Homework 1

Solve the following periodic heat equation using Finite Difference Method(FDM), Finite Element Method(FEM), and Fourier Spectral Method(FSM)

$$\begin{cases} u_t = u_{xx}, \\ u(x,0) = u_0(x) = e^{\sin(x)}, \\ u(x+2\pi) = u(x). \end{cases} \tag{1}$$

Solution: We use implicit Euler time discretization for both FDM and FEM, exact time integration for Fourier spectral method. Given $N > 0$, denote by $h = 2\pi/N$, $x_j = 2j\,\pi/N$. Let δt by the time step, $t^n = n\ \delta t$, given $U^0 = (u_0(x_0),\ \ldots,\ u_0(x_{N-1}))^T$, we need to calculate $U^n = (u_0^n, \ldots, u_{N-1}^n)^T$ as the approximate solution of $(u(x_0, t^n), \ldots, u(x_{N-1}, t^n))^T$ using FDM, FEM and FSM.

- *Finite difference method.*

  $$(I - \delta t\, D) U^{n+1} = U^n, \tag{2}$$

  where D is

  $$D = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{pmatrix} \in \mathbb{R}^{N\times N}$$

  (2) is equivalent to (need no loops in matlab)

  $$U^n = (I - \delta t\, D)^{-n} U^0.$$

- *Finite element method.*

  $$(M + \delta t\, S)\, U^{n+1} = M U^n, \tag{3}$$

  which is equivalent to

  $$U^n = (I + \delta t\, M^{-1} S)^{-n}\, U^0.$$

  Here both M and S are $N \times N$ Toeplitz matrix, and they are given by

  $$M = \frac{h}{6}\begin{pmatrix} 4 & 1 & & & 1 \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ 1 & & & 1 & 4 \end{pmatrix}, \quad S = \frac{1}{h}\begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix}.$$

- *Fourier spectral method.*
  We only need three steps to solve the heat equation using Fourier spectral method.

  1. Transform U^0 to $\{\hat{u}_k(0)\}$ by Fast Fourier Transform
  2. Exact time integration

     $$\hat{u}_k(t^n) = \hat{u}_k(0)\, e^{-k^2 t^n},$$

  3. Transform $\{\hat{u}_k(t^n)\}$ back to U^n by inverse Fast Fourier Transform

Octave code:

```
% p11f.m - Comparison among FDM, FEM and FSM solving periodic heat equation
clear;
u0=inline('exp(sin(x))','x');
% Use the spectral procedure to calculate a solution as exact solution
function [u, x] = EvalSolEx(uinit, N, t)
   h=2*pi/N; x=(0:h:2*pi-h)'; k=[0:N/2-1 N/2 -N/2+1:-1]';
   ui=feval(uinit, x);
   v=fft(ui, N); v=v.*exp(-k.^2 * t); u=real(ifft(v,N));
end
function errsp = spm(uinit, Nvec, t)
   errsp = [];
   [ue, xe] = EvalSolEx(uinit, 2048, t);
   for N=Nvec
      [u, x] = EvalSolEx(uinit, N, t);
      solu = interp1(xe, ue, x, 'spline');
      err = norm(u-solu, 'inf');  errsp = [errsp err];
   end
end

function errfd = fdm(uinit, Nvec, dt, nt)
   errfd = [];
   [ue, xe] = EvalSolEx(uinit, 1024, dt*nt);
   for N=Nvec
      h =2*pi/N; x=(0:h:2*pi-h)'; a=dt/h^2;
      u0 = feval(uinit, x);
      D = toeplitz([2 -1 zeros(1,N-3) -1]);
      un = (eye(N)+a*D)^(-nt)*u0;
      solu = interp1(xe, ue, x, 'spline');
      err = norm(un-solu, 'inf'); errfd = [errfd err];
   end
end
function errfe = fem(uinit, Nvec, dt, nt)
   errfe = [];
   [ue, xe] = EvalSolEx(uinit, 1024, dt*nt);
   for N=Nvec
      h = 2*pi/N; x=(0:h:2*pi-h)'; a=dt/h^2;
      u0 = feval(uinit, x);
      M = toeplitz([2/3 1/6 zeros(1,N-3) 1/6]);
      S = toeplitz([2 -1 zeros(1,N-3) -1]);
      un = (eye(N)+a*(M\S))^(-nt)*u0;
      solu = interp1(xe, ue, x, 'spline');
      err = norm(un-solu, 'inf');  errfe = [errfe err];
    end
end

figure('Position',[0 0 1000 500] ), subplot(1,2,1);
Nvec = 2.^(3:9); dt = .5e-4; nt = 10000;
errfd = fdm(u0, Nvec, dt, nt); errfe = fem(u0, Nvec, dt, nt);
loglog(Nvec, errfd, 'r+', 'markersize', 14), hold on
loglog(Nvec, errfe, 'bo', 'markersize', 14);
title('The maximum error of FD(+) and FE(o) methods');
grid on, xlabel N, ylabel error

subplot(1,2,2);  Nvec = 4:2:16;
errsp= spm(u0, Nvec, dt*nt);
semilogy(Nvec, errsp, 'r+', 'markersize', 14), hold on
grid on, xlabel N, ylabel error
title('The maximum error of the Fourier spectral method');
```

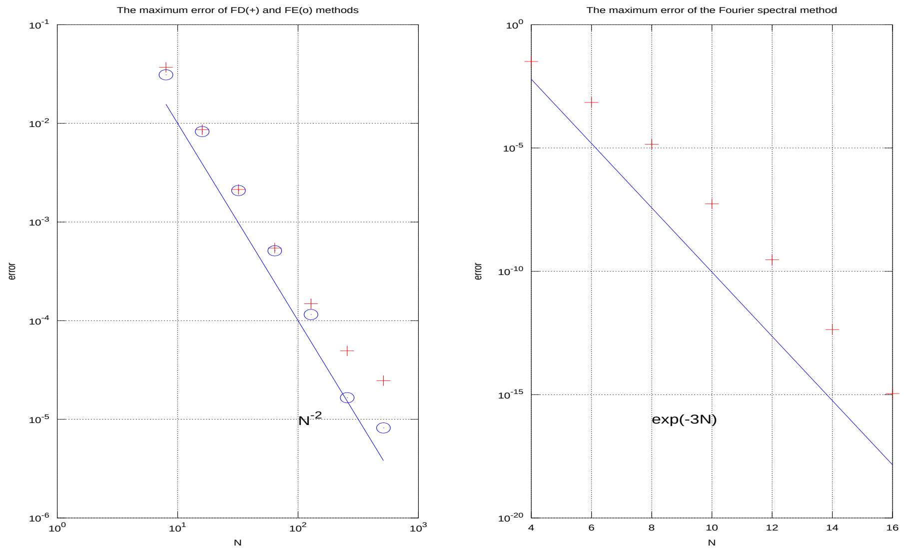

Figure 1. The convergence rate of FDM, FEM (left), and FSM (right) in solving the periodic heat equation with initial condition $u_0(x) = e^{\sin(x)}$. The plotted error is maximum norm error for $t = 0.5$. The time step of the implicit Euler for FDM and FEM is 0.5×10^{-4}.
From this figure, we can conclude that the Fourier spectral method converges geometrically, while both the FDM and FEM converge algebraicly.