{4} \sum_{k=\pm N} |2\hat{u}_k - 2\tilde{u}_k|^2 \\
&\lesssim \sum_{|k|<N} |\hat{u}_k - \tilde{u}_k|^2 + \frac{1}{2} \sum_{k=\pm N} |\hat{u}_k - 2\tilde{u}_k|^2 + \frac{1}{2} \sum_{k=\pm N} |\hat{u}_k|^2 \\
&\lesssim \sum_{|k| \leqslant N} |\hat{u}_k - \tilde{u}_k|^2 + \frac{1}{2} \sum_{k=\pm N} |\hat{u}_k|^2
\end{aligned}$$

The last term is bounded by

$$|\hat{u}_N|^2 + |\hat{u}_{-N}|^2 \leqslant N^{-2m} \sum_{|k|=N}^{\infty} |k|^{2m} |\hat{u}_k|^2 \leqslant N^{-2m} |u|_m^2,$$

and the first term can be estimated by using the aliasing formula and the Cauchy–Schwarz inequality:

$$\begin{aligned}
\sum_{|k| \leqslant N} |\hat{u}_k - c_k \tilde{u}_k|^2 &= \sum_{|k| \leqslant N} \left| \sum_{|p|>0}^{\infty} \hat{u}_{k+2pN} \right|^2 \\
&\leqslant \sum_{|k| \leqslant N} \left\{ \left\{ \sum_{|p|>0}^{\infty} |k + 2pN|^{-2m} \right\} \left\{ \sum_{|p|>0}^{\infty} |k + 2pN|^{2m} |\hat{u}_{k+2pN}|^2 \right\} \right\} \\
&\leqslant \max_{|k| \leqslant N} \left\{ \sum_{|p|>0}^{\infty} |k + 2pN|^{-2m} \right\} \times \left\{ \sum_{|k| \leqslant N} \sum_{|p|>0}^{\infty} |k + 2pN|^{2m} |\hat{u}_{k+2pN}|^2 \right\}
\end{aligned}$$

It is clear that ($m > 1/2$)

$$\max_{|k| \leqslant N} \left\{ \sum_{|p|>0}^{\infty} |k + 2pN|^{-2m} \right\} \leqslant \frac{1}{N^{2m}} \sum_{|p|>0}^{\infty} \frac{1}{|2p-1|^{2m}} \lesssim N^{-2m},$$

and

$$\sum_{|k| \leqslant N} \sum_{|p|>0}^{\infty} |k + 2pN|^{2m} |\hat{u}_{k+2pN}|^2 \leqslant 2|u|_m^2.$$

Hence, a combination of the above estimate leads to

$$\|P_N u - I_N u\| \lesssim N^{-m} |u|_m$$

By the inverse inequality (21)*,*

$$\|\partial_x^l (P_N u - I_N u)\| \lesssim N^l \|P_N u - I_N u\| \lesssim N^{l-m} |u|_m,$$

Finally, by the triangular inequality and and Theorem (8) yields

$$\|\partial_x^l (I_N u - u)\| \leqslant \|\partial_x^l (P_N u - I_N u)\| + \|\partial_x^l (P_N u - u)\| \lesssim N^{l-m} |u|_m.$$

□

3 Application

We look at the elliptic equation

$$\alpha\, u(x) - u''(x) = f, \quad x \in [0, 2\pi) \tag{24}$$

with periodic boundary condition

$$u(x+2\pi) = u(x).$$

We use Fourier method on N equispaced grid points $\{\, x_j : x_j = j2\pi/N,\, j = 0, \ldots, N-1\,\}$.

1. Collocation method. Approximating equation (24) on $\{\, x_j\}$ by using $D^{(2)}u(x_j)$ to approximate $u''(x_j)$, we get

$$\alpha u(x_k) - D^{(2)}_{kj} u(x_j) = f(x_k), \quad \forall k = 0, 1, \ldots, N-1.$$

   Denote by $U = (u(x_0), u(x_1), \ldots, u(x_{N-1}))^T$, and $F = (f(x_0), f(x_1), \ldots, f(x_{N-1}))^T$, we have

$$\alpha\, U - D^{(2)} U = F, \quad i.e. \quad \big(\alpha I - D^{(2)}\big) U = F \tag{25}$$

   The above linear system is a dense matrix. To solve it, we need use some matrix factorization skill or Gauss elimination method.

2. Galerkine method. In equation (24), using $u_N = \sum_{k=-N/2}^{N/2} v_k\, e^{ikx}$ to approximate the solution, and taking inner product the result equation with e^{ijx}, we get

$$\alpha\, v_k\, \delta_{kj} + v_k k^2\, \delta_{kj} = f_j, \quad f_k = (I_N f, e^{ikx}) = \langle f, e^{ikx}\rangle_N, \quad j = 0, 1, \ldots, \pm N/2,$$

   i.e.

$$\alpha\, v_j + j^2 v_j = f_j, \quad j = 0, 1, \ldots, \pm N/2,$$

   so we get

$$v_j = \frac{f_j}{\alpha + j^2}, \quad j = 0, 1, \ldots, \pm N/2. \tag{26}$$

   Thus the numerical solution is

$$u_N = \sum_{k=-N/2}^{N/2} v_j\, e^{ikx} = \sum_{k=-N/2}^{N/2} \frac{f_j}{\alpha + j^2} e^{ikx}.$$

   We can done this by three steps.

   a. calculate $f_k, k = 0, 1, \ldots, \pm N/2$, by FFT
   b. calulate v_j using (26).
   c. calculate u_N by the inverse FFT.

3.1 Error estimate

Suppose that $u \in H^m_p(I)$, then by first Strang lemma (Theorem 1.3 in ShenTangWang's book)

$$\|u - u_N\|_1 \lesssim \|u - P_N u\|_1 + \sup_{v_N \in X_N} \frac{|(f - I_N f, v_N)|}{\|v_N\|_1} \lesssim N^{1-m} |u|_m + \|f - I_N f\|_{-1} \tag{27}$$

Since this equation is so simple, we know that the regularity of u is 2-order higher than f, i.e. if $f \in H^s_p(I)$, then $u \in H^{s+2}_p(I)$.

3.2 Numerical results

We use several f with different regularity to verify the convergence rate. For $\alpha = 1$, the right hand side

$$f^k(x) = \begin{cases} 10 \sin^3|x - \pi| - 6 \sin|x - \pi|, & k = 1, \\ e^{\sin(x)}\, (\sin^2(x) + \sin(x)), & k = 2 \end{cases}$$

corresponds to solution

$$u^k(x) = \begin{cases} \sin^3|x-\pi|, & k=1, \\ e^{\sin(x)}, & k=2. \end{cases}$$

We use following octave code (octave is a free clone of Matlab, http://www.gnu.org/software/octave/) to verify the convergence rate (27) of the numerical solution.

```
% lec2.m - convergence of the Fourier method for linear elliptic equations
a=1;
% algebraic convergence for f1
Nvec = 2.^(3:12);
clf, subplot(1,2,1);
for N=Nvec
    h=2*pi/N; x=0:h:2*pi-h;
    ii=1:N; kk=N/2-abs(ii-N/2-1);
    ue=sin(abs(x-pi)).^3;  f=10*sin(abs(x-pi)).^3 - 6*sin(abs(x-pi));
% three steps to solve the linear system
    v=fft(f, N);
    v=v./(a+kk.^2);
    u=ifft(v,N);
% plot the L2 and H1 norm error
    uerr=fft(u-ue,N)/N;
    err0 = norm(abs(uerr), 2);                % L^2 norm
    err1 = norm(abs(uerr).*sqrt(a+kk.^2), 2); % H^1 norm
    loglog(N, err0, 'r+', 'markersize', 14), hold on
    loglog(N, err1, 'go', 'markersize', 14), hold on
end
grid on, xlabel N, ylabel error
title('The L^2(+) and H^1(o) convergence for f^1');
loglog(Nvec, Nvec.^(-2), '--');

% spectral convergence for f2
Nvec = 4:2:28; subplot(1,2,2);
for N=Nvec
    h=2*pi/N; x=0:h:2*pi-h;
    ii=1:N; kk=N/2-abs(ii-N/2-1);
    ue=exp(sin(x)); f=exp(sin(x)).*(sin(x).^2+sin(x));
% three steps to solve the linear system
    v=fft(f, N);
    v=v./(a+kk.^2);
    u=ifft(v,N);
% plot the L2 and H1 norm error
    uerr=fft(u-ue,N)/N;
    err0 = norm(abs(uerr), 2); % L^2 norm
    err1 = norm(abs(uerr).*sqrt(a+kk.^2), 2); % H^1 norm
    semilogy(N, err0, 'r+', 'markersize', 14), hold on
    semilogy(N, err1, 'go', 'markersize', 14), hold on
end
grid on, xlabel N, ylabel error
title('The L^2(+) and H^1(o) convergence for f^2');
semilogy(Nvec, exp(-Nvec*1.5), '--');
```

The numerical results show that for the first right hand side f^1, the convergence rate is N^{-2}, for the right hand side f^2, the convergence rate is $\exp(-1.5N)$, this is consistent to the regularity of f, u and the error estimate (27).

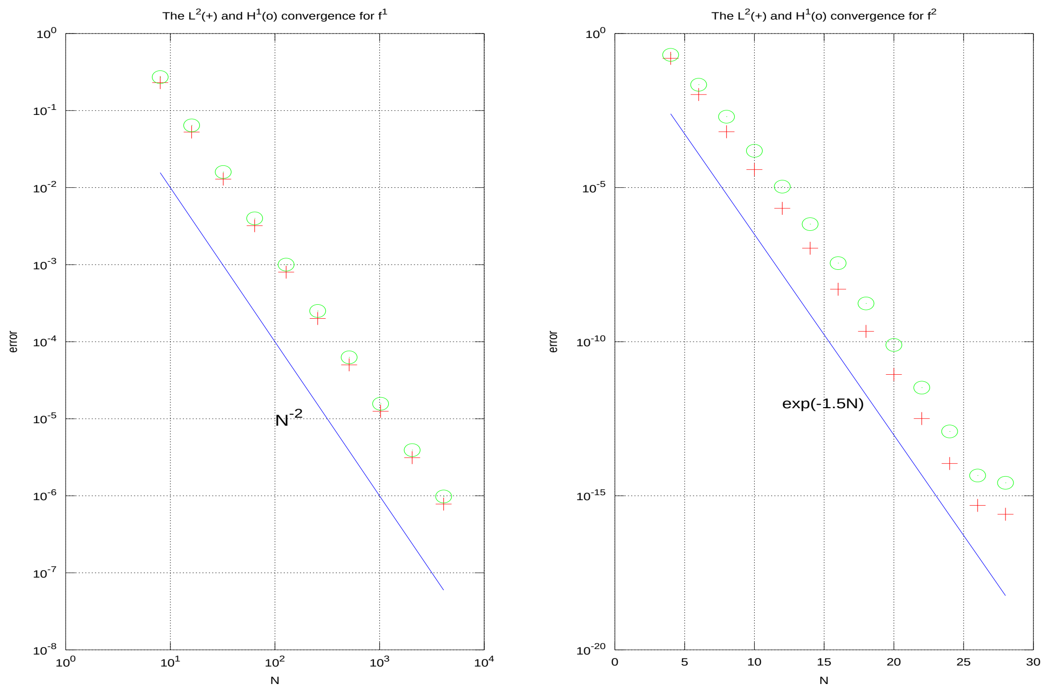

Figure 1. The convergence rate of the Fourier spectral method for equation (24) for $f = f^1$ (Left) and $f = f^2$ (right).

4 Homework

1. Let

$$s(x) = \begin{cases} 1/2, & |x| \leqslant 1, \\ 0, & |x| > 1. \end{cases}$$

calculate the Fourier transform of $s, s{*}s, s{*}s{*}s$, where $*$ denote **convolution** operation defined by

$$(u{*}v)(x) = \int_{-\infty}^{\infty} u(y)v(x-y)\,\mathrm{d}y = \int_{-\infty}^{\infty} v(y)u(x-y)\,\mathrm{d}y.$$

2. Calculate the Fourier series of the periodic function (period$=2\pi$)

$$\delta(x) = \begin{cases} \dfrac{N}{2\pi}, & x \in [-\pi/N, \pi/N], \quad N > 1. \\ 0, & \text{otherwise}. \end{cases}$$

3. Prove the aliasing formula (23).