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Invariant measures of stochastic Maxwell equations and ergodic numerical approximations ☆

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Abstract

This paper studies the existence and uniqueness of the invariant measure for a class of stochastic Maxwell equations and proposes a novel kind of ergodic numerical approximations to inherit the intrinsic properties. The key to proving the ergodicity lies in the uniform regularity estimates of the exact and numerical solutions with respect to time, which are established by analyzing some important physical quantities. By introducing an auxiliary process, we show that the mean-square convergence order of the discontinuous Galerkin full discretization is $\frac{1}{2}$ in the temporal direction and $\frac{1}{2}$ in the spatial direction, which provides the convergence order of the numerical invariant measure to the exact one in L^2-Wasserstein distance.

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1. Introduction

Stochastic Maxwell equations play an important role in many fields, including statistical radiophysics and stochastic electromagnetism, whose stochasticity may come from the random medium or the stochastic source (see e.g. [17]). In this paper, we consider the following stochastic Maxwell equations in an isotropic conductive medium

$$\begin{cases} \mathrm{d}\mathbf{E}(t) = \nabla \times \mathbf{H}(t)\mathrm{d}t - \sigma \mathbf{E}(t)\mathrm{d}t - \lambda_1 \mathbf{H}(t) \circ \mathrm{d}W_1(t) + \boldsymbol{\lambda}_2 \mathrm{d}W_2(t), & (t,\mathbf{x}) \in \mathbb{R}_+ \times D, \\ \mathrm{d}\mathbf{H}(t) = -\nabla \times \mathbf{E}(t)\mathrm{d}t - \sigma \mathbf{H}(t)\mathrm{d}t + \lambda_1 \mathbf{E}(t) \circ \mathrm{d}W_1(t) + \boldsymbol{\lambda}_2 \mathrm{d}W_2(t), & (t,\mathbf{x}) \in \mathbb{R}_+ \times D, \\ \mathbf{E}(0,\mathbf{x}) = \mathbf{E}_0(\mathbf{x}),\ \mathbf{H}(0,\mathbf{x}) = \mathbf{H}_0(\mathbf{x}), & \mathbf{x} \in D, \\ \mathbf{n} \times \mathbf{E} = 0,\ \mathbf{n} \cdot \mathbf{H} = 0, & (t,\mathbf{x}) \in \mathbb{R}_+ \times \partial D, \end{cases} \tag{1.1}$$

where $D = (x_L, x_R) \times (y_L, y_R) \times (z_L, z_R) \subset \mathbb{R}^3$ is a cuboid, $\mathbf{n}$ is the outer unit normal, the notation $\circ$ means Stratonovich integral, $\nabla\times$ is the curl operator, $\mathbf{E} = (E_1, E_2, E_3)^\top$ and $\mathbf{H} = (H_1, H_2, H_3)^\top$ are the electromagnetic field, $\lambda_1 \in \mathbb{R}$ and $\boldsymbol{\lambda}_2 = (\lambda_2^{(1)}, \lambda_2^{(2)}, \lambda_2^{(3)})^\top$. Here, $\{W_1(t)\}_{t\geq 0}$ and $\{W_2(t)\}_{t\geq 0}$ are two independent Wiener processes with respect to a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, which characterize the randomness from the medium and the source, respectively. The damping terms $\sigma\mathbf{E}$ and $\sigma\mathbf{H}$ may be induced by conductivity of the medium or by the perfectly matched layer technique (see e.g. [16,19]). Due to the existence of these damping terms, the properties of stochastic Maxwell equations change tremendously. The aim of this paper is to analyze the longtime properties of (1.1) and further construct numerical discretizations to inherit these properties.

As we all know, ergodicity is an important longtime property of stochastic partial differential equations (SPDEs). There have been many works concentrating on the study of their ergodicity and ergodic numerical approximations; see e.g., [15] for the stochastic nonlinear Schrödinger equation and [8] for parabolic SPDEs. However, to our best knowledge, there is no result on the ergodicity of stochastic Maxwell equations. The main difficulty lies in the uniform estimates of the exact solution with respect to time. By analyzing some important physical quantities of (1.1), we prove that the solution is bounded uniformly in $L^2(\Omega, H^1(D)^6)$ in time. This result ensures that the $H^1(D)^6$-norm of the solution is a proper choice for Lyapunov function, which leads to the existence of the invariant measure for (1.1). Furthermore, we obtain the continuous dependence of the solution on the initial data with exponential decay rate. As a consequence, (1.1) possesses a unique invariant measure π^* which is ergodic and exponentially mixing. Moreover, we show that the phase flow of (1.1) possesses the stochastic conformal multi-symplectic structure.

It is meaningful and important to design structure-preserving numerical discretizations since they have remarkable superiority in the longtime computation. In recent years, various symplectic and multi-symplectic numerical methods have been proposed and analyzed for stochastic Maxwell equations without damping terms. We refer the interested readers to [6,13,14] for the stochastic multi-symplectic methods, to [4] for the symplectic Runge–Kutta method, to [1,21,20] for the symplectic and multi-symplectic discontinuous Galerkin methods, to [3,7] for the semi-implicit and exponential Euler methods. For stochastic Maxwell equations with damping terms, we do not find any relative works on constructing numerical discretizations to inherit the ergodicity up to now.

To this end, we first propose a novel temporal semi-discretization for (1.1) which is a modification of the midpoint method. This temporal semi-discretization is specially constructed to preserve the ergodicity and the stochastic conformal multi-symplecticity simultaneously. Compared with the continuous case, the proof of the ergodicity of the temporal semi-discretization is more complicated since the curl and divergence of the numerical solution must be estimated together. By establishing the uniform boundedness of the numerical solution in $L^2(\Omega, H^1(D)^6)$ with respect to time, we show that the temporal semi-discretization is ergodic with a unique invariant measure $\pi^{\Delta t}$. The mean-square convergence order of the temporal semi-discretization is shown to be $\frac{1}{2}$, which provides the convergence order of the numerical invariant measure $\pi^{\Delta t}$ to the exact one π^* in L^2-Wasserstein distance.

Further, we apply the discontinuous Galerkin (dG) method to discretize the temporal semi-discretization in space. The ergodicity of the dG numerical solution is obtained by establishing the uniform boundedness of the numerical solution in $L^2(\Omega, \mathbb{H})$, where $\mathbb{H} := L^2(D)^3 \times L^2(D)^3$. The mean-square convergence analysis of the dG full discretization is more challenging due to the low regularity of the numerical solution. To solve this problem, we introduce an auxiliary process in our convergence analysis, which allows us to take full advantage of the $H^1(D)^6$-regularity of the exact solution. We show that the mean-square convergence order of the dG full discretization is $\frac{1}{2}$ both in the temporal and spatial directions. As a byproduct, the L^2-Wasserstein distance between the numerical invariant measure $\pi^{\Delta t,h}$ and the exact one π^* is estimated. We remark that there exist many alternative choices of the spatial discretization. For example, we also use a finite difference method to discretize the temporal semi-discretization in space. We prove that this full discretization preserves the ergodicity by deriving the uniform boundedness of the averaged discrete energy, and meanwhile it possesses the discrete stochastic conformal multi-symplectic conservation law. Numerical experiments are given to verify the performance of the proposed full discretization.

The rest of this paper is organized as follows. In Section 2, we first introduce some notations and focus on studying the uniformly boundedness, ergodicity and stochastic conformal multi-symplecticity of (1.1). In Section 3, an ergodic modified midpoint temporal semi-discretization of (1.1) is proposed. We establish the mean-square convergence of the numerical scheme. Section 4 is devoted to designing ergodic full discretizations. Finally, numerical results are shown to support the theoretical analysis in Section 5. The details of the proof to a priori estimates of some operators are provided in Appendices.

2. Properties of stochastic Maxwell equations

In this section, we investigate the regularities, ergodicity and the stochastic conformal multi-symplecticity of the solution of (1.1), which make preparations for the numerical approximations in the rest sections. Throughout this paper, C will be used to denote a generic positive constant independent of time.

Let $W^{k,p}(D)$ be the standard Sobolev space. Especially, we denote $H^k(D) := W^{k,2}(D)$. Denote the Euclidean norm in $\mathbb{R}^6$ by $|\cdot|$. We define the Maxwell operator by

$$M = \begin{pmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{pmatrix} \tag{2.1}$$

with domain

$$\begin{aligned}\mathcal{D}(M) &= \left\{ \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in \mathbb{H} : M \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \nabla \times \mathbf{H} \\ -\nabla \times \mathbf{E} \end{pmatrix} \in \mathbb{H},\ \mathbf{n} \times \mathbf{E}|_{\partial D} = \mathbf{0} \right\} \\ &= H_0(\mathrm{curl}, D) \times H(\mathrm{curl}, D),\end{aligned} \tag{2.2}$$

where the curl-spaces are defined by

$$H(\mathrm{curl}, D) := \{v \in L^2(D)^3 : \nabla \times v \in L^2(D)^3\}$$

and

$$H_0(\mathrm{curl}, D) := \{v \in H(\mathrm{curl}, D) : \mathbf{n} \times v|_{\partial D} = \mathbf{0}\}.$$

The corresponding graph norm is $\|v\|_{\mathcal{D}(M)} := (\|v\|_{\mathbb{H}}^2 + \|Mv\|_{\mathbb{H}}^2)^{\frac{1}{2}}$ and the Maxwell operator M is skew-adjoint on $\mathbb{H}$ (see e.g., [3]). Denote by $HS(U, H)$ the Banach space of all Hilbert–Schmidt operators from one separable Hilbert space U to another separable Hilbert space H, equipped with the norm

$$\|\Gamma\|_{HS(U,H)} = \Big(\sum_{j=1}^{\infty} \|\Gamma q_j\|_H^2 \Big)^{\frac{1}{2}} \quad \forall \Gamma \in HS(U, H),$$

where $\{q_j\}_{j\in\mathbb{N}_+}$ is an orthonormal basis of U. For the Wiener processes, we give the following assumption.

Assumption 2.1. For $i = 1, 2$, assume that $W_i(t)$ is a Q_i-Wiener process on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, which can be represented as

$$W_i(t) = \sum_{k=1}^{\infty} \sqrt{\eta_k^{(i)}} q_k \beta_k^{(i)}(t), \quad t \geq 0,$$

where $Q_i q_k = \eta_k^{(i)} q_k$ with $\eta_k^{(i)} \geq 0$ and $\{q_k\}_{k\in\mathbb{N}_+}$ being the orthonormal basis of $L^2(D)$. In addition, assume that $Q_i^{\frac{1}{2}} \in HS(L^2(D), H^{\gamma_i}(D)) =: \mathcal{L}_2^{\gamma_i}$ for some $\gamma_i \geq 0$.

Assumption 2.2. Assume that $\sigma \in W^{1,\infty}(D)$ and $\sigma \geq \sigma_0 > 0$ for a constant σ_0.

Note that there exist some functions satisfying Assumption 2.2, such as the positive constant functions, the positive piecewise constant functions and polynomials with positive lower bound (see e.g., [19]).

Let $u = (\mathbf{E}^\top, \mathbf{H}^\top)^\top$ and $u_0 = (\mathbf{E}_0^\top, \mathbf{H}_0^\top)^\top$. We can rewrite (1.1) as a stochastic evolution equation

$$\begin{cases} \mathrm{d}u(t) = \Big(Mu(t) - \sigma u(t) \Big)\mathrm{d}t + \lambda_1 J u(t) \circ \mathrm{d}W_1(t) + \widetilde{\boldsymbol{\lambda}}_2 \mathrm{d}W_2(t), & t > 0, \\ u(0) = u_0, \end{cases} \tag{2.3}$$

where $\widetilde{\boldsymbol{\lambda}}_2 = (\boldsymbol{\lambda}_2^\top, \boldsymbol{\lambda}_2^\top)^\top$, $J = \begin{pmatrix} 0 & -I_3 \\ I_3 & 0 \end{pmatrix}$ with I_3 being the identity matrix on $\mathbb{R}^{3\times 3}$. The equivalent Itô formulation of (2.3) reads as

$$\mathrm{d}u(t) = \Big(Mu(t) - \sigma u(t) - \frac{1}{2}\lambda_1^2 F_{Q_1} u(t)\Big)\mathrm{d}t + \lambda_1 J u(t)\mathrm{d}W_1(t) + \widetilde{\boldsymbol{\lambda}}_2 \mathrm{d}W_2(t) \tag{2.4}$$

for $t > 0$, where $F_{Q_1} = \sum_{k=1}^{\infty} \eta_k^{(1)} (q_k)^2$.

The following proposition states the well-posedness and the uniform boundedness in $L^2(\Omega, \mathbb{H})$ of the solution of (2.4).

Proposition 2.3. *Let Assumption 2.1 hold with $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$, and let $u_0 \in L^2(\Omega, \mathbb{H})$. Then the system (2.4) is well-posed. Moreover, there exists a positive constant $C_1 := C_1(\sigma_0, \widetilde{\boldsymbol{\lambda}}_2, \mathrm{tr}(Q_2))$ such that*

$$\mathbb{E}\big[\|u(t)\|_{\mathbb{H}}^2\big] \leq e^{-2\sigma_0 t}\mathbb{E}\big[\|u_0\|_{\mathbb{H}}^2\big] + C_1\big(1 - e^{-2\sigma_0 t}\big).$$

Proof. The well-posedness of (2.4) follows similarly to [3, Theorem 2.1]. By applying the Itô formula to $\|u\|_{\mathbb{H}}^2$, we have

$$\begin{aligned} \mathrm{d}\|u(t)\|_{\mathbb{H}}^2 &= -2\langle u(t), \sigma u(t)\rangle_{\mathbb{H}}\mathrm{d}t + |\widetilde{\boldsymbol{\lambda}}_2|^2 \mathrm{tr}(Q_2)\mathrm{d}t + 2\langle u(t), \widetilde{\boldsymbol{\lambda}}_2 \mathrm{d}W_2(t)\rangle_{\mathbb{H}} \\ &\leq -2\sigma_0\|u(t)\|_{\mathbb{H}}^2\mathrm{d}t + |\widetilde{\boldsymbol{\lambda}}_2|^2 \mathrm{tr}(Q_2)\mathrm{d}t + 2\langle u(t), \widetilde{\boldsymbol{\lambda}}_2 \mathrm{d}W_2(t)\rangle_{\mathbb{H}} \end{aligned} \tag{2.5}$$

due to the skew-adjointness of M. By taking the expectation on both sides of (2.5) and using the Gronwall inequality, we have

$$\begin{aligned} \mathbb{E}\big[\|u(t)\|_{\mathbb{H}}^2\big] &\leq e^{-2\sigma_0 t}\mathbb{E}\big[\|u_0\|_{\mathbb{H}}^2\big] + (1 - e^{-2\sigma_0 t})\frac{|\widetilde{\boldsymbol{\lambda}}_2|^2 \mathrm{tr}(Q_2)}{2\sigma_0} \\ &=: e^{-2\sigma_0 t}\mathbb{E}\big[\|u_0\|_{\mathbb{H}}^2\big] + (1 - e^{-2\sigma_0 t})C_1. \end{aligned}$$

Thus we finish the proof. □

2.1. Ergodicity

In this part, we investigate the ergodicity of (1.1), that is, the existence and uniqueness of the invariant measure. Let $P_t\varphi(x) := \mathbb{E}[\varphi(u(t))]$, $t \geq 0$ which is the Markov transition semigroup associated to the solution u of (1.1). Denote by $\mathcal{P}(\mathbb{H})$ the space of all Borel probability measures on $\mathbb{H}$. A probability measure $\pi \in \mathcal{P}(\mathbb{H})$ is said to be invariant for u if $\int_{\mathbb{H}} P_t\varphi(x)\pi(\mathrm{d}x) = \int_{\mathbb{H}} \varphi \mathrm{d}\pi =: \pi(\varphi)$ for any Borel bounded mapping φ and $t \geq 0$. Further, u is said to be ergodic on $\mathbb{H}$ if $\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}[\varphi(u(t))]\mathrm{d}t = \pi(\varphi)$ in $L^2(\mathbb{H}, \pi)$ for all $\varphi \in L^2(\mathbb{H}, \pi)$; u is said to be exponentially mixing on $\mathbb{H}$ if there exist a positive constant ρ and a positive function $C(\cdot)$ such that for any bounded Lipschitz continuous function φ on $\mathbb{H}$, all $t > 0$ and all $u_0 \in \mathbb{H}$, $\big|P_t\varphi(u_0) - \pi(\varphi)\big| \leq C(u_0)L_\varphi e^{-\rho t}$ with L_φ being the Lipschitz constant of φ.

In order to obtain the ergodicity, we first derive the uniform boundedness of the solution of (1.1) in $L^2(\Omega, H^1(D)^6)$ in the following lemma.

Lemma 2.4. *Let Assumption 2.1 hold with $\gamma_1 > \frac{5}{2}$ and $\gamma_2 \geq 1$, let $u_0 \in L^2(\Omega, H^1(D)^6)$ and $F_{Q_1} \in W^{1,\infty}(D)$. Then the solution of (1.1) is bounded uniformly in time, i.e.,*

$$\mathbb{E}\big[\|u(t)\|^2_{H^1(D)^6}\big] \leq C_2 e^{-\sigma_0 t}\mathbb{E}\big[\|u_0\|^2_{H^1(D)^6}\big] + C_3, \tag{2.6}$$

where the positive constant C_2 depends on $|D|$, and the positive constant C_3 depends on σ_0, $\|\sigma\|_{W^{1,\infty}(D)}$, $\|F_{Q_1}\|_{W^{1,\infty}(D)}$, C_1, $\mathbb{E}\big[\|u_0\|^2_{\mathbb{H}}\big]$, λ_1, $\widetilde{\boldsymbol{\lambda}}_2$, $\|Q_1^{\frac{1}{2}}\|_{\mathcal{L}_2^{\gamma_1}}$ and $\|Q_2^{\frac{1}{2}}\|_{\mathcal{L}_2^{\gamma_2}}$.

Proof. *Step 1. Uniform boundedness of the curl of the solution.*

Applying the Itô formula to $\|Mu\|^2_{\mathbb{H}}$, we have

$$\begin{aligned}\mathrm{d}\mathbb{E}\big[\|Mu(t)\|^2_{\mathbb{H}}\big] =& -2\mathbb{E}\big[\langle Mu(t), M\big((\sigma + \frac{1}{2}\lambda_1^2 F_{Q_1})u(t)\big)\rangle_{\mathbb{H}}\big]\mathrm{d}t \\ &+ \lambda_1^2\mathbb{E}\Big[\sum_{k=1}^{\infty}\big\|M(Ju(t)Q_1^{\frac{1}{2}}q_k)\big\|^2_{\mathbb{H}}\Big]\mathrm{d}t + \sum_{k=1}^{\infty}\big\|M(\widetilde{\boldsymbol{\lambda}}_2 Q_2^{\frac{1}{2}}q_k)\big\|^2_{\mathbb{H}}\mathrm{d}t. \end{aligned}\tag{2.7}$$

We note that

$$\begin{aligned}-2\mathbb{E}\Big[\Big\langle Mu, M\big((\sigma + \frac{1}{2}\lambda_1^2 F_{Q_1})u\big)\Big\rangle_{\mathbb{H}}\Big] =& -2\mathbb{E}\Big[\Big\langle Mu, (\sigma + \frac{1}{2}\lambda_1^2 F_{Q_1})Mu\Big\rangle_{\mathbb{H}}\Big] \\ &- 2\mathbb{E}\Big[\Big\langle Mu, \begin{pmatrix}\nabla(\sigma + \frac{1}{2}\lambda_1^2 F_{Q_1}) \times \mathbf{H} \\ -\nabla(\sigma + \frac{1}{2}\lambda_1^2 F_{Q_1}) \times \mathbf{E}\end{pmatrix}\Big\rangle_{\mathbb{H}}\Big] \\ \leq& -\frac{3}{2}\sigma_0\mathbb{E}\big[\|Mu\|^2_{\mathbb{H}}\big] - \lambda_1^2\mathbb{E}\big[\langle Mu, F_{Q_1}Mu\rangle_{\mathbb{H}}\big] + C,\end{aligned}\tag{2.8}$$

where we use the fact that

$$\begin{aligned}-2\mathbb{E}\Big[\Big\langle Mu, \begin{pmatrix}\nabla(\sigma + \frac{1}{2}\lambda_1^2 F_{Q_1}) \times \mathbf{H} \\ -\nabla(\sigma + \frac{1}{2}\lambda_1^2 F_{Q_1}) \times \mathbf{E}\end{pmatrix}\Big\rangle_{\mathbb{H}}\Big] &\leq 4\mathbb{E}\big[\|Mu\|_{\mathbb{H}}\|u\|_{\mathbb{H}}\big]\|\nabla(\sigma + \frac{1}{2}\lambda_1^2 F_{Q_1})\|_{L^\infty(D)^3} \\ &\leq \frac{1}{2}\sigma_0\mathbb{E}[\|Mu\|^2_{\mathbb{H}}] + C\end{aligned}$$

due to the Young inequality, the assumption $\sigma \in W^{1,\infty}(D, \mathbb{R})$ and Proposition 2.3.

By using the Sobolev embedding $H^\gamma(D) \hookrightarrow L^\infty(D)$ for $\gamma > \frac{3}{2}$, the Hölder inequality, the Young inequality and Proposition 2.3, it holds that

$$\begin{aligned}&\lambda_1^2\mathbb{E}\Big[\sum_{k=1}^{\infty}\|M(JuQ_1^{\frac{1}{2}}q_k)\|^2_{\mathbb{H}}\Big] + \sum_{k=1}^{\infty}\|M(\widetilde{\boldsymbol{\lambda}}_2 Q_2^{\frac{1}{2}}q_k)\|^2_{\mathbb{H}} \\ &= \lambda_1^2\mathbb{E}[\langle Mu, F_{Q_1}Mu\rangle_{\mathbb{H}}] + \lambda_1^2\mathbb{E}\left[\sum_{k=1}^{\infty}\left\|\begin{pmatrix}\nabla(Q_1^{\frac{1}{2}}q_k) \times \mathbf{E} \\ \nabla(Q_1^{\frac{1}{2}}q_k) \times \mathbf{H}\end{pmatrix}\right\|^2_{\mathbb{H}}\right] \\ &\quad + 2\lambda_1^2\mathbb{E}\left[\sum_{k=1}^{\infty}\left\langle\begin{pmatrix}(Q_1^{\frac{1}{2}}q_k)\nabla \times \mathbf{E} \\ (Q_1^{\frac{1}{2}}q_k)\nabla \times \mathbf{H}\end{pmatrix}, \begin{pmatrix}\nabla(Q_1^{\frac{1}{2}}q_k) \times \mathbf{E} \\ \nabla(Q_1^{\frac{1}{2}}q_k) \times \mathbf{H}\end{pmatrix}\right\rangle_{\mathbb{H}}\right]\end{aligned}\tag{2.9}$$

$$+2\sum_{k=1}^{\infty}\|\boldsymbol{\lambda}_2\times\nabla(Q_2^{\frac{1}{2}}q_k)\|^2_{L^2(D)^3}$$
$$\le \lambda_1^2\mathbb{E}\big[\langle Mu, F_{Q_1}Mu\rangle_{\mathbb{H}}\big]+\frac{1}{2}\sigma_0\mathbb{E}\big[\|Mu\|^2_{\mathbb{H}}\big]+C.$$

Substituting (2.8) and (2.9) into (2.7), we obtain

$$\mathrm{d}\mathbb{E}\big[\|Mu(t)\|^2_{\mathbb{H}}\big]\le -\sigma_0\mathbb{E}\big[\|Mu(t)\|^2_{\mathbb{H}}\big]\mathrm{d}t+C\mathrm{d}t,$$

which combining the Gronwall inequality yields

$$\mathbb{E}\big[\|Mu(t)\|^2_{\mathbb{H}}\big]\le e^{-\sigma_0 t}\mathbb{E}\big[\|Mu_0\|^2_{\mathbb{H}}\big]+C.$$

Step 2. Uniform boundedness of the divergence of the solution.

Applying the Itô formula to $\|\nabla\cdot\mathbf{E}\|^2_{L^2(D)}$ and taking the expectation, we arrive at

$$\mathrm{d}\mathbb{E}[\|\nabla\cdot\mathbf{E}(t)\|^2_{L^2(D)}]=-2\mathbb{E}\Big[\Big\langle\nabla\cdot\mathbf{E}(t),\nabla\cdot\Big((\sigma+\frac{1}{2}\lambda_1^2F_{Q_1})\mathbf{E}(t)\Big)\Big\rangle_{L^2(D)}\Big]\mathrm{d}t$$
$$+\lambda_1^2\mathbb{E}\Big[\sum_{k=1}^{\infty}\|\nabla\cdot(\mathbf{H}(t)Q_1^{\frac{1}{2}}q_k)\|^2_{L^2(D)}\Big]\mathrm{d}t+\sum_{k=1}^{\infty}\|\nabla\cdot(\boldsymbol{\lambda}_2Q_2^{\frac{1}{2}}q_k)\|^2_{L^2(D)}\mathrm{d}t. \tag{2.10}$$

By the Sobolev embedding $H^{\gamma}(D)\hookrightarrow L^{\infty}(D)$ for $\gamma>\frac{3}{2}$, the Hölder inequality, the Young inequality, the assumption $\sigma, F_{Q_1}\in W^{1,\infty}(D)$ and Proposition 2.3, we have

$$-2\mathbb{E}\Big[\Big\langle\nabla\cdot\mathbf{E},\nabla\cdot\big((\sigma+\frac{1}{2}\lambda_1^2F_{Q_1})\mathbf{E}\big)\Big\rangle_{L^2(D)}\Big]$$
$$=-2\mathbb{E}\Big[\Big\langle\nabla\cdot\mathbf{E},\nabla\big(\sigma+\frac{1}{2}\lambda_1^2F_{Q_1}\big)\cdot\mathbf{E}\Big\rangle_{L^2(D)}\Big]-2\mathbb{E}\Big[\Big\langle\nabla\cdot\mathbf{E},(\sigma+\frac{1}{2}\lambda_1^2F_{Q_1})\nabla\cdot\mathbf{E}\Big\rangle_{L^2(D)}\Big] \tag{2.11}$$
$$\le-\frac{3}{2}\sigma_0\mathbb{E}[\|\nabla\cdot\mathbf{E}\|^2_{L^2(D)}]-\lambda_1^2\mathbb{E}\Big[\langle\nabla\cdot\mathbf{E},F_{Q_1}\nabla\cdot\mathbf{E}\rangle_{L^2(D)}\Big]+C$$

and

$$\lambda_1^2\mathbb{E}\Big[\sum_{k=1}^{\infty}\|\nabla\cdot(\mathbf{H}Q_1^{\frac{1}{2}}q_k)\|^2_{L^2(D)}\Big]=\lambda_1^2\mathbb{E}\Big[\Big\langle\nabla\cdot\mathbf{H},F_{Q_1}\nabla\cdot\mathbf{H}\Big\rangle_{L^2(D)}\Big]$$
$$+\lambda_1^2\mathbb{E}\Big[\sum_{k=1}^{\infty}\|\mathbf{H}\cdot\nabla(Q_1^{\frac{1}{2}}q_k)\|^2_{L^2(D)}\Big]$$
$$+2\lambda_1^2\mathbb{E}\Big[\sum_{k=1}^{\infty}\Big\langle Q_1^{\frac{1}{2}}q_k\nabla\cdot\mathbf{H},\mathbf{H}\cdot\nabla(Q_1^{\frac{1}{2}}q_k)\Big\rangle_{L^2(D)}\Big]$$
$$\le\lambda_1^2\mathbb{E}\big[\langle\nabla\cdot\mathbf{H},F_{Q_1}\nabla\cdot\mathbf{H}\rangle_{L^2(D)}\big]+\frac{1}{2}\sigma_0\mathbb{E}\big[\|\nabla\cdot\mathbf{H}\|^2_{L^2(D)}\big]+C. \tag{2.12}$$

Using the fact that $\sum_{k=1}^{\infty}\|\nabla\cdot(\boldsymbol{\lambda}_2 Q_2^{\frac12}q_k)\|_{L^2(D)}^2\le C(|\boldsymbol{\lambda}_2|,\|Q_2^{\frac12}\|_{\mathcal{L}_2^{\gamma_2}})$ and substituting (2.11)–(2.12) into (2.10), we have

$$\begin{aligned}\mathrm{d}\mathbb{E}[\|\nabla\cdot\mathbf{E}(t)\|_{L^2(D)}^2]\le&-\frac{3}{2}\sigma_0\mathbb{E}[\|\nabla\cdot\mathbf{E}(t)\|_{L^2(D)}^2]\mathrm{d}t+\frac{1}{2}\sigma_0\mathbb{E}[\|\nabla\cdot\mathbf{H}(t)\|_{L^2(D)}^2]\mathrm{d}t\\&-\lambda_1^2\mathbb{E}\Big[\Big\langle\nabla\cdot\mathbf{E}(t),F_{Q_1}\nabla\cdot\mathbf{E}(t)\Big\rangle_{L^2(D)}\Big]\mathrm{d}t\\&+\lambda_1^2\mathbb{E}\Big[\Big\langle\nabla\cdot\mathbf{H}(t),F_{Q_1}\nabla\cdot\mathbf{H}(t)\Big\rangle_{L^2(D)}\Big]\mathrm{d}t+C\mathrm{d}t.\end{aligned}$$

Similarly, one gets

$$\begin{aligned}\mathrm{d}\mathbb{E}[\|\nabla\cdot\mathbf{H}(t)\|_{L^2(D)}^2]\le&-\frac{3}{2}\sigma_0\mathbb{E}[\|\nabla\cdot\mathbf{H}(t)\|_{L^2(D)}^2]\mathrm{d}t+\frac{1}{2}\sigma_0\mathbb{E}[\|\nabla\cdot\mathbf{E}(t)\|_{L^2(D)}^2]\mathrm{d}t\\&-\lambda_1^2\mathbb{E}\Big[\Big\langle\nabla\cdot\mathbf{H}(t),F_{Q_1}\nabla\cdot\mathbf{H}(t)\Big\rangle_{L^2(D)}\Big]\mathrm{d}t\\&+\lambda_1^2\mathbb{E}\Big[\Big\langle\nabla\cdot\mathbf{E}(t),F_{Q_1}\nabla\cdot\mathbf{E}(t)\Big\rangle_{L^2(D)}\Big]\mathrm{d}t+C\mathrm{d}t.\end{aligned}$$

Combining them together, we have

$$\begin{aligned}\mathrm{d}\mathbb{E}\Big[\|\nabla\cdot\mathbf{E}(t)\|_{L^2(D)}^2+\|\nabla\cdot\mathbf{H}(t)\|_{L^2(D)}^2\Big]\le&-\sigma_0\mathbb{E}\Big[\|\nabla\cdot\mathbf{E}(t)\|_{L^2(D)}^2+\|\nabla\cdot\mathbf{H}(t)\|_{L^2(D)}^2\Big]\mathrm{d}t\\&+C\mathrm{d}t,\end{aligned}$$

which by the Gronwall inequality implies

$$\mathbb{E}\Big[\|\nabla\cdot\mathbf{E}(t)\|_{L^2(D)}^2+\|\nabla\cdot\mathbf{H}(t)\|_{L^2(D)}^2\Big]\le e^{-\sigma_0 t}\mathbb{E}\Big[\|\nabla\cdot\mathbf{E}_0\|_{L^2(D)}^2+\|\nabla\cdot\mathbf{H}_0\|_{L^2(D)}^2\Big]+C.$$

Step 3. Proof of (2.6).

By utilizing the fact that $v\in H(\mathrm{curl},D)\cap H(\mathrm{div},D)$ belongs to $H^1(D)^3$ if $\mathbf{n}\times v|_{\partial D}=0$ or $\mathbf{n}\cdot v|_{\partial D}=0$, we can get the time-independent $H^1(D)^6$-regularity of the solution of (1.1). Combining *Step 1*, *Step 2* and Proposition 2.3, we obtain

$$\begin{aligned}\mathbb{E}\big[\|u(t)\|_{H^1(D)^6}^2\big]\le&\,C\Big(\mathbb{E}\big[\|u(t)\|_{\mathbb{H}}^2\big]+\mathbb{E}\big[\|\nabla\times\mathbf{E}(t)\|_{L^2(D)^3}^2\big]+\mathbb{E}\big[\|\nabla\cdot\mathbf{E}(t)\|_{L^2(D)}^2\big]\\&+\mathbb{E}\big[\|\nabla\times\mathbf{H}(t)\|_{L^2(D)^3}^2\big]+\mathbb{E}\big[\|\nabla\cdot\mathbf{H}(t)\|_{L^2(D)}^2\big]\Big)\\\le&\,C_2e^{-\sigma_0 t}\mathbb{E}\big[\|u_0\|_{H^1(D)^6}^2\big]+C_3.\end{aligned}$$

Thus we finish the proof. □

By applying the Itô formula to $\|u(t)\|_{\mathbb{H}}^{2p}$ and $\|Mu(t)\|_{\mathbb{H}}^{2p}$, respectively, we can similarly obtain the uniform boundedness of $\mathbb{E}[\|u(t)\|_{\mathcal{D}(M)}^{2p}]$ in the following proposition.

Proposition 2.5. *Let Assumption 2.1 hold with $\gamma_1 > \frac{5}{2}$ and $\gamma_2 \geq 1$, let $F_{Q_1} \in W^{1,\infty}(D)$ and $u_0 \in L^{2p}(\Omega, \mathcal{D}(M))$ with $p \in \mathbb{N}_+$. Then the solution of (1.1) is bounded uniformly in time, i.e.,*

$$\mathbb{E}\big[\|u(t)\|_{\mathcal{D}(M)}^{2p}\big] \leq \widetilde{C}_{2,p} e^{-p\sigma_0 t}\mathbb{E}\big[\|u_0\|_{\mathcal{D}(M)}^{2p}\big] + \widetilde{C}_{3,p}, \tag{2.13}$$

where the positive constant $\widetilde{C}_{2,p}$ depends on $p, |D|$, and the positive constant $\widetilde{C}_{3,p}$ depends on p, σ_0, $\|\sigma\|_{W^{1,\infty}(D)}$, $\|F_{Q_1}\|_{W^{1,\infty}(D)}$, C_1, $\mathbb{E}\big[\|u_0\|_{\mathbb{H}}^{2p}\big]$, λ_1, $\widetilde{\lambda}_2$, $\|Q_1^{\frac{1}{2}}\|_{\mathcal{L}_2^{\gamma_1}}$ and $\|Q_2^{\frac{1}{2}}\|_{\mathcal{L}_2^{\gamma_2}}$.

Let $\pi \in \mathcal{P}(\mathbb{H})$ and denote the transpose operator of P_t by P_t^*. Based on Lemma 2.4, we obtain the ergodicity of stochastic Maxwell equations and the convergence of $P_t^*\pi$ towards the invariant measure in the L^2-Wasserstein distance as $t \to \infty$ in the following proposition.

Proposition 2.6. *Under the conditions in Lemma 2.4, the following statements hold.*
(i) *The solution u of (1.1) possesses a unique invariant measure $\pi^* \in \mathcal{P}_2(\mathbb{H})$, where $\mathcal{P}_2(\mathbb{H}) = \{\mu \in \mathcal{P}(\mathbb{H}) : \int_{\mathbb{H}} \|x\|_{\mathbb{H}}^2 \mu(\mathrm{d}x) < \infty\}$. Thus u is ergodic. Moreover, u is exponentially mixing.*
(ii) *For any distribution $\pi \in \mathcal{P}_2(\mathbb{H})$,*

$$\mathcal{W}_2(P_t^*\pi, \pi^*) \leq e^{-\sigma_0 t}\mathcal{W}_2(\pi, \pi^*).$$

Proof. (i) Let u and $\widetilde{u}$ be solutions of (1.1) with initial data u_0 and $\widetilde{u}_0$, respectively. Similarly to Proposition 2.3, we get

$$\mathbb{E}\big[\|u(t) - \widetilde{u}(t)\|_{\mathbb{H}}^2\big] \leq \mathbb{E}\big[\|u(0) - \widetilde{u}(0)\|_{\mathbb{H}}^2\big] - 2\sigma_0 \int_0^t \mathbb{E}\big[\|u(s) - \widetilde{u}(s)\|_{\mathbb{H}}^2\big]\mathrm{d}s$$

which by the Gronwall inequality yields

$$\mathbb{E}\big[\|u(t) - \widetilde{u}(t)\|_{\mathbb{H}}^2\big] \leq e^{-2\sigma_0 t}\mathbb{E}\big[\|u_0 - \widetilde{u}_0\|_{\mathbb{H}}^2\big]. \tag{2.14}$$

By Lemma 2.4, we can choose $\|\cdot\|_{H^1(D)^6}$ as a proper Lyapunov function. We note that $H^1(D)^6$ is compactly embedded in $\mathbb{H}$. Therefore, the level set $K_\alpha := \{v \in \mathbb{H} : \|v\|_{H^1(D)^6} \leq \alpha\}$ is compact for any constant $\alpha > 0$. By using the Krylov–Bogoliubov theorem (see [9, Proposition 7.10]), the general Harris' theorem (see [11, Theorem 4.8]) and (2.14), we get the existence and the uniqueness of invariant measure π^*. By Proposition 2.3 and [10, Proposition 4.24], it holds that

$$\int_{\mathbb{H}} \|\mathbf{r}\|_{\mathbb{H}}^2 \pi^*(\mathrm{d}\mathbf{r}) < C_1, \tag{2.15}$$

which implies that $\pi^* \in \mathcal{P}_2(\mathbb{H})$.

For any bounded Lipschitz continuous function φ on $\mathbb{H}$ and all $u_0 \in \mathbb{H}$,

$$\big|P_t\varphi(u_0) - \pi^*(\varphi)\big| = \Big|\int_{\mathbb{H}} \mathbb{E}\big[\varphi(u(t,u_0)) - \varphi(u(t,\mathbf{r}))\big]\pi^*(\mathrm{d}\mathbf{r})\Big|$$

$$\leq L_\varphi \int\limits_{\mathbb{H}} \mathbb{E}\big[\|u(t,u_0)-u(t,\mathbf{r})\|_{\mathbb{H}}\big]\pi^*(\mathrm{d}\mathbf{r}) \leq L_\varphi \int\limits_{\mathbb{H}} \big(\mathbb{E}\big[\|u(t,u_0)-u(t,\mathbf{r})\|^2_{\mathbb{H}}\big]\big)^{\frac{1}{2}}\pi^*(\mathrm{d}\mathbf{r})$$
$$\leq L_\varphi e^{-\sigma_0 t}\int\limits_{\mathbb{H}} \|u_0-\mathbf{r}\|_{\mathbb{H}}\pi^*(\mathrm{d}\mathbf{r})$$
$$\leq \Big(\|u_0\|_{\mathbb{H}} + \int\limits_{\mathbb{H}} \|\mathbf{r}\|_{\mathbb{H}}\pi^*(\mathrm{d}\mathbf{r})\Big)L_\varphi e^{-\sigma_0 t} \leq (\|u_0\|_{\mathbb{H}}+C_1)L_\varphi e^{-\sigma_0 t}$$

due to the Hölder inequality, (2.14) and (2.15). Hence the exponentially mixing property is proved.

(ii) We fix initial values $(u_0,\widetilde{u}_0)\in\mathbb{H}\times\mathbb{H}$ and denote initial distributions by δ_{u_0} and $\delta_{\widetilde{u}_0}$. Let u and $\widetilde{u}$ be solutions of (1.1) with initial data u_0 and $\widetilde{u}_0$, respectively. Denote the joint distribution of $(u(t),\widetilde{u}(t))$ by $\mathcal{J}(P_t^*\delta_{u_0},P_t^*\delta_{\widetilde{u}_0})$. By (2.14), we have

$$\mathcal{W}_2(P_t^*\delta_{u_0},P_t^*\delta_{\widetilde{u}_0}) \leq \Bigg(\int\limits_{\mathbb{H}\times\mathbb{H}} \|\mathbf{r}_1-\mathbf{r}_2\|^2_{\mathbb{H}}\mathcal{J}(P_t^*\delta_{u_0},P_t^*\delta_{\widetilde{u}_0})(\mathrm{d}\mathbf{r}_2,\mathrm{d}\mathbf{r}_2)\Bigg)^{\frac{1}{2}}$$
$$=\Bigg(\int\limits_{\Omega}\|u(t)-\widetilde{u}(t)\|^2_{\mathbb{H}}\mathrm{d}\mathbb{P}\Bigg)^{\frac{1}{2}} \leq e^{-\sigma_0 t}\|u_0-\widetilde{u}_0\|_{\mathbb{H}} = e^{-\sigma_0 t}\mathcal{W}_2(\delta_{u_0},\delta_{\widetilde{u}_0}).$$

Combining with the convexity of L^2-Wasserstein distance (see e.g., [22, Theorem 4.8]) and the Hölder inequality, we obtain that for any coupling γ of π and π^*,

$$\mathcal{W}_2(P_t^*\pi,P_t^*\pi^*) \leq \int\limits_{\mathbb{H}\times\mathbb{H}} \mathcal{W}_2(P_t^*\delta_{\mathbf{r}_1},P_t^*\delta_{\mathbf{r}_2})\gamma(\mathrm{d}\mathbf{r}_1,\mathrm{d}\mathbf{r}_2)$$
$$\leq e^{-\sigma_0 t}\int\limits_{\mathbb{H}\times\mathbb{H}} \mathcal{W}_2(\delta_{\mathbf{r}_1},\delta_{\mathbf{r}_2})\gamma(\mathrm{d}\mathbf{r}_1,\mathrm{d}\mathbf{r}_2) \leq e^{-\sigma_0 t}\Big(\int\limits_{\mathbb{H}\times\mathbb{H}}\|\mathbf{r}_1-\mathbf{r}_2\|^2_{\mathbb{H}}\gamma(\mathrm{d}\mathbf{r}_1,\mathrm{d}\mathbf{r}_2)\Big)^{\frac{1}{2}}.$$

Thus we finish the proof. □

2.2. *Stochastic conformal multi-symplecticity*

This part is devoted to studying the stochastic conformal multi-symplecticity of (1.1). We set $S_1(u)=\frac{\lambda_1}{2}\big(|\mathbf{E}|^2+|\mathbf{H}|^2\big)$, $S_2(u)=\boldsymbol{\lambda}_2\cdot\mathbf{E}-\boldsymbol{\lambda}_2\cdot\mathbf{H}$ and

$$F=\begin{pmatrix}0 & I_3\\ -I_3 & 0\end{pmatrix},\quad K_s=\begin{pmatrix}\mathcal{D}_s & 0\\ 0 & \mathcal{D}_s\end{pmatrix},\ s=1,2,3$$

with

$$\mathcal{D}_1=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix},\quad \mathcal{D}_2=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\quad \mathcal{D}_3=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}.$$

Then, (1.1) can be rewritten into the following form of damped stochastic Hamiltonian PDE:

$$F\mathrm{d}u + K_1\partial_x u\mathrm{d}t + K_2\partial_y u\mathrm{d}t + K_3\partial_z u\mathrm{d}t = -\sigma F u\mathrm{d}t + \nabla_u S_1(u)\circ \mathrm{d}W_1(t) + \nabla_u S_2(u)\circ \mathrm{d}W_2(t).$$

Theorem 2.7. *The system* (1.1) *possesses the stochastic conformal multi-symplectic conservation law*

$$\mathrm{d}w + \partial_x\kappa_1\mathrm{d}t + \partial_y\kappa_2\mathrm{d}t + \partial_z\kappa_3\mathrm{d}t = -2\sigma w\mathrm{d}t, \qquad \mathbb{P}\text{-}a.s.,$$

i.e.,

$$\begin{aligned}
&\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1}\omega(t_1,x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z + \int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{t_0}^{t_1}\kappa_1(t,x_1,y,z)\mathrm{d}t\mathrm{d}y\mathrm{d}z\\
&+\int_{z_0}^{z_1}\int_{x_0}^{x_1}\int_{t_0}^{t_1}\kappa_2(t,x,y_1,z)\mathrm{d}t\mathrm{d}x\mathrm{d}z + \int_{y_0}^{y_1}\int_{x_0}^{x_1}\int_{t_0}^{t_1}\kappa_3(t,x,y,z_1)\mathrm{d}t\mathrm{d}x\mathrm{d}y\\
&-\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1}\omega(t_0,x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z - \int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{t_0}^{t_1}\kappa_1(t,x_0,y,z)\mathrm{d}t\mathrm{d}y\mathrm{d}z\\
&-\int_{z_0}^{z_1}\int_{x_0}^{x_1}\int_{t_0}^{t_1}\kappa_2(t,x,y_0,z)\mathrm{d}t\mathrm{d}x\mathrm{d}z - \int_{y_0}^{y_1}\int_{x_0}^{x_1}\int_{t_0}^{t_1}\kappa_3(t,x,y,z_0)\mathrm{d}t\mathrm{d}x\mathrm{d}y\\
&= -2\int_{t_0}^{t_1}\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1}\sigma\omega(t,x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t,
\end{aligned}$$

where $\omega(t,x,y,z) = \frac{1}{2}du\wedge Fdu$, $\kappa_s(t,x,y,z) = \frac{1}{2}du\wedge K_s du$ $(s=1,2,3)$ *are the differential 2-forms associated with the skew-symmetric matrices* F *and* K_s*, respectively, and* $(t_0,t_1)\times(x_0,x_1)\times(y_0,y_1)\times(z_0,z_1)$ *is the local domain of* $u(t,x,y,z)$*.*

Proof. The proof is similar to that of [18, Theorem 1], thus we omit it here. □

3. Ergodic temporal semi-discretization

In this section, we propose a temporal semi-discretization to inherit the ergodicity and stochastic conformal multi-symplecticity of (1.1). Moreover, the mean-square convergence order of the temporal semi-discretization is derived.

By introducing a uniform partition in time interval $[0,+\infty)$ with time step-size Δt, we propose a modified midpoint method to discretize (2.3) in the temporal direction

$$u^{n+1} - e^{-\sigma\Delta t}u^n = \Delta t M\frac{u^{n+1}+e^{-\sigma\Delta t}u^n}{2} + \lambda_1 J\frac{u^{n+1}+e^{-\sigma\Delta t}u^n}{2}\Delta\overline{W}_1^n + \widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n \tag{3.1}$$

for $n \in \mathbb{N}$. Here, $\Delta W_2^n := W_2(t_{n+1}) - W_2(t_n)$. Since the diffusion term of (3.1) is implicit and the noise could be unbounded for arbitrary small time step-size, we truncate the noise ΔW_1^n by another random variable

$$\Delta \overline{W}_1^n := \sqrt{\Delta t} \sum_{i=1}^{\infty} \zeta_i^{(1),n} Q_1^{\frac{1}{2}} q_i,$$

where

$$\zeta_i^{(1),n} = \begin{cases} A_{\Delta t} & \xi_i^{(1),n} > A_{\Delta t}, \\ \xi_i^{(1),n} & |\xi_i^{(1),n}| \leq A_{\Delta t}, \\ -A_{\Delta t} & \xi_i^{(1),n} < -A_{\Delta t} \end{cases}$$

with $\{\xi_i^{(1),n}\}_{i \in \mathbb{N}_+}$ being a family of independent standard normal random variables and $A_{\Delta t} = \sqrt{2b|\ln \Delta t|}$, $b \geq 4$. Similarly to [2], it holds that

$$\mathbb{E}\big[|\zeta_i^{(1),n} - \xi_i^{(1),n}|^{2p}\big] \leq \mathbb{E}\big[|\xi_i^{(1),n}|^{2p}\big]\Delta t^b \qquad \forall\, p \in \mathbb{N}_+,$$

which implies that for $Q_1^{\frac{1}{2}} \in \mathcal{L}_2^{\gamma_1}$ with $\gamma_1 \geq 0$,

$$\mathbb{E}\big[\|\Delta \overline{W}_1^n - \Delta W_1^n\|_{H^{\gamma_1}(D)}^{2p}\big] \leq C\Delta t^{b+p} \qquad \forall\, p \in \mathbb{N}_+ \tag{3.2}$$

and

$$\mathbb{E}\big[\|(\Delta \overline{W}_1^k)^2 - (\Delta W_1^k)^2\|_{H^{\gamma_1}(D)}^2\big] \leq C\Delta t^{b+2}(1 + A_{\Delta t} + A_{\Delta t}^2) \leq C\Delta t^{b+1} \tag{3.3}$$

due to the fact that $\Delta t(1 + A_{\Delta t} + A_{\Delta t}^2) < 1$ for sufficiently small Δt.

The well-posedness and the uniform boundedness of the numerical solution of (3.1) in $L^p(\Omega, \mathbb{H})$ are stated in the following proposition.

Proposition 3.1. *Let Assumption 2.1 hold with $\gamma_1 \geq 1$ and $\gamma_2 \geq 1$, and let $u_0 \in L^{2p}(\Omega, \mathbb{H})$ with $p \in \mathbb{N}_+$. Then for sufficiently small $\Delta t > 0$, there uniquely exists a family of $\mathbb{H}$-valued and $\{\mathcal{F}_{t_n}\}_{n\in\mathbb{N}}$-adapted solution $\{u^n\}_{n\in\mathbb{N}}$ of (3.1), which satisfies*

$$\mathbb{E}\big[\|u^n\|_{\mathbb{H}}^{2p}\big] \leq e^{-(2p-1)\sigma_0 n\Delta t}\mathbb{E}\big[\|u_0\|_{\mathbb{H}}^{2p}\big] + C_{4,p} \quad \forall\, n \in \mathbb{N}, \tag{3.4}$$

where the positive constant $C_{4,p} := C\big(p, \sigma_0, \lambda_1, \widetilde{\lambda}_2, \mathbb{E}\big[\|u_0\|_{\mathbb{H}}^{2p}\big], \|Q_1^{\frac{1}{2}}\|_{\mathcal{L}_2^{\gamma_1}}, \|Q_2^{\frac{1}{2}}\|_{\mathcal{L}_2^{\gamma_2}}\big)$.

Proof. The proof of the existence and uniqueness of the numerical solution is similar to that of [3, Lemma 4.1], so we omit it here. In the following we focus on proving the assertion (3.4).

Step 1: Case $p = 1$.

We apply $\langle \cdot, u^{n+1} + e^{-\sigma\Delta t}u^n\rangle_{\mathbb{H}}$ on both sides of (3.1) to get

$$\|u^{n+1}\|_{\mathbb{H}}^2 - e^{-2\sigma_0\Delta t}\|u^n\|_{\mathbb{H}}^2 \leq \langle u^{n+1} + e^{-\sigma\Delta t}u^n, \widetilde{\lambda}_2 \Delta W_2^n\rangle_{\mathbb{H}} \tag{3.5}$$

due to the skew-adjointness of the Maxwell operator M. By taking the expectation and using the fact that ΔW_2^n is independent of $\mathcal{F}_{t_n}$, we have

$$\mathbb{E}\big[\|u^{n+1}\|_{\mathbb{H}}^2\big] \le e^{-2\sigma_0\Delta t}\mathbb{E}\big[\|u^n\|_{\mathbb{H}}^2\big] + \mathbb{E}\big[\langle u^{n+1} - e^{-\sigma\Delta t}u^n, \widetilde{\boldsymbol{\lambda}}_2\Delta W_2\rangle_{\mathbb{H}}\big]. \tag{3.6}$$

Substituting (3.1) into the second term of the right side of (3.6) leads to

$$\begin{aligned}\mathbb{E}\big[\|u^{n+1}\|_{\mathbb{H}}^2\big] \le{}& e^{-2\sigma_0\Delta t}\mathbb{E}\big[\|u^n\|_{\mathbb{H}}^2\big] + \frac{\Delta t}{2}\mathbb{E}\big[\langle Mu^{n+1}, \widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n\rangle_{\mathbb{H}}\big]\\ &+ \frac{\lambda_1}{2}\mathbb{E}\big[\langle Ju^{n+1}\Delta\overline{W}_1^n, \widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n\rangle_{\mathbb{H}}\big] + \mathbb{E}\big[\|\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n\|_{\mathbb{H}}^2\big].\end{aligned} \tag{3.7}$$

For the second term on the right side of (3.7), using the skew-adjointness of M and the Young inequality, we have

$$\begin{aligned}\frac{\Delta t}{2}\mathbb{E}\big[\langle Mu^{n+1}, \widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n\rangle_{\mathbb{H}}\big] &\le \frac{\sigma_0}{8}\Delta t\mathbb{E}\big[\|u^{n+1}\|_{\mathbb{H}}^2\big] + C(\sigma_0)\Delta t\mathbb{E}\big[\|M(\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n)\|_{\mathbb{H}}^2\big]\\ &\le \frac{\sigma_0}{8}\Delta t\mathbb{E}\big[\|u^{n+1}\|_{\mathbb{H}}^2\big] + C\Delta t^2.\end{aligned}$$

Based on the Hölder inequality, the Young inequality and the Sobolev embedding $H^1(D) \hookrightarrow L^4(D)$, the third term of the right side of (3.7) is estimated as follows

$$\begin{aligned}\frac{\lambda_1}{2}\mathbb{E}\big[\langle Ju^{n+1}\Delta\overline{W}_1^n, \widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n\rangle_{\mathbb{H}}\big] \le{}& \frac{\sigma_0}{8}\Delta t\mathbb{E}\big[\|u^{n+1}\|_{\mathbb{H}}^2\big]\\ &+ \frac{C(\sigma_0,\lambda_1,\widetilde{\boldsymbol{\lambda}}_2)}{\Delta t}\mathbb{E}\big[\|\Delta\overline{W}_1^n\|_{H^1(D)}^2\|\Delta W_2^n\|_{H^1(D)}^2\big]\\ \le{}& \frac{\sigma_0}{8}\Delta t\mathbb{E}\big[\|u^{n+1}\|_{\mathbb{H}}^2\big] + C\Delta t.\end{aligned} \tag{3.8}$$

Noting that $\mathbb{E}[||\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n||_{\mathbb{H}}^2] \le C(\widetilde{\boldsymbol{\lambda}}_2, \mathrm{tr}(Q_2))\Delta t$ and combining (3.7)–(3.8), we get

$$\mathbb{E}\big[\|u^{n+1}\|_{\mathbb{H}}^2\big] \le e^{-2\sigma_0\Delta t}\mathbb{E}\big[\|u^n\|_{\mathbb{H}}^2\big] + \frac{\sigma_0}{4}\Delta t\mathbb{E}\big[\|u^{n+1}\|_{\mathbb{H}}^2\big] + C\Delta t.$$

There exists a $\Delta t^* > 0$ (e.g., $\Delta t^* = \frac{3}{\sigma_0}$), such that for any $\Delta t \in (0, \Delta t^*]$,

$$\frac{1}{1-\sigma_0\Delta t/4} \le 1 + \sigma_0\Delta t \le e^{\sigma_0\Delta t}.$$

Then the Gronwall inequality implies the assertion for the case $p = 1$.

Step 2: Case $p \ge 2$.

We give the proof of the case $p = 2$ since the proofs for the cases $p > 2$ are similar. By multiplying $\|u^{n+1}\|_{\mathbb{H}}^2$ on both sides of (3.5) and taking the expectation, it yields

$$\begin{aligned}&\frac{1}{2}\mathbb{E}\big[\|u^{n+1}\|_{\mathbb{H}}^4\big]-\frac{1}{2}e^{-4\sigma_0\Delta t}\mathbb{E}\big[\|u^n\|_{\mathbb{H}}^4\big]+\frac{1}{2}\mathbb{E}\big[(\|u^{n+1}\|_{\mathbb{H}}^2-e^{-2\sigma_0\Delta t}\|u^n\|_{\mathbb{H}}^2)^2\big]\\&\le\mathbb{E}\big[\langle u^{n+1}-e^{-\sigma\Delta t}u^n,\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n\rangle_{\mathbb{H}}\|u^{n+1}\|_{\mathbb{H}}^2\big]d+2\mathbb{E}\big[\langle e^{-\sigma\Delta t}u^n,\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n\rangle_{\mathbb{H}}\|u^{n+1}\|_{\mathbb{H}}^2\big]\\&=:I+II.\end{aligned}\tag{3.9}$$

For the first term I, we substitute (3.1) into it, and use the Hölder inequality and the Young inequality to get

$$\begin{aligned}I=&\frac{\Delta t}{2}\mathbb{E}\big[\langle M(u^{n+1}+e^{-\sigma\Delta t}u^n),\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n\rangle_{\mathbb{H}}\|u^{n+1}\|_{\mathbb{H}}^2\big]\\&+\frac{\lambda_1}{2}\mathbb{E}\big[\langle J(u^{n+1}+e^{-\sigma\Delta t}u^n)\Delta\overline{W}_1^n,\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n\rangle_{\mathbb{H}}\|u^{n+1}\|_{\mathbb{H}}^2\big]+\mathbb{E}\big[\|\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n\|_{\mathbb{H}}^2\|u^{n+1}\|_{\mathbb{H}}^2\big]\\\le&\frac{\Delta t}{2}\mathbb{E}\big[\|u^{n+1}+e^{-\sigma\Delta t}u^n\|_{\mathbb{H}}\|M(\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n)\|_{\mathbb{H}}\|u^{n+1}\|_{\mathbb{H}}^2\big]\\&+C\mathbb{E}\big[\|u^{n+1}+e^{-\sigma\Delta t}u^n\|_{\mathbb{H}}\|\Delta\overline{W}_1^n\|_{L^4(D)}\|\Delta W_2^n\|_{L^4(D)}\|u^{n+1}\|_{\mathbb{H}}^2\big]\\&+\mathbb{E}\big[\|\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n\|_{\mathbb{H}}^2\|u^{n+1}\|_{\mathbb{H}}^2\big]\\\le&\frac{\sigma_0}{8}\Delta t\mathbb{E}\big[\|u^{n+1}\|_{\mathbb{H}}^4\big]+\frac{\sigma_0}{16}\Delta te^{-4\sigma_0\Delta t}\mathbb{E}\big[\|u^n\|_{\mathbb{H}}^4\big]+C\Delta t,\end{aligned}$$

where in the last step we use the Sobolev embedding $H^1(D)\hookrightarrow L^4(D)$. For the second term II, it yields

$$\begin{aligned}II&=2\mathbb{E}\big[\langle e^{-\sigma\Delta t}u^n,\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n\rangle_{\mathbb{H}}(\|u^{n+1}\|_{\mathbb{H}}^2-e^{-2\sigma_0\Delta t}\|u^n\|_{\mathbb{H}}^2)\big]\\&\le\frac{1}{2}\mathbb{E}\big[(\|u^{n+1}\|_{\mathbb{H}}^2-e^{-2\sigma_0\Delta t}\|u^n\|_{\mathbb{H}}^2)^2\big]+\frac{\sigma_0}{16}\Delta te^{-4\sigma_0\Delta t}\mathbb{E}\big[\|u^n\|_{\mathbb{H}}^4\big]+C\Delta t.\end{aligned}$$

Combining (3.9) and the estimates of I and II, we obtain

$$\mathbb{E}\big[\|u^{n+1}\|_{\mathbb{H}}^4\big]\le e^{-4\sigma_0\Delta t}\mathbb{E}\big[\|u^n\|_{\mathbb{H}}^4\big]+\frac{\sigma_0}{4}\Delta t\mathbb{E}\big[\|u^{n+1}\|_{\mathbb{H}}^4\big]+\frac{\sigma_0}{4}\Delta te^{-4\sigma_0\Delta t}\mathbb{E}\big[\|u^n\|_{\mathbb{H}}^4\big]+C\Delta t.$$

There exists a $\Delta t^*>0$ (e.g., $\Delta t^*=\frac{2}{\sigma_0}$), such that for any $\Delta t\in(0,\Delta t^*]$,

$$\frac{1+\sigma_0\Delta t/4}{1-\sigma_0\Delta t/4}\le1+\sigma_0\Delta t\le e^{\sigma_0\Delta t}.$$

By the Gronwall inequality, (3.4) holds for the case $p=2$. Thus we complete the proof. □

3.1. Properties of the temporal semi-discretization

We are now in the position to study the ergodicity and stochastic multi-symplecticity of (3.1).

3.1.1. Ergodicity

In order to show the ergodicity of (3.1), we first give the uniform boundedness of the numerical solution in $L^2(\Omega, H^1(D)^6)$.

Lemma 3.2. *Let Assumption 2.1 hold with $\gamma_1 > \frac{7}{2}$ and $\gamma_2 \geq 2$, and let $\sum_{i=1}^{\infty} \|Q_1^{\frac{1}{2}} e_i\|_{H^{\gamma_1 - 1}(D)} < \infty$ and $u_0 \in L^2(\Omega, H^1(D)^6)$. Then there exist positive constants C_5 and C_6 such that for sufficiently small $\Delta t > 0$,*

$$\sup_{n\geq 0} \mathbb{E}\big[\|u^n\|_{H^1(D)^6}^2\big] \leq C_5 e^{-\sigma_0 n \Delta t} \mathbb{E}\big[\|u_0\|_{H^1(D)^6}^2\big] + C_6, \tag{3.10}$$

where C_5 depends on $|D|$, and C_6 depends on $\sigma_0, C_{4,1}, C_{4,2}, \lambda_1, \widetilde{\boldsymbol{\lambda}}_2, \mathbb{E}\big[\|u_0\|_{\mathbb{H}}^4\big], \|Q_1^{\frac{1}{2}}\|_{\mathcal{L}_2^{\gamma_1}}$ and $\|Q_2^{\frac{1}{2}}\|_{\mathcal{L}_2^{\gamma_2}}$.

Proof. By utilizing the fact that $v \in H(\mathrm{curl}, D) \cap H(\mathrm{div}, D)$ belongs to $H^1(D)^3$ if $\mathbf{n} \times v|_{\partial D} = 0$ or $\mathbf{n} \cdot v|_{\partial D} = 0$, it is sufficient to show

$$\begin{aligned} &\mathbb{E}\big[\|Mu^n\|_{\mathbb{H}}^2 + \|\nabla \cdot \mathbf{E}^n\|_{L^2(D)}^2 + \|\nabla \cdot \mathbf{H}^n\|_{L^2(D)}^2\big] \\ &\leq e^{-\sigma_0 t} \mathbb{E}\big[\|Mu_0\|_{\mathbb{H}}^2 + \|\nabla \cdot \mathbf{E}_0\|_{L^2(D)}^2 + \|\nabla \cdot \mathbf{H}_0\|_{L^2(D)}^2\big] + C. \end{aligned} \tag{3.11}$$

We prove (3.11) in the following two steps.

Step 1. Estimate of $\mathbb{E}\big[\|\nabla \cdot \mathbf{E}^n\|_{L^2(D)}^2 + \|\nabla \cdot \mathbf{H}^n\|_{L^2(D)}^2\big]$.

It follows from (3.1) that

$$\mathbf{E}^{n+1} - e^{-\sigma \Delta t}\mathbf{E}^n = \frac{\Delta t}{2} \nabla \times \big(\mathbf{H}^{n+1} + e^{-\sigma \Delta t}\mathbf{H}^n\big) - \frac{\lambda_1}{2}\big(\mathbf{H}^{n+1} + e^{-\sigma \Delta t}\mathbf{H}^n\big)\Delta \overline{W}_1^n + \boldsymbol{\lambda}_2 \Delta W_2^n, \tag{3.12}$$

$$\mathbf{H}^{n+1} - e^{-\sigma \Delta t}\mathbf{H}^n = -\frac{\Delta t}{2} \nabla \times \big(\mathbf{E}^{n+1} + e^{-\sigma \Delta t}\mathbf{E}^n\big) + \frac{\lambda_1}{2}\big(\mathbf{E}^{n+1} + e^{-\sigma \Delta t}\mathbf{E}^n\big)\Delta \overline{W}_1^n + \boldsymbol{\lambda}_2 \Delta W_2^n. \tag{3.13}$$

After applying $\nabla\cdot$ on both sides of (3.12) and (3.13), we have

$$\begin{aligned} \nabla \cdot \mathbf{E}^{n+1} - e^{-\sigma \Delta t}\big(\nabla \cdot \mathbf{E}^n\big) = \big(\nabla e^{-\sigma \Delta t}\big) \cdot \mathbf{E}^n - \frac{\lambda_1}{2} \nabla \cdot \big[\big(\mathbf{H}^{n+1} + e^{-\sigma \Delta t}\mathbf{H}^n\big)\Delta \overline{W}_1^n\big] \\ + \boldsymbol{\lambda}_2 \cdot \nabla(\Delta W_2^n), \end{aligned} \tag{3.14}$$

$$\begin{aligned} \nabla \cdot \mathbf{H}^{n+1} - e^{-\sigma \Delta t}\big(\nabla \cdot \mathbf{H}^n\big) = \big(\nabla e^{-\sigma \Delta t}\big) \cdot \mathbf{H}^n + \frac{\lambda_1}{2} \nabla \cdot \big[\big(\mathbf{E}^{n+1} + e^{-\sigma \Delta t}\mathbf{E}^n\big)\Delta \overline{W}_1^n\big] \\ + \boldsymbol{\lambda}_2 \cdot \nabla(\Delta W_2^n). \end{aligned} \tag{3.15}$$

Next we apply $\langle \cdot, \nabla \cdot \mathbf{E}^{n+1} + e^{-\sigma \Delta t}\nabla \cdot \mathbf{E}^n\rangle_{L^2(D)}$ and $\langle \cdot, \nabla \cdot \mathbf{H}^{n+1} + e^{-\sigma \Delta t}\nabla \cdot \mathbf{H}^n\rangle_{L^2(D)}$ on both sides of (3.14) and (3.15), respectively, to get

$$
\begin{aligned}
&\|\nabla\cdot\mathbf{E}^{n+1}\|_{L^2(D)}^2+\|\nabla\cdot\mathbf{H}^{n+1}\|_{L^2(D)}^2-\|e^{-\sigma\Delta t}(\nabla\cdot\mathbf{E}^{n})\|_{L^2(D)}^2-\|e^{-\sigma\Delta t}(\nabla\cdot\mathbf{H}^{n})\|_{L^2(D)}^2\\
&=\langle(\nabla e^{-\sigma\Delta t})\cdot\mathbf{E}^n,\nabla\cdot\mathbf{E}^{n+1}+e^{-\sigma\Delta t}\nabla\cdot\mathbf{E}^n\rangle_{L^2(D)}\\
&\quad+\langle(\nabla e^{-\sigma\Delta t})\cdot\mathbf{H}^n,\nabla\cdot\mathbf{H}^{n+1}+e^{-\sigma\Delta t}\nabla\cdot\mathbf{H}^n\rangle_{L^2(D)}\\
&\quad+\langle\boldsymbol{\lambda}_2\cdot\nabla(\Delta W_2^n),\nabla\cdot\mathbf{E}^{n+1}+e^{-\sigma\Delta t}\nabla\cdot\mathbf{E}^n\rangle_{L^2(D)}\\
&\quad+\langle\boldsymbol{\lambda}_2\cdot\nabla(\Delta W_2^n),\nabla\cdot\mathbf{H}^{n+1}+e^{-\sigma\Delta t}\nabla\cdot\mathbf{H}^n\rangle_{L^2(D)}\\
&\quad-\frac{\lambda_1}{2}\langle\nabla\cdot[(\mathbf{H}^{n+1}+e^{-\sigma\Delta t}\mathbf{H}^n)\Delta\overline{W}_1^n],\nabla\cdot\mathbf{E}^{n+1}+e^{-\sigma\Delta t}\nabla\cdot\mathbf{E}^n\rangle_{L^2(D)}\\
&\quad+\frac{\lambda_1}{2}\langle\nabla\cdot[(\mathbf{E}^{n+1}+e^{-\sigma\Delta t}\mathbf{E}^n)\Delta\overline{W}_1^n],\nabla\cdot\mathbf{H}^{n+1}+e^{-\sigma\Delta t}\nabla\cdot\mathbf{H}^n\rangle_{L^2(D)}\\
&=:\sum_{k=1}^{6}A_k.
\end{aligned}
\tag{3.16}
$$

For terms A_1 and A_2, by the Young inequality, Proposition 3.1 and $\nabla e^{-\sigma\Delta t}=-\Delta t e^{-\sigma\Delta t}\nabla\sigma$, we have

$$
\begin{aligned}
&\mathbb{E}\big[A_1+A_2\big]\\
&\le\mathbb{E}\big[\|(\nabla e^{-\sigma\Delta t})\cdot\mathbf{E}^n\|_{L^2(D)}(\|\nabla\cdot\mathbf{E}^{n+1}\|_{L^2(D)}+\|e^{-\sigma\Delta t}(\nabla\cdot\mathbf{E}^n)\|_{L^2(D)})\big]\\
&\quad+\mathbb{E}\big[\|(\nabla e^{-\sigma\Delta t})\cdot\mathbf{H}^n\|_{L^2(D)}(\|\nabla\cdot\mathbf{H}^{n+1}\|_{L^2(D)}+\|e^{-\sigma\Delta t}(\nabla\cdot\mathbf{H}^n)\|_{L^2(D)})\big]\\
&\le\frac{\sigma_0}{16}\Delta t\mathbb{E}\big[\|\nabla\cdot\mathbf{E}^{n+1}\|_{L^2(D)}^2+\|\nabla\cdot\mathbf{H}^{n+1}\|_{L^2(D)}^2\big]+C\Delta t\\
&\quad+\frac{\sigma_0}{16}e^{-2\sigma_0\Delta t}\Delta t\mathbb{E}\big[\|\nabla\cdot\mathbf{E}^n\|_{L^2(D)}^2+\|\nabla\cdot\mathbf{H}^n\|_{L^2(D)}^2\big].
\end{aligned}
$$

By using the fact that $\Delta\overline{W}_1^n$ and ΔW_2^n are independent of $\mathcal{F}_{t_n}$, and substituting (3.14) and (3.15) into A_3 and A_4, respectively, it yields

$$
\begin{aligned}
&\mathbb{E}\big[A_3+A_4\big]\\
&=2\mathbb{E}\big[\|\boldsymbol{\lambda}_2\cdot\nabla(\Delta W_2^n)\|_{L^2(D)}^2\big]\\
&\quad-\frac{\lambda_1}{2}\mathbb{E}\big[\langle\boldsymbol{\lambda}_2\cdot\nabla(\Delta W_2^n),\Delta\overline{W}_1^n(\nabla\cdot\mathbf{H}^{n+1})+\mathbf{H}^{n+1}\cdot\nabla(\Delta\overline{W}_1^n)\rangle_{L^2(D)}\big]\\
&\quad+\frac{\lambda_1}{2}\mathbb{E}\big[\langle\boldsymbol{\lambda}_2\cdot\nabla(\Delta W_2^n),\Delta\overline{W}_1^n(\nabla\cdot\mathbf{E}^{n+1})+\mathbf{E}^{n+1}\cdot\nabla(\Delta\overline{W}_1^n)\rangle_{L^2(D)}\big]\\
&\le\frac{\sigma_0}{16}\Delta t\mathbb{E}\big[\|\nabla\cdot\mathbf{E}^{n+1}\|_{L^2(D)}^2+\|\nabla\cdot\mathbf{H}^{n+1}\|_{L^2(D)}^2\big]+C\Delta t
\end{aligned}
$$

in view of the Sobolev embedding $H^\gamma(D)\hookrightarrow L^\infty(D)$ for $\gamma>\frac{3}{2}$, the Young inequality and Proposition 3.1.

For terms A_5 and A_6, we note that

$$\begin{aligned}&\nabla\cdot\left[(\mathbf{H}^{n+1}+e^{-\sigma\Delta t}\mathbf{H}^n)\Delta\overline{W}_1^n\right]\\&=\Delta\overline{W}_1^n\big(\nabla\cdot\mathbf{H}^{n+1}+e^{-\sigma\Delta t}(\nabla\cdot\mathbf{H}^n)\big)+\mathbf{H}^{n+1}\cdot\nabla(\Delta\overline{W}_1^n)\\&\quad+e^{-\sigma\Delta t}\mathbf{H}^n\cdot\nabla(\Delta\overline{W}_1^n)+\Delta\overline{W}_1^n(\nabla e^{-\sigma\Delta t})\cdot\mathbf{H}^n\end{aligned}$$

and

$$\begin{aligned}&\nabla\cdot\left[(\mathbf{E}^{n+1}+e^{-\sigma\Delta t}\mathbf{E}^n)\Delta\overline{W}_1^n\right]\\&=\Delta\overline{W}_1^n\big(\nabla\cdot\mathbf{E}^{n+1}+e^{-\sigma\Delta t}(\nabla\cdot\mathbf{E}^n)\big)+\mathbf{E}^{n+1}\cdot\nabla(\Delta\overline{W}_1^n)\\&\quad+e^{-\sigma\Delta t}\mathbf{E}^n\cdot\nabla(\Delta\overline{W}_1^n)+\Delta\overline{W}_1^n(\nabla e^{-\sigma\Delta t})\cdot\mathbf{E}^n.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}[A_5+A_6]=&-\frac{\lambda_1}{2}\mathbb{E}\Big[\langle\mathbf{H}^{n+1}\cdot\nabla(\Delta\overline{W}_1^n),\nabla\cdot\mathbf{E}^{n+1}+e^{-\sigma\Delta t}(\nabla\cdot\mathbf{E}^n)\rangle_{L^2(D)}\Big]\\&-\frac{\lambda_1}{2}\mathbb{E}\Big[\langle e^{-\sigma\Delta t}\mathbf{H}^n\cdot\nabla(\Delta\overline{W}_1^n)+\Delta\overline{W}_1^n\mathbf{H}^n\cdot(\nabla e^{-\sigma\Delta t}),\nabla\cdot\mathbf{E}^{n+1}\rangle_{L^2(D)}\Big]\\&+\frac{\lambda_1}{2}\mathbb{E}\Big[\langle\mathbf{E}^{n+1}\cdot\nabla(\Delta\overline{W}_1^n),\nabla\cdot\mathbf{H}^{n+1}+e^{-\sigma\Delta t}(\nabla\cdot\mathbf{H}^n)\rangle_{L^2(D)}\Big]\\&+\frac{\lambda_1}{2}\mathbb{E}\Big[\langle e^{-\sigma\Delta t}\mathbf{E}^n\cdot\nabla(\Delta\overline{W}_1^n)+\Delta\overline{W}_1^n\mathbf{E}^n\cdot(\nabla e^{-\sigma\Delta t}),\nabla\cdot\mathbf{H}^{n+1}\rangle_{L^2(D)}\Big]\\=:&A_{5,1}+A_{5,2}+A_{6,1}+A_{6,2}\end{aligned}$$

due to the fact that $\Delta\overline{W}_1^n$ and ΔW_2^n are independent of $\mathcal{F}_{t_n}$. First, we consider the term $A_{5,1}$,

$$\begin{aligned}A_{5,1}=&-\frac{\lambda_1}{2}\mathbb{E}\Big[\langle\mathbf{H}^{n+1}\cdot\nabla(\Delta\overline{W}_1^n),\nabla\cdot\mathbf{E}^{n+1}-e^{-\sigma\Delta t}(\nabla\cdot\mathbf{E}^n)\rangle_{L^2(D)}\Big]\\&-\lambda_1\mathbb{E}\Big[\langle(\mathbf{H}^{n+1}-e^{-\sigma\Delta t}\mathbf{H}^n)\cdot\nabla(\Delta\overline{W}_1^n),e^{-\sigma\Delta t}(\nabla\cdot\mathbf{E}^n)\rangle_{L^2(D)}\Big]\\=:&\,A_{5,1,1}+A_{5,1,2}.\end{aligned}$$

Substituting (3.14) into the term $A_{5,1,1}$ and using the Sobolev embedding $H^\gamma(D)\hookrightarrow L^\infty(D)$ for $\gamma>\frac{3}{2}$ lead to

$$\begin{aligned}A_{5,1,1}=&-\frac{\lambda_1}{2}\mathbb{E}\Big[\langle\mathbf{H}^{n+1}\cdot\nabla(\Delta\overline{W}_1^n),(\nabla e^{-\sigma\Delta t})\cdot\mathbf{E}^n+\boldsymbol{\lambda}_2\cdot\nabla(\Delta W_2^n)\rangle_{L^2(D)}\Big]\\&+\frac{\lambda_1^2}{4}\mathbb{E}\Big[\langle\mathbf{H}^{n+1}\cdot(\nabla\Delta\overline{W}_1^n),\Delta\overline{W}_1^n(\nabla\cdot\mathbf{H}^{n+1})\rangle_{L^2(D)}+\|\mathbf{H}^{n+1}\cdot\nabla(\Delta\overline{W}_1^n)\|_{L^2(D)}^2\Big]\\&+\frac{\lambda_1^2}{4}\mathbb{E}\Big[\Big\langle\mathbf{H}^{n+1}\cdot\nabla(\Delta\overline{W}_1^n),\Delta\overline{W}_1^ne^{-\sigma\Delta t}(\nabla\cdot\mathbf{H}^n)\\&+\Big(e^{-\sigma\Delta t}\nabla(\Delta\overline{W}_1^n)+\Delta\overline{W}_1^n(\nabla e^{-\sigma\Delta t})\Big)\cdot\mathbf{H}^n\Big\rangle_{L^2(D)}\Big]\end{aligned}$$

$$\begin{aligned}
&\leq C\mathbb{E}\Big[\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}\|\mathbf{H}^{n+1}\|_{L^2(D)^3}\Big(\Delta t\|\mathbf{E}^n\|_{L^2(D)^3}+\|\Delta W_2^n\|_{H^1(D)}\Big)\Big]\\
&\quad+C\mathbb{E}\Big[\|\Delta\overline{W}_1^n\|^2_{H^{\gamma_1-1}(D)}\|\mathbf{H}^{n+1}\|_{L^2(D)^3}\Big(\|\mathbf{H}^{n+1}\|_{L^2(D)^3}+\|\nabla\cdot\mathbf{H}^{n+1}\|_{L^2(D)}\Big)\Big]\\
&\quad+C\mathbb{E}\Big[\|\Delta\overline{W}_1^n\|^2_{H^{\gamma_1-1}(D)}\|\mathbf{H}^{n+1}\|_{L^2(D)^3}\Big((1+\Delta t)\|\mathbf{H}^n\|_{L^2(D)^3}+e^{-\sigma_0\Delta t}\|\nabla\cdot\mathbf{H}^n\|_{L^2(D)}\Big)\Big]\\
&\leq\frac{\sigma_0}{16}\Delta t\mathbb{E}\Big[\|\nabla\cdot\mathbf{H}^{n+1}\|^2_{L^2(D)}\Big]+\frac{\sigma_0}{16}\Delta te^{-2\sigma_0\Delta t}\mathbb{E}\Big[\|\nabla\cdot\mathbf{H}^n\|^2_{L^2(D)}\Big]+C\Delta t,
\end{aligned}$$

where in the last step we use the Young inequality and Proposition 3.1. We substitute (3.13) into $A_{5,1,2}$ and use a similar argument as $A_{5,1,1}$ to get

$$A_{5,1,2}\leq\frac{\sigma_0}{20}\Delta t\mathbb{E}\Big[\|\nabla\times\mathbf{E}^{n+1}\|^2_{L^2(D)^3}\Big]+(C\Delta t+\frac{\sigma_0}{8})\Delta te^{-2\sigma_0\Delta t}\mathbb{E}\Big[\|\nabla\cdot\mathbf{E}^n\|^2_{L^2(D)}\Big]+C\Delta t.$$

Combining the estimates of $A_{5,1,1}$ and $A_{5,1,2}$, it holds that

$$\begin{aligned}
A_{5,1}\leq&\frac{\sigma_0}{20}\Delta t\mathbb{E}\Big[\|\nabla\times\mathbf{E}^{n+1}\|^2_{L^2(D)^3}\Big]+(C\Delta t+\frac{\sigma_0}{8})\Delta te^{-2\sigma_0\Delta t}\mathbb{E}\Big[\|\nabla\cdot\mathbf{E}^n\|^2_{L^2(D)}\Big]\\
&+\frac{\sigma_0}{16}\Delta t\mathbb{E}\Big[\|\nabla\cdot\mathbf{H}^{n+1}\|^2_{L^2(D)}\Big]+\frac{\sigma_0}{16}\Delta te^{-2\sigma_0\Delta t}\mathbb{E}\Big[\|\nabla\cdot\mathbf{H}^n\|^2_{L^2(D)}\Big]+C\Delta t.
\end{aligned}$$

Similar to the estimate of $A_{5,1,1}$, for the term $A_{5,2}$ we obtain

$$\begin{aligned}
A_{5,2}=&-\frac{\lambda_1}{2}\mathbb{E}\Big[\langle e^{-\sigma\Delta t}\mathbf{H}^n\cdot\nabla(\Delta\overline{W}_1^n),\nabla\cdot\mathbf{E}^{n+1}-e^{-\sigma\Delta t}\nabla\cdot\mathbf{E}^n\rangle_{L^2(D)}\Big]\\
&-\frac{\lambda_1}{2}\mathbb{E}\Big[\langle\Delta\overline{W}_1^n(\nabla e^{-\sigma\Delta t})\cdot\mathbf{H}^n,\nabla\cdot\mathbf{E}^{n+1}\rangle_{L^2(D)}\Big]\\
\leq&\frac{\sigma_0}{16}\Delta t\mathbb{E}\Big[\|\nabla\cdot\mathbf{H}^{n+1}\|^2_{L^2(D)}\Big]+\frac{\sigma_0}{16}\Delta te^{-2\sigma_0\Delta t}\mathbb{E}\Big[\|\nabla\cdot\mathbf{H}^n\|^2_{L^2(D)}\Big]+C\Delta t.
\end{aligned}$$

Since terms $A_{6,1}$ and $A_{6,2}$ can be similarly estimated as terms $A_{5,1}$ and $A_{5,2}$ respectively, one gets

$$\begin{aligned}
\mathbb{E}[A_5+A_6]\leq&\frac{\sigma_0}{20}\Delta t\mathbb{E}\Big[\|Mu^{n+1}\|^2_{\mathbb{H}}\Big]\\
&+(C\Delta t+\frac{\sigma_0}{8})\Delta te^{-2\sigma_0\Delta t}\mathbb{E}\Big[\|\nabla\cdot\mathbf{E}^n\|^2_{L^2(D)}+\|\nabla\cdot\mathbf{H}^n\|^2_{L^2(D)}\Big]\\
&+\frac{\sigma_0}{8}\Delta t\mathbb{E}\Big[\|\nabla\cdot\mathbf{E}^{n+1}\|^2_{L^2(D)}+\|\nabla\cdot\mathbf{H}^{n+1}\|^2_{L^2(D)}\Big].
\end{aligned}$$

Combining the estimates of A_1–A_6, we arrive at

$$\begin{aligned}
&\mathbb{E}\Big[\|\nabla\cdot\mathbf{H}^{n+1}\|^2_{L^2(D)}+\|\nabla\cdot\mathbf{E}^{n+1}\|^2_{L^2(D)}\Big]\\
&\leq e^{-2\sigma_0\Delta t}\mathbb{E}\Big[\|\nabla\cdot\mathbf{E}^n\|^2_{L^2(D)}+\|\nabla\cdot\mathbf{H}^n\|^2_{L^2(D)}\Big]+\frac{\sigma_0}{4}\Delta t\mathbb{E}\Big[\|\nabla\cdot\mathbf{E}^{n+1}\|^2_{L^2(D)}+\|\nabla\cdot\mathbf{H}^{n+1}\|^2_{L^2(D)}\Big]\\
&\quad+(C\Delta t+\frac{3\sigma_0}{16})\Delta te^{-2\sigma_0\Delta t}\mathbb{E}\Big[\|\nabla\cdot\mathbf{E}^n\|^2_{L^2(D)}+\|\nabla\cdot\mathbf{H}^n\|^2_{L^2(D)}\Big]+\frac{\sigma_0}{20}\Delta t\mathbb{E}\Big[\|Mu^{n+1}\|^2_{\mathbb{H}}\Big]\\
&\quad+C\Delta t.
\end{aligned}$$

Step 2. Estimate of $\mathbb{E}\Big[\|Mu^n\|_{\mathbb{H}}^2\Big]$.

We apply $\langle\cdot, M(u^{n+1}-e^{-\sigma\Delta t}u^n)\rangle_{\mathbb{H}}$ on both sides of (3.1) and get

$$\begin{aligned}
&\|M(u^{n+1}\|_{\mathbb{H}}^2-\|M(e^{-\sigma\Delta t}u^n)\|_{\mathbb{H}}^2\\
&=-\frac{\lambda_1}{\Delta t}\langle J(u^{n+1}+e^{-\sigma\Delta t}u^n)\Delta\overline{W}_1^n, M(u^{n+1}-e^{-\sigma\Delta t}u^n)\rangle_{\mathbb{H}}\\
&\quad-\frac{2}{\Delta t}\langle\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n, M(u^{n+1}-e^{-\sigma\Delta t}u^n)\rangle_{\mathbb{H}}.
\end{aligned}\tag{3.17}$$

By the skew-adjointness of the Maxwell operator M, we substitute (3.1) into the two terms on the right side of (3.17) and take the expectation to obtain

$$\begin{aligned}
\mathbb{E}\Big[\|Mu^{n+1}\|_{\mathbb{H}}^2\Big]&=\mathbb{E}\Big[\|M(e^{-\sigma\Delta t}u^n)\|_{\mathbb{H}}^2\Big]+\mathbb{E}\Big[\langle M(\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n), M(u^{n+1}+e^{-\sigma\Delta t}u^n)\rangle_{\mathbb{H}}\Big]\\
&\quad+\frac{\lambda_1}{2}\mathbb{E}\Big[\langle M(J(u^{n+1}+e^{-\sigma\Delta t}u^n)\Delta\overline{W}_1^n), M(u^{n+1}+e^{-\sigma\Delta t}u^n)\rangle_{\mathbb{H}}\Big]\\
&=: B_1+B_2+B_3.
\end{aligned}$$

For the term B_1, we note that $M(e^{-\sigma\Delta t}u^n)=e^{-\sigma\Delta t}Mu^n+R_\sigma^{\Delta t}u^n$, where

$$R_\sigma^{\Delta t}:=\begin{pmatrix}0 & (\nabla e^{-\sigma\Delta t})\times\\ -(\nabla e^{-\sigma\Delta t})\times & 0\end{pmatrix}.$$

Since

$$\|R_\sigma^{\Delta t}u^n\|_{\mathbb{H}}\le 2\Delta t e^{-\sigma_0\Delta t}\|\sigma\|_{W^{1,\infty}(D)}\|u^n\|_{\mathbb{H}}\le C\Delta t e^{-\sigma_0\Delta t}\|u^n\|_{\mathbb{H}},$$

we derive that

$$\begin{aligned}
B_1&=\mathbb{E}\Big[\|e^{-\sigma\Delta t}Mu^n\|_{\mathbb{H}}^2\Big]+\mathbb{E}\Big[\|R_\sigma^{\Delta t}u^n\|_{\mathbb{H}}^2\Big]+2\mathbb{E}\Big[\langle e^{-\sigma\Delta t}Mu^n, R_\sigma u^n\rangle_{\mathbb{H}}\Big]\\
&\le e^{-2\sigma_0\Delta t}\mathbb{E}\Big[\|Mu^n\|_{\mathbb{H}}^2\Big]+\frac{\sigma_0}{16}\Delta t e^{-2\sigma_0\Delta t}\mathbb{E}\Big[\|Mu^n\|_{\mathbb{H}}^2\Big]+C\Delta t,
\end{aligned}$$

where we use the Young inequality and Proposition 3.1.

Using the fact that ΔW_2^n is independent of $\mathcal{F}_{t_n}$, the skew-adjointness of M and substituting (3.1) into the term B_2, we get

$$\begin{aligned}
B_2&=\mathbb{E}\Big[\langle M(\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n), M(u^{n+1}-e^{-\sigma\Delta t}u^n)\rangle_{\mathbb{H}}\Big]\\
&=-\frac{\Delta t}{2}\mathbb{E}\Big[\langle M^2(\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n), Mu^{n+1}\rangle_{\mathbb{H}}\Big]+\mathbb{E}\Big[\|M(\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n)\|_{\mathbb{H}}^2\Big]\\
&\quad-\frac{\lambda_1}{2}\mathbb{E}\Big[\langle M^2(\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n), Ju^{n+1}\Delta\overline{W}_1^n\rangle_{\mathbb{H}}\Big]\\
&\le\frac{\sigma_0}{20}\Delta t\mathbb{E}\Big[\|Mu^{n+1}\|_{\mathbb{H}}^2\Big]+C\Delta t
\end{aligned}$$

due the Young inequality, Sobolev embedding $H^\gamma(D) \hookrightarrow L^\infty(D)$ for $\gamma > \frac{3}{2}$ and Proposition 3.1.

Let $H_W^1 := \begin{pmatrix} (\nabla(\Delta \overline{W}_1^n))\times & 0 \\ 0 & (\nabla(\Delta \overline{W}_1^n))\times \end{pmatrix}$. For the term B_3, we have

$$\begin{aligned} B_3 &= \frac{\lambda_1}{2}\mathbb{E}\Big[\langle J\Delta \overline{W}_1^n M(u^{n+1}+e^{-\sigma\Delta t}u^n), M(u^{n+1}+e^{-\sigma\Delta t}u^n)\rangle_{\mathbb{H}}\Big] \\ &\quad + \frac{\lambda_1}{2}\mathbb{E}\Big[\langle H_W^1(u^{n+1}+e^{-\sigma\Delta t}u^n), M(u^{n+1}+e^{-\sigma\Delta t}u^n)\rangle_{\mathbb{H}}\Big] \\ &= \frac{\lambda_1}{2}\mathbb{E}\Big[\langle H_W^1(u^{n+1}-e^{-\sigma\Delta t}u^n), M(u^{n+1}+e^{-\sigma\Delta t}u^n)\rangle_{\mathbb{H}}\Big] \\ &\quad + \lambda_1\mathbb{E}\Big[\langle H_W^1 e^{-\sigma\Delta t}u^n, M(u^{n+1}-e^{-\sigma\Delta t}u^n)\rangle_{\mathbb{H}}\Big] \\ &=: B_{3,1}+B_{3,2}. \end{aligned}$$

For the term $B_{3,1}$, we substitute (3.1) into it and obtain

$$\begin{aligned} B_{3,1} &= \frac{\lambda_1\Delta t}{4}\mathbb{E}\Big[\langle H_W^1 M(u^{n+1}+e^{-\sigma\Delta t}u^n), M(u^{n+1}+e^{-\sigma\Delta t}u^n)\rangle_{\mathbb{H}}\Big] \\ &\quad + \frac{\lambda_1^2}{4}\mathbb{E}\Big[\langle H_W^1 J(u^{n+1}+e^{-\sigma\Delta t}u^n)\Delta\overline{W}_1^n, M(u^{n+1}+e^{-\sigma\Delta t}u^n)\rangle_{\mathbb{H}}\Big] \\ &\quad + \frac{\lambda_1}{2}\mathbb{E}\Big[\langle H_W^1\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n, M(u^{n+1}+e^{-\sigma\Delta t}u^n)\rangle_{\mathbb{H}}\Big] \\ &=: B_{3,1,1}+B_{3,1,2}+B_{3,1,3}. \end{aligned}$$

It follows from the Sobolev embedding $H^\gamma(D) \hookrightarrow L^\infty(D)$ for $\gamma > \frac{3}{2}$, the Young inequality and Proposition 3.1 that

$$\begin{aligned} B_{3,1,1} &\le C\Delta t\mathbb{E}\Big[\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}\|Mu^{n+1}\|_{\mathbb{H}}^2\Big] \\ &\quad + C\Delta t\mathbb{E}\Big[\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}\|Mu^{n+1}\|_{\mathbb{H}}\Big(e^{-\sigma_0\Delta t}\|Mu^n\|_{\mathbb{H}}+C\Delta t e^{-\sigma_0\Delta t}\|u^n\|_{\mathbb{H}}\Big)\Big] \\ &\le C\Delta t^{\frac{3}{2}}A_{\Delta t}\mathbb{E}\Big[\|Mu^{n+1}\|_{\mathbb{H}}^2\Big] \\ &\quad + C\Delta t\mathbb{E}\Big[\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}\|Mu^{n+1}\|_{\mathbb{H}}\Big(e^{-\sigma_0\Delta t}\|Mu^n\|_{\mathbb{H}}+C\Delta t e^{-\sigma_0\Delta t}\|u^n\|_{\mathbb{H}}\Big)\Big] \end{aligned}$$

since $\sum_{i=1}^{\infty}\|Q_1^{\frac{1}{2}}e_i\|_{H^{\gamma_1-1}(D)} < \infty$ implies $\|\Delta\overline{W}_1^n\| \le C\Delta t^{\frac{1}{2}}A_{\Delta t}$. Therefore, using the Young inequality and the fact that $\Delta\overline{W}_1^n$ is independent of $\mathcal{F}_{t_n}$, we obtain

$$\begin{aligned} B_{3,1,1} &\le C\Delta t^{\frac{3}{2}}A_{\Delta t}\mathbb{E}\Big[\|Mu^{n+1}\|_{\mathbb{H}}^2\Big] + \frac{\sigma_0}{20}\Delta t\mathbb{E}\Big[\|Mu^{n+1}\|_{\mathbb{H}}^2\Big] \\ &\quad + C\Delta t e^{-2\sigma_0\Delta t}\mathbb{E}\Big[\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}^2\Big]\mathbb{E}\Big[\|Mu^n\|_{\mathbb{H}}^2\Big] \\ &\quad + C\Delta t^3\mathbb{E}\Big[\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}^2\Big]\mathbb{E}\Big[\|u^n\|_{\mathbb{H}}^2\Big] \end{aligned}$$

$$\leq (C\Delta t^{\frac{1}{2}} A_{\Delta t} + \frac{\sigma_0}{20})\Delta t \mathbb{E}\Big[\|Mu^{n+1}\|_{\mathbb{H}}^2\Big] + C\Delta t^2 e^{-2\sigma_0 \Delta t}\mathbb{E}\Big[\|Mu^n\|_{\mathbb{H}}^2\Big] + C\Delta t^4.$$

For terms $B_{3,1,2}$ and $B_{3,1,3}$, we have

$$B_{3,1,2} + B_{3,1,3} \leq \frac{\sigma_0}{20}\Delta t \mathbb{E}[\|Mu^{n+1}\|_{\mathbb{H}}^2] + \frac{\sigma_0}{16}\Delta t e^{-2\sigma_0\Delta t}\mathbb{E}[\|Mu^n\|_{\mathbb{H}}^2] + C\Delta t.$$

Hence,

$$B_{3,1} \leq (C\Delta t^{\frac{1}{2}} A_{\Delta t} + \frac{\sigma_0}{10})\Delta t \mathbb{E}\Big[\|Mu^{n+1}\|_{\mathbb{H}}^2\Big] + (C\Delta t + \frac{\sigma_0}{16})\Delta t e^{-2\sigma_0 \Delta t}\mathbb{E}\Big[\|Mu^n\|_{\mathbb{H}}^2\Big] + C\Delta t.$$

For the term $B_{3,2}$, we use the skew-adjointness of M and then substitute (3.1) into it to obtain

$$\begin{aligned}
B_{3,2} &= -\lambda_1 \mathbb{E}\Big[\langle e^{-\sigma\Delta t} M(H_W^1 u^n), u^{n+1} - e^{-\sigma \Delta t} u^n\rangle_{\mathbb{H}}\Big] - \lambda_1 \mathbb{E}\Big[\langle R_\sigma^{\Delta t}(H_W^1 u^n), u^{n+1}\rangle_{\mathbb{H}}\Big] \\
&= -\frac{\lambda_1 \Delta t}{2}\mathbb{E}\Big[\langle e^{-\sigma\Delta t} M(H_W^1 u^n), Mu^{n+1}\rangle_{\mathbb{H}}\Big] \\
&\quad - \frac{\lambda_1^2}{2}\mathbb{E}\Big[\langle e^{-\sigma\Delta t} M(H_W^1 u^n), J(u^{n+1} + e^{-\sigma\Delta t} u^n)\Delta \overline{W}_1^n\rangle_{\mathbb{H}}\Big] \\
&\quad - \lambda_1 \mathbb{E}\Big[\langle R_\sigma^{\Delta t}(H_W^1 u^n), u^{n+1}\rangle_{\mathbb{H}}\Big].
\end{aligned}$$

Notice that

$$\begin{aligned}
M(H_W^1 u^n) &= \begin{pmatrix} (\nabla\cdot \mathbf{H}^n)\nabla(\Delta\overline{W}_1^n) \\ -(\nabla\cdot\mathbf{E}^n)\nabla(\Delta\overline{W}_1^n)\end{pmatrix} + \begin{pmatrix} (\mathbf{H}^n\cdot\nabla)\nabla(\Delta\overline{W}_1^n) \\ -(\mathbf{E}^n\cdot\nabla)\nabla(\Delta\overline{W}_1^n)\end{pmatrix} \\
&\quad + (\nabla\cdot(\nabla(\Delta\overline{W}_1^n)))Ju^n + (\nabla(\Delta\overline{W}_1^n)\cdot\nabla)Ju^n,
\end{aligned}$$

which leads to

$$\|M(H_W^1 u^n)\|_{\mathbb{H}}^2 \leq C\|\Delta\overline{W}_1^n\|_{H^{\gamma_1}(D)}^2\Big(\|\nabla\cdot\mathbf{H}^n\|_{L^2(D)}^2 + \|\nabla\cdot\mathbf{E}^n\|_{L^2(D)}^2 + \|u^n\|_{\mathbb{H}}^2 + \|Mu^n\|_{\mathbb{H}}^2\Big)$$

due to the Sobolev embedding $H^\gamma(D) \hookrightarrow L^\infty(D)$ for $\gamma > \frac{3}{2}$ and the Young inequality. Thus, one gets

$$\begin{aligned}
B_{3,2} &\leq \big(C\Delta t^{\frac{1}{2}} + \frac{\sigma_0}{16}\big)\Delta t e^{-2\sigma_0\Delta t}\mathbb{E}\Big[\|\nabla\cdot\mathbf{E}^n\|_{L^2(D)}^2 + \|\nabla\cdot\mathbf{H}^n\|_{L^2(D)}^2\Big] \\
&\quad + \frac{\sigma_0}{20}\Delta t\mathbb{E}\Big[\|Mu^{n+1}\|_{\mathbb{H}}^2\Big] + \big(C\Delta t + \frac{\sigma_0}{16}\big)\Delta t e^{-2\sigma_0\Delta t}\mathbb{E}\Big[\|Mu^n\|_{\mathbb{H}}^2\Big] + C\Delta t.
\end{aligned}$$

Combining the estimates of $B_{3,1}$ and $B_{3,2}$, it yields

$$\begin{aligned}
B_3 &\leq \big(C\Delta t^{\frac{1}{2}} A_{\Delta t} + \frac{3\sigma_0}{20}\big)\Delta t\mathbb{E}\Big[\|Mu^{n+1}\|_{\mathbb{H}}^2\Big] + \big(C\Delta t + \frac{\sigma_0}{8}\big)\Delta t e^{-2\sigma_0\Delta t}\mathbb{E}\Big[\|Mu^n\|_{\mathbb{H}}^2\Big] \\
&\quad + \big(C\Delta t^{\frac{1}{2}} + \frac{\sigma_0}{16}\big)\Delta t e^{-2\sigma_0\Delta t}\mathbb{E}\Big[\|\nabla\cdot\mathbf{E}^n\|_{L^2(D)}^2 + \|\nabla\cdot\mathbf{H}^n\|_{L^2(D)}^2\Big] + C\Delta t.
\end{aligned}$$

Putting all estimates of B_1–B_3 together, we arrive at

$$\begin{aligned}\mathbb{E}\big[\|Mu^{n+1}\|_{\mathbb{H}}^2\big] \leq & e^{-2\sigma_0\Delta t}\mathbb{E}\big[\|Mu^n\|_{\mathbb{H}}^2\big] + \big(\frac{3\sigma_0}{16} + C\Delta t\big)\Delta t e^{-2\sigma_0\Delta t}\mathbb{E}\big[\|Mu^n\|_{\mathbb{H}}^2\big] \\ & + \big(C\Delta t^{\frac{1}{2}}A_{\Delta t} + \frac{\sigma_0}{5}\big)\Delta t\mathbb{E}\big[\|Mu^{n+1}\|_{\mathbb{H}}^2\big] + C\Delta t \\ & + \big(C\Delta t^{\frac{1}{2}} + \frac{\sigma_0}{16}\big)\Delta t e^{-2\sigma_0\Delta t}\mathbb{E}\big[\|\nabla\cdot\mathbf{E}^n\|_{L^2(D)}^2 + \|\nabla\cdot\mathbf{H}^n\|_{L^2(D)}^2\big].\end{aligned}$$

Step 3. Proof of (3.10).

Combining *Step 1* and *Step 2*, it holds that

$$\begin{aligned}&\mathbb{E}\Big[\|Mu^{n+1}\|_{\mathbb{H}}^2 + \|\nabla\cdot\mathbf{H}^{n+1}\|_{L^2(D)}^2 + \|\nabla\cdot\mathbf{E}^{n+1}\|_{L^2(D)}^2\Big] \\ &\leq e^{-2\sigma_0\Delta t}\mathbb{E}\Big[\|Mu^n\|_{\mathbb{H}}^2 + \|\nabla\cdot\mathbf{H}^n\|_{L^2(D)}^2 + \|\nabla\cdot\mathbf{E}^n\|_{L^2(D)}^2\Big] \\ &\quad + \big(C\Delta t^{\frac{1}{2}} + \frac{\sigma_0}{4}\big)\Delta t e^{-2\sigma_0\Delta t}\mathbb{E}\Big[\|Mu^n\|_{\mathbb{H}}^2 + \|\nabla\cdot\mathbf{H}^n\|_{L^2(D)}^2 + \|\nabla\cdot\mathbf{E}^n\|_{L^2(D)}^2\Big] \\ &\quad + \big(C\Delta t^{\frac{1}{2}}A_{\Delta t} + \frac{\sigma_0}{4}\big)\Delta t\mathbb{E}\Big[\|Mu^{n+1}\|_{\mathbb{H}}^2 + \|\nabla\cdot\mathbf{H}^{n+1}\|_{L^2(D)}^2 + \|\nabla\cdot\mathbf{E}^{n+1}\|_{L^2(D)}^2\Big] + C\Delta t.\end{aligned}$$

Since $\lim_{\Delta t\to 0}\Delta t^{\frac{1}{2}}A_{\Delta t} = 0$, there exists a $\Delta t^* > 0$ such that for all $\Delta t \in (0, \Delta t^*]$, we have

$$C\big(\Delta t^{\frac{1}{2}} + \Delta t^{\frac{1}{2}}A_{\Delta t}(1 + \sigma_0\Delta t)\big) + \frac{\sigma_0^2}{4}\Delta t \leq \frac{\sigma_0}{2},$$

which implies that

$$\frac{1 + (C\Delta t^{\frac{1}{2}} + \frac{\sigma_0}{4})\Delta t}{1 - (C\Delta t^{\frac{1}{2}}A_{\Delta t} + \frac{\sigma_0}{4})\Delta t} \leq 1 + \sigma_0\Delta t \leq e^{\sigma_0\Delta t}.$$

By the Gronwall inequality, we finish the proof. □

Remark 3.3. In view of the Sobolev embedding $H^\gamma(D) \hookrightarrow L^\infty(D)$ for $\gamma > \frac{3}{2}$, it is clear that $F_{Q_1} \in W^{1,\infty}(D)$ if $\sum_{i=1}^\infty \|Q_1^{\frac{1}{2}}e_i\|_{H^\gamma} < \infty$ with $\gamma > \frac{5}{2}$.

For the fourth moment estimates of the curl and divergence of the numerical solution u^n, by multiplying $\|\nabla\cdot\mathbf{E}^{n+1}\|_{L^2(D)}^2 + \|\nabla\cdot\mathbf{H}^{n+1}\|_{L^2(D)}^2$ and $\|Mu^{n+1}\|_{\mathbb{H}}^2$ on both sides of (3.16) and (3.17), respectively, we can derive the following result. The proof is similar to that of Lemma 3.2.

Corollary 3.4. *Let Assumption 2.1 hold with $\gamma_1 > \frac{7}{2}$ and $\gamma_2 \geq 2$, and let $\sum_{i=1}^\infty \|Q_1^{\frac{1}{2}}e_i\|_{H^{\gamma_1-1}(D)} < \infty$ and $u_0 \in L^4(\Omega, H^1(D)^6)$. There exist positive constants C_7 and C_8 such that for sufficiently small $\Delta t > 0$,*

$$\mathbb{E}\big[\|u^n\|_{H^1(D)^6}^4\big] \leq C_7 e^{-3\sigma_0 t}\mathbb{E}\big[\|u_0\|_{H^1(D)^6}^4\big] + C_8,$$

where C_7 depends on $|D|$, and C_8 depends on $\sigma_0, C_{4,1}, C_{4,2}, \lambda_1, \widetilde{\lambda}_2, \mathbb{E}\big[\|u_0\|_{\mathbb{H}}^4\big], \|Q_1^{\frac{1}{2}}\|_{\mathcal{L}_2^{\gamma_1}}$ and $\|Q_2^{\frac{1}{2}}\|_{\mathcal{L}_2^{\gamma_2}}$.

Let $\{P_n\}_{n\in\mathbb{N}}$ be the Markov transition semigroup associated to the numerical solution $\{u^n\}_{n\in\mathbb{N}}$. Based on Lemma 3.2, the ergodicity of $\{u^n\}_{n\in\mathbb{N}}$ and the convergence of $P_n^*\pi$ to the numerical invariant measure in the L^2-Wasserstein distance are similar to that of Proposition 2.6, which are stated below.

Theorem 3.5. *Under the conditions in Lemma 3.2, the following statements hold.*
(i) *The numerical solution $\{u^n\}_{n\in\mathbb{N}}$ of (3.1) has a unique invariant measure $\pi^{\Delta t} \in \mathcal{P}_2(\mathbb{H})$ for sufficiently small $\Delta t > 0$. Thus $\{u^n\}_{n\in\mathbb{N}}$ is ergodic. Moreover, $\{u^n\}_{n\in\mathbb{N}}$ is exponentially mixing.*
(ii) *For any distribution $\pi \in \mathcal{P}_2(\mathbb{H})$,*

$$\mathcal{W}_2(P_n^*\pi, \pi^{\Delta t}) \le e^{-\sigma_0 t_n}\mathcal{W}_2(\pi, \pi^{\Delta t}).$$

Proof. Notice that $u^n - \widetilde{u}^n$ solves

$$\begin{aligned}(u^n - \widetilde{u}^n) - e^{-\sigma\Delta t}(u^{n-1} - \widetilde{u}^{n-1}) =& \Delta t M\frac{(u^n - \widetilde{u}^n) + e^{-\sigma\Delta t}(u^{n-1} - \widetilde{u}^{n-1})}{2} \\ &+ \lambda_1 J\frac{(u^n - \widetilde{u}^n) + e^{-\sigma\Delta t}(u^{n-1} - \widetilde{u}^{n-1})}{2}\Delta\overline{W}_1^{n-1}.\end{aligned}$$

We apply $\langle\cdot, (u^n - \widetilde{u}^n) + e^{-\sigma\Delta t}(u^{n-1} - \widetilde{u}^{n-1})\rangle_{\mathbb{H}}$ to both sides of the above equation and take expectation to get

$$\mathbb{E}\big[\|u^n - \widetilde{u}^n\|_{\mathbb{H}}^2\big] \le e^{-2\sigma_0\Delta t}\mathbb{E}\big[\|u^{n-1} - \widetilde{u}^{n-1}\|_{\mathbb{H}}^2\big] \le \cdots \le e^{-2\sigma_0 t_n}\mathbb{E}\big[\|u_0 - \widetilde{u}_0\|_{\mathbb{H}}^2\big],$$

by which assertions (i) and (ii) can be obtained similarly to Proposition 2.6. Thus we finish the proof. □

3.1.2. Stochastic conformal multi-symplecticity

Now we turn to the stochastic conformal multi-symplecticity of the temporal semi-discretization (3.1). Let

$$\delta_t^\sigma u^n = \frac{u^{n+1} - e^{-\sigma\Delta t}u^n}{\Delta t}, \quad A_t^\sigma u^n = \frac{u^{n+1} + e^{-\sigma\Delta t}u^n}{2}.$$

Then (3.1) can be transformed into the following compact form

$$F\delta_t^\sigma u^n + K_1\partial_x A_t^\sigma u^n + K_2\partial_y A_t^\sigma u^n + K_3\partial_z A_t^\sigma u^n = \nabla_u S_1(A_t^\sigma u^n)\frac{\Delta\overline{W}_1^n}{\Delta t} + \nabla_u S_2(A_t^\sigma u^n)\frac{\Delta W_2^n}{\Delta t}.$$

Similar to [6, Theorem 3.1] and [14, Theorem 3.1], we can obtain the following result.

Proposition 3.6. *The temporal semi-discretization* (3.1) *possesses the stochastic conformal multi-symplectic conservation law*

$$\delta_t^{2\sigma}\omega^n + \partial_x\kappa_1^n + \partial_y\kappa_2^n + \partial_z\kappa_3^n = 0 \qquad \mathbb{P}\text{-}a.s.,$$

that is,

$$\begin{aligned}&\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1}\omega^{n+1}(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z - \int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1}e^{-2\sigma\Delta t}\omega^n(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\&+\Delta t\int_{z_0}^{z_1}\int_{y_0}^{y_1}\kappa_1^n(x_1,y,z)\mathrm{d}y\mathrm{d}z - \Delta t\int_{z_0}^{z_1}\int_{y_0}^{y_1}\kappa_1^n(x_0,y,z)\mathrm{d}y\mathrm{d}z\\&+\Delta t\int_{z_0}^{z_1}\int_{x_0}^{x_1}\kappa_2^n(x,y_1,z)\mathrm{d}x\mathrm{d}z - \Delta t\int_{z_0}^{z_1}\int_{x_0}^{x_1}\kappa_2^n(x,y_0,z)\mathrm{d}x\mathrm{d}z\\&+\Delta t\int_{y_0}^{y_1}\int_{x_0}^{x_1}\kappa_3^n(x,y,z_1)\mathrm{d}x\mathrm{d}y - \Delta t\int_{y_0}^{y_1}\int_{x_0}^{x_1}\kappa_3^n(x,y,z_0)\mathrm{d}x\mathrm{d}y = 0,\end{aligned}$$

where $\omega^n = \frac{1}{2}du^n \wedge Fdu^n, \kappa_s^n = \frac{1}{2}(dA_t^\sigma u^n)\wedge K_s d(A_t^\sigma u^n)$*,* $s = 1,2,3$ *are differential 2-forms associated with the skew-symmetric matrices* F *and* K_s*, respectively, and* $(x_0,x_1)\times(y_0,y_1)\times(z_0,z_1)$ *is the local domain of* $u^n(x,y,z)$*.*

3.2. Error analysis of the temporal semi-discretization

In this section, we study the mean-square convergence order of the temporal semi-discretization (3.1). To this end, we rewrite (3.1) into the following form

$$u^{n+1} = \hat{S}_{\Delta t}u^n + \lambda_1 T_{\Delta t} J A_t^\sigma u^n \Delta\overline{W}_1^n + T_{\Delta t}\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n, \quad n \geq 0, \tag{3.18}$$

where

$$\hat{S}_{\Delta t} = \Big(I - \frac{\Delta t}{2}M\Big)^{-1}\Big(I + \frac{\Delta t}{2}M\Big)e^{-\sigma\Delta t}, \quad T_{\Delta t} = \Big(I - \frac{\Delta t}{2}M\Big)^{-1}.$$

In order to derive the error estimate of (3.1), we need to introduce the following lemma, whose proof is given in Appendix A.

Lemma 3.7. *Let* $\hat{S}(t) := e^{t(M-\sigma I)}$*,* $t \geq 0$*. There exist positive constants* C *such that*
(1) $\|\hat{S}(t)\|_{\mathcal{L}(\mathbb{H},\mathbb{H})} \leq e^{-\sigma_0 t}$ *and* $\|(\hat{S}_{\Delta t})^n\|_{\mathcal{L}(\mathbb{H},\mathbb{H})} \leq e^{-\sigma_0 n\Delta t}$*.*
(2) $\|\hat{S}(t_n) - (\hat{S}_{\Delta t})^n\|_{\mathcal{L}(D(M),\mathbb{H})} \leq Ce^{-\sigma_0 n\Delta t/2}\Delta t^{1/2}$*.*
(3) $\|\hat{S}(t) - I\|_{\mathcal{L}(D(M),\mathbb{H})} \leq Ct \quad \forall t \geq 0$ *and* $\|\hat{S}(t) - e^{-\sigma\Delta t}\|_{\mathcal{L}(D(M),\mathbb{H})} \leq C\Delta t \quad \forall t \in [0,\Delta t]$*.*
(4) $\|T_{\Delta t}\|_{\mathcal{L}(\mathbb{H},\mathbb{H})} \leq 1$ *and* $\|I - T_{\Delta t}\|_{\mathcal{L}(\mathcal{D}(M),\mathbb{H})} \leq C\Delta t$*.*
(5) $\|\hat{S}(t_n - r) - (\hat{S}_{\Delta t})^{n-k-1}T_{\Delta t}\|_{\mathcal{L}(D(M),\mathbb{H})} \leq Ce^{-\sigma_0(n-k-1)\Delta t/2}\Delta t^{1/2} \quad \forall r \in [t_k, t_{k+1}]$*,* $k = 0,\dots,n-1$*.*

The error of (3.1) is estimated in the following theorem.

Theorem 3.8. *Under the conditions in Corollary 3.4, there exists a positive constant C_9 such that for sufficiently small $\Delta t > 0$,*

$$\max_{n\geq 0}\left(\mathbb{E}\left[\|u(t_n)-u^n\|_{\mathbb{H}}^2\right]\right)^{1/2}\leq C_9\Delta t^{1/2},$$

where C_9 depends on σ_0, λ_1, $\widetilde{\boldsymbol{\lambda}}_2$, C_1–C_3, $C_{4,1}$, $C_{4,2}$, C_5–C_8, $\mathbb{E}\left[\|u_0\|^4_{H^1(D)^6}\right]$, $\|Q_1^{\frac{1}{2}}\|_{\mathcal{L}_2^{\gamma_1}}$, $\|Q_2^{\frac{1}{2}}\|_{\mathcal{L}_2^{\gamma_2}}$, $\|\sigma\|_{W^{1,\infty}(D)}$.

Proof. For $n \in \mathbb{N}$, we note that the mild solution of (1.1) is

$$\begin{aligned}u(t_n)=&\hat{S}(t_n)u_0-\frac{1}{2}\lambda_1^2\int_0^{t_n}\hat{S}(t_n-s)F_{Q_1}u(s)\mathrm{d}s\\&+\lambda_1\int_0^{t_n}\hat{S}(t_n-s)Ju(r)\mathrm{d}W_1(s)+\int_0^{t_n}\hat{S}(t_n-s)\widetilde{\boldsymbol{\lambda}}_2\mathrm{d}W_2(s).\end{aligned}\tag{3.19}$$

From (3.18), we have

$$u^n=\left(\hat{S}_{\Delta t}\right)^n u_0+\lambda_1\sum_{k=0}^{n-1}\left(\hat{S}_{\Delta t}\right)^{n-k-1}T_{\Delta t}JA_t^{\sigma}u^k\Delta\overline{W}_1^k+\sum_{k=0}^{n-1}\left(\hat{S}_{\Delta t}\right)^{n-1-k}T_{\Delta t}\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^k.\tag{3.20}$$

Let $e_n := u(t_n) - u^n$. Subtracting (3.20) from (3.19), it yields

$$e_n =: I + II + III$$

with

$$\begin{aligned}I=&\left[\hat{S}(t_n)-(\hat{S}_{\Delta t})^n\right]u_0,\\II=&\int_0^{t_n}\hat{S}(t_n-r)\widetilde{\boldsymbol{\lambda}}_2\mathrm{d}W_2(r)-\sum_{k=0}^{n-1}\left(\hat{S}_{\Delta t}\right)^{n-1-k}T_{\Delta t}\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^k,\\III=&\lambda_1\int_0^{t_n}\hat{S}(t_n-r)Ju(r)\mathrm{d}W_1(r)-\frac{1}{2}\lambda_1^2\int_0^{t_n}\hat{S}(t_n-r)F_{Q_1}u(r)\mathrm{d}r\\&-\lambda_1\sum_{k=0}^{n-1}\left(\hat{S}_{\Delta t}\right)^{n-1-k}T_{\Delta t}JA_t^{\sigma}u^k\Delta\overline{W}_1^k.\end{aligned}$$

By Lemma 3.7(2), we have

$$\sup_{n\geq 0}\mathbb{E}\big[\|I\|_{\mathbb{H}}^2\big]\leq C\Delta t.$$

Using the Itô isometry and Lemma 3.7(5), it holds that

$$\begin{aligned}\mathbb{E}\big[\|II\|_{\mathbb{H}}^2\big]&=\mathbb{E}\Big[\Big\|\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(\hat{S}(t_n-r)-(\hat{S}_{\Delta t})^{n-k-1}T_{\Delta t})\widetilde{\boldsymbol{\lambda}}_2\mathrm{d}W_2(r)\Big\|_{\mathbb{H}}^2\Big]\\&=\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\big\|(\hat{S}(t_n-r)-(\hat{S}_{\Delta t})^{n-k-1}T_{\Delta t})\widetilde{\boldsymbol{\lambda}}_2Q_2^{\frac12}\big\|_{HS(L^2(D),\mathbb{H})}^2\mathrm{d}r\\&\leq\sum_{k=0}^{n-1}Ce^{-\sigma_0(n-k-1)\Delta t}\Delta t^2=C\Delta t,\end{aligned}$$

where in the last step we use the fact $\frac{1-e^{-\sigma_0 n\Delta t}}{1-e^{-\sigma_0\Delta t}}\leq\frac{1}{\sigma_0\Delta t}$.

For the third term III, we substitute (3.1) into it and obtain

$$\begin{aligned}III=&\sum_{k=0}^{n-1}\Big[\lambda_1\int_{t_k}^{t_{k+1}}\hat{S}(t_n-r)Ju(r)\mathrm{d}W_1(r)-\frac12\lambda_1^2\int_{t_k}^{t_{k+1}}\hat{S}(t_n-r)F_{Q_1}u(r)\mathrm{d}r\\&-\lambda_1(\hat{S}_{\Delta t})^{n-1-k}T_{\Delta t}JA_t^{\sigma}u^k\Delta\overline{W}_1^k\Big]=:\sum_{i=1}^4\sum_{k=0}^{n-1}A_{i,k},\end{aligned}$$

where

$$\begin{aligned}A_{1,k}=&\frac{\lambda_1^2}{2}\Big[\big(\hat{S}_{\Delta t}\big)^{n-1-k}T_{\Delta t}e^{-\sigma\Delta t}u^k(\Delta W_1^k)^2-\int_{t_k}^{t_{k+1}}\hat{S}(t_n-r)F_{Q_1}u(r)\mathrm{d}r\Big],\\A_{2,k}=&\lambda_1\big(\hat{S}_{\Delta t}\big)^{n-1-k}T_{\Delta t}Je^{-\sigma\Delta t}u^k\big(\Delta W_1^k-\Delta\overline{W}_1^k\big)\\&-\frac{\lambda_1\Delta t}{4}\big(\hat{S}_{\Delta t}\big)^{n-1-k}T_{\Delta t}JM(u^{k+1}+e^{-\sigma\Delta t}u^k)\Delta\overline{W}_1^k\\&+\frac{\lambda_1^2}{4}\big(\hat{S}_{\Delta t}\big)^{n-1-k}T_{\Delta t}\big(u^{k+1}-e^{-\sigma\Delta t}u^k\big)(\Delta\overline{W}_1^k)^2\\&+\frac{\lambda_1^2}{2}\big(\hat{S}_{\Delta t}\big)^{n-1-k}T_{\Delta t}e^{-\sigma\Delta t}u^k\big[(\Delta\overline{W}_1^k)^2-(\Delta W_1^k)^2\big],\\A_{3,k}=&\lambda_1\int_{t_k}^{t_{k+1}}\Big(\hat{S}(t_n-r)Ju(r)-\big(\hat{S}_{\Delta t}\big)^{n-1-k}T_{\Delta t}Je^{-\sigma\Delta t}u^k\Big)\mathrm{d}W_1(r),\end{aligned}$$

$$A_{4,k} = -\frac{\lambda_1}{2}\big(\hat{S}_{\Delta t}\big)^{n-1-k} T_{\Delta t} J \widetilde{\boldsymbol{\lambda}}_2 \Delta \overline{W}_1^k \Delta W_2^k.$$

(i) *Estimate of the term* $\sum_{k=0}^{n-1} A_{1,k}$.
Notice that

$$\begin{aligned}
A_{1,k} &= \lambda_1^2 \int_{t_k}^{t_{k+1}} \int_{t_k}^{r} \big(\hat{S}_{\Delta t}\big)^{n-1-k} T_{\Delta t} e^{-\sigma \Delta t} u^k \mathrm{d}W_1(\rho) \mathrm{d}W_1(r) \\
&\qquad + \frac{\lambda_1^2}{2}\Big[\big(\hat{S}_{\Delta t}\big)^{n-1-k} T_{\Delta t} F_{Q_1} e^{-\sigma \Delta t} u^k \Delta t - \int_{t_k}^{t_{k+1}} \hat{S}(t_n - r) F_{Q_1} u(r) \mathrm{d}r\Big] \\
&= \lambda_1^2 \int_{t_k}^{t_{k+1}} \int_{t_k}^{r} \big(\hat{S}_{\Delta t}\big)^{n-1-k} T_{\Delta t} e^{-\sigma \Delta t} u^k \mathrm{d}W_1(\rho) \mathrm{d}W_1(r) \\
&\quad + \frac{\lambda_1^2}{2} \int_{t_k}^{t_{k+1}} \Big[\big(\hat{S}_{\Delta t}\big)^{n-1-k} T_{\Delta t} - \hat{S}(t_n - r)\Big] (F_{Q_1} e^{-\sigma \Delta t} u^k) \mathrm{d}r \\
&\quad - \frac{\lambda_1^2}{2} \int_{t_k}^{t_{k+1}} \hat{S}(t_n - r)\big(F_{Q_1} e^{-\sigma \Delta t} e_k\big) \mathrm{d}r \\
&\quad - \frac{\lambda_1^2}{2} \int_{t_k}^{t_{k+1}} \hat{S}(t_n - r) F_{Q_1}\big(u(r) - e^{-\sigma \Delta t} u(t_k)\big) \mathrm{d}r \\
&=: A_{1,k,1} + A_{1,k,2} + A_{1,k,3} + A_{1,k,4}.
\end{aligned}$$

For the term $A_{1,k,1}$, it follows from the Itô isometry, Sobolev embedding $H^\gamma(D) \hookrightarrow L^\infty(D)$ for $\gamma > \frac{3}{2}$, Lemma 3.7(1)(4) and Proposition 3.1 that

$$\begin{aligned}
\mathbb{E}\big[\|A_{1,k,1}\|_{\mathbb{H}}^2\big] &\le C \int_{t_k}^{t_{k+1}} \int_{t_k}^{r} \mathbb{E}\big[\|(\hat{S}_{\Delta t})^{n-k-1} T_{\Delta t} e^{-\sigma \Delta t} u^k Q_1^{\frac{1}{2}}\|_{HS(L^2(D),\mathbb{H})}^2\big] \mathrm{d}\rho \mathrm{d}r \\
&\le C \int_{t_k}^{t_{k+1}} \int_{t_k}^{r} \mathbb{E}\big[\|(\hat{S}_{\Delta t})^{n-k-1} T_{\Delta t} e^{-\sigma \Delta t} u^k\|_{\mathbb{H}}^2\big] \mathrm{d}\rho \mathrm{d}r \le C e^{-2\sigma_0 (n-k)\Delta t} \Delta t^2,
\end{aligned}$$

which implies

$$\mathbb{E}\Big[\Big\|\sum_{k=0}^{n-1} A_{1,k,1}\Big\|_{\mathbb{H}}^2\Big] = \sum_{k=0}^{n-1} \mathbb{E}\Big[\Big\|A_{1,k,1}\Big\|_{\mathbb{H}}^2\Big] \le C \Delta t.$$

For the term $A_{1,k,2}$, we use Proposition 3.1 and Lemma 3.7(5) to obtain

$$\mathbb{E}\big[\|A_{1,k,2}\|_{\mathbb{H}}^2\big] \le C\Delta t \int_{t_k}^{t_{k+1}} \|(\hat{S}_{\Delta t})^{n-k-1}T_{\Delta t} - \hat{S}(t_n - r)\|_{\mathcal{L}(\mathcal{D}(M),\mathbb{H})}^2 \mathbb{E}\big[\|F_{Q_1}e^{-\sigma\Delta t}u^k\|_{\mathcal{D}(M)}^2\big]\mathrm{d}r$$
$$\le Ce^{-\sigma_0(n-k)}\Delta t^3,$$

which leads to

$$\mathbb{E}\Big[\Big\|\sum_{k=0}^{n-1}A_{1,k,2}\Big\|_{\mathbb{H}}^2\Big] = \sum_{i,j=0}^{n-1}\mathbb{E}\big[\langle A_{1,i,2}, A_{1,j,2}\rangle_{\mathbb{H}}\big] \le \sum_{i,j=0}^{n-1}\Big(\mathbb{E}\big[\|A_{1,i,2}\|_{\mathbb{H}}^2\big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\big[\|A_{1,j,2}\|_{\mathbb{H}}^2\big]\Big)^{\frac{1}{2}}$$
$$= \bigg(\sum_{k=0}^{n-1}\Big(\mathbb{E}\big[\|A_{1,k,2}\|_{\mathbb{H}}^2\big]\Big)^{\frac{1}{2}}\bigg)^2 = C\bigg(\Delta t\sum_{k=0}^{n-1}e^{-\sigma_0(n-k)\Delta t/2}\bigg)^2\Delta t \le C\Delta t.$$

For the term $A_{1,k,3}$, the Hölder inequality and Lemma 3.7(1) imply

$$\mathbb{E}\big[\|A_{1,k,3}\|_{\mathbb{H}}^2\big] \le C\Delta t\int_{t_k}^{t_{k+1}}\mathbb{E}\big[\big\|\hat{S}(t_n - r)\big(F_{Q_1}e^{-\sigma\Delta t}e_k\big)\big\|_{\mathbb{H}}^2\big]\mathrm{d}r \le Ce^{-2\sigma_0(n-k)\Delta t}\Delta t^2\mathbb{E}\big[\|e_k\|_{\mathbb{H}}^2\big],$$

by which we get

$$\mathbb{E}\Big[\Big\|\sum_{k=0}^{n-1}A_{1,k,3}\Big\|_{\mathbb{H}}^2\Big] \le \bigg(\sum_{k=0}^{n-1}\Big(\mathbb{E}\big[\|A_{1,k,3}\|_{\mathbb{H}}^2\big]\Big)^{\frac{1}{2}}\bigg)^2 \le \bigg(\sum_{k=0}^{n-1}Ce^{-\sigma_0(n-k)\Delta t}\Delta t\Big(\mathbb{E}\big[\|e_k\|_{\mathbb{H}}^2\big]\Big)^{\frac{1}{2}}\bigg)^2$$
$$\le C\bigg(\sum_{k=0}^{n-1}\Big(\Delta t^{\frac{1}{2}}e^{-\sigma_0(n-k)\Delta t/2}\Big)\Big(\Delta t^{\frac{1}{2}}e^{-\sigma_0(n-k)\Delta t/2}\big(\mathbb{E}\big[\|e_k\|_{\mathbb{H}}^2\big]\big)^{\frac{1}{2}}\Big)\bigg)^2$$
$$\le C\Big(\sum_{k=0}^{n-1}\Delta te^{-\sigma_0(n-k)\Delta t}\Big)\Big(\sum_{k=0}^{n-1}\Delta te^{-\sigma_0(n-k)\Delta t}\mathbb{E}\big[\|e_k\|_{\mathbb{H}}^2\big]\Big) \le C\Delta t\sum_{k=0}^{n-1}e^{-\sigma_0(n-k)\Delta t}\mathbb{E}\big[\|e_k\|_{\mathbb{H}}^2\big].$$

For $r \in [t_k, t_{k+1}]$, by (3.19) we have

$$\begin{aligned}
&\mathbb{E}\big[\|u(r) - e^{-\sigma\Delta t}u(t_k)\|_{\mathbb{H}}^2\big]\\
&\le C\big\|\hat{S}(r - t_k) - e^{-\sigma\Delta t}\big\|_{\mathcal{L}(D(M),\mathbb{H})}^2\mathbb{E}\big[\|u(t_k)\|_{D(M)}^2\big] + C\Delta t\int_{t_k}^{r}\mathbb{E}\big[\|\hat{S}(r-\rho)F_{Q_1}u(\rho)\|_{\mathbb{H}}^2\big]\mathrm{d}\rho\\
&\quad + C\int_{t_k}^{r}\mathbb{E}\big[\|\hat{S}(r-\rho)Ju(\rho)Q_1^{\frac{1}{2}}\|_{HS(L^2(D),\mathbb{H})}^2\big]\mathrm{d}\rho + C\int_{t_k}^{r}\mathbb{E}\big[\|\hat{S}(r-\rho)\widetilde{\boldsymbol{\lambda}}_2Q_2^{\frac{1}{2}}\|_{HS(L^2(D),\mathbb{H})}^2\big]\mathrm{d}\rho\\
&\le C\Delta t,
\end{aligned} \tag{3.21}$$

where we use the Itô isometry, Lemma 3.7(1)(3) and Proposition 2.3. Therefore, for the term $A_{1,k,4}$, one obtains

$$\mathbb{E}\big[\|A_{1,k,4}\|_{\mathbb{H}}^2\big] \le C\Delta t e^{-2\sigma_0(n-k)\Delta t}\int_{t_k}^{t_{k+1}} \mathbb{E}\big[\|u(r)-e^{-\sigma\Delta t}u(t_k)\|_{\mathbb{H}}^2\big]\mathrm{d}r \le Ce^{-2\sigma_0(n-k)\Delta t}\Delta t^3,$$

from which we have

$$\mathbb{E}\Big[\Big\|\sum_{k=1}^{n-1} A_{1,k,4}\Big\|_{\mathbb{H}}^2\Big] \le C\Delta t.$$

Combining the above estimates of $A_{1,k,1}$–$A_{1,k,4}$, it holds that

$$\mathbb{E}\Big[\Big\|\sum_{k=0}^{n-1} A_{1,k}\Big\|_{\mathbb{H}}^2\Big] \le C\Delta t + C\Delta t\sum_{k=0}^{n-1} e^{-\sigma_0(n-k)\Delta t}\mathbb{E}\big[\|e_k\|_{\mathbb{H}}^2\big].$$

(ii) *Estimate of the term* $\sum_{k=0}^{n-1} A_{2,k}$.
Substituting (3.1) into the third term of $A_{2,k}$ and Lemma 3.7(1)(4), we have

$$\begin{aligned}\mathbb{E}\big[\|A_{2,k}\|_{\mathbb{H}}^2\big] \le{}& Ce^{-2\sigma_0(n-k)\Delta t}\mathbb{E}\big[\|u^k\|_{\mathbb{H}}^2\|\Delta W_1^k-\Delta\overline{W}_1^k\|_{H^{\gamma_1-2}(D)}^2\big]\\ &+C\Delta t^2 e^{-2\sigma_0(n-k)\Delta t}\mathbb{E}\big[\|M(u^{k+1}+e^{-\sigma\Delta t}u^k)\|_{\mathbb{H}}^2\|\Delta\overline{W}_1^k\|_{H^{\gamma_1-2}(D)}^2\big]\\ &+Ce^{-2\sigma_0(n-k)\Delta t}\mathbb{E}\big[\|(u^{k+1}+e^{-\sigma\Delta t}u^k)(\Delta\overline{W}_1^k)^3\|_{\mathbb{H}}^2\big]\\ &+Ce^{-2\sigma_0(n-k)\Delta t}\mathbb{E}\big[\|\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^k(\Delta\overline{W}_1^k)^2\|_{\mathbb{H}}^2\big]\\ &+Ce^{-2\sigma(n-k)\Delta t}\mathbb{E}\big[\|u^k\|_{\mathbb{H}}^2\|(\Delta\overline{W}_1^k)^2-(\Delta W_1^k)^2\|_{H^{\gamma_2-\frac{1}{2}}(D)}^2\big],\end{aligned}$$

which along with Proposition 3.1, Lemma 3.2, Corollary 3.4 and (3.3), leads to

$$\mathbb{E}\Big[\Big\|\sum_{k=0}^{n-1} A_{2,k}\Big\|_{\mathbb{H}}^2\Big] \le C\Delta t.$$

(iii) *Estimate of the term* $\sum_{k=0}^{n-1} A_{3,k}$.
We use the Itô isometry, (3.21), Lemma 2.4 and Lemma 3.7(1)(4)(5) to obtain

$$\begin{aligned}\mathbb{E}\big[\|A_{3,k}\|_{\mathbb{H}}^2\big] ={}& \lambda_1^2\int_{t_k}^{t_{k+1}} \mathbb{E}\Big[\Big\|\big(\hat{S}(t_n-r)Ju(r)-\big(\hat{S}_{\Delta t}\big)^{n-1-k}T_{\Delta t}Je^{-\sigma\Delta t}u^k\big)Q_1^{\frac{1}{2}}\Big\|_{HS(L^2(D),\mathbb{H})}^2\Big]\mathrm{d}r\\ \le{}& C\int_{t_k}^{t_{k+1}} \mathbb{E}\Big[\Big\|\hat{S}(t_n-r)Ju(r)-\big(\hat{S}_{\Delta t}\big)^{n-k-1}T_{\Delta t}Ju(r)\Big\|_{\mathbb{H}}^2\Big]\mathrm{d}r\end{aligned}$$

$$
\begin{aligned}
&+C\int_{t_k}^{t_{k+1}}\mathbb{E}\big[\big\|\big(\hat{S}_{\Delta t}\big)^{n-k-1}T_{\Delta t}Ju(r)-\big(\hat{S}_{\Delta t}\big)^{n-k-1}T_{\Delta t}Je^{-\sigma\Delta t}u^k\big\|_{\mathbb{H}}^2\big]\mathrm{d}r\\
&\le C\int_{t_k}^{t_{k+1}}\big\|\hat{S}(t_n-r)-\big(\hat{S}_{\Delta t}\big)^{n-k-1}T_{\Delta t}\big\|_{\mathcal{L}(\mathcal{D}(M),\mathbb{H})}^2\mathbb{E}\big[\|u(r)\|_{\mathcal{D}(M)}^2\big]\mathrm{d}r\\
&+C\int_{t_k}^{t_{k+1}}\|(\hat{S}_{\Delta t})^{n-1-k}T_{\Delta t}\|_{\mathcal{L}(\mathbb{H},\mathbb{H})}^2\mathbb{E}\big[\|u(r)-e^{-\sigma\Delta t}u(t_k)\|_{\mathbb{H}}^2+\|e^{-\sigma\Delta t}e_k\|_{\mathbb{H}}^2\big]\mathrm{d}r\\
&\le Ce^{-\sigma_0(n-k)\Delta t}\Delta t\Big(\Delta t+\mathbb{E}\big[\|e_k\|_{\mathbb{H}}^2\big]\Big),
\end{aligned}
$$

by which

$$
\mathbb{E}\Big[\Big\|\sum_{k=0}^{n-1}A_{3,k}\Big\|_{\mathbb{H}}^2\Big]=\sum_{k=0}^{n-1}\mathbb{E}\big[\|A_{3,k}\|_{\mathbb{H}}^2\big]\le C\Delta t+C\Delta t\sum_{k=0}^{n-1}e^{-\sigma_0(n-k)\Delta t}\mathbb{E}\big[\|e_k\|_{\mathbb{H}}^2\big].
$$

(iv) *Estimate of the term* $\sum_{k=0}^{n-1}A_{4,k}$.
By the independence of increments of W_1 and W_2, we have

$$
\mathbb{E}\Big[\Big\|\sum_{k=0}^{n-1}A_{4,k}\Big\|_{\mathbb{H}}^2\Big]=\sum_{k=0}^{n-1}\mathbb{E}\big[\|A_{4,k}\|_{\mathbb{H}}^2\big]\le C\sum_{k=0}^{n-1}e^{-\sigma_0(n-k-1)}\Delta t\mathbb{E}\big[\|\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^k\Delta\overline{W}_1^k\|_{\mathbb{H}}^2\big]\le C\Delta t.
$$

Combining (i)–(iv), it yields

$$
\mathbb{E}\big[\|III\|_{\mathbb{H}}^2\big]\le C\Delta t+C\Delta t\sum_{k=0}^{n-1}e^{-\sigma_0(n-k)\Delta t}\mathbb{E}\big[\|e_k\|_{\mathbb{H}}^2\big].
$$

Altogether, we conclude that

$$
\mathbb{E}\big[\|e_n\|_{\mathbb{H}}^2\big]\le C\Delta t\exp\Big(\sum_{k=0}^{n-1}\Delta te^{-\sigma_0(n-k)\Delta t}\Big)\le C\Delta t
$$

due to the Gronwall inequality. Thus we finish the proof. □

Similar to Proposition 2.6(ii), the error of the invariant measure between the exact solution and the numerical solution in L^2-Wasserstein distance can be estimated via Theorem 3.8.

Corollary 3.9. *Under the conditions in Lemma 2.4 and Theorem 3.8, there exists a positive constant C_{10} such that*

$$
\mathcal{W}_2(\pi^*,\pi^{\Delta t})\le C_{10}\Delta t^{\frac{1}{2}}.
$$

Proof. By Proposition 2.6 and Theorem 3.8, we have

$$\mathcal{W}_2(\pi^*, \pi^{\Delta t}) \le \mathcal{W}_2(P_n^*\pi^{\Delta t}, P_{t_n}^*\pi^{\Delta t}) + \mathcal{W}_2(P_{t_n}^*\pi^{\Delta t}, P_{t_n}^*\pi^*) \le C_9\Delta t^{\frac{1}{2}} + e^{-\sigma_0 t_n}\mathcal{W}_2(\pi^*, \pi^{\Delta t}),$$

which leads to the assertion by letting $n \to \infty$. □

4. Ergodic full discretizations

This section focuses on the study of ergodic full discretizations for (1.1) which are based on the further discretizations on the temporal semi-discretization by the dG method and the finite difference method in the spatial direction, respectively.

4.1. Ergodic dG full discretization

We first apply the dG method to discretize (3.1) in space and obtain a dG full discretization for (1.1). For the dG full discretization, we prove the ergodicity based on the analysis of the uniform boundedness of the numerical solution in $L^2(\Omega, \mathbb{H})$. Moreover, the mean-square convergence order of the dG full discretization in both temporal and spatial directions is shown. As a result, the L^2-Wasserstein distance between the numerical invariant measure and the exact one is estimated.

To this end, we introduce some basic notations and properties of the dG method. Let $\mathcal{T}_h = \{K\}$ be a simplicial, shape- and contact-regular mesh of the domain D consisting of elements K, i.e., $D = \bigcup K$. The index h refers to the maximum diameter of all elements of $\mathcal{T}_h$. We denote the restriction of a function v to an element K by $v_K := v|_K$. The dG space with respect to the mesh $\mathcal{T}_h$ is taken to be the set of piecewise linear functions, i.e., $\mathbb{H}_h := \{v_h \in L^2(D) : v_h|_K \in \mathbb{P}_1(K)\}^6$, where $\mathbb{P}_1(K)$ denotes the set of continuous piecewise polynomials of degree up to 1. The set of faces is denoted by $\mathcal{G}_h = \mathcal{G}_h^{\text{int}} \cup \mathcal{G}_h^{\text{ext}}$, where $\mathcal{G}_h^{\text{int}}$ and $\mathcal{G}_h^{\text{ext}}$ consist of all interior and all exterior faces, respectively. We denote the unit normal of a face $F \in \mathcal{G}_h^{\text{int}}$ by $\mathbf{n}_F$, where the orientation of $\mathbf{n}_F$ is fixed once and forever for each interior face. For a face $F \in \mathcal{G}_h^{\text{ext}}$, $\mathbf{n}_F$ is the outward normal vector. Jumps of v_h on an interior face F with normal vector $\mathbf{n}_F$ pointing from K to K_F are defined as $[[v_h]]_F := (v_{K_F})|_F - (v_K)|_F$. Note that the sign of the jump on face F is fixed by the direction of the normal vector $\mathbf{n}_F$. Define the broken Sobolev spaces by

$$H^k(\mathcal{T}_h) := \{v \in L^2(D) : v_K \in H^k(K) \text{ for all } K \in \mathcal{T}_h\}, \quad k \in \mathbb{N},$$

with seminorm and norm being $|v|^2_{H^k(\mathcal{T}_h)} := \sum_{K\in\mathcal{T}_h} |v|^2_{H^k(K)}$ and $\|v\|^2_{H^k(\mathcal{T}_h)} := \sum_{j=0}^{k} |v|^2_{H^j(\mathcal{T}_h)}$, respectively. Let $\pi_h : \mathbb{H} \to \mathbb{H}_h$ be the L^2-orthogonal projection operator on $\mathbb{H}_h$, where the projection acts componentwise for vector fields. For the projection operator π_h, we have

$$\|\pi_h v\|_{\mathbb{H}} \le \|v\|_{\mathbb{H}} \qquad \forall v \in \mathbb{H} \tag{4.1}$$

and

$$\langle v - \pi_h v, u_h\rangle_{\mathbb{H}} = 0 \qquad \forall u_h \in \mathbb{H}_h. \tag{4.2}$$

Moreover, for all $v \in H^1(\mathcal{T}_h)^6$, it holds that

$$\|v-\pi_h v\|_{\mathbb{H}} \le Ch|v|_{H^1(\mathcal{T}_h)^6} \tag{4.3}$$

and

$$\sum_{F\in\mathcal{G}_h} \|v-\pi_h v\|^2_{L^2(F)^6} \le Ch|v|^2_{H^1(\mathcal{T}_h)^6}. \tag{4.4}$$

For the parameter σ, we give the following assumption in this subsection.

Assumption 4.1. Suppose that $\sigma \equiv \sigma_0$ is constant on D.

Now we are in the position to propose the following dG full discretization:

$$\begin{aligned} u_h^{n+1} - e^{-\sigma_0\Delta t}u_h^n &= \frac{\Delta t}{2}\big(M_h u_h^{n+1} + e^{-\sigma_0\Delta t}M_h u_h^n\big) \\ &\quad + \frac{\lambda_1}{2}\pi_h\Big[J\big(u_h^{n+1}+e^{-\sigma_0\Delta t}u_h^n\big)\Delta\overline{W}_1^n\Big] + \pi_h(\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n), \end{aligned} \tag{4.5}$$

where the discrete Maxwell operator $M_h : \mathbb{H}_h \to \mathbb{H}_h$ is given as

$$\begin{aligned} \langle M_h u_h, v_h\rangle_{\mathbb{H}} &:= \sum_{K\in\mathcal{T}_h} \Big(\langle \nabla\times\mathbf{H}_h, \psi_h\rangle_{L^2(K)^3} - \langle\nabla\times\mathbf{E}_h,\phi_h\rangle_{L^2(K)^3}\Big) \\ &+ \frac{1}{2}\sum_{F\in\mathcal{G}_h^{\mathrm{int}}} \Big(\langle \mathbf{n}_F\times[[\mathbf{H}_h]]_F, \psi_K+\psi_{K_F}\rangle_{L^2(F)^3} - \langle\mathbf{n}_F\times[[\mathbf{E}_h]]_F,\phi_K+\phi_{K_F}\rangle_{L^2(F)^3} \\ &\qquad - \langle\mathbf{n}_F\times[[\mathbf{E}_h]]_F, \mathbf{n}_F\times[[\psi_h]]_F\rangle_{L^2(F)^3} - \langle\mathbf{n}_F\times[[\mathbf{H}_h]]_F,\mathbf{n}_F\times[[\phi_h]]_F\rangle_{L^2(F)^3}\Big) \\ &+ \sum_{F\in\mathcal{G}_h^{\mathrm{ext}}}\Big(\langle\mathbf{n}_F\times\mathbf{E}_h,\phi_h\rangle_{L^2(F)^3} - \langle\mathbf{n}_F\times\mathbf{E}_h,\mathbf{n}_F\times\psi_h\rangle_{L^2(F)^3}\Big) \end{aligned}$$

with $u_h = (\mathbf{E}_h^\top, \mathbf{H}_h^\top)^\top$, $v_h = (\psi_h^\top,\phi_h^\top)^\top \in \mathbb{H}_h$.

Proposition 4.2. *Let Assumption 2.1 hold with $\gamma_1 \ge 1$ and $\gamma_2 \ge 1$, and let $Q_2^{\frac{1}{2}} \in HS(L^2(D), H_0^{\gamma_2}(D))$ and $u_0 \in L^2(\Omega,\mathbb{H})$. There exists a constant C_{11} independent of time and h such that for sufficiently small $\Delta t > 0$,*

$$\sup_{n\ge 0}\mathbb{E}\big[\|u_h^n\|^2_{\mathbb{H}}\big] \le C_{11},$$

where the positive constant C_{11} depends on σ_0, $\|Q_1^{\frac{1}{2}}\|_{\mathcal{L}_2^{\gamma_1}}$, $\|Q_2^{\frac{1}{2}}\|_{\mathcal{L}_2^{\gamma_2}}$, $\mathbb{E}\big[\|u_0\|^2_{\mathbb{H}}\big]$, λ_1 and $\widetilde{\boldsymbol{\lambda}}_2$.

Proof. We apply $\langle\cdot, u_h^{n+1}+e^{-\sigma_0\Delta t}u_h^n\rangle_{\mathbb{H}}$ on both sides of (4.5) and take the expectation to get

$$\begin{aligned}
&\mathbb{E}\big[\|u_h^{n+1}\|_{\mathbb{H}}^2\big]-e^{-2\sigma_0\Delta t}\mathbb{E}\big[\|u_h^n\|_{\mathbb{H}}^2\big]\\
&=\frac{\Delta t}{2}\mathbb{E}\big[\langle M_h(u_h^{n+1}+e^{-\sigma_0\Delta t}u_h^n), u_h^{n+1}+e^{-\sigma_0\Delta t}u_h^n\rangle_{\mathbb{H}}\big]\\
&\quad+\frac{\lambda_1}{2}\mathbb{E}\big[\langle\pi_h(J(u_h^{n+1}+e^{-\sigma_0\Delta t}u_h^n)\Delta\overline{W}_1^n), u_h^{n+1}+e^{-\sigma_0\Delta t}u_h^n\rangle_{\mathbb{H}}\big]\\
&\quad+\mathbb{E}\big[\langle\pi_h(\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n), u_h^{n+1}+e^{-\sigma_0\Delta t}u_h^n\rangle_{\mathbb{H}}\big]\\
&\le\mathbb{E}\big[\langle\pi_h(\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n), u_h^{n+1}+e^{-\sigma_0\Delta t}u_h^n\rangle_{\mathbb{H}}\big],
\end{aligned}\tag{4.6}$$

where in the last step we use the dissipative property of M_h (see e.g., [1, Proposition 4.4(ii)]) and (4.2). By using the fact that ΔW_2^n is independent of $\mathcal{F}_{t_n}$, we substitute (4.5) into (4.6) to obtain

$$\begin{aligned}
&\mathbb{E}\big[\|u_h^{n+1}\|_{\mathbb{H}}^2\big]-e^{-2\sigma_0\Delta t}\mathbb{E}\big[\|u_h^n\|_{\mathbb{H}}^2\big]\le\mathbb{E}\big[\langle\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n, u_h^{n+1}-e^{-\sigma_0\Delta t}u_h^n\rangle_{\mathbb{H}}\big]\\
&=\frac{\Delta t}{2}\mathbb{E}\big[\langle\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n, M_hu_h^{n+1}\rangle_{\mathbb{H}}\big]\\
&\quad+\mathbb{E}\big[\langle\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n, \frac{\lambda_1}{2}\pi_h(Ju_h^{n+1}\Delta\overline{W}_1^n)+\pi_h(\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n)\rangle_{\mathbb{H}}\big]=:A_1+A_2.
\end{aligned}$$

For the term A_1, we use [12, Lemma A.4] to obtain

$$A_1=\frac{\Delta t}{2}\mathbb{E}\big[\langle M_hu_h^{n+1}, \pi_h(\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n)\rangle_{\mathbb{H}}\big]\le\frac{\sigma_0}{8}\Delta t\mathbb{E}\big[\|u_h^{n+1}\|_{\mathbb{H}}^2\big]+C\Delta t^2.$$

For the term A_2, we have

$$\begin{aligned}
A_2&\le C\mathbb{E}\Big[\|\Delta W_2^n\|_{L^4(D)}\|\Delta\overline{W}_1^n\|_{L^4(D)}\|u_h^{n+1}\|_{\mathbb{H}}\Big]+\mathbb{E}\big[\|\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^n\|_{\mathbb{H}}^2\big]\\
&\le\frac{\sigma_0}{8}\Delta t\mathbb{E}\big[\|u^{n+1}\|_{\mathbb{H}}^2\big]+C\Delta t
\end{aligned}$$

in view of (4.1) and the Sobolev embedding $L^4(D)\hookrightarrow H^1(D)$.

Combining A_1 and A_2, we have

$$\mathbb{E}\big[\|u_h^{n+1}\|_{\mathbb{H}}^2\big]\le e^{-2\sigma_0\Delta t}\mathbb{E}\big[\|u_h^n\|_{\mathbb{H}}^2\big]+\frac{\sigma_0}{4}\Delta t\mathbb{E}\big[\|u_h^{n+1}\|_{\mathbb{H}}^2\big]+C\Delta t.$$

Then the Gronwall inequality implies the assertion for any $\Delta t\in(0,\frac{3}{\sigma_0}]$. □

We denote by $\{P_h^n\}_{n\in\mathbb{N}}$ the Markov transition semigroup associated to the numerical solution $\{u_h^n\}_{n\in\mathbb{N}}$. The following proposition gives the ergodicity of $\{u_h^n\}_{n\in\mathbb{N}}$ and the convergence of $(P_h^n)^*\pi$ towards the numerical invariant measure in the L^2-Wasserstein distance.

Theorem 4.3. *Under the conditions in Proposition 4.2, the following statements hold.*
(i) *The numerical solution $\{u_h^n\}_{n\in\mathbb{N}}$ of (4.5) has a unique invariant measure $\pi^{\Delta t,h}\in\mathcal{P}_2(\mathbb{H})$ for sufficiently small $\Delta t>0$. Thus $\{u_h^n\}_{n\in\mathbb{N}}$ is ergodic. Moreover, $\{u_h^n\}_{n\in\mathbb{N}}$ is exponentially mixing.*

(ii) *For any distribution* $\pi \in \mathcal{P}_2(\mathbb{H})$,

$$\mathcal{W}_2((P_h^n)^*\pi, \pi^{\Delta t,h}) \leq e^{-\sigma_0 t_n}\mathcal{W}_2(\pi, \pi^{\Delta t,h}).$$

Proof. The proof is similar to that of Theorem 3.5. The main difference lies in the proof of the continuous dependence of the solution on the initial data. Since $u^n - \widetilde{u}^n$ solves

$$(u_h^n - \widetilde{u}_h^n) - e^{-\sigma_0\Delta t}(u_h^{n-1} - \widetilde{u}_h^{n-1}) = \Delta t M_h \frac{(u_h^n - \widetilde{u}_h^n) + e^{-\sigma_0\Delta t}(u_h^{n-1} - \widetilde{u}_h^{n-1})}{2}$$
$$+ \lambda_1\pi_h\Big[J\frac{(u_h^n - \widetilde{u}_h^n) + e^{-\sigma_0\Delta t}(u_h^{n-1} - \widetilde{u}_h^{n-1})}{2}\Delta\overline{W}_1^{n-1}\Big],$$

we apply $\langle\cdot, (u_h^n - \widetilde{u}_h^n) + e^{-\sigma_0\Delta t}(u_h^{n-1} - \widetilde{u}_h^{n-1})\rangle_{\mathbb{H}}$ and take the expectation to get

$$\mathbb{E}\big[\|u_h^n - \widetilde{u}_h^n\|_{\mathbb{H}}^2\big] \leq e^{-2\sigma_0\Delta t}\mathbb{E}\big[\|u_h^{n-1} - \widetilde{u}_h^{n-1}\|_{\mathbb{H}}^2\big] \leq \cdots \leq e^{-2\sigma_0 t_n}\mathbb{E}\big[\|u_h^0 - \widetilde{u}_h^0\|_{\mathbb{H}}^2\big]$$

due to the dissipative property of M_h given in [1, Proposition 4.4(ii)]. □

4.1.1. Error analysis of the dG full discretization

In this subsection, we aim to give the mean-square convergence analysis of the dG full discretization (4.5). Let $\widehat{S}_{h,\Delta t} := (I - \frac{\Delta t}{2}M_h)^{-1}(I + \frac{\Delta t}{2}M_h)e^{-\sigma_0\Delta t}$, $T_{h,\Delta t} := (I - \frac{\Delta t}{2}M_h)^{-1}\pi_h$, then the dG full discretization can be rewritten as

$$u_h^n = (\widehat{S}_{h,\Delta t})^n u_h^0 + \lambda_1\sum_{k=0}^{n-1}(\widehat{S}_{h,\Delta t})^{n-k-1}T_{h,\Delta t}(JA_t^{\sigma_0}u_h^k\Delta\overline{W}_1^k) + \sum_{k=0}^{n-1}(\widehat{S}_{h,\Delta t})^{n-k-1}T_{h,\Delta t}\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^k$$

for $n \in \mathbb{N}_+$.

In order to estimate the mean-square error of (4.5), we need to introduce the following lemma, whose proof is given in Appendix B.

Lemma 4.4. *Let* $v = (v_1^\top, v_2^\top)^\top \in H^1(D)^6$ *with* $\mathbf{n}\times v_1|_{\partial D} = 0$ *and* $\mathbf{n}\cdot v_2|_{\partial D} = 0$. *There exist positive constants* C *such that*
(1) $\|\widehat{S}_{h,\Delta t}\|_{\mathcal{L}(\mathbb{H}_h,\mathbb{H}_h)} \leq e^{-\sigma_0\Delta t}$ *and* $\|T_{h,\Delta t}\|_{\mathcal{L}(\mathbb{H},\mathbb{H}_h)} \leq 1$.
(2) $\|(\widehat{S}_{\Delta t})^n v\|_{H^1(D)^6} \leq Ce^{-\sigma_0 n\Delta t/2}\|v\|_{H^1(D)^6}$.
(3) $\|(\pi_h(\widehat{S}_{\Delta t})^n - (\widehat{S}_{h,\Delta t})^n\pi_h)v\|_{\mathbb{H}} \leq Ch^{\frac{1}{2}}e^{-\sigma_0 n\Delta t/2}\|v\|_{H^1(D)^6}$.
(4) $\|(\widehat{S}(t_n) - (\widehat{S}_{h,\Delta t})^n\pi_h)v\|_{\mathbb{H}} \leq Ce^{-\sigma_0 n\Delta t/2}(\Delta t^{\frac{1}{2}} + h^{\frac{1}{2}})\|v\|_{H^1(D)^6}$.
(5) $\|(\widehat{S}(t_n - r) - (\widehat{S}_{h,\Delta t})^{n-k-1}T_{h,\Delta t})v\|_{\mathbb{H}} \leq Ce^{-\sigma_0(n-k-1)\Delta t/2}(\Delta t^{\frac{1}{2}} + h^{\frac{1}{2}})\|v\|_{H^1(D)^6}$, *for* $r \in [t_k, t_{k+1}]$ *and* $k = 0, 1, \cdots, n-1$.

Theorem 4.5. *Let* $Q_1^{\frac{1}{2}} \in HS(L^2(D), H_0^{\gamma_1}(D))$, $Q_2^{\frac{1}{2}} \in HS(L^2(D), H_0^{\gamma_2}(D))$ *hold with* $\gamma_1 > \frac{5}{2}$ *and* $\gamma_2 \geq 1$, *and let* $u_0 \in L^4(\Omega, H^1(D)^6)$ *and* $F_{Q_1} \in W^{1,\infty}(D)$. *There exists a positive constant* C_{12} *independent of* Δt *and* h *such that*

$$\sup_{n\geq 0}\big(\mathbb{E}[\|u_h^n - u(t_n)\|_{\mathbb{H}}^2]\big)^{\frac{1}{2}} \leq C_{12}(\Delta t^{\frac{1}{2}} + h^{\frac{1}{2}}),$$

where the positive constant C_{12} depends on $|D|$, $\|F_{Q_1}\|_{W^{1,\infty}(D)}$, C_1, $\mathbb{E}\big[\|u_0\|^4_{H^1(D)^6}\big]$, λ_1, $\widetilde{\boldsymbol{\lambda}}_2$, $\|Q_1^{\frac{1}{2}}\|_{\mathcal{L}_2^{\gamma_1}}$ and $\|Q_2^{\frac{1}{2}}\|_{\mathcal{L}_2^{\gamma_2}}$.

Proof. For $n \in \mathbb{N}_+$, we introduce an auxiliary process

$$\widetilde{u}_h^n = \big(\widehat{S}_{h,\Delta t}\big)^n u_h^0 + \lambda_1 \sum_{k=0}^{n-1} \big(\widehat{S}_{h,\Delta t}\big)^{n-k-1} T_{h,\Delta t}\big(J A_t^{\sigma_0} u(t_k) \Delta \overline{W}_1^k\big) + \sum_{k=0}^{n-1} \big(\widehat{S}_{h,\Delta t}\big)^{n-k-1} T_{h,\Delta t}\big(\widetilde{\boldsymbol{\lambda}}_2 \Delta W_2^k\big), \tag{4.7}$$

that is,

$$\widetilde{u}_h^{n+1} - e^{-\sigma_0 \Delta t}\widetilde{u}_h^n = \frac{\Delta t}{2}\big(M_h \widetilde{u}_h^{n+1} + e^{-\sigma_0 \Delta t} M_h \widetilde{u}_h^n\big) + \frac{\lambda_1}{2}\pi_h\Big[J\big(u(t_{n+1}) + e^{-\sigma_0\Delta t}u(t_n)\big)\Delta \overline{W}_1^n\Big] + \pi_h\big(\widetilde{\boldsymbol{\lambda}}_2 \Delta W_2^n\big). \tag{4.8}$$

Let $e_n^{full} := u_h^n - u(t_n) = \widetilde{e}_n + \widehat{e}_n$, where $\widetilde{e}_n := u_h^n - \widetilde{u}_h^n$ and $\widehat{e}_n := \widetilde{u}_h^n - u(t_n)$.

Step 1. Estimate of $\mathbb{E}\big[\|\widehat{e}_{n+1}\|^2_{\mathbb{H}}\|\Delta \overline{W}_1^n\|^{2p}_{H^{\gamma_1-1}(D)}\big]$, $p = 0, 1, 2$.

We use $\widehat{e}_{n+1} = \widetilde{u}_h^{n+1} - u(t_{n+1})$, (3.19) and (4.7) to obtain

$$\begin{aligned}
&\mathbb{E}\big[\|\widehat{e}_{n+1}\|^2_{\mathbb{H}}\|\Delta \overline{W}_1^n\|^{2p}_{H^{\gamma_1-1}(D)}\big] \\
&\le 3\mathbb{E}\Big[\big\|\big((\widehat{S}_{h,\Delta t})^{n+1}\pi_h - \widehat{S}(t_{n+1})\big)u_0\big\|^2_{\mathbb{H}}\Big]\mathbb{E}\big[\|\Delta \overline{W}_1^n\|^{2p}_{H^{\gamma_1-1}(D)}\big] \\
&\quad + 3\mathbb{E}\Big[\Big\|\sum_{k=0}^{n}(\widehat{S}_{h,\Delta t})^{n-k}T_{h,\Delta t}(\widetilde{\boldsymbol{\lambda}}_2\Delta W_2^k) - \int_0^{t_{n+1}}\widehat{S}(t_{n+1}-r)\widetilde{\boldsymbol{\lambda}}_2 \mathrm{d}W_2(r)\Big\|^2_{\mathbb{H}}\Big]\mathbb{E}\big[\|\Delta \overline{W}_1^n\|^{2p}_{H^{\gamma_1-1}(D)}\big] \\
&\quad + 3\mathbb{E}\Big[\Big\|\lambda_1\sum_{k=0}^{n}\big(\widehat{S}_{h,\Delta t}\big)^{n-k}T_{h,\Delta t}\big(JA_t^{\sigma_0}u(t_k)\Delta\overline{W}_1^k\big) - \lambda_1\int_0^{t_{n+1}}\widehat{S}(t_{n+1}-r)Ju(r)\mathrm{d}W_1(r) \\
&\qquad + \frac{1}{2}\lambda_1^2\int_0^{t_{n+1}}\widehat{S}(t_{n+1}-r)\big(F_{Q_1}u(r)\big)\mathrm{d}r\Big\|^2_{\mathbb{H}}\|\Delta \overline{W}_1^n\|^{2p}_{H^{\gamma_1-1}(D)}\Big] \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

For the term I_1, Lemma 4.4(4) yields

$$I_1 \le C\big(\Delta t + h\big)\Delta t^p.$$

For the term I_2, it follows from Lemma 3.7(5) and $Q_2^{\frac{1}{2}} \in HS(L^2(D), H_0^{\gamma_2}(D))$ that

$$I_2 \leq C\Delta t^p \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \|(\widehat{S}(t_{n+1}-r) - (\widehat{S}_{h,\Delta t})^{n-k} T_{h,\Delta t})\widetilde{\boldsymbol{\lambda}}_2 Q_2^{\frac{1}{2}}\|^2_{HS(L^2(D),\mathbb{H})} \mathrm{d}r$$
$$\leq C\Delta t^p \sum_{k=0}^{n} e^{-\sigma_0(n+1-k)\Delta t}\big(\Delta t + h\big)\Delta t \leq C\Delta t^p \big(\Delta t + h\big).$$

For the term I_3, notice that

$$\lambda_1 \sum_{k=0}^{n} \big(\widehat{S}_{h,\Delta t}\big)^{n-k} T_{h,\Delta t}\big(J A_t^{\sigma_0} u(t_k)\Delta \overline{W}_1^k\big) - \lambda_1 \int_0^{t_{n+1}} \widehat{S}(t_{n+1}-r) J u(r)\mathrm{d}W_1(r)$$
$$+ \frac{1}{2}\lambda_1^2 \int_0^{t_{n+1}} \widehat{S}(t_{n+1}-r)\big(F_{Q_1} u(r)\big)\mathrm{d}r$$
$$= \sum_{k=0}^{n} \Big[\lambda_1 \big(\widehat{S}_{h,\Delta t}\big)^{n-k} T_{h,\Delta t}\big(J e^{-\sigma_0 \Delta t} u(t_k)\big(\Delta \overline{W}_1^k - \Delta W_1^k\big)\big)\Big]$$
$$+ \sum_{k=0}^{n} \Big[\lambda_1 \int_{t_k}^{t_{k+1}} \Big[\big(\widehat{S}_{h,\Delta t}\big)^{n-k} T_{h,\Delta t} J\big(e^{-\sigma_0 \Delta t} u(t_k)\big) - \widehat{S}(t_{n+1}-r) J u(r)\Big]\mathrm{d}W_1(r)\Big]$$
$$+ \sum_{k=0}^{n} \Big[\lambda_1 (\widehat{S}_{h,\Delta t})^{n-k} T_{h,\Delta t}\Big(J \frac{u(t_{k+1}) - e^{-\sigma_0 \Delta t} u(t_k)}{2} \Delta \overline{W}_1^k\Big)$$
$$+ \frac{1}{2}\lambda_1^2 \int_{t_k}^{t_{k+1}} \widehat{S}(t_{n+1}-r)\big(F_{Q_1} u(r)\big)\mathrm{d}r\Big]$$
$$=: \sum_{k=0}^{n} A_{1,k} + \sum_{k=0}^{n} A_{2,k} + \sum_{k=0}^{n} A_{3,k}.$$

Therefore,

$$I_3 \leq C\mathbb{E}\Big[\Big\|\sum_{k=0}^{n} A_{1,k}\|^2_{\mathbb{H}} \Big\|\Delta \overline{W}_1^n\|^{2p}_{H^{\gamma_1-1}(D)}\Big] + C\mathbb{E}\Big[\Big\|\sum_{k=0}^{n} A_{2,k}\Big\|^2_{\mathbb{H}} \|\Delta \overline{W}_1^n\|^{2p}_{H^{\gamma_1-1}(D)}\Big]$$
$$+ C\mathbb{E}\Big[\Big\|\sum_{k=0}^{n} A_{3,k}\Big\|^2_{\mathbb{H}} \|\Delta \overline{W}_1^n\|^{2p}_{H^{\gamma_1-1}(D)}\Big]$$
$$=: I_{3,1} + I_{3,2} + I_{3,3}.$$

For the term $I_{3,1}$, Lemma 4.4(1), Proposition 2.5, Sobolev embedding $H^\gamma \hookrightarrow L^\infty(D)$ for $\gamma > \frac{3}{2}$ and (3.2) with $b \geq 4$ yield

$$\mathbb{E}\big[\|A_{1,k}\|_{\mathbb{H}}^4\big] \le Ce^{-4\sigma_0(n-k)\Delta t}\mathbb{E}\big[\|u(t_k)\|_{\mathbb{H}}^4\big]\mathbb{E}\big[\|\Delta\overline{W}_1^k - \Delta W_1^k\|_{H^{\gamma_1-1}(D)}^4\big] \le Ce^{-4\sigma_0(n-k)\Delta t}\Delta t^6,$$

which implies

$$\mathbb{E}\Big[\Big\|\sum_{k=0}^{n} A_{1,k}\Big\|_{\mathbb{H}}^4\Big] \le \Big(\sum_{k=0}^{n}\big(\mathbb{E}\big[\|A_{1,k}\|_{\mathbb{H}}^4\big]\big)^{\frac{1}{4}}\Big)^4 \le \Big(\sum_{k=0}^{n} Ce^{-\sigma_0(n-k)\Delta t}\Delta t^{\frac{3}{2}}\Big)^4 \le C\Delta t^2.$$

Hence

$$I_{3,1} \le C\Big(\mathbb{E}\Big[\Big\|\sum_{k=0}^{n} A_{1,k}\Big\|_{\mathbb{H}}^4\Big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\big[\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}^{4p}\big]\Big)^{\frac{1}{2}} \le C\Delta t^{1+p}.$$

For the term $I_{3,2}$, notice that

$$\begin{aligned}
&\mathbb{E}\Big[\Big\|\sum_{k=0}^{n-1} A_{2,k}\Big\|_{\mathbb{H}}^2\Big] \\
&\le C\sum_{k=0}^{n-1}\mathbb{E}\int_{t_k}^{t_{k+1}}\sum_{i=1}^{\infty}\big\|\big[\big(\widehat{S}_{h,\Delta t}\big)^{n-k}T_{h,\Delta t}J\big(e^{-\sigma_0\Delta t}u(t_k)\big) - \widehat{S}(t_{n+1}-r)Ju(r)\big]Q_1^{\frac{1}{2}}q_i\big\|_{\mathbb{H}}^2\mathrm{d}r \\
&\le C\Big(\sum_{k=0}^{n-1}\mathbb{E}\int_{t_k}^{t_{k+1}}\Big[\sum_{i=1}^{\infty}\big\|\big(\widehat{S}_{h,\Delta t}\big)^{n-k}T_{h,\Delta t}J\big(\big(e^{-\sigma_0\Delta t}u(t_k) - u(r)\big)Q_1^{\frac{1}{2}}q_i\big)\big\|_{\mathbb{H}}^2 \\
&\qquad + \sum_{i=1}^{\infty}\big\|\big(\big(\widehat{S}_{h,\Delta t}\big)^{n-k}T_{h,\Delta t} - \widehat{S}(t_{n+1}-r)\big)J\big(u(r)Q_1^{\frac{1}{2}}q_i\big)\big\|_{\mathbb{H}}^2\Big]\mathrm{d}r\Big) \\
&\le C\sum_{k=0}^{n-1} e^{-(n-k)\sigma_0\Delta t}(\Delta t + h)\Delta t \le C(\Delta t + h),
\end{aligned}$$

where we use (3.21), Lemma 2.4, the Itô isometry, Sobolev embedding $H^{\gamma}(D) \hookrightarrow L^{\infty}(D)$ for $\gamma > \frac{3}{2}$ and Lemma 4.4(1)(5). Similar to the estimate of (3.21), for $r \in [t_n, t_{n+1}]$, we have

$$\mathbb{E}\big[\|u(r) - e^{-\sigma_0\Delta t}u(t_n)\|_{\mathbb{H}}^{2p}\big] \le C\Delta t^p. \tag{4.9}$$

By utilizing the Burkholder–Davis–Gundy-type inequality for stochastic integrals, (4.9), Proposition 2.5, Lemma 3.7(1) and Lemma 4.4(1), it yields that

$$\mathbb{E}\big[\|A_{2,n}\|_{\mathbb{H}}^4\big] \le C\Delta t^2.$$

Consequently,

$$I_{3,2} \leq C\mathbb{E}\Big[\|\sum_{k=0}^{n-1} A_{2,k}\|_{\mathbb{H}}^2\Big]\mathbb{E}\big[\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}^{2p}\big] + C\Big(\mathbb{E}\big[\|A_{2,n}\|_{\mathbb{H}}^4\big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\big[\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}^{4p}\big]\Big)^{\frac{1}{2}}$$
$$\leq C(\Delta t + h)\Delta t^p.$$

For the term $I_{3,3}$, we substitute (3.19) into $A_{3,k}$ and further split it to obtain

$$A_{3,k} := A_{3,k,1} + A_{3,k,2} + A_{3,k,3} + A_{3,k,4} + A_{3,k,5},$$

where

$$\begin{aligned}
A_{3,k,1} &= \frac{\lambda_1}{2}(\widehat{S}_{h,\Delta t})^{n-k}T_{h,\Delta t}\Big[J\big((\widehat{S}(\Delta t) - e^{\sigma_0\Delta t})u(t_k)\big)\Delta\overline{W}_1^k\Big] \\
&\quad - \frac{\lambda_1^3}{4}(\widehat{S}_{h,\Delta t})^{n-k}T_{h,\Delta t}\Big[J\Big(\int_{t_k}^{t_{k+1}}\widehat{S}(t_{k+1}-r)\big(F_{Q_1}u(r)\big)\mathrm{d}r\Big)\Delta\overline{W}_1^k\Big] \\
&\quad + \frac{\lambda_1^2}{2}(\widehat{S}_{h,\Delta t})^{n-k}T_{h,\Delta t}\Big[J\Big(\int_{t_k}^{t_{k+1}}\widehat{S}(t_{k+1}-r)J\big(u(r) - e^{-\sigma_0\Delta t}u(t_k)\big)\mathrm{d}W_1(r)\Big)\Delta\overline{W}_1^k\Big] \\
&\quad + \frac{\lambda_1^2}{2}(\widehat{S}_{h,\Delta t})^{n-k}T_{h,\Delta t}\Big[J\Big(\int_{t_k}^{t_{k+1}}\widehat{S}(t_{k+1}-r)J\big(e^{-\sigma_0\Delta t}u(t_k)\big)\mathrm{d}W_1(r)\Big)\big(\Delta\overline{W}_1^k - \Delta W_1^k\big)\Big],
\end{aligned}$$

$$\begin{aligned}
A_{3,k,2} &= \frac{\lambda_1^2}{2}\big(\widehat{S}_{h,\Delta t}\big)^{n-k}T_{h,\Delta t}\Big[J\Big(\int_{t_k}^{t_{k+1}}\big(\widehat{S}(t_{k+1}-r) - I\big)J\big(e^{-\sigma_0\Delta t}u(t_k)\big)\mathrm{d}W_1(r)\Big)\Delta W_1^k\Big] \\
&\quad - \frac{\lambda_1^2}{2}\big(\widehat{S}_{h,\Delta t}\big)^{n-k}T_{h,\Delta t}\int_{t_k}^{t_{k+1}}F_{Q_1}\big(e^{-\sigma_0\Delta t}u(t_k) - u(r)\big)\mathrm{d}r \\
&\quad + \frac{\lambda_1^2}{2}\int_{t_k}^{t_{k+1}}\big(\widehat{S}(t_{n+1}-r) - (\widehat{S}_{h,\Delta t})^{n-k}T_{h,\Delta t}\big)\big(F_{Q_1}\big(u(r) - e^{-\sigma_0\Delta t}u(t_k)\big)\big)\mathrm{d}r,
\end{aligned}$$

$$A_{3,k,3} = \frac{\lambda_1}{2}(\widehat{S}_{h,\Delta t})^{n-k}T_{h,\Delta t}\Big[J\Big(\int_{t_k}^{t_{k+1}}\widehat{S}(t_{k+1}-r)\widetilde{\lambda}_2\mathrm{d}W_2(r)\Big)\Delta\overline{W}_1^k\Big],$$

$$A_{3,k,4} = -\lambda_1^2(\widehat{S}_{h,\Delta t})^{n-k}T_{h,\Delta t}\Big(\int_{t_k}^{t_{k+1}}\int_{t_k}^{r}e^{-\sigma_0\Delta t}u(t_k)\mathrm{d}W_1(\rho)\mathrm{d}W_1(r)\Big),$$

$$A_{3,k,5} = \frac{\lambda_1^2}{2}\int_{t_k}^{t_{k+1}}\Big(\widehat{S}(t_{n+1}-r) - (\widehat{S}_{h,\Delta t})^{n-k}T_{h,\Delta t}\Big)\big(F_{Q_1}e^{-\sigma_0\Delta t}u(t_k)\big)\mathrm{d}r.$$

For the term $A_{3,k,1}$, it follows from Proposition 2.5, Sobolev embedding $H^\gamma \hookrightarrow L^\infty(D)$ for $\gamma > \frac{3}{2}$, Lemma 3.7(1)(3), Lemma 4.4(1), (4.9), the Burkholder–Davis–Gundy-type inequality and (3.2) that

$$\begin{aligned}
&\mathbb{E}\big[\|A_{3,k,1}\|^2_{\mathbb{H}}\big] \\
&\le Ce^{-2(n-k)\sigma_0\Delta t}\Delta t^2\mathbb{E}\big[\|u(t_k)\|^2_{D(M)}\big]\mathbb{E}\big[\|\Delta\overline{W}_1^k\|^2_{H^{\gamma_1-1}(D)}\big] \\
&\quad + Ce^{-2(n-k)\sigma_0\Delta t}\mathbb{E}\Big[\Big\|\int_{t_k}^{t_{k+1}}\widehat{S}(t_{k+1}-r)(F_{Q_1}u(r))\mathrm{d}r\Big\|^2_{\mathbb{H}}\|\Delta\overline{W}_1^k\|^2_{H^{\gamma_1-1}(D)}\Big] \\
&\quad + Ce^{-2(n-k)\sigma_0\Delta t}\mathbb{E}\Big[\Big\|\int_{t_k}^{t_{k+1}}\widehat{S}(t_{k+1}-r)J(u(r)-e^{-\sigma_0\Delta t}u(t_k))\mathrm{d}W_1(r)\Big\|^2_{\mathbb{H}}\|\Delta\overline{W}_1^k\|^2_{H^{\gamma_1-1}(D)}\Big] \\
&\quad + Ce^{-2(n-k)\sigma_0\Delta t}\mathbb{E}\Big[\Big\|\int_{t_k}^{t_{k+1}}\widehat{S}(t_{k+1}-r)J(e^{-\sigma_0\Delta t}u(t_k))\mathrm{d}W_1(r)\Big\|^2_{\mathbb{H}}\|\Delta\overline{W}_1^k-\Delta W_1^k\|^2_{H^{\gamma_1-1}(D)}\Big] \\
&\le Ce^{-2(n-k)\sigma_0\Delta t}\Delta t^3.
\end{aligned}$$

Similarly, we have

$$\mathbb{E}\big[\|A_{3,n,1}\|^2_{\mathbb{H}}\|\Delta\overline{W}_1^n\|^{2p}_{H^{\gamma_1-1}(D)}\big] \le C\Delta t^{3+p}.$$

Therefore,

$$\mathbb{E}\Big[\Big\|\sum_{k=0}^{n}A_{3,k,1}\Big\|^2_{\mathbb{H}}\|\Delta\overline{W}_1^n\|^{2p}_{H^{\gamma_1-1}(D)}\Big] \le C\Delta t^{1+p}.$$

For the term $A_{3,k,2}$, Lemma 3.7(3), (4.9) and the Burkholder–Davis–Gundy-type inequality yield

$$\begin{aligned}
&\mathbb{E}[\|A_{3,k,2}\|^4_{\mathbb{H}}] \\
&\le Ce^{-4\sigma_0(n-k)\Delta t}\mathbb{E}\Big[\Big\|\int_{t_k}^{t_{k+1}}\Big(\widehat{S}(t_{k+1}-r)-I\Big)J(e^{-\sigma_0\Delta t}u(t_k))\mathrm{d}W_1(r)\Big\|^4_{\mathbb{H}}\|\Delta W_1\|^4_{H^{\gamma_1-1}(D)}\Big] \\
&\quad + Ce^{-4(n-k)\sigma_0\Delta t}\mathbb{E}\Big[\Big\|\int_{t_k}^{t_{k+1}}F_{Q_1}(u(r)-e^{-\sigma_0\Delta t}u(t_k))\mathrm{d}r\Big\|^4_{\mathbb{H}}\Big] \\
&\quad + C\Delta t^3\int_{t_k}^{t_{k+1}}e^{-4\sigma_0(n-k)\Delta t}\mathbb{E}[\|F_{Q_1}(u(r)-e^{-\sigma_0\Delta t}u(t_k))\|^4_{\mathbb{H}}]\mathrm{d}r \\
&\le Ce^{-4(n-k)\sigma_0\Delta t}\Delta t^6,
\end{aligned}$$

which implies that

$$\mathbb{E}\Big[\Big\|\sum_{k=0}^{n} A_{3,k,2}\Big\|_{\mathbb{H}}^{2}\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}^{2p}\Big] \le \Big(\mathbb{E}\Big[\Big\|\sum_{k=0}^{n} A_{3,k,2}\Big\|_{\mathbb{H}}^{4}\Big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\big[\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}^{4p}\big]\Big)^{\frac{1}{2}}$$
$$\le C\Delta t^{1+p}.$$

For the term $A_{3,k,3}$, notice that

$$\mathbb{E}\big[\|A_{3,k,3}\|_{\mathbb{H}}^4\big] \le Ce^{-4(n-k)\sigma_0\Delta t}\mathbb{E}\Big[\Big\|\int_{t_k}^{t_{k+1}} \widehat{S}(t_{k+1}-r)\widetilde{\boldsymbol{\lambda}}_2 \mathrm{d}W_2(r)\Big\|_{\mathbb{H}}^4\Big]\mathbb{E}\big[\|\Delta\overline{W}_1^k\|_{H^{\gamma_1-1}(D)}^4\big]$$
$$\le Ce^{-4(n-k)\sigma_0\Delta t}\Delta t^4,$$

which leads to

$$\mathbb{E}\Big[\Big\|\sum_{k=0}^{n} A_{3,k,3}\Big\|_{\mathbb{H}}^{4}\Big] \le C\sum_{i=0}^{n}\mathbb{E}\big[\|A_{3,i,3}\|_{\mathbb{H}}^4\big] + C\sum_{i\neq j}\mathbb{E}\big[\|A_{3,i,3}\|_{\mathbb{H}}^2\|A_{3,j,3}\|_{\mathbb{H}}^2\big]$$
$$\le C\Big(\sum_{i=0}^{n}\big(\mathbb{E}\big[\|A_{3,i,3}\|_{\mathbb{H}}^4\big]\big)^{\frac{1}{2}}\Big)^2 \le C\Big(\sum_{k=0}^{n} e^{-2(n-k)\sigma_0\Delta t}\Delta t^2\Big)^2 \le C\Delta t^2.$$

Hence

$$\mathbb{E}\Big[\Big\|\sum_{k=0}^{n} A_{3,k,3}\Big\|_{\mathbb{H}}^{2}\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}^{2p}\Big] \le \Big(\mathbb{E}\Big[\Big\|\sum_{k=0}^{n} A_{3,k,3}\Big\|_{\mathbb{H}}^{4}\Big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\big[\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}^{4p}\big]\Big)^{\frac{1}{2}}$$
$$\le C\Delta t^{1+p}.$$

For the term $A_{3,k,4}$, it holds that

$$\mathbb{E}\Big[\Big\|\sum_{k=0}^{n-1} A_{3,k,4}\Big\|_{\mathbb{H}}^{2}\Big] \le C\Big(\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} e^{-2\sigma_0(n-k)\Delta t}\mathbb{E}\Big[\Big\|\int_{t_k}^{r} e^{-\sigma_0\Delta t}u(t_k)\mathrm{d}W_1(\rho)\Big\|_{\mathbb{H}}^2\Big]\mathrm{d}r\Big)$$
$$\le C\sum_{k=0}^{n-1} e^{-2\sigma_0(n-k)\Delta t}\Delta t^2 \le C\Delta t$$

and

$$\mathbb{E}\big[\|A_{3,n,4}\|_{\mathbb{H}}^4\big] \le C\mathbb{E}\Big[\Big(\int_{t_k}^{t_{k+1}}\Big\|\int_{t_k}^{r} e^{-\sigma_0\Delta t}u(t_k)\mathrm{d}W_1(\rho)\Big\|_{\mathbb{H}}^2\mathrm{d}r\Big)^2\Big]$$

$$\leq C\Delta t \int_{t_k}^{t_{k+1}} \mathbb{E}\Big[\Big\| \int_{t_k}^{r} u(t_k)\mathrm{d}W_1(\rho)\Big\|_{\mathbb{H}}^4\Big]\mathrm{d}r \leq C\Delta t^4,$$

which imply that

$$\mathbb{E}\Big[\Big\| \sum_{k=0}^{n} A_{3,k,4}\Big\|_{\mathbb{H}}^2 \|\Delta \overline{W}_1^n\|_{H^{\gamma_1 - 1}(D)}^{2p}\Big] \leq C\Delta t^{1+p}.$$

For the term $A_{3,k,5}$, notice that

$$\mathbb{E}\big[\|A_{3,k,5}\|_{\mathbb{H}}^2\big] \leq C\Delta t \int_{t_k}^{t_{k+1}} \mathbb{E}\big[\|(\widehat{S}(t_{n+1}-r) - (S_{h,\Delta t})^{n-k}T_{h,\Delta t})(F_{Q_1}e^{-\sigma_0\Delta t}u(t_k))\|_{\mathbb{H}}^2\big]\mathrm{d}r$$
$$\leq C\Delta t^2(\Delta t + h)e^{-\sigma_0(n-k)\Delta t},$$

which yields

$$\mathbb{E}\Big[\Big\| \sum_{k=0}^{n} A_{3,k,5}\Big\|_{\mathbb{H}}^2 \|\Delta \overline{W}_1^n\|_{H^{\gamma_1 - 1}(D)}^{2p}\Big] \leq C(\Delta t + h)\Delta t^{p}.$$

Hence

$$I_{3,3} \leq C\sum_{i=1}^{5} \mathbb{E}\Big[\Big\| \sum_{k=0}^{n} A_{3,k,i}\Big\|_{\mathbb{H}}^2 \|\Delta \overline{W}_1^n\|_{H^{\gamma_1 - 1}(D)}^{2p}\Big] \leq C(\Delta t + h)\Delta t^{p}.$$

Combining $I_{3,1}$–$I_{3,3}$, we have $I_3 \leq C(\Delta t + h)\Delta t^p$. Further, we have

$$\mathbb{E}\big[\|\widehat{e}_{n+1}\|_{\mathbb{H}}^2 \|\Delta \overline{W}_1^n\|_{H^{\gamma_1 - 1}(D)}^{2p}\big] \leq C(\Delta t + h)\Delta t^{p}. \tag{4.10}$$

Step 2. Estimate of $\sup_{n\in\mathbb{N}} \mathbb{E}\big[\|\widetilde{e}_n\|_{\mathbb{H}}^2\big]$.

Subtracting (4.8) from (4.5) leads to

$$\widetilde{e}_{n+1} - e^{-\sigma_0\Delta t}\widetilde{e}_n = \frac{\Delta t}{2}M_h(\widetilde{e}_{n+1} + e^{-\sigma_0\Delta t}\widetilde{e}_n) + \frac{\lambda_1}{2}\pi_h\big[J(e_{n+1}^{full} + e^{-\sigma_0\Delta t}e_n^{full})\Delta \overline{W}_1^n\big], \tag{4.11}$$

which is equivalent to

$$\widetilde{e}_{n+1} = \widehat{S}_{h,\Delta t}\widetilde{e}_n + \frac{\lambda_1}{2}T_{h,\Delta t}\big[J(e_{n+1}^{full} + e^{-\sigma_0\Delta t}e_n^{full})\Delta \overline{W}_1^n\big]. \tag{4.12}$$

We apply $\langle \cdot, \widetilde{e}_{n+1} + e^{-\sigma_0\Delta t}\widetilde{e}_n\rangle_{\mathbb{H}}$ to both sides of (4.11) and take the expectation to get

$$\mathbb{E}\big[\|\widetilde{e}_{n+1}\|_{\mathbb{H}}^2\big] \leq e^{-2\sigma_0\Delta t}\mathbb{E}\big[\|\widetilde{e}_n\|_{\mathbb{H}}^2\big] + \frac{\lambda_1}{2}\mathbb{E}\big[\langle J(\widehat{e}_{n+1} + e^{-\sigma_0\Delta t}\widehat{e}_n)\Delta \overline{W}_1^n, \widetilde{e}_{n+1} + e^{-\sigma_0\Delta t}\widetilde{e}_n\rangle_{\mathbb{H}}\big]$$

due to the dissipative property of M_h in [1, Proposition 4.4(ii)].

By (4.12) and using the fact that $\Delta\overline{W}_1^n$ is independent of $\mathcal{F}_{t_n}$, we obtain

$$\begin{aligned}
&\frac{\lambda_1}{2}\mathbb{E}\big[\langle J(\widehat{e}_{n+1}+e^{-\sigma_0\Delta t}\widehat{e}_n)\Delta\overline{W}_1^n,\widetilde{e}_{n+1}+e^{-\sigma_0\Delta t}\widetilde{e}_n\rangle_{\mathbb{H}}\big]\\
&=\frac{\lambda_1}{2}\mathbb{E}\big[\langle J(\widehat{e}_{n+1}+e^{-\sigma_0\Delta t}\widehat{e}_n)\Delta\overline{W}_1^n,\widetilde{e}_{n+1}-e^{-\sigma_0\Delta t}\widetilde{e}_n\rangle_{\mathbb{H}}\big]\\
&\quad+\lambda_1\mathbb{E}\big[\langle J(\widehat{e}_{n+1}-e^{-\sigma_0\Delta t}\widehat{e}_n)\Delta\overline{W}_1^n,e^{-\sigma_0\Delta t}\widetilde{e}_n\rangle_{\mathbb{H}}\big]+2\lambda_1\mathbb{E}\big[\langle Je^{-\sigma_0\Delta t}\widehat{e}_n\Delta\overline{W}_1^n,e^{-\sigma_0\Delta t}\widetilde{e}_n\rangle_{\mathbb{H}}\big]\\
&=\frac{\lambda_1}{2}\mathbb{E}\big[\langle J(\widehat{e}_{n+1}-e^{-\sigma_0\Delta t}\widehat{e}_n)\Delta\overline{W}_1^n,(\widehat{S}_{h,\Delta t}+e^{-\sigma_0\Delta t})\widetilde{e}_n\rangle_{\mathbb{H}}\big]\\
&\quad+\frac{\lambda_1^2}{4}\mathbb{E}\big[\langle J(\widehat{e}_{n+1}+e^{-\sigma_0\Delta t}\widehat{e}_n)\Delta\overline{W}_1^n,T_{h,\Delta t}(J(e_{n+1}^{full}+e^{-\sigma_0\Delta t}e_n^{full})\Delta\overline{W}_1^n)\rangle_{\mathbb{H}}\big]\\
&=:I_4+I_5.
\end{aligned}$$

Noticing that

$$\begin{aligned}
&\widehat{e}_{n+1}-e^{-\sigma_0\Delta t}\widehat{e}_n\\
&=\big(\widehat{S}_{h,\Delta t}-e^{-\sigma_0\Delta t}\big)\big(\widehat{S}_{h,\Delta t}\big)^n\pi_h u_0-\big(\widehat{S}(\Delta t)-e^{-\sigma_0\Delta t}\big)\widehat{S}(t_n)u_0\\
&\quad+\int_{t_n}^{t_{n+1}}\big(T_{h,\Delta t}-\widehat{S}(t_{n+1}-r)\big)\widetilde{\boldsymbol{\lambda}}_2\mathrm{d}W_2(r)\\
&\quad+\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\Big[\big(\widehat{S}_{h,\Delta t}-e^{-\sigma_0\Delta t}\big)\big(\widehat{S}_{h,\Delta t}\big)^{n-1-k}T_{h,\Delta t}-\big(\widehat{S}(\Delta t)-e^{-\sigma_0\Delta t}\big)\widehat{S}(t_n-r)\Big]\widetilde{\boldsymbol{\lambda}}_2\mathrm{d}W_2(r)\\
&\quad+\lambda_1T_{h,\Delta t}\big(JA_t^{\sigma_0}u(t_n)\Delta\overline{W}_1^n\big)-\lambda_1\int_{t_n}^{t_{n+1}}\widehat{S}(t_{n+1}-r)Ju(r)\mathrm{d}W_1(r)\\
&\quad+\frac{1}{2}\lambda_1^2\int_{t_n}^{t_{n+1}}\widehat{S}(t_{n+1}-r)\big(F_{Q_1}u(r)\big)\mathrm{d}r\\
&\quad+\lambda_1\big(\widehat{S}_{h,\Delta t}-e^{-\sigma_0\Delta t}\big)\sum_{k=0}^{n-1}\big(\widehat{S}_{h,\Delta t}\big)^{n-k-1}T_{h,\Delta t}\big(JA_t^{\sigma_0}u(t_k)\Delta\overline{W}_1^k\big)\\
&\quad-\lambda_1\big(\widehat{S}(\Delta t)-e^{-\sigma_0\Delta t}\big)\int_0^{t_n}\widehat{S}(t_n-r)Ju(r)\mathrm{d}W_1(r)\\
&\quad+\frac{1}{2}\lambda_1^2\big(\widehat{S}(\Delta t)-e^{-\sigma_0\Delta t}\big)\int_0^{t_n}\widehat{S}(t_n-r)\big(F_{Q_1}u(r)\big)\mathrm{d}r,
\end{aligned}$$

and using the fact that $\Delta \overline{W}_1^n$ is independent of $\mathcal{F}_{t_n}$ and $W_2(t)$, we get

$$
\begin{aligned}
I_4 =& \frac{\lambda_1}{2}\mathbb{E}\Big[\langle J\Big(\lambda_1 T_{h,\Delta t}(J A_t^{\sigma_0} u(t_n)\Delta \overline{W}_1^n) - \lambda_1 \int_{t_n}^{t_{n+1}} \widehat{S}(t_{n+1}-r) J u(r) \mathrm{d}W_1(r) \\
&+ \frac{\lambda_1^2}{2}\int_{t_n}^{t_{n+1}} \widehat{S}(t_{n+1}-r)(F_{Q_1}u(r))\mathrm{d}r\Big)\Delta \overline{W}_1^n, (\widehat{S}_{h,\Delta t} + e^{-\sigma_0 \Delta t})\widetilde{e}_n\rangle_{\mathbb{H}}\Big] \\
=& \frac{\lambda_1^2}{2}\mathbb{E}\Big[\langle J\big(T_{h,\Delta t}\big[J A_t^{\sigma_0} u(t_n)(\Delta \overline{W}_1^n - \Delta W_1^n)\big]\big)\Delta \overline{W}_1^n, \big(\widehat{S}_{h,\Delta t} + e^{-\sigma_0 \Delta t}\big)\widetilde{e}_n\rangle_{\mathbb{H}}\Big] \\
&+ \frac{\lambda_1^2}{2}\mathbb{E}\Big[\langle J \int_{t_n}^{t_{n+1}} \big[T_{h,\Delta t}(J e^{-\sigma_0 \Delta t} u(t_n)) - \widehat{S}(t_{n+1}-r) J u(r)\big]\mathrm{d}W_1(r)\Delta \overline{W}_1^n, \big(\widehat{S}_{h,\Delta t} + e^{-\sigma_0 \Delta t}\big)\widetilde{e}_n\rangle_{\mathbb{H}}\Big] \\
&+ \frac{\lambda_1^2}{2}\mathbb{E}\Big[\langle J\big[T_{h,\Delta t}\big(J\frac{u(t_{n+1}) - e^{-\sigma_0 \Delta t}u(t_n)}{2}\Delta W_1^n\big)\big]\Delta \overline{W}_1^n, \big(\widehat{S}_{h,\Delta t} + e^{-\sigma_0 \Delta t}\big)\widetilde{e}_n\rangle_{\mathbb{H}}\Big] \\
&+ \frac{\lambda_1^3}{4}\mathbb{E}\Big[\langle J \int_{t_n}^{t_{n+1}} \widehat{S}(t_{n+1}-r)(F_{Q_1}u(r))\mathrm{d}r\,\Delta \overline{W}_1^n, \big(\widehat{S}_{h,\Delta t} + e^{-\sigma_0 \Delta t}\big)\widetilde{e}_n\rangle_{\mathbb{H}}\Big] \\
=:& I_{4,1} + I_{4,2} + I_{4,3} + I_{4,4}.
\end{aligned}
$$

For the term $I_{4,1}$, it holds that

$$
\begin{aligned}
I_{4,1} &\le C\mathbb{E}\big[\|A_t^{\sigma_0} u(t_n)\|_{\mathbb{H}}\|\Delta \overline{W}_1^n - \Delta W_1^n\|_{H^{\gamma_1 - 1}(D)}\|\Delta \overline{W}_1^n\|_{H^{\gamma_1 - 1}(D)} e^{-\sigma_0 \Delta t}\|\widetilde{e}_n\|_{\mathbb{H}}\big] \\
&\le \frac{\sigma_0}{16} e^{-2\sigma_0 \Delta t}\Delta t \mathbb{E}\big[\|\widetilde{e}_n\|_{\mathbb{H}}^2\big] + C\Delta t^2
\end{aligned}
$$

due to Proposition 2.5, the Sobolev embedding $H^{\gamma}(D) \hookrightarrow L^{\infty}(D)$ for $\gamma > \frac{3}{2}$, Lemma 4.4(1) and (3.2).

For the term $I_{4,2}$, we use (3.21), Lemma 4.4(1)(5), $Q_1^{\frac{1}{2}} \in HS(L^2(D), H_0^{\gamma_1}(D))$, Lemma 2.4 and Sobolev embedding $H^{\gamma}(D) \hookrightarrow L^{\infty}(D)$ for $\gamma > \frac{3}{2}$ to obtain

$$
\begin{aligned}
I_{4,2} \le& C\mathbb{E}\Big[\Big\| \int_{t_n}^{t_{n+1}} \big[T_{h,\Delta t}\big(J e^{-\sigma_0 \Delta t} u(t_n)\big) \\
&- \widehat{S}(t_{n+1}-r)\big(J u(r)\big)\big]\mathrm{d}W_1(r)\Big\|_{\mathbb{H}}\|\Delta \overline{W}_1^n\|_{H^{\gamma_1 - 1}(D)} e^{-\sigma_0 \Delta t}\|\widetilde{e}_n\|_{\mathbb{H}}\Big] \\
\le& C\mathbb{E}\Big[\Big\| \int_{t_n}^{t_{n+1}} \big[T_{h,\Delta t} J\big(e^{-\sigma_0 \Delta t}u(t_n) - u(r)\big) + \big(T_{h,\Delta t} - \widehat{S}(t_{n+1}-r)\big) J u(r)\big]\mathrm{d}W_1(r)\Big\|_{\mathbb{H}}^2\Big] \\
&+ \frac{\sigma_0}{16}\Delta t e^{-2\sigma_0 \Delta t}\mathbb{E}\big[\|\widetilde{e}_n\|_{\mathbb{H}}^2\big]
\end{aligned}
$$

$$\leq C\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\sum_{k=1}^{\infty}\big\|T_{h,\Delta t}J\big(u(r)-e^{-\sigma_0\Delta t}u(t_n)\big)Q_1^{\frac12}q_k\big\|_{\mathbb{H}}^2\mathrm{d}r\Big]$$

$$+C\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\sum_{k=1}^{\infty}\big\|\big(\widehat{S}(t_{n+1}-r)-T_{h,\Delta t}\big)Ju(r)Q_1^{\frac12}q_k\big\|_{\mathbb{H}}^2\mathrm{d}r\Big]+\frac{\sigma_0}{16}\Delta te^{-2\sigma_0\Delta t}\mathbb{E}\big[\|\widetilde{e}_n\|_{\mathbb{H}}^2\big]$$

$$\leq C(\Delta t+h)\Delta t+\frac{\sigma_0}{16}\Delta te^{-2\sigma_0\Delta t}\mathbb{E}\big[\|\widetilde{e}_n\|_{\mathbb{H}}^2\big].$$

For the term $I_{4,3}$, Lemma 4.4(1), (4.9) and Sobolev embedding $H^\gamma(D)\hookrightarrow L^\infty(D)$ for $\gamma>\frac32$ yield

$$\begin{aligned}I_{4,3}&\leq C\mathbb{E}\big[\|u(t_{n+1})-e^{-\sigma_0\Delta t}u(t_n)\|_{\mathbb{H}}\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}\|\Delta W_1^n\|_{H^{\gamma_1-1}(D)}e^{-\sigma_0\Delta t}\|\widetilde{e}_n\|_{\mathbb{H}}\big]\\&\leq\frac{\sigma_0}{16}\Delta te^{-2\sigma_0\Delta t}\mathbb{E}\big[\|\widetilde{e}_n\|_{\mathbb{H}}^2\big]\\&\quad+\frac{C}{\Delta t}\mathbb{E}\big[\|u(t_{n+1})-e^{-\sigma_0\Delta t}u(t_n)\|_{\mathbb{H}}^2\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}^2\|\Delta W_1^n\|_{H^{\gamma_1-1}(D)}^2\big]\\&\leq\frac{\sigma_0}{16}\Delta te^{-2\sigma_0\Delta t}\mathbb{E}\big[\|\widetilde{e}_n\|_{\mathbb{H}}^2\big]+C\Delta t^2.\end{aligned}$$

For the term $I_{4,4}$, one has

$$\begin{aligned}I_{4,4}&\leq C\mathbb{E}\Big[\Big\|\int_{t_n}^{t_{n+1}}\widehat{S}(t_{n+1}-r)(F_{Q_1}u(r))\mathrm{d}r\Big\|_{\mathbb{H}}\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}e^{-\sigma_0\Delta t}\|\widetilde{e}_n\|_{\mathbb{H}}\Big]\\&\leq\frac{\sigma_0}{16}\Delta te^{-2\sigma_0\Delta t}\mathbb{E}\big[\|\widetilde{e}_n\|_{\mathbb{H}}^2\big]+C\Delta t^2,\end{aligned}$$

where we use Sobolev embedding $H^\gamma(D)\hookrightarrow L^\infty(D)$ for $\gamma>\frac32$, Lemma 3.7(1), Lemma 4.4(1) and Proposition 2.3. Altogether,

$$I_4\leq\frac{\sigma_0}{4}\Delta te^{-2\sigma_0\Delta t}\mathbb{E}\big[\|\widetilde{e}_n\|_{\mathbb{H}}^2\big]+C\Delta t(\Delta t+h).$$

For the term I_5, we use (4.10) to obtain

$$\begin{aligned}I_5&\leq C\mathbb{E}\big[\|\widehat{e}_{n+1}+e^{-\sigma_0\Delta t}\widehat{e}_n\|_{\mathbb{H}}\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}^2\|\widetilde{e}_{n+1}+\widehat{e}_{n+1}+e^{-\sigma_0\Delta t}\widetilde{e}_n+e^{-\sigma_0\Delta t}\widehat{e}_n\|_{\mathbb{H}}\big]\\&\leq C\mathbb{E}\big[\|\widehat{e}_{n+1}+e^{-\sigma_0\Delta t}\widehat{e}_n\|_{\mathbb{H}}^2\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}^2\big]+\frac{\sigma_0}{4}\Delta t\mathbb{E}\big[\|\widetilde{e}_{n+1}\|_{\mathbb{H}}^2\big]\\&\quad+\frac{\sigma_0}{4}\Delta te^{-2\sigma_0\Delta t}\mathbb{E}\big[\|\widetilde{e}_n\|_{\mathbb{H}}^2\big]+\frac{C}{\Delta t}\mathbb{E}\big[\|\widehat{e}_{n+1}+e^{-\sigma_0\Delta t}\widehat{e}_n\|_{\mathbb{H}}^2\|\Delta\overline{W}_1^n\|_{H^{\gamma_1-1}(D)}^4\big]\\&\leq\frac{\sigma_0}{4}\Delta te^{-2\sigma_0\Delta t}\mathbb{E}\big[\|\widetilde{e}_n\|_{\mathbb{H}}^2\big]+\frac{\sigma_0}{4}\Delta t\mathbb{E}\big[\|\widetilde{e}_{n+1}\|_{\mathbb{H}}^2\big]+C\Delta t\big(\Delta t+h\big).\end{aligned}$$

Combining I_4 and I_5, we have

$$\mathbb{E}\big[\|\widetilde{e}_{n+1}\|_{\mathbb{H}}^2\big] \le e^{-2\sigma_0\Delta t}\mathbb{E}\big[\|\widetilde{e}_n\|_{\mathbb{H}}^2\big] + \frac{\sigma_0}{2}\Delta t e^{-2\sigma_0\Delta t}\mathbb{E}\big[\|\widetilde{e}_n\|_{\mathbb{H}}^2\big] + \frac{\sigma_0}{4}\Delta t\mathbb{E}\big[\|\widetilde{e}_{n+1}\|_{\mathbb{H}}^2\big] + C\Delta t\big(\Delta t + h\big).$$

There exists a $\Delta t^* := \frac{1}{\sigma_0}$ such that for any $\Delta t \in (0, \Delta t^*]$,

$$\sup_{n\in\mathbb{N}} \mathbb{E}\big[\|\widetilde{e}_n\|_{\mathbb{H}}^2\big] \le C\big(\Delta t + h\big).$$

Therefore, it concludes that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\big[\|e_n^{full}\|_{\mathbb{H}}^2\big] \le 2\sup_{n\in\mathbb{N}} \mathbb{E}\big[\|\widetilde{e}_n\|_{\mathbb{H}}^2\big] + 2\sup_{n\in\mathbb{N}} \mathbb{E}\big[\|\widehat{e}_n\|_{\mathbb{H}}^2\big] \le C\big(\Delta t + h\big).$$

The proof is thus finished. □

Similar to Proposition 2.6(ii), we state the following corollary.

Corollary 4.6. *Under the conditions in Proposition 4.2 and Theorem 4.5, there exists a positive constant C_{13} such that*

$$\mathcal{W}_2(\pi^*, \pi^{\Delta t,h}) \le C_{13}(\Delta t^{\frac{1}{2}} + h^{\frac{1}{2}}).$$

Proof. By Proposition 2.6 and Theorem 4.5, we have

$$\begin{aligned}\mathcal{W}_2(\pi^*, \pi^{\Delta t,h}) &\le \mathcal{W}_2((P_h^n)^*\pi^{\Delta t,h}, P_{t_n}^*\pi^{\Delta t,h}) + \mathcal{W}_2(P_{t_n}^*\pi^{\Delta t,h}, P_{t_n}^*\pi^*)\\ &\le C\big(\Delta t^{\frac{1}{2}} + h^{\frac{1}{2}}\big) + e^{-\sigma_0 t_n}\mathcal{W}_2(\pi^*, \pi^{\Delta t,h}),\end{aligned}$$

which gives the desired result by letting $n \to \infty$. □

4.2. Ergodic FD full discretization

For the temporal semi-discretization (3.1), we can apply many kinds of numerical methods to discretize the spatial direction to obtain full discretizations. In this part, we use a finite difference (FD) method to discretize the temporal semi-discretization in space.

We introduce a uniform partition with Δx, Δy and Δz being the mesh sizes in x, y and z directions, respectively. For $i = 0, 1, \ldots, I_x$, $j = 0, 1, \ldots, J_y$ and $k = 0, 1, \ldots, K_z$, we define $x_i = x_L + i\Delta x$, $y_j = y_L + j\Delta y$, $z_k = z_L + k\Delta z$, and $Z_{i,j,k}^n$ is the approximation of $Z(t, x, y, z)$ at node (t_n, x_i, y_j, z_k) and let $\sigma_{i,j,k} = \sigma(x_i, y_j, z_k)$. Denote

$$\delta_x Z_{i,j,k}^n = \frac{Z_{i+1,j,k}^n - Z_{i,j,k}^n}{\Delta x}, \quad \delta_y Z_{i,j,k}^n = \frac{Z_{i,j+1,k}^n - Z_{i,j,k}^n}{\Delta y}, \quad \delta_z Z_{i,j,k}^n = \frac{Z_{i,j,k+1}^n - Z_{i,j,k}^n}{\Delta z}.$$

We now propose the following full discretization for (1.1) by applying the midpoint method to (3.1) in the spatial direction:

$$\delta_t^{\sigma_{i,j,k}}(E_1)^n_{\bar{i},\bar{j},\bar{k}} = \delta_y A_t^{\sigma_{i,j,k}}(H_3)^n_{\bar{i},j,\bar{k}} - \delta_z A_t^{\sigma_{i,j,k}}(H_2)^n_{\bar{i},\bar{j},k}$$
$$- \lambda_1 A_t^{\sigma_{i,j,k}}(H_1)^n_{\bar{i},\bar{j},\bar{k}}(\dot{\overline{W}}_1)^n_{\bar{i},\bar{j},\bar{k}} + \lambda_2^{(1)}(\dot{W}_2)^n_{\bar{i},\bar{j},\bar{k}}, \tag{4.13a}$$

$$\delta_t^{\sigma_{i,j,k}}(E_2)^n_{\bar{i},\bar{j},\bar{k}} = \delta_z A_t^{\sigma_{i,j,k}}(H_1)^n_{i,\bar{j},k} - \delta_x A_t^{\sigma_{i,j,k}}(H_3)^n_{i,\bar{j},\bar{k}}$$
$$- \lambda_1 A_t^{\sigma_{i,j,k}}(H_2)^n_{\bar{i},\bar{j},\bar{k}}(\dot{\overline{W}}_1)^n_{\bar{i},\bar{j},\bar{k}} + \lambda_2^{(2)}(\dot{W}_2)^n_{\bar{i},\bar{j},\bar{k}}, \tag{4.13b}$$

$$\delta_t^{\sigma_{i,j,k}}(E_3)^n_{\bar{i},\bar{j},\bar{k}} = \delta_x A_t^{\sigma_{i,j,k}}(H_2)^n_{i,\bar{j},\bar{k}} - \delta_y A_t^{\sigma_{i,j,k}}(H_1)^n_{\bar{i},j,\bar{k}}$$
$$- \lambda_1 A_t^{\sigma_{i,j,k}}(H_3)^n_{\bar{i},\bar{j},\bar{k}}(\dot{\overline{W}}_1)^n_{\bar{i},\bar{j},\bar{k}} + \lambda_2^{(3)}(\dot{W}_2)^n_{\bar{i},\bar{j},\bar{k}}, \tag{4.13c}$$

$$\delta_t^{\sigma_{i,j,k}}(H_1)^n_{\bar{i},\bar{j},\bar{k}} = \delta_z A_t^{\sigma_{i,j,k}}(E_2)^n_{i,\bar{j},k} - \delta_y A_t^{\sigma_{i,j,k}}(E_3)^n_{\bar{i},j,\bar{k}}$$
$$+ \lambda_1 A_t^{\sigma_{i,j,k}}(E_1)^n_{\bar{i},\bar{j},\bar{k}}(\dot{\overline{W}}_1)^n_{\bar{i},\bar{j},\bar{k}} + \lambda_2^{(1)}(\dot{W}_2)^n_{\bar{i},\bar{j},\bar{k}}, \tag{4.13d}$$

$$\delta_t^{\sigma_{i,j,k}}(H_2)^n_{\bar{i},\bar{j},\bar{k}} = \delta_x A_t^{\sigma_{i,j,k}}(E_3)^n_{i,\bar{j},\bar{k}} - \delta_z A_t^{\sigma_{i,j,k}}(E_1)^n_{\bar{i},\bar{j},k}$$
$$+ \lambda_1 A_t^{\sigma_{i,j,k}}(E_2)^n_{\bar{i},\bar{j},\bar{k}}(\dot{\overline{W}}_1)^n_{\bar{i},\bar{j},\bar{k}} + \lambda_2^{(2)}(\dot{W}_2)^n_{\bar{i},\bar{j},\bar{k}}, \tag{4.13e}$$

$$\delta_t^{\sigma_{i,j,k}}(H_3)^n_{\bar{i},\bar{j},\bar{k}} = \delta_y A_t^{\sigma_{i,j,k}}(E_1)^n_{\bar{i},j,\bar{k}} - \delta_x A_t^{\sigma_{i,j,k}}(E_2)^n_{i,\bar{j},\bar{k}}$$
$$+ \lambda_1 A_t^{\sigma_{i,j,k}}(E_3)^n_{\bar{i},\bar{j},\bar{k}}(\dot{\overline{W}}_1)^n_{\bar{i},\bar{j},\bar{k}} + \lambda_2^{(3)}(\dot{W}_2)^n_{\bar{i},\bar{j},\bar{k}}, \tag{4.13f}$$

where $\bar{i} = i + \frac{1}{2}$, $\bar{j} = j + \frac{1}{2}$, $\bar{k} = k + \frac{1}{2}$ and

$$(\dot{\overline{W}}_1)^n_{\bar{i},\bar{j},\bar{k}} := \frac{(\Delta \overline{W}_1)^n_{\bar{i},\bar{j},\bar{k}}}{\Delta t} = \frac{\overline{W}_1(t_{n+1}, x_{\bar{i}}, y_{\bar{j}}, z_{\bar{k}}) - \overline{W}_1(t_n, x_{\bar{i}}, y_{\bar{j}}, z_{\bar{k}})}{\Delta t},$$
$$(\dot{W}_2)^n_{\bar{i},\bar{j},\bar{k}} := \frac{(\Delta W_2)^n_{\bar{i},\bar{j},\bar{k}}}{\Delta t} = \frac{W_2(t_{n+1}, x_{\bar{i}}, y_{\bar{j}}, z_{\bar{k}}) - W_2(t_n, x_{\bar{i}}, y_{\bar{j}}, z_{\bar{k}})}{\Delta t}.$$

Denote the discrete energy by

$$\Phi(t_n) := \Delta x \Delta y \Delta z \sum_{i,j,k} \Big(|\mathbf{E}^n_{\bar{i},\bar{j},\bar{k}}|^2 + |\mathbf{H}^n_{\bar{i},\bar{j},\bar{k}}|^2 \Big), \quad n \in \mathbb{N}. \tag{4.14}$$

Similarly to [6, Theorem 3.2], we obtain the following discrete energy evolution law for (4.13a)–(4.13f):

$$\Phi(t_{n+1}) = \Delta x \Delta y \Delta z \sum_{i,j,k} e^{-2\sigma_{i,j,k}\Delta t} \Big(|\mathbf{E}^n_{\bar{i},\bar{j},\bar{k}}|^2 + |\mathbf{H}^n_{\bar{i},\bar{j},\bar{k}}|^2 \Big) + 2\Delta x \Delta y \Delta z \sum_{i,j,k} \Big[\Upsilon^n_{\bar{i},\bar{j},\bar{k}} (\Delta W_2)^n_{\bar{i},\bar{j},\bar{k}} \Big] \tag{4.15}$$

under the periodic boundary condition. Here, $\Upsilon^n_{\bar{i},\bar{j},\bar{k}} := \widetilde{\boldsymbol{\lambda}}_2 \cdot A_t^{\sigma_{i,j,k}} \big((\mathbf{E}^n_{\bar{i},\bar{j},\bar{k}})^\top, (\mathbf{H}^n_{\bar{i},\bar{j},\bar{k}})^\top \big)^\top$.

We now investigate the ergodicity and stochastic multi-symplecticity of (4.13).

4.2.1. Ergodicity

To get the ergodicity, we first establish the uniform boundedness of the averaged discrete energy in the following proposition.

Proposition 4.7. *Assume that* $\mathbf{E}_0, \mathbf{H}_0 \in L^2(\Omega; L^2(D)^3)$, $q_m \in C^1(D), m \in \mathbb{N}$ *and let Assumption 2.1 hold with* $\gamma_1 > \frac{3}{2}$ *and* $\gamma_2 > \frac{5}{2}$. *Then there exists a positive constant* Δt^* *such that when* $\Delta t \in (0, \Delta t^*]$, *the averaged discrete energy is uniformly bounded under the periodic boundary condition, i.e.,*

$$\mathbb{E}\big[\Phi(t_n)\big] \le e^{-\sigma_0 n \Delta t}\mathbb{E}\big[\Phi(t_0)\big] + C_{14}, \quad n \in \mathbb{N},$$

where the positive constant C_{14} *depends on* $\sigma_0, \lambda_1, \widetilde{\boldsymbol{\lambda}}_2, |D|$, $\|Q_1^{\frac{1}{2}}\|_{\mathcal{L}_2^{\gamma_1}}$ *and* $\|Q_2^{\frac{1}{2}}\|_{\mathcal{L}_2^{\gamma_2}}$.

Proof. For the first term on the right side of (4.15), we have

$$\Delta x \Delta y \Delta z \sum_{i,j,k} e^{-2\sigma_{i,j,k}\Delta t}\big(|\mathbf{E}^n_{\bar{i},\bar{j},\bar{k}}|^2 + |\mathbf{H}^n_{\bar{i},\bar{j},\bar{k}}|^2\big) \le e^{-2\sigma_0 \Delta t}\Phi(t_n). \tag{4.16}$$

For the second term on the right side of (4.15), which contains six sub-terms, the estimate is more technical. We consider the first sub-term. It holds that

$$\mathbb{E}\Big[\sum_{i,j,k} A_t^{\sigma_{i,j,k}}(E_1)^n_{\bar{i},\bar{j},\bar{k}}(\Delta W_2)^n_{\bar{i},\bar{j},\bar{k}}\Big] = \frac{1}{2}\mathbb{E}\Bigg[\sum_{i,j,k}\Big[(E_1)^{n+1}_{\bar{i},\bar{j},\bar{k}} - e^{-\sigma_{i,j,k}\Delta t}(E_1)^n_{\bar{i},\bar{j},\bar{k}}\Big](\Delta W_2)^n_{\bar{i},\bar{j},\bar{k}}\Bigg]. \tag{4.17}$$

Substituting (4.13a) into (4.17), we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{i,j,k} A_t^{\sigma_{i,j,k}}(E_1)^n_{\bar{i},\bar{j},\bar{k}}(\Delta W_2)^n_{\bar{i},\bar{j},\bar{k}}\Big] \\
&= \frac{1}{2}\mathbb{E}\Big[\sum_{i,j,k}\Big[-\Delta t A_t^{\sigma_{i,j,k}}(H_3)^n_{\bar{i},\bar{j},\bar{k}}\delta_y(\Delta W_2)^n_{\bar{i},j,\bar{k}} + \Delta t A_t^{\sigma_{i,j,k}}(H_2)^n_{\bar{i},\bar{j},\bar{k}}\delta_z(\Delta W_2)^n_{\bar{i},\bar{j},k}\Big]\Big] \\
&\quad + \frac{1}{2}\mathbb{E}\Big[\sum_{i,j,k}\Big[-\lambda_1 A_t^{\sigma_{i,j,k}}(H_1)^n_{\bar{i},\bar{j},\bar{k}}(\Delta W_1)^n_{\bar{i},\bar{j},\bar{k}}(\Delta W_2)^n_{\bar{i},\bar{j},\bar{k}} + \lambda_2^{(1)}\big[(\Delta W_2)^n_{\bar{i},\bar{j},\bar{k}}\big]^2\Big]\Big],
\end{aligned} \tag{4.18}$$

where we use the fact

$$\sum_{i,j,k}\delta_y A_t^{\sigma_{i,j,k}}(H_3)^n_{\bar{i},j,\bar{k}}(\Delta W_2)^n_{\bar{i},\bar{j},\bar{k}} = -\sum_{i,j,k}A_t^{\sigma_{i,j,k}}(H_3)^n_{\bar{i},\bar{j},\bar{k}}\delta_y(\Delta W_2)^n_{\bar{i},j,\bar{k}}$$

and

$$\sum_{i,j,k}\delta_z A_t^{\sigma_{i,j,k}}(H_2)^n_{\bar{i},\bar{j},k}(\Delta W_2)^n_{\bar{i},\bar{j},\bar{k}} = -\sum_{i,j,k}A_t^{\sigma_{i,j,k}}(H_2)^n_{\bar{i},\bar{j},\bar{k}}\delta_z(\Delta W_2)^n_{\bar{i},\bar{j},k}$$

due to the periodic boundary condition. Notice that

$$\begin{aligned}
&\Delta x\Delta y\Delta z\sum_{i,j,k}\mathbb{E}\big[|(\Delta W_2)^n_{\bar{i},\bar{j},\bar{k}}|^2\big]\\
&=\Delta x\Delta y\Delta z\sum_{i,j,k}\mathbb{E}\Big[\Big|\sum_{m=1}^{\infty}\sqrt{\eta_m^{(2)}}q_m(x_{\bar{i}},y_{\bar{j}},z_{\bar{k}})(\beta_m^{(2)}(t_{n+1})-\beta_m^{(2)}(t_n))\Big|^2\Big]\\
&\le \Delta x\Delta y\Delta z\Delta t\sum_{i,j,k}\sum_{m=1}^{\infty}\eta_m^{(2)}\|q_m\|^2_{L^\infty(D)}\le C|D|\Delta t\sum_{m=1}^{\infty}\eta_m^{(2)}\|q_m\|^2_{H^{\gamma_1}(D)}=C\Delta t
\end{aligned}\tag{4.19}$$

and similarly

$$\Delta x\Delta y\Delta z\sum_{i,j,k}\Big(\mathbb{E}\Big[\big|(\Delta W_1)^n_{\bar{i},\bar{j},\bar{k}}\big|^2\Big]\mathbb{E}\Big[\big|(\Delta W_2)^n_{\bar{i},\bar{j},\bar{k}}\big|^2\Big]\Big)\le C\Delta t^2,\tag{4.20}$$

where we use the Sobolev embedding $H^\gamma(D)\hookrightarrow L^\infty(D)$ for $\gamma>\frac{3}{2}$. Thus, for (4.18), the Hölder inequality, the Young inequality and (4.19)–(4.20) lead to

$$\begin{aligned}
&\Delta x\Delta y\Delta z\lambda_2^{(1)}\mathbb{E}\Big[\sum_{i,j,k}A_t^{\sigma_{i,j,k}}(E_1)^n_{\bar{i},\bar{j},\bar{k}}(\Delta W_2)^n_{\bar{i},\bar{j},\bar{k}}\Big]\\
&\le\frac{\sigma_0}{8}\Delta t\Delta x\Delta y\Delta z\mathbb{E}\Big[\sum_{i,j,k}\Big[\big|A_t^{\sigma_{i,j,k}}(H_3)^n_{\bar{i},\bar{j},\bar{k}}\big|^2+\big|A_t^{\sigma_{i,j,k}}(H_2)^n_{\bar{i},\bar{j},\bar{k}}\big|^2\Big]\Big]\\
&\quad+\frac{|\lambda_2^{(1)}|^2}{2\sigma_0}\Delta t\Delta x\Delta y\Delta z\mathbb{E}\Big[\sum_{i,j,k}\Big[\big|\delta_y(\Delta W_2)^n_{\bar{i},j,\bar{k}}\big|^2+\big|\delta_z(\Delta W_2)^n_{\bar{i},\bar{j},k}\big|^2\Big]\Big]\\
&\quad+\frac{1}{2}\Delta x\Delta y\Delta z|\lambda_2^{(1)}|\sum_{i,j,k}\mathbb{E}\Big[\big|(\Delta W_2)^n_{\bar{i},\bar{j},\bar{k}}\big|^2\Big]+\frac{\sigma_0}{4}\Delta t\Delta x\Delta y\Delta z\mathbb{E}\Big[\sum_{i,j,k}\big|A_t^{\sigma_{i,j,k}}(H_1)^n_{\bar{i},\bar{j},\bar{k}}\big|^2\Big]\\
&\quad+\frac{|\lambda_1\lambda_2^{(1)}|^2}{4\sigma_0\Delta t}\Delta x\Delta y\Delta z\sum_{i,j,k}\mathbb{E}\Big[\big|(\Delta W_1)^n_{\bar{i},\bar{j},\bar{k}}\big|^2\Big]\mathbb{E}\Big[\big|(\Delta W_2)^n_{\bar{i},\bar{j},\bar{k}}\big|^2\Big]\\
&\le\frac{\sigma_0}{4}\Delta t\Delta x\Delta y\Delta z\mathbb{E}\Big[\sum_{i,j,k}\Big[\frac{1}{4}\big|(H_3)^{n+1}_{\bar{i},\bar{j},\bar{k}}\big|^2+\frac{1}{4}|(H_2)^{n+1}_{\bar{i},\bar{j},\bar{k}}|^2+\frac{1}{2}|(H_1)^{n+1}_{\bar{i},\bar{j},\bar{k}}|^2\Big]\Big]\\
&\quad+\frac{\sigma_0}{4}\Delta t\Delta x\Delta y\Delta z\mathbb{E}\Big[\sum_{i,j,k}e^{-2\sigma_{i,j,k}\Delta t}\Big[\frac{1}{4}\big|(H_3)^n_{\bar{i},\bar{j},\bar{k}}\big|^2+\frac{1}{4}|(H_2)^n_{\bar{i},\bar{j},\bar{k}}|^2+\frac{1}{2}\big|(H_1)^n_{\bar{i},\bar{j},\bar{k}}\big|^2\Big]\Big]\\
&\quad+\frac{|\lambda_2^{(1)}|^2}{2\sigma_0}\Delta t\Delta x\Delta y\Delta z\mathbb{E}\Big[\sum_{i,j,k}\Big[\big|\delta_y(\Delta W_2)^n_{\bar{i},j,\bar{k}}\big|^2+\big|\delta_z(\Delta W_2)^n_{\bar{i},\bar{j},k}\big|^2\Big]\Big]+C\Delta t.
\end{aligned}$$

Applying the similar approach to estimate the other five sub-terms on the second term on the right side of (4.15), we get

$$\Delta x\Delta y\Delta z\sum_{i,j,k}\mathbb{E}\Big[\Upsilon^n_{\bar{i},\bar{j},\bar{k}}(\Delta W_2)^n_{\bar{i},\bar{j},\bar{k}}\Big]$$

$$\leq \frac{\sigma_0}{4}\Delta t\mathbb{E}[\Phi(t_{n+1})] + \frac{\sigma_0}{4}\Delta t e^{-2\sigma_0\Delta t}\mathbb{E}[\Phi(t_n)] + C\Delta t$$
$$+ \frac{|\boldsymbol{\lambda}_2|^2}{\sigma_0}\Delta t\Delta x\Delta y\Delta z\mathbb{E}\Big[\sum_{i,j,k}\Big[|\delta_x(\Delta W_2)^n_{i,\bar{j},\bar{k}}|^2 + |\delta_y(\Delta W_2)^n_{\bar{i},j,\bar{k}}|^2 + |\delta_z(\Delta W_2)^n_{\bar{i},\bar{j},k}|^2\Big]\Big].$$

It follows from the Sobolev embedding $H^\gamma \hookrightarrow L^\infty(D)$ for $\gamma > \frac{3}{2}$ that

$$\mathbb{E}\Big[\sum_{i,j,k}\big|\delta_x(\Delta W_2)^n_{i,\bar{j},\bar{k}}\big|^2\Big]$$
$$= \sum_{i,j,k}\mathbb{E}\Big[\Big|\sum_{m=1}^\infty\sqrt{\eta_m^{(2)}}\frac{q_m(x_{i+1},y_{\bar{j}},z_{\bar{k}}) - q_m(x_i,y_{\bar{j}},z_{\bar{k}})}{\Delta x}\Big(\beta_m^{(2)}(t_{n+1}) - \beta_m^{(2)}(t_n)\Big)\Big|^2\Big]$$
$$= \Delta t\sum_{i,j,k}\sum_{m=1}^\infty\eta_m^{(2)}\big|\partial_x q_m(x_i+\theta_i\Delta x, y_{\bar{j}}, z_{\bar{k}})\big|^2 \leq \Delta t\sum_{i,j,k}\sum_{m=1}^\infty\eta_m^{(2)}\|q_m\|^2_{W^{1,\infty}(D)}$$
$$\leq C\Delta t\sum_{i,j,k}\sum_{m=1}^\infty\eta_m^{(2)}\|q_m\|^2_{H^{\gamma_2}(D)} \leq C\Delta t I_x J_y K_z,$$

where $\theta_i \in [0,1]$. Therefore, we get

$$\frac{|\boldsymbol{\lambda}_2|^2}{4\alpha}\Delta t\Delta x\Delta y\Delta z\mathbb{E}\Big[\sum_{i,j,k}|\delta_x(\Delta W_2)^n_{i,\bar{j},\bar{k}}|^2\Big] \leq C\Delta t^2.$$

Estimating terms concerning $|\delta_y(\Delta W_2)^n_{\bar{i},j,\bar{k}}|^2$ and $|\delta_z(\Delta W_2)^n_{\bar{i},\bar{j},k}|^2$ similarly, we conclude that

$$\Delta x\Delta y\Delta z\sum_{i,j,k}\mathbb{E}\Big[\Upsilon^n_{\bar{i},\bar{j},\bar{k}}(\Delta W_2)^n_{\bar{i},\bar{j},\bar{k}}\Big] \leq \frac{\sigma_0}{4}\Delta t\mathbb{E}[\Phi(t_{n+1})] + \frac{\sigma_0}{4}\Delta t e^{-2\sigma_0\Delta t}\mathbb{E}[\Phi(t_n)] + C\Delta t. \tag{4.21}$$

Consequently, combining (4.16) and (4.21) yields

$$\mathbb{E}[\Phi(t_{n+1})] \leq e^{-2\sigma_0\Delta t}\mathbb{E}[\Phi(t_n)] + \frac{\sigma_0}{4}\Delta t\mathbb{E}[\Phi(t_{n+1})] + \frac{\sigma_0}{4}\Delta t e^{-2\sigma_0\Delta t}\mathbb{E}[\Phi(t_n)] + C\Delta t.$$

By the Gronwall inequality, it can be shown that for any $\Delta t \in (0, \frac{2}{\sigma_0}]$, the conclusion of the proposition holds. □

Remark 4.8. Especially, if the orthonormal basis of $L^2(D)$ is chosen as

$$q_m(x,y,z) = q_{m_1}(x)q_{m_2}(y)q_{m_3}(z), \quad m = (m_1,m_2,m_3) \in \mathbb{N}^3$$

with

$$q_{m_1}(x)=\sqrt{\frac{2}{x_R-x_L}}\sin\Big(\frac{2m_1\pi(x-x_L)}{x_R-x_L}\Big),$$

and $q_{m_2}(y), q_{m_3}(z)$ being defined similarly, the assumption of Proposition 4.7 can be weakened as $\gamma_1 \geq 0$ and $\gamma_2 \geq 1$ since the condition $q_m \in C^1(D)$ is automatically satisfied in this case.

Let

$$U^n=\Big((E_1)^n_{\bar 1,\bar 1,\bar 1},(E_1)^n_{\bar 2,\bar 1,\bar 1},\ldots,(E_1)^n_{\bar I_x,\bar 1,\bar 1},(E_1)^n_{\bar 1,\bar 2,\bar 1},\ldots,(E_1)^n_{\bar I_x,\bar 2,\bar 1},\ldots,(E_1)^n_{\bar I_x,\bar J_y,\bar K_z},$$
$$(E_2)^n_{\bar 1,\bar 1,\bar 1},\ldots,(E_2)^n_{\bar I_x,\bar J_y,\bar K_z},(E_3)^n_{\bar 1,\bar 1,\bar 1},\ldots,(E_3)^n_{\bar I_x,\bar J_y,\bar K_z},(H_1)^n_{\bar 1,\bar 1,\bar 1},\ldots,(H_3)^n_{\bar I_x,\bar J_y,\bar K_z}\Big)^\top,$$

then the discrete energy Φ given by (4.14) can be rewritten into

$$\Phi(t_n)=\Delta x\Delta y\Delta z|U^n|^2,\quad n\in\mathbb{N}. \tag{4.22}$$

By taking (4.22) as a Lyapunov function, the ergodicity of (4.13) can be proved similar to that of Proposition 2.6, which is stated below.

Theorem 4.9. *Under the conditions in Proposition 4.7, for sufficiently small $\Delta t>0$, the numerical solution $\{U^n\}_{n\in\mathbb{N}}$ of (4.13) has a unique invariant measure π. Thus $\{U^n\}_{n\in\mathbb{N}}$ is ergodic. Moreover, $\{U^n\}_{n\in\mathbb{N}}$ is exponentially mixing.*

4.2.2. Stochastic conformal multi-symplecticity

Now we turn to the stochastic conformal multi-symplecticity of (4.13). By using the definitions of skew-symmetric matrices F, K_s, $s=1,2,3$, the full discretization (4.13) can be read as

$$F\left(\delta_t^{\sigma_{i,j,k}}u^n_{\bar i,\bar j,\bar k}\right)+K_1\left(\delta_x A_t^{\sigma_{i,j,k}}u^n_{i,\bar j,\bar k}\right)+K_2\left(\delta_y A_t^{\sigma_{i,j,k}}u^n_{\bar i,j,\bar k}\right)+K_3\left(\delta_z A_t^{\sigma_{i,j,k}}u^n_{\bar i,\bar j,k}\right)$$
$$=\nabla_u S_1\left(A_t^{\sigma_{i,j,k}}u^n_{\bar i,\bar j,\bar k}\right)(\dot W_1)^n_{\bar i,\bar j,\bar k}+\nabla_u S_2\left(A_t^{\sigma_{i,j,k}}u^n_{\bar i,\bar j,\bar k}\right)(\dot W_2)^n_{\bar i,\bar j,\bar k}.$$

Performing the wedge product on both sides of the above equation yields the following result.

Theorem 4.10. *The full discretization (4.13) preserves the discrete stochastic conformal multi-symplectic conservation law*

$$\delta_t^{2\sigma_{i,j,k}}\omega^n_{\bar i,\bar j,\bar k}+\delta_x(\kappa_1)^{n,\sigma_{i,j,k}}_{i,\bar j,\bar k}+\delta_y(\kappa_2)^{n,\sigma_{i,j,k}}_{\bar i,j,\bar k}+\delta_z(\kappa_3)^{n,\sigma_{i,j,k}}_{\bar i,\bar j,k}=0,\quad \mathbb{P}\text{-}a.s.,$$

where $\omega^n_{i,j,k}=\frac12 du^n_{i,j,k}\wedge F du^n_{i,j,k}$, and $(\kappa_s)^{n,\sigma_{i,j,k}}_{i,j,k}=\frac12 d(A_t^{\sigma_{i,j,k}}u^n_{i,j,k})\wedge K_s d(A_t^{\sigma_{i,j,k}}u^n_{i,j,k})$, $s=1,2,3$.

5. Numerical experiments

In this section we provide numerical examples to illustrate several properties of the fully discretization (4.13) of the stochastic Maxwell equations (1.1). In the sequel, we consider the stochastic Maxwell equations (1.1) with $\lambda_1 = 0.5$ and $\widetilde{\boldsymbol{\lambda}}_2 = (1, 1, 1, 1, 1, 1)^\top$ on the domain $D = [-1, 1]^3$. We take the truncated parameter $A_{\Delta t}$, the eigenvalues $\{\eta^{(m)}_{i,j,k}\}_{i,j,k\in\mathbb{N}_+}, m = 1, 2$, and the orthonormal basis $\{q_{i,j,k}\}_{i,j,k\in\mathbb{N}_+}$ of $L^2(D)$) in this section as

$$A_{\Delta t} = \sqrt{8|\ln \Delta t|}, \qquad \eta^{(1)}_{i,j,k} = \eta^{(2)}_{i,j,k} = \frac{1}{i^3 + j^3 + k^3},$$

$$q_{i,j,k}(x, y, z) = \sin(i\pi x)\sin(j\pi y)\sin(k\pi z), \quad (x, y, z) \in D.$$

5.1. Ergodic limit

This subsection provides numerical experiments to test the longtime dynamical behavior of (4.13) for stochastic Maxwell equations with various coefficients σ. We take $\Delta t = 2^{-4}$ and $\Delta x = \Delta y = \Delta z = \frac{2}{100}$. Fig. 1 and Fig. 2 present the temporal averages $\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}[f(U^n)]$ for the full discretization (4.13) with various σ for f being (a) $f(U) = \sin(U)$, (b) $f(U) = \sin(\|U\|^2)$, and (c) $f(U) = \sin(\|U\|^4)$ starting from the following seven different initial values:

- Initial(1):

$$(E_1)^0_{i,j,k} = \begin{cases} 1, & i = j = k = 1, \\ 0, & otherwise, \end{cases} \qquad (E_2)^0 = (E_3)^0 = (H_1)^0 = (H_2)^0 = (H_3)^0 = (E_1)^0.$$

- Initial(2):

$$(E_1)^0_{i,j,k} \equiv 1, \quad i, j, k \in \mathcal{K}, \qquad (E_2)^0 = (E_3)^0 = (H_2)^0 = (H_3)^0 = (E_1)^0.$$

- Initial(3):

$$(E_1)^0_{i,j,k} = \begin{cases} \sin(\frac{\pi}{2} i\Delta x), & i \in \mathcal{K}, j = k = 1, \\ 0, & otherwise, \end{cases} \qquad (E_2)^0 = (E_3)^0 = (E_1)^0,$$

$$(H_1)^0_{i,j,k} = \begin{cases} \cos(\frac{\pi}{2} i\Delta x), & i \in \mathcal{K}, j = k = 1, \\ 0, & otherwise, \end{cases} \qquad (H_2)^0 = (H_3)^0 = (H_1)^0.$$

- Initial(4):

$$
\begin{aligned}
&(E_1)^0 = (E_3)^0 = \mathrm{rand}(101^3, 1), \qquad (H_3)^0 = 3\mathrm{rand}(101^3, 1),\\
&(E_2)^0_{i,j,k} = e^{-(i\Delta x-0.5)^2-(j\Delta y-0.5)^2-(k\Delta z-0.5)^2}, \quad i, j, k \in \mathcal{K},\\
&(H_1)^0_{i,j,k} = \begin{cases} 2i\Delta x, & i \in \mathcal{K}, j = k = 1,\\ 0, & otherwise,\end{cases}\\
&(H_2)^0_{i,j,k} = \sqrt{2}\cos(\pi i\Delta x)\sin(\pi j\Delta y)\cos(\pi k\Delta z), \quad i, j, k \in \mathcal{K}.
\end{aligned}
$$

- Initial(5):

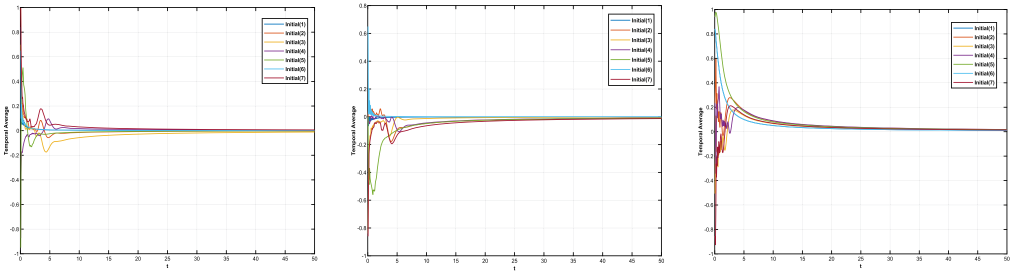
$$
\begin{aligned}
&(E_1)^0_{i,j,k} \equiv 0, \quad i, j, k \in \mathcal{K}, \qquad (E_2)^0 = (E_1)^0,\\
&(E_3)^0_{i,j,k} = \begin{cases} \Big(1-\sqrt{\frac{\pi}{2}(e^{\frac{1}{4}}-1)}\Big)(1-e^{i\Delta x(1-i\Delta x)}, & i \in \mathcal{K}, j = k = 1,\\ 0, & otherwise,\end{cases}\\
&(H_1)^0 = \mathrm{rand}(101^3, 1), \qquad (H_2)^0 = \mathrm{rand}(101^3, 1),\\
&(H_3)^0_{i,j,k} = \begin{cases} 0.002, & i = j = k = 101,\\ 0, & otherwise.\end{cases}
\end{aligned}
$$

- Initial(6):

$$
\begin{aligned}
&(E_1)^0_{i,j,k} \equiv 0, \quad i, j, k \in \mathcal{K}, \qquad (E_2)^0 = \mathrm{rand}(101^3, 1),\\
&(E_3)^0_{i,j,k} = \begin{cases} \mathrm{sech}(\frac{i\Delta x}{\sqrt{2}})e^{i\Delta x/2}, & i \in \mathcal{K}, j, k = 1,\\ 0, & otherwise,\end{cases}\\
&(H_1)^0_{i,j,k} = \begin{cases} e^{\pi i\Delta x/4}, & i \in \mathcal{K}, j, k = 1,\\ 0, & otherwise,\end{cases}\\
&(H_2)^0_{i,j,k} \equiv 0, \quad i, j, k \in \mathcal{K}, \qquad (H_3)^0 = \mathrm{rand}(101^3, 1).
\end{aligned}
$$

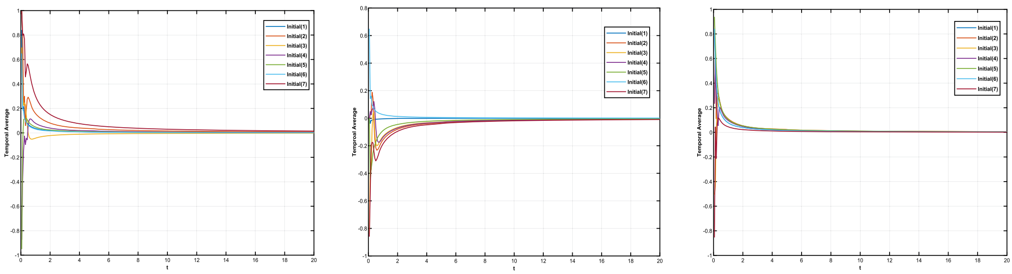
- Initial(7):

$$
\begin{aligned}
&(E_1)^0_{i,j,k} = \frac{5}{\sqrt{14}}\cos(\pi i\Delta x)\sin(2\pi j\Delta y)\sin(-3\pi k\Delta z), \quad i, j, k \in \mathcal{K},\\
&(E_2)^0_{i,j,k} = \frac{-4}{\sqrt{14}}\sin(\pi i\Delta x)\cos(2\pi j\Delta y)\sin(-3\pi k\Delta z), \quad i, j, k \in \mathcal{K},\\
&(E_3)^0_{i,j,k} = \frac{-1}{\sqrt{14}}\sin(\pi i\Delta x)\sin(2\pi j\Delta y)\cos(-3\pi k\Delta z), \quad i, j, k \in \mathcal{K},\\
&(H_1)^0_{i,j,k} = \sin(\pi i\Delta x)\cos(2\pi j\Delta y)\cos(-3\pi k\Delta z), \quad i, j, k \in \mathcal{K},\\
&(H_2)^0_{i,j,k} = \cos(\pi i\Delta x)\sin(2\pi j\Delta y)\cos(-3\pi k\Delta z), \quad i, j, k \in \mathcal{K},
\end{aligned}
$$



(a) $f(U) = \sin(U)$ (b) $f(U) = \sin(\|U\|^2)$ (c) $f(U) = \sin(\|U\|^4)$

Fig. 1. The temporal averages $\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}[f(U^n)]$ for the full discretization (4.13) with (a) $f(U) = \sin(U)$, (b) $f(U) = \sin(\|U\|^2)$, and (c) $f(U) = \sin(\|U\|^4)$ starting from different initial values over 100 realizations ($\sigma = 1$, $T = 50$). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)



(a) $f(U) = \sin(U)$ (b) $f(U) = \sin(\|U\|^2)$ (c) $f(U) = \sin(\|U\|^4)$

Fig. 2. The temporal averages $\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}[f(U^n)]$ for the full discretization (4.13) with (a) $f(U) = \sin(U)$, (b) $f(U) = \sin(\|U\|^2)$, and (c) $f(U) = \sin(\|U\|^4)$ starting from different initial values over 100 realizations ($\sigma = 10$, $T = 20$).

$$(H_3)^0_{i,j,k} = \cos(\pi i \Delta x)\cos(2\pi j \Delta y)\sin(-3\pi k \Delta z), \quad i, j, k \in \mathcal{K}.$$

Here, $\mathcal{K} = \{1, 2, \cdots, 101\}$.

We can observe from Fig. 1 and Fig. 2 that the temporal averages starting from different initial values converge to the same value, which implies the ergodicity of the full discretization for stochastic Maxwell equations.

5.2. Mean-square convergence

In this subsection, we provide the numerical experiment to test the mean-square convergence order in temporal direction of the full discretization (4.13) given in Theorem 3.8. We fix the spatial mesh sizes sufficiently small such that errors stemming from the spatial approximation are negligible. We compute the errors at the final time $T = 0.25$ with mesh sizes $\Delta x = \Delta y = \Delta z = \frac{1}{256}$. The initial data read

$$\begin{aligned}
&(E_1)^0_{i,j,k} = \cos(2\pi(i\Delta x + j\Delta y + k\Delta z)), \quad i, j, k \in \widetilde{\mathcal{K}},\\
&(E_2)^0 = -2(E_1)^0, \ \ (E_3)^0 = (E_1)^0,\\
&(H_1)^0 = \sqrt{3}(E_1)^0, \ \ (H_2)^0 = 0, \ \ (H_3)^0 = -(H_1)^0,
\end{aligned}$$

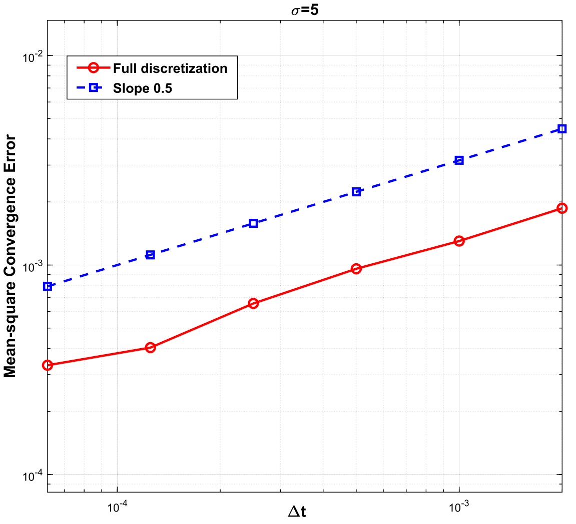


Fig. 3. The mean-square convergence order in temporal direction of the full discretization for stochastic Maxwell equations with $\sigma = 5$.

where $\widetilde{\mathcal{K}} = \{1, \cdots, 513\}$. Since we do not know the explicit form of the solution to (1.1), we take the full discretization with small time step-size $\Delta t = 2^{-14}$ as the reference solution. We then compare it with the full discretizations evaluated with time step-sizes $\{2^1\Delta t, 2^2\Delta t, 2^3\Delta t, 2^4\Delta t, 2^5\Delta t, 2^6\Delta t\}$ in order to estimate the order of convergence. As is displayed in Fig. 3, the mean-square convergence order of the full discretization is shown to be 1/2 which coincides with the theoretical result.

Appendix A. The proof of Lemma 3.7

To derive the estimates of the operators, we introduce the following two deterministic systems:

$$\begin{cases} \mathrm{d}\widetilde{u}(t) = (M - \sigma I)\widetilde{u}(t)\mathrm{d}t, \quad t > 0, \\ \widetilde{u}(0) = v, \end{cases} \tag{A.1}$$

and

$$\begin{cases} \mathrm{d}\overline{u}(t) = -\sigma \overline{u}(t)\mathrm{d}t, \quad t > 0, \\ \overline{u}(0) = v. \end{cases} \tag{A.2}$$

(1) Notice that $\widetilde{u}(t) = e^{t(M-\sigma I)}v$, $t \geq 0$. Then

$$\|\widetilde{u}(t)\|_{\mathbb{H}} \leq \|e^{tM}\|_{\mathcal{L}(\mathbb{H},\mathbb{H})}\|e^{-\sigma I}v\|_{\mathbb{H}} \leq e^{-\sigma_0 t}\|v\|_{\mathbb{H}}, \tag{A.3}$$

which leads to the first assertion of (1).

Let $\widetilde{u}^n := [(I - \frac{\Delta t}{2}M)^{-1}(I + \frac{\Delta t}{2}M)e^{-\sigma\Delta t}]^n v, n \in \mathbb{N}$. Then

$$\widetilde{u}^n = (I - \frac{\Delta t}{2}M)^{-1}(I + \frac{\Delta t}{2}M)e^{-\sigma\Delta t}\widetilde{u}^{n-1}, \quad n \geq 1$$

implies

$$\widetilde{u}^n - e^{-\sigma\Delta t}\widetilde{u}^{n-1} = \frac{\Delta t}{2}M(\widetilde{u}^n + e^{-\sigma\Delta t}\widetilde{u}^{n-1}). \tag{A.4}$$

We apply $\langle \cdot, \widetilde{u}^n + e^{-\sigma\Delta t}\widetilde{u}^{n-1}\rangle_{\mathbb{H}}$ on both sides of (A.4) and get

$$\|\widetilde{u}^n\|^2_{\mathbb{H}} - \|e^{-\sigma\Delta t}\widetilde{u}^{n-1}\|^2_{\mathbb{H}} = 0.$$

Then we have

$$\|\widetilde{u}^n\|_{\mathbb{H}} \leq e^{-\sigma_0\Delta t}\|\widetilde{u}^{n-1}\|_{\mathbb{H}} \leq \cdots \leq e^{-n\Delta t\sigma_0}\|\widetilde{u}^0\|_{\mathbb{H}},$$

which leads to the second assertion of (1).

(2) To estimate $\|\hat{S}(t_n) - (\hat{S}_{\Delta t})^n\|_{\mathcal{L}(D(M),\mathbb{H})}$, we proceed now in two steps.

Step 1. Estimate of $\|\widetilde{u}^n\|_{\mathcal{D}(M)}$ and $\|\widetilde{u}(t)\|_{\mathcal{D}(M)}$.

Applying $\langle \cdot, M^2(\widetilde{u}^n + e^{-\sigma\Delta t}\widetilde{u}^{n-1})\rangle_{\mathbb{H}}$ on both sides of (A.4) obtains

$$\|M\widetilde{u}^n\|^2_{\mathbb{H}} - \|M(e^{-\sigma\Delta t}\widetilde{u}^n)\|^2_{\mathbb{H}} = 0.$$

Hence

$$\begin{aligned}
&\|M\widetilde{u}^n\|_{\mathbb{H}} = \|M(e^{-\sigma\Delta t}\widetilde{u}^{n-1})\|_{\mathbb{H}} \leq e^{-\sigma_0\Delta t}\|M\widetilde{u}^{n-1}\|_{\mathbb{H}} + 2\|\nabla\sigma\|_{L^\infty(D)^3}\Delta t e^{-\sigma_0\Delta t}\|\widetilde{u}^{n-1}\|_{\mathbb{H}}\\
&\leq e^{-n\Delta t\sigma_0}\|M\widetilde{u}^0\|_{\mathbb{H}} + 2\|\nabla\sigma\|_{L^\infty(D)^3}\Delta t\Big(\sum_{k=1}^n e^{-\sigma_0 k\Delta t}e^{-(n-k)\Delta t\sigma_0}\|\widetilde{u}^0\|_{\mathbb{H}}\Big)\\
&\leq Ce^{-\sigma_0 n\Delta t/2}\|\widetilde{u}^0\|_{\mathcal{D}(M)},
\end{aligned}$$

from which we get

$$\|\widetilde{u}^n\|_{\mathcal{D}(M)} \leq \Big(Ce^{-2\sigma_0 n\Delta t}\|\widetilde{u}^0\|^2_{\mathbb{H}} + Ce^{-\sigma_0 n\Delta t}\|\widetilde{u}^0\|^2_{\mathcal{D}(M)}\Big)^{\frac{1}{2}} \leq Ce^{-\sigma_0 n\Delta t/2}\|\widetilde{u}^0\|_{\mathcal{D}(M)}. \tag{A.5}$$

It follows from

$$\frac{\partial}{\partial t}\|M\widetilde{u}(t)\|^2_{\mathbb{H}} = 2\langle M\frac{\partial\widetilde{u}(t)}{\partial t}, M\widetilde{u}(t)\rangle_{\mathbb{H}} = -2\langle M(\sigma\widetilde{u}(t)), M\widetilde{u}(t)\rangle_{\mathbb{H}}$$

that

$$\begin{aligned}\mathrm{d}\|M\widetilde{u}(t)\|_{\mathbb{H}}^2 &= -2\langle \sigma M\widetilde{u}(t), M\widetilde{u}(t)\rangle_{\mathbb{H}}\mathrm{d}t - 2\Big\langle \begin{pmatrix} 0 & \nabla\sigma\times \\ -\nabla\sigma\times & 0\end{pmatrix}\widetilde{u}(t), M\widetilde{u}(t)\Big\rangle_{\mathbb{H}}\mathrm{d}t \\ &\le -2\sigma_0\|M\widetilde{u}(t)\|_{\mathbb{H}}^2\mathrm{d}t + 4\|M\widetilde{u}(t)\|_{\mathbb{H}}\|\widetilde{u}(t)\|_{\mathbb{H}}\|\nabla\sigma\|_{L^\infty(D)^3}\mathrm{d}t \\ &\le -\sigma_0\|M\widetilde{u}(t)\|_{\mathbb{H}}^2\mathrm{d}t + Ce^{-2\sigma_0 t}\|\widetilde{u}(0)\|_{\mathbb{H}}^2\mathrm{d}t.\end{aligned}$$

By [5, Proposition A.6], we get

$$\|M\widetilde{u}(t)\|_{\mathbb{H}}^2 \le e^{-\sigma_0 t}\|M\widetilde{u}(0)\|_{\mathbb{H}}^2 + C\int_0^t e^{-\sigma_0(t-s)}e^{-2\sigma_0 s}\|\widetilde{u}(0)\|_{\mathbb{H}}^2\mathrm{d}s \le Ce^{-\sigma_0 t}\|\widetilde{u}(0)\|_{\mathcal{D}(M)}^2. \tag{A.6}$$

Combining (A.3) and (A.6) yields

$$\|\widetilde{u}(t)\|_{\mathcal{D}(M)} \le Ce^{-\sigma_0 t/2}\|\widetilde{u}(0)\|_{\mathcal{D}(M)}.$$

Step 2. Estimate of $\|\widetilde{u}(t_n) - \widetilde{u}^n\|_{\mathbb{H}}$.

Let $\widetilde{e}_n := \widetilde{u}(t_n) - \widetilde{u}^n$, then

$$\begin{aligned}\widetilde{e}_{n+1} - e^{-\sigma\Delta t}\widetilde{e}_n &= \big[\widetilde{u}(t_{n+1}) - e^{-\sigma\Delta t}\widetilde{u}(t_n)\big] - \big[\widetilde{u}^{n+1} - e^{-\sigma\Delta t}\widetilde{u}^n\big] \\ &:= \xi^{n+1} + \frac{\Delta t}{2}M\big(\widetilde{e}_{n+1} + e^{-\sigma\Delta t}\widetilde{e}_n\big),\end{aligned} \tag{A.7}$$

where $\xi^{n+1} := \widetilde{u}(t_{n+1}) - e^{-\sigma\Delta t}\widetilde{u}(t_n) - \frac{\Delta t}{2}M\big(\widetilde{u}(t_{n+1}) + e^{-\sigma\Delta t}\widetilde{u}(t_n)\big)$.

Applying $\langle\cdot, \widetilde{e}_{n+1} + e^{-\sigma\Delta t}\widetilde{e}^n\rangle_{\mathbb{H}}$ on both sides of (A.7) yields

$$\|\widetilde{e}_{n+1}\|_{\mathbb{H}}^2 = \|e^{-\sigma\Delta t}\widetilde{e}_n\|_{\mathbb{H}}^2 + \langle\xi^{n+1}, \widetilde{e}_{n+1}\rangle_{\mathbb{H}} + \langle\xi^{n+1}, e^{-\sigma\Delta t}\widetilde{e}_n\rangle_{\mathbb{H}}.$$

Using the fact that $\widetilde{u}(t_{n+1}) = e^{-\sigma\Delta t}\widetilde{u}(t_n) + \int_{t_n}^{t_{n+1}} e^{-\sigma(t_{n+1}-s)}M\widetilde{u}(s)\mathrm{d}s$, we obtain

$$\begin{aligned}\langle\xi^{n+1}, \widetilde{e}_{n+1}\rangle_{\mathbb{H}} &= \frac{1}{2}\int_{t_n}^{t_{n+1}}\int_s^{t_{n+1}}\langle e^{-\sigma(t_{n+1}-r)}M\widetilde{u}(r), M\widetilde{e}_{n+1}\rangle_{\mathbb{H}}\mathrm{d}r\mathrm{d}s - \int_{t_n}^{t_{n+1}}\langle\widetilde{u}(s), R_\sigma^{t_{n+1}-s}\widetilde{e}_{n+1}\rangle_{\mathbb{H}}\mathrm{d}s \\ &\quad - \frac{1}{2}\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\langle e^{-\sigma(t_{n+1}-r)}M\widetilde{u}(r), M\widetilde{e}_{n+1}\rangle_{\mathbb{H}}\mathrm{d}s \\ &\le \frac{1}{2}\int_{t_n}^{t_{n+1}}\int_s^{t_{n+1}}\|e^{-\sigma(t_{n+1}-r)}M\widetilde{u}(r)\|_{\mathbb{H}}\|M\widetilde{e}_{n+1}\|_{\mathbb{H}}\mathrm{d}r\mathrm{d}s \\ &\quad + \frac{1}{2}\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\|e^{-\sigma(t_{n+1}-r)}M\widetilde{u}(r)\|_{\mathbb{H}}\|M\widetilde{e}_{n+1}\|_{\mathbb{H}}\mathrm{d}r\mathrm{d}s\end{aligned}$$

$$+ 2e^{-\sigma_0 \Delta t} \|\nabla\sigma\|_{L^\infty(D)^3} \int_{t_n}^{t_{n+1}} (t_{n+1} - s) \|\widetilde{u}(s)\|_{\mathbb{H}} \|\widetilde{e}_{n+1}\|_{\mathbb{H}} \mathrm{d}s$$
$$\leq Ce^{-\sigma_0 n \Delta t} \Delta t^2 \|\widetilde{u}^0\|^2_{\mathcal{D}(M)}.$$

Similarly, we have $\langle \xi^{n+1}, e^{-\sigma\Delta t}\widetilde{e}_n \rangle_{\mathbb{H}} \leq C\Delta t^2 \|\widetilde{u}^0\|^2_{\mathcal{D}(M)} e^{-\sigma_0 n\Delta t}$. Thus,

$$\|\widetilde{e}_n\|^2_{\mathbb{H}} \leq e^{-2n\Delta t\sigma_0} \|\widetilde{e}_0\|^2_{\mathbb{H}} + C\Delta t^2 e^{-\sigma_0(n-1)\Delta t} \|\widetilde{u}^0\|^2_{\mathcal{D}(M)} \frac{1 - e^{-2\sigma_0 n\Delta t}}{1 - e^{-2\sigma_0\Delta t}} \leq Ce^{-n\Delta t\sigma_0} \Delta t \|\widetilde{u}^0\|^2_{\mathcal{D}(M)},$$

which gives the assertion.

(3) It follows from (A.1) that

$$\|(\hat{S}(t) - I)v\|_{\mathbb{H}} = \|\widetilde{u}(t) - v\|_{\mathbb{H}} \leq \int_0^t \|(M - \sigma I)\widetilde{u}(s)\|_{\mathbb{H}} \mathrm{d}s$$
$$\leq (1 + \|\sigma\|_{L^\infty(D)}) \int_0^t \|\widetilde{u}(s)\|_{\mathcal{D}(M)} \mathrm{d}s \leq Ct\|v\|_{\mathcal{D}(M)} \quad \forall\, v \in \mathcal{D}(M),$$

which leads to the first assertion of (3).

For any $t \in [0, \Delta t]$, we combine (A.1), (A.2) and *Step 1* to get

$$\left\|(e^{t(M-\sigma I)} - e^{-\sigma\Delta t})v\right\|_{\mathbb{H}} = \|\widetilde{u}(t) - \overline{u}(\Delta t)\|_{\mathbb{H}}$$
$$= \left\| \int_0^t M\widetilde{u}(s)\mathrm{d}s + \int_0^t \sigma(\overline{u}(s) - \widetilde{u}(s))\mathrm{d}s + \int_t^{\Delta t} \sigma\overline{u}(s)\mathrm{d}s \right\|_{\mathbb{H}} \leq C\Delta t\|v\|_{\mathcal{D}(M)},$$

which yields the second assertion of (3).

(4) For the proof, see [1, Lemma 3.5, Lemma 5.2].

(5) From (1)–(4), it holds that

$$\|\hat{S}(t_n - r) - (\hat{S}_{\Delta t})^{n-k-1} T_{\Delta t}\|_{\mathcal{L}(\mathcal{D}(M),\mathbb{H})}$$
$$\leq \|\hat{S}(t_n - t_{k+1})(\hat{S}(t_{k+1} - r) - I)\|_{\mathcal{L}(\mathcal{D}(M),\mathbb{H})} + \|\hat{S}(t_n - t_{k+1}) - (\hat{S}_{\Delta t})^{n-k-1}\|_{\mathcal{L}(\mathcal{D}(M),\mathbb{H})}$$
$$+ \|(\hat{S}_{\Delta t})^{n-k-1}(I - T_{\Delta t})\|_{\mathcal{L}(\mathcal{D}(M),\mathbb{H})}$$
$$\leq Ce^{-\sigma_0(n-k-1)\Delta t/2} \Delta t^{1/2}.$$

Thus we finish the proof.

Appendix B. The proof of Lemma 4.4

Proof. (1) For the proof, see [1, Lemma 5.2].
(2) Let $q^{n+1} := (\widehat{S}_{\Delta t})^{n+1}v$ with $q^n := ((q_1^n)^\top, (q_2^n)^\top)^\top$ and $q^0 = v$. Then

$$q^{n+1} - e^{-\sigma_0 \Delta t} q^n = \frac{\Delta t}{2}\big(Mq^{n+1} + M(e^{-\sigma_0 \Delta t} q^n)\big), \tag{B.1}$$

which implies

$$\nabla \cdot q_1^{n+1} = e^{-\sigma_0 \Delta t} \nabla \cdot q_1^n, \qquad \nabla \cdot q_2^{n+1} = e^{-\sigma_0 \Delta t} \nabla \cdot q_2^n.$$

Hence we obtain

$$\|\nabla \cdot q_1^n\|_{L^2(D)}^2 + \|\nabla \cdot q_2^n\|_{L^2(D)}^2 = e^{-2n\sigma_0 \Delta t}\big(\|\nabla \cdot v_1\|_{L^2(D)}^2 + \|\nabla \cdot v_2\|_{L^2(D)}^2\big).$$

Combining (A.5) and using the fact that $f \in H(\mathrm{curl}, D) \cap H(\mathrm{div}, D)$ belongs to $H^1(D)^3$ if $\mathbf{n} \times f|_{\partial D} = 0$ or $\mathbf{n} \cdot f|_{\partial D} = 0$, we have

$$\|q^n\|_{H^1(D)^6} \le C e^{-\sigma_0 n \Delta t/2} \|v\|_{H^1(D)^6}.$$

(3) Let $p_h^n := \big(\widehat{S}_{h,\Delta t}\big)^n \pi_h v$, that is,

$$\begin{cases} p_h^n = \big(I - \frac{\Delta t}{2} M_h\big)^{-1}\big(I + \frac{\Delta t}{2} M_h\big) e^{-\sigma_0 \Delta t} p_h^{n-1} \\ p_h^0 = \pi_h v, \end{cases}$$

which yields

$$p_h^{n+1} - e^{-\sigma_0 \Delta t} p_h^n = \frac{\Delta t}{2}\big(M_h p_h^{n+1} + e^{-\sigma_0 \Delta t} M_h p_h^n\big). \tag{B.2}$$

By applying π_h on both sides of (B.1) and using [1, Proposition 4.4(i)], we obtain

$$\pi_h q^{n+1} - e^{\sigma_0 \Delta t} \pi_h q^n = \frac{\Delta t}{2}\big(M_h q^{n+1} + e^{-\sigma_0 \Delta t} M_h q^n\big). \tag{B.3}$$

Let $e_h^n := p_h^n - \pi_h q^n$ and $e_\pi^n := \pi_h q^n - q^n$. Subtracting (B.3) from (B.2) leads to

$$e_h^{n+1} - e^{-\sigma_0 \Delta t} e_h^n = \frac{\Delta t}{2}\Big[M_h\big(e_h^{n+1} + e^{-\sigma_0 \Delta t} e_h^n\big) + M_h\big(e_\pi^{n+1} + e^{-\sigma_0 \Delta t} e_\pi^n\big)\Big]. \tag{B.4}$$

We apply $\langle \cdot, e_h^{n+1} + e^{-\sigma_0 \Delta t} e_h^n \rangle_{\mathbb{H}}$ on both sides of (B.4) to obtain

$$\begin{aligned} \|e_h^{n+1}\|_{\mathbb{H}}^2 - e^{-2\sigma_0 \Delta t}\|e_h^n\|_{\mathbb{H}}^2 = &\frac{\Delta t}{2}\langle M_h(e_h^{n+1} + e^{-\sigma_0 \Delta t} e_h^n), e_h^{n+1} + e^{-\sigma_0 \Delta t} e_h^n \rangle_{\mathbb{H}} \\ &+ \frac{\Delta t}{2}\langle M_h(e_\pi^{n+1} + e^{-\sigma_0 \Delta t} e_\pi^n), e_h^{n+1} + e^{-\sigma_0 \Delta t} e_h^n \rangle_{\mathbb{H}}. \end{aligned}$$

It follows from [1, Proposition 4.4(iii)] that

$$\begin{aligned}
&\langle M_h(e_\pi^{n+1}+e^{-\sigma_0\Delta t}e_\pi^n), e_h^{n+1}+e^{-\sigma_0\Delta t}e_h^n\rangle_{\mathbb{H}} \\
&=\frac{1}{2}\sum_{F\in\mathcal{G}_h^{\rm int}}\Big(\Big\langle (e_{\pi,\mathbf{H}}^{n+1}+e^{-\sigma_0\Delta t}e_{\pi,\mathbf{H}}^n)|_{K_F}+(e_{\pi,\mathbf{H}}^{n+1}+e^{-\sigma_0\Delta t}e_{\pi,\mathbf{H}}^n)|_K \\
&\qquad -\mathbf{n}_F\times[[e_{\pi,\mathbf{E}}^{n+1}+e^{-\sigma_0\Delta t}e_{\pi,\mathbf{E}}^n]]_F, \mathbf{n}_F\times[[e_{h,\mathbf{E}}^{n+1}+e^{-\sigma_0\Delta t}e_{h,\mathbf{E}}^n]]_F\Big\rangle_{L^2(F)^3}\Big) \\
&\quad -\frac{1}{2}\sum_{F\in\mathcal{G}_h^{\rm int}}\Big(\Big\langle (e_{\pi,\mathbf{E}}^{n+1}+e^{-\sigma_0\Delta t}e_{\pi,\mathbf{E}}^n)|_{K_F}+(e_{\pi,\mathbf{E}}^{n+1}+e^{-\sigma_0\Delta t}e_{\pi,\mathbf{E}}^n)|_K \\
&\qquad +\mathbf{n}_F\times[[e_{\pi,\mathbf{H}}^{n+1}+e^{-\sigma_0\Delta t}e_{\pi,\mathbf{H}}^n]]_F, \mathbf{n}_F\times[[e_{h,\mathbf{H}}^{n+1}+e^{-\sigma_0\Delta t}e_{h,\mathbf{H}}^n]]_F\Big\rangle_{L^2(F)^3}\Big) \\
&\quad -\sum_{F\in\mathcal{G}_h^{\rm ext}}\Big(\langle (e_{\pi,\mathbf{H}}^{n+1}+e^{-\sigma_0\Delta t}e_{\pi,\mathbf{H}}^n), \mathbf{n}_F\times(e_{h,\mathbf{E}}^{n+1}+e^{-\sigma_0\Delta t}e_{h,\mathbf{E}}^n)\rangle_{L^2(F)^3} \\
&\qquad +\langle \mathbf{n}_F\times(e_{\pi,\mathbf{E}}^{n+1}+e^{-\sigma_0\Delta t}e_{\pi,\mathbf{E}}^n), \mathbf{n}_F\times(e_{h,\mathbf{E}}^{n+1}+e^{-\sigma_0\Delta t}e_{h,\mathbf{E}}^n)\rangle_{L^2(F)^3}\Big),
\end{aligned}$$

where $e_\pi^n=((e_{\pi,\mathbf{E}}^n)^\top,(e_{\pi,\mathbf{H}}^n)^\top)^\top$ and $e_h^n=((e_{h,\mathbf{E}}^n)^\top,(e_{h,\mathbf{H}}^n)^\top)^\top$. Then the Cauchy–Schwarz and Young inequalities lead to

$$\begin{aligned}
&\langle M_h(e_\pi^{n+1}+e^{-\sigma_0\Delta t}e_\pi^n), e_h^{n+1}+e^{-\sigma_0\Delta t}e_h^n\rangle_{\mathbb{H}} \\
&\le \frac{1}{2}\sum_{F\in\mathcal{G}_h^{\rm ext}}\|\mathbf{n}_F\times(e_{h,\mathbf{E}}^{n+1}+e^{-\sigma_0\Delta t}e_{h,\mathbf{E}}^n)\|_{L^2(F)^3}^2 \\
&\quad +\frac{1}{4}\sum_{F\in\mathcal{G}_h^{\rm int}}\Big(\|\mathbf{n}_F\times[[e_{h,\mathbf{H}}^{n+1}+e^{-\sigma_0\Delta t}e_{h,\mathbf{H}}^n]]_F\|_{L^2(F)^3}^2+\|\mathbf{n}_F\times[[e_{h,\mathbf{E}}^{n+1}+e^{-\sigma_0\Delta t}e_{h,\mathbf{E}}^n]]_F\|_{L^2(F)^3}^2\Big) \\
&\quad +\frac{1}{4}\sum_{F\in\mathcal{G}_h^{\rm int}}\Big(\|(e_{\pi,\mathbf{H}}^{n+1}+e^{-\sigma_0\Delta t}e_{\pi,\mathbf{H}}^n)|_{K_F}+(e_{\pi,\mathbf{H}}^{n+1}+e^{-\sigma_0\Delta t}e_{\pi,\mathbf{H}}^n)|_K-\mathbf{n}_F \\
&\qquad \times[[e_{\pi,\mathbf{E}}^{n+1}+e^{-\sigma_0\Delta t}e_{\pi,\mathbf{E}}^n]]_F\|_{L^2(F)^3}^2\Big) \\
&\quad +\frac{1}{4}\sum_{F\in\mathcal{G}_h^{\rm int}}\Big(\|(e_{\pi,\mathbf{E}}^{n+1}+e^{-\sigma_0\Delta t}e_{\pi,\mathbf{E}}^n)|_{K_F}+(e_{\pi,\mathbf{E}}^{n+1}+e^{-\sigma_0\Delta t}e_{\pi,\mathbf{E}}^n)|_K+\mathbf{n}_F \\
&\qquad \times[[e_{\pi,\mathbf{H}}^{n+1}+e^{-\sigma_0\Delta t}e_{\pi,\mathbf{H}}^n]]_F\|_{L^2(F)^3}^2\Big) \\
&\quad +\sum_{F\in\mathcal{G}_h^{\rm ext}}\Big(\|e_{\pi,\mathbf{H}}^{n+1}+e^{-\sigma_0\Delta t}e_{\pi,\mathbf{H}}^n\|_{L^2(F)^3}^2+\|\mathbf{n}_F\times(e_{\pi,\mathbf{E}}^{n+1}+e^{-\sigma_0\Delta t}e_{\pi,\mathbf{E}}^n)\|_{L^2(F)^3}^2\Big) \\
&\le -\frac{1}{2}\langle M_h(e_h^{n+1}+e^{-\sigma_0\Delta t}e_h^n), e_h^{n+1}+e^{-\sigma_0\Delta t}e_h^n\rangle_{\mathbb{H}}+Che^{-\sigma_0 n\Delta t}\|v\|_{H^1(D)^6},
\end{aligned}$$

where in the last step we use [1, Proposition 4.4(ii)], (4.4) and the assertion (2). Hence, we have

$$\|e_h^{n+1}\|_{\mathbb{H}}^2 \leq e^{-2\sigma_0\Delta t}\|e_h^n\|_{\mathbb{H}}^2 + Che^{-\sigma_0 n\Delta t}\Delta t\|v\|_{H^1(D)^6}^2 \leq Che^{-\sigma_0 n\Delta t}\|v\|_{H^1(D)^6}^2.$$

(4) Let $p(t_n) := \widehat{S}(t_n)v$. Notice that

$$\begin{aligned}\|p(t_n) - p_h^n\|_{\mathbb{H}} &= \|(\widehat{S}(t_n) - (S_{h,\Delta t})^n\pi_h)v\|_{\mathbb{H}} \\ &\leq \|(\widehat{S}(t_n) - (\widehat{S}_{\Delta t})^n)v\|_{\mathbb{H}} + \|(I-\pi_h)(\widehat{S}_{\Delta t})^n v\|_{\mathbb{H}} \\ &\quad + \|(\pi_h(\widehat{S}_{\Delta t})^n - (\widehat{S}_{h,\Delta t})^n\pi_h)v\|_{\mathbb{H}} \\ &\leq Ce^{-\sigma_0 n\Delta t/2}\big(\Delta t^{1/2} + h^{1/2}\big)\end{aligned}$$

due to Lemma 3.7(2), (4.3) and the assertion (3).
(5) We know that

$$\begin{aligned}\widehat{S}(t_n - r) - (\widehat{S}_{h,\Delta t})^{n-k-1}T_{h,\Delta t} &= \widehat{S}(t_n - t_{k+1})(\widehat{S}(t_{k+1} - r) - I) \\ &\quad + \Big(\widehat{S}(t_n - t_{k+1}) - (\widehat{S}_{h,\Delta t})^{n-k-1}\pi_h\Big) \\ &\quad + (\widehat{S}_{h,\Delta t})^{n-k-1}(\pi_h - T_{h,\Delta t}).\end{aligned}$$

By Lemma 3.7(1)(3), one gets

$$\begin{aligned}&\|\widehat{S}(t_n - t_{k+1})(\widehat{S}(t_{k+1} - r) - I)v\|_{\mathbb{H}} \\ &\leq \|\widehat{S}(t_n - t_{k+1})\|_{\mathcal{L}(\mathbb{H},\mathbb{H})}\|\widehat{S}(t_{k+1} - r) - I\|_{\mathcal{L}(\mathcal{D}(M),\mathbb{H})}\|v\|_{\mathcal{D}(M)} \\ &\leq C\Delta t e^{-\sigma_0(n-k-1)\Delta t}\|v\|_{H^1(D)^6},\end{aligned}$$

which by the assertion (4) leads to

$$\|(\widehat{S}(t_n - t_{k+1}) - (\widehat{S}_{h,\Delta t})^{n-k-1}\pi_h)v\|_{\mathbb{H}} \leq Ce^{-\sigma_0(n-k-1)\Delta t/2}\big(\Delta t^{\frac{1}{2}} + h^{\frac{1}{2}}\big)\|v\|_{H^1(D)^6}.$$

Let $\widetilde{v} := \big(I - (I - \frac{\Delta t}{2}M_h)^{-1}\big)\pi_h v$, namely,

$$\widetilde{v} - \frac{\Delta t}{2}M_h\widetilde{v} = -\frac{\Delta t}{2}M_h\pi_h v. \tag{B.5}$$

We apply $\langle\cdot,\widetilde{v}\rangle_{\mathbb{H}}$ on both sides of (B.5) to obtain

$$\|\widetilde{v}\|_{\mathbb{H}}^2 \leq -\frac{\Delta t}{2}\langle M_h\pi_h v,\widetilde{v}\rangle_{\mathbb{H}} = -\frac{\Delta t}{2}\langle M_h\pi_h v,\pi_h\widetilde{v}\rangle_{\mathbb{H}} \leq \frac{\Delta t}{2}\Big(\frac{1}{2}\|\widetilde{v}\|_{\mathbb{H}}^2 + C\|v\|_{H^1(D)^6}^2\Big)$$

due to [12, Lemma A.4]. For sufficient small Δt, we have

$$\|\widetilde{v}\|_{\mathbb{H}} \leq C\Delta t^{\frac{1}{2}}\|v\|_{H^1(D)^6}.$$

Hence, it holds that

$$\|(\widehat{S}(t_n - r) - (\widehat{S}_{h,\Delta t})^{n-k-1} T_{h,\Delta t})v\|_{\mathbb{H}} \le C e^{-\sigma_0 (n-k-1)\Delta t/2} \big(\Delta t^{\frac{1}{2}} + h^{\frac{1}{2}}\big) \|v\|_{H^1(D)^6}. \quad \square$$

Data availability

No data was used for the research described in the article.

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