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# On numerical discretizations that preserve probabilistic limit behaviors for time-homogeneous Markov processes

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For the general time-homogeneous Markov process, do numerical discretizations exactly preserve its probabilistic limit behaviors, in particular the strong LLN and the CLT? This paper gives a positive answer to the question by proposing a unified and transparent approach to investigating probabilistic limit behaviors of numerical discretizations of time-homogeneous Markov processes. Once the properties of uniformly mixing and convergence for numerical discretizations are satisfied, it is shown that the time-averages of numerical discretizations converge to the ergodic limit in the almost surely sense and that the normalized time-averages converge in distribution to a normal distribution. The limits coincide with the ones for the underlying Markov process. Our results have the merit of the flexible application to numerical discretizations for a large class of stochastic differential equations. Notably, the preservation of these probabilistic limit behaviors of the full discretization of the stochastic Allen–Cahn equation and numerical discretizations of stochastic functional differential equations are obtained for the first time.

*Keywords:* Central limit theorem; Markov process; numerical discretization; probabilistic limit behaviors; strong law of large numbers

## 1. Introduction

It is known that solutions of a large class of stochastic differential equations can be regarded as time-homogeneous Markov processes, for instance, solutions of autonomous stochastic ordinary differential equations (SODEs) and stochastic partial differential equations (SPDEs), and functional solutions of autonomous stochastic functional differential equations (SFDEs). One of the fundamental studies in the Markov process theory focuses on the probabilistic limit behaviors of the time-average $\frac{1}{T}\int_0^T f(X_t)\mathrm{d}t$ as $T \to \infty$ for the Markov process $\{X_t\}_{t\geq 0}$, which is strongly connected with the invariant measure $\mu$ of the process (see e.g. Hong and Wang (2019)). It is well known that for the time-homogeneous Markov process with strong mixing, the time-average $\frac{1}{T}\int_0^T f(X_t)\mathrm{d}t$ converges to the ergodic limit $\mu(f) := \int f\mathrm{d}\mu$, and the normalized fluctuations around $\mu(f)$ can be described by a centered Gaussian random variable, i.e., $\{X_t\}_{t\geq 0}$ fulfills the strong law of large numbers (LLN)

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X_t)\mathrm{d}t = \mu(f) \quad \text{a.s.},$$

and the central limit theorem (CLT)

$$\lim_{T\to\infty}\frac{1}{\sqrt{T}}\int_0^T (f(X_t)-\mu(f))\mathrm{d}t = \mathscr{N}(0,v^2) \quad \text{in distribution}$$

1350-7265 

with some variance $v^2$; see e.g. Komorowski and Walczuk (2012), Kulik (2018), Shirikyan (2006). Such probabilistic limit behaviors have wide applications in statistical mechanics, biochemistry, and the machine learning; see e.g. Duncan, Lelièvre and Pavliotis (2016) and references therein. In many circumstances, it is unavoidable to approximate the ergodic limit by using the time-average $\frac{1}{k}\sum_{i=0}^{k-1} f(Y_i)$ of a numerical discretization $\{Y_i\}_{i\in\mathbb{N}}$ for the Markov process (see e.g. Hong and Wang (2019), Hong, Wang and Zhang (2017), Hong, Sun and Wang (2017)). It is interesting to study the inheritance of the probabilistic limit behaviors in particular the strong LLN and the CLT of the underlying Markov process by numerical discretizations.

For some concrete autonomous stochastic differential equations, e.g., the SODEs and the SPDEs, there have been some works on the study of the strong LLN and the CLT of numerical discretizations. For example, for SODEs with globally Lipschitz continuous coefficients, the strong LLN for Euler-type methods is derived in Bréhier and Vilmart (2016), Mattingly, Stuart and Tretyakov (2010); Pagès and Panloup (2012) obtains the CLT for the Euler–Maruyama (EM) scheme with decreasing step of a wide class of Brownian ergodic diffusions; Lu, Tan and Xu (2022) proves the CLT and the self-normalized Cramér-type moderate deviation of the time-average for the EM method. For SODEs with non-globally Lipschitz continuous coefficients, Jin (2025) establishes the strong LLN and the CLT of the backward EM (BEM) method. For the case of SPDEs, a first attempt is Chen et al. (2023), where the strong LLN and the CLT for approximating the ergodic limit via a full discretization are established for the parabolic SPDE with the linearly growing drift coefficient. More recently, for the stochastic reaction-diffusion equation near sharp interface limit, Cui and Sun (2023) establishes a CLT for a temporal semi-discretization. It is natural to ask the following questions:

(i) For a class of SPDEs with superlinearly growing coefficients, e.g. the stochastic Allen–Cahn equation, can we present the probabilistic limit behaviors of its full discretizations?
(ii) For the SFDE, the solution is no longer Markovian due to the dependence on the history, and hence the ergodicity of the underlying equation is described by the functional solution in the infinite dimensional state space. For this type of equations whose coefficients are highly degenerate, is there a numerical discretization that inherits the probabilistic limit behaviors of the original equation?
(iii) Further, for the general time-homogeneous Markov processes, can we present a unified result on the inheritance of probabilistic limit behaviors by numerical discretizations?

To answer these questions, we focus on the general time-homogeneous Markov processes and study probabilistic limit behaviors of numerical discretizations of the underlying process, which are expected to be applicable to numerical discretizations of a large class of stochastic differential equations, including the stochastic Allen–Cahn equation and the SFDE.

Our results are based on the uniformly mixing property and convergence in the strong or weak sense for numerical discretizations, which are fundamental research topics in the longtime numerical analysis, see e.g. Cui, Hong and Sun (2021), Fang and Giles (2020) and references therein. For the general time-homogeneous Markov processes, we show that once these properties are satisfied, the time-averages of numerical discretizations converge to the ergodic limit in the almost surely sense and that the normalized time-averages converge in distribution to a normal distribution. The limits coincide with the ones for the underlying Markov process. Namely,

$$\text{strong LLN:} \lim_{|\Delta|\to 0}\lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1} f(Y_{t_i}^{x^h,\Delta}) = \mu(f) \quad \text{a.s.,}$$

and

$$\text{CLT: } \lim_{\tau\to 0} \frac{1}{\sqrt{k\tau}} \sum_{i=0}^{k-1} \big(f(Y_{t_i}^{x^h,\Delta}) - \mu(f)\big)\tau = \mathscr{N}(0, v^2) \quad \text{in distribution,}$$

where parameters $k$ and $\Delta := (h, \tau)^\top$ satisfy some relation in the CLT, and the limit variance $v^2 = 2\mu((f - \mu(f)) \int_0^\infty (P_t f - \mu(f))\mathrm{d}t)$; see Theorems 3.1 and 3.2 for details. Based on these results, the computational cost of the time-average of numerical discretizations is obtained in Remark 3 for approximating the ergodic limit. We would like to mention that in the strong LLN and the CLT, the regularity assumption on the test functional $f$ is moderate, which is only of polynomial growth and has certain weighted Hölder continuity.

As expected, our results are well applicable to numerical discretizations of a wide variety of stochastic differential equations, which is supported by discussions of SODEs, SPDEs, and SFDEs. Here we stress the main merit of our result by showing the application to the stochastic Allen–Cahn equation and the SFDE. For the stochastic Allen–Cahn equation whose drift coefficient grows superlinearly, we show that the strong LLN and the CLT of the full discretization hold for test functionals with lower regularity, which has not been reported before to our knowledge. In addition, for the SFDE, we obtain the strong LLN and the CLT of the numerical discretization for the first time.

For the proof of the numerical CLT, a prerequisite is to construct a suitable discrete martingale term such that it contains the essential contribution for the convergence of the normalized time-average of the numerical discretization. For the case of the concrete stochastic differential equation, the discrete martingale term is extracted by means of the associated Poisson equation of the original equation. By contrast, a similar analysis of the Poisson equation for the general time-homogeneous Markov process can be technically challenging. This paper introduces a different approach to constructing a new discrete martingale term from the normalized time-average by fully utilizing the Markov property and the uniformly mixing property for numerical discretizations of the time-homogeneous Markov process. It is worth mentioning that the approach for this construction is beneficial to effectively lowering the regularity of test functionals. With the detailed analysis, we show that the martingale term converges to a normal distribution with the variance being the same as that of the underlying Markov process and the remainder converges to zero in probability.

The paper is organized as follows. In the next section, some preliminaries are introduced, and the probabilistic limit behaviors including the strong LLN and the CLT of the time-homogeneous Markov processes are presented. Section 3 presents our main results, including the strong LLN and the CLT of numerical discretizations as well as the applications to stochastic differential equations. Section 4 and Section 5 are devoted to presenting proofs of the strong LLN and the CLT of numerical discretizations, respectively, based on the convergence of the numerical invariant measure and the decomposition of the normalized time-average of numerical discretizations.

## 2. Preliminaries

In this section, we give some preliminaries for the study of the Banach space-valued time-homogeneous Markov process $\{X_t^x\}_{t\geq 0}$. Throughout this article, we use $K$ to denote a generic positive constant independent of the initial datum and the step-size $\Delta$, which may take different values at different appearances. Denote by $K(a, b)$ specific constant which depends on parameters $a, b$. Let $\lfloor a \rfloor$ denote the integer part of a real number $a$ and $\lceil a \rceil$ denote the minimal integer greater than or equal to $a$. Let $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$ for real numbers $a$ and $b$. Denote by $\xrightarrow{d}$, $\xrightarrow{\mathbb{P}}$, and $\xrightarrow{a.s.}$ the convergence in distribution, in probability, and in the almost sure sense, respectively.

## 2.1. Settings

Let $(E, \|\cdot\|)$ be a real-valued separable Banach space and $\mathfrak{B}(E)$ be the Borel $\sigma$-algebra. Denote by $\mathcal{P}(E)$ the family of probability measures on $(E, \mathfrak{B}(E))$ and by $\nu(f) := \int_E f(u)\mathrm{d}\nu(u)$ the integral of the measurable functional $f$ with respect to $\nu \in \mathcal{P}(E)$. For fixed positive constants $p \geq 1$ and $\gamma \in (0,1]$, define the quasi-metric $d_{p,\gamma}$ on $E$ by

$$d_{p,\gamma}(u_1, u_2) := (1 \wedge \|u_1 - u_2\|^\gamma)(1 + \|u_1\|^p + \|u_2\|^p)^{\frac{1}{2}}. \tag{1}$$

Then we introduce a test functional space related to this quasi-metric. Define $C_{p,\gamma} := C_{p,\gamma}(E;\mathbb{R})$ the set of continuous functionals on $E$ endowed with the following norm

$$\|f\|_{p,\gamma} := \sup_{u\in E} \frac{|f(u)|}{1 + \|u\|^{\frac{p}{2}}} + \sup_{\substack{u_1,u_2\in E\\ u_1\neq u_2}} \frac{|f(u_1) - f(u_2)|}{d_{p,\gamma}(u_1, u_2)}. \tag{2}$$

It can be seen from (2) that the test functional $f$ is of polynomial growth and has certain weighted Hölder continuity. It is worthwhile pointing out that $C_{p_1,\gamma} \subset C_{p_2,\gamma}$ for $p_2 \geq p_1 \geq 1$ and $C_{p,\gamma_2} \subset C_{p,\gamma_1}$ for $0 < \gamma_1 \leq \gamma_2 \leq 1$, see e.g. Shirikyan (2006) for a more general form of the class of spaces. Noting that the Lipschitz continuous functionals belong to $C_{2,1}$, we call the test functional in $C_{p,\gamma}$ is of low regularity. The Wasserstein quasi-distance induced by the quasi-metric is defined by

$$\mathbb{W}_{p,\gamma}(\nu_1, \nu_2) := \inf_{\pi\in\Pi(\nu_1,\nu_2)} \int_{E\times E} d_{p,\gamma}(u_1, u_2)\pi(\mathrm{d}u_1, \mathrm{d}u_2), \tag{3}$$

where $\nu_1, \nu_2 \in \mathcal{P}_{p,\gamma}(E) := \{\nu \in \mathcal{P}(E) : \int_{E\times E} d_{p,\gamma}(u_1, u_2)\mathrm{d}\nu(u_1)\mathrm{d}\nu(u_2) < \infty\}$ and $\Pi(\nu_1, \nu_2)$ denotes the collection of all probability measures on $E \times E$ with marginal measures $\nu_1$ and $\nu_2$.

Note that when $\nu_1(\|\cdot\|^p) < \infty$ and $\nu_2(\|\cdot\|^p) < \infty$,

$$\begin{aligned}
&\mathbb{W}_{p,\gamma}(\nu_1, \nu_2)\\
\leq&\big(1 + \nu_1(\|\cdot\|^p) + \nu_2(\|\cdot\|^p)\big)^{\frac{1}{2}} \Big(\inf_{\pi\in\Pi(\nu_1,\nu_2)} \int_{E\times E} 1 \wedge \|u_1 - u_2\|^2 \pi(\mathrm{d}u_1, \mathrm{d}u_2)\Big)^{\frac{\gamma}{2}}\\
=&\big(1 + \nu_1(\|\cdot\|^p) + \nu_2(\|\cdot\|^p)\big)^{\frac{1}{2}} (\mathbb{W}_2(\nu_1, \nu_2))^\gamma,
\end{aligned}$$

where (1) and the Hölder inequality are used, and $\mathbb{W}_p$ is the bounded-Wasserstein distance defined by

$$\mathbb{W}_p(\nu_1, \nu_2) := \Big(\inf_{\pi\in\Pi(\nu_1,\nu_2)} \int_{E\times E} 1 \wedge \|u_1 - u_2\|^p \pi(\mathrm{d}u_1, \mathrm{d}u_2)\Big)^{\frac{1}{p}}.$$

This leads to that for any $f \in C_{p,\gamma}$,

$$\begin{aligned}
|\nu_1(f) - \nu_2(f)| &= \inf_{\pi\in\Pi(\nu_1,\nu_2)} \Big|\int_{E\times E} (f(u_1) - f(u_2))\pi(\mathrm{d}u_1, \mathrm{d}u_2)\Big|\\
&\leq \|f\|_{p,\gamma}\mathbb{W}_{p,\gamma}(\nu_1, \nu_2) \leq \|f\|_{p,\gamma}\big(1 + \nu_1(\|\cdot\|^p) + \nu_2(\|\cdot\|^p)\big)^{\frac{1}{2}} (\mathbb{W}_2(\nu_1, \nu_2))^\gamma.
\end{aligned} \tag{4}$$

Let $\{X_t^x\}_{t\geq 0}$ be the $E$-valued time-homogeneous Markov process with the deterministic initial value $X_0^x = x \in E$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, in which the corresponding expectation is denoted by $\mathbb{E}$. The temporal semi-discretization is considered for the finite or infinite dimensional

**Table 1.** Definitions of the uniform step-size $\Delta$ and the state space $E^h$

| | temporal semi-discretization for $h=0$ | spatial-temporal full discretization for $h \neq 0$ |
|---|---|---|
| step-size $\Delta$ | $\Delta := (0,\tau)^\top$ | $\Delta := (h,\tau)^\top$ |
| state space $E^h$ | $E^0 := E$ | $E^h (\subset E)$ with norm $\|\cdot\|_h$ satisfying $\|u^h\|_h \le K\|u^h\|$ for $u^h \in E^h$ |

case and the spatial-temporal full discretization is considered for the infinite dimensional case. Let $h \in [0,1]$ and $\tau \in (0,1]$ be step-sizes in spatial and temporal directions, respectively. To unify the notation, we define the uniform step-size $\Delta$ and the state space $E^h$ for the temporal semi-discretization or spatial-temporal full discretization; see Table 1.

When the initial datum $x \in E^h$, we always take $x^h = x$. Denote $t_k := k\tau$ for $k \in \mathbb{N}$. For each given $\Delta$, let $\{Y_{t_k}^{x^h,\Delta}\}_{k\in\mathbb{N}}$ denote the $E^h$-valued discretization of $\{X_t^x\}_{t\ge 0}$ satisfying $Y_0^{x^h,\Delta} = x^h$, and define the time-homogeneous Markov process on the whole time horizon by

$$Y_t^{x^h,\Delta} = \sum_{k=0}^{\infty} Y_{t_k}^{x^h,\Delta} \mathbf{1}_{[t_k,t_{k+1})}(t), \quad t > 0.$$

Denote by $\mu_t^x$ and $\mu_t^{x^h,\Delta}$ the probability measures generated by $X_t^x$ and $Y_t^{x^h,\Delta}$, respectively, i.e., for any $A \in \mathfrak{B}(E)$, $\mu_t^x(A) = \mathbb{P}\{\omega \in \Omega : X_t^x \in A\}$, $\mu_t^{x^h,\Delta}(A) = \mathbb{P}\{\omega \in \Omega : Y_t^{x^h,\Delta} \in A\}$. Denote by $\mathcal{B}_b := \mathcal{B}_b(E;\mathbb{R})$ (resp. $C_b := C_b(E;\mathbb{R})$) the family of bounded Borel measurable (resp. bounded and continuous) functions on $E$. When there is no confusion, we also denote $\mathcal{B}_b(E^h;\mathbb{R}), C_b(E^h;\mathbb{R}), C_{p,\gamma}(E^h;\mathbb{R})$ by $\mathcal{B}_b$, $C_b$, $C_{p,\gamma}$, respectively.

Define the linear operators $P_t$ and $P_t^\Delta$ generated by $X_t^\cdot$ and $Y_t^{\cdot,\Delta}$, respectively, as

$$P_t : \mathcal{B}_b \to \mathcal{B}_b, \quad P_t f(x) := \mathbb{E}\big[f(X_t^x)\big] = \int_E f(u)\mathrm{d}\mu_t^x(u) \quad \forall\, x \in E,$$

$$P_t^\Delta : \mathcal{B}_b \to \mathcal{B}_b, \quad P_t^\Delta f(x^h) := \mathbb{E}\big[f(Y_t^{x^h,\Delta})\big] = \int_E f(u)\mathrm{d}\mu_t^{x^h,\Delta}(u) \quad \forall\, x \in E^h.$$

According to the Markov property of $X_t^x$, one deduces that $P_t$ is a Markov semigroup. If $X_t^x$ (resp. $Y_t^{x^h,\Delta}$) admits a unique invariant measure $\mu$ (resp. $\mu^\Delta$) satisfying $\mu(\|\cdot\|^l) < \infty$ (resp. $\mu^\Delta(\|\cdot\|^l) < \infty$) for some constant $l > 0$, it follows from (Da Prato, 2006, Theorem 5.8) that $P_t$ (resp. $P_t^\Delta$) is uniquely extendible to a bounded linear operator on $L^l(E,\mu)$ which is still denoted by $P_t$ (resp. $P_t^\Delta$). Here, denote by $L^l(E,\mu)$ the family of Borel measurable functions $f : E \to \mathbb{R}$ such that $\int_E |f(u)|^l \mathrm{d}\mu(u) < \infty$. In addition, for the space $C_{p,\gamma}$ with some $p \ge 1$ and $\gamma \in (0,1]$, if there are functions $\Phi : E \to [0,+\infty)$ and $\bar{\rho} : [0,+\infty) \to [0,+\infty)$ satisfying $\int_0^\infty \bar{\rho}(t)\mathrm{d}t < \infty$ such that for any $f \in C_{p,\gamma}$,

$$|P_t f(x) - \mu(f)| \le K\|f\|_{p,\gamma}\Phi(x)\bar{\rho}(t) \quad \forall\, x \in E,\, t \ge 0,$$

then the Markov process $\{X_t^x\}_{t\ge 0}$ is said to be *uniformly mixing* for the space $C_{p,\gamma}$; see (Shirikyan, 2006, Definition 2.5). This property characterizes that the process "forgets" its initial states as time goes on.

## 2.2. Probabilistic limit behaviors of time-homogeneous Markov processes

We first introduce the assumption for the time-homogeneous Markov process to ensure that the process admits a unique invariant measure and that the probabilistic limit behaviors including the strong LLN and the CLT hold.

**Assumption 1.** Assume that the time-homogeneous Markov process $\{X_t^x\}_{t\geq 0}$ satisfies the following conditions.

(i) There exist constants $r \geq 2, \tilde{r} \geq 1$, and $L_1 > 0$ such that for any $x \in E$,

$$\sup_{t\geq 0} \mathbb{E}[\|X_t^x\|^r] \leq L_1(1+\|x\|^{\tilde{r}r}).$$

(ii) There exist constants $\gamma_1 \in (0,1]$, $\beta \in [0, r-1]$, $L_2 > 0$, and a continuous function $\rho : [0,+\infty) \to [0,+\infty)$ with $\int_0^\infty \rho^{\gamma_1}(t)\mathrm{d}t < \infty$ such that for any $x, y \in E$,

$$\big(\mathbb{E}[\|X_t^x - X_t^y\|^2]\big)^{\frac{1}{2}} \leq L_2\|x-y\|(1+\|x\|^\beta + \|y\|^\beta)\rho(t).$$

The probabilistic limit behaviors including the LLN and the CLT of Markov processes have been well-studied in the monograph Kulik (2018). For our convenience, we list a modified version of (Kulik, 2018, Proposition 5.3.5) in the following proposition, which can be proved in a similar manner.

**Proposition 2.1.** *Let Assumption 1 hold. Then $\{X_t^x\}_{t\geq 0}$ admits a unique invariant measure $\mu \in \mathcal{P}(E)$. Moreover, if $\gamma \in [\gamma_1, 1]$ and $p$ satisfy $2\tilde{r}\big(p\tilde{r} + (2+3\beta)\gamma\big) \leq r$, then the strong LLN and the CLT hold: for any $f \in C_{p,\gamma}$,*

$$\frac{1}{T}\int_0^T f(X_t^x)\mathrm{d}t \xrightarrow{a.s.} \mu(f) \quad \text{as } T \to \infty,$$

$$\frac{1}{\sqrt{T}}\int_0^T (f(X_t^x) - \mu(f))\mathrm{d}t \xrightarrow{d} \mathcal{N}(0, v^2) \quad \text{as } T \to \infty,$$

*where $v^2 := 2\mu\big((f-\mu(f))\int_0^\infty (P_t f - \mu(f))\mathrm{d}t\big)$.*

We remark that by Assumption 1, the Markov process $\{X_t^x\}_{t\geq 0}$ is uniformly mixing for the space $C_{p,\gamma}$ with $\gamma \in [\gamma_1, 1]$ and $p\tilde{r} + 2(1+\beta)\gamma \leq r$, which follows from

$$\begin{aligned} |P_t f(x) - \mu(f)| = |\mu_t^x(f) - \mu(f)| &\leq K\|f\|_{p,\gamma}(1+\mathbb{E}[\|X_t^x\|^p] + \mu(\|\cdot\|^p))^{\frac{1}{2}}\big(\mathbb{W}_2(\mu_t^x, \mu)\big)^\gamma \\ &\leq K\|f\|_{p,\gamma}(1+\|x\|^{\frac{p\tilde{r}}{2}+(1+\beta)\gamma})\rho^\gamma(t), \end{aligned} \tag{5}$$

due to (4). The function $\rho$ characterizes the decay speed in time, ensuring that the influence on the initial state diminishes rapidly over long time.

# 3. Main results and applications

In this section, we first present the main results of this paper, namely, the preservation of probabilistic limit behaviors, in particular the strong LLN and the CLT for time-homogeneous Markov processes by numerical discretizations. Then we apply the main results to numerical discretizations of a large class of autonomous stochastic differential equations, including SODEs, SPDEs, and SFDEs.

## 3.1. Main results

We first propose sufficient conditions on numerical discretizations to ensure the preservation of the strong LLN and the CLT of time-homogeneous Markov processes. These conditions are related to moment boundedness, contractivity, and strong convergence of numerical discretizations.

**Assumption 2.** Assume that there exists $\tilde{\Delta} = (\tilde{h}, \tilde{\tau})^\top \in [0,1] \times (0,1]$ such that for any $h \in [0, \tilde{h}]$, $\tau \in (0, \tilde{\tau}]$, and $x^h \in E^h$, $\{Y_{t_k}^{x^h,\Delta}\}_{k\in\mathbb{N}}$ is time-homogeneous Markovian and satisfies the following conditions.

(i) There exist constants $q \geq 2, \tilde{q} \geq 1$, and $L_3 > 0$ such that

$$\sup_{k\geq 0} \mathbb{E}[\|Y_{t_k}^{x^h,\Delta}\|^q] \leq L_3(1 + \|x^h\|^{\tilde{q}q}).$$

(ii) There exist constants $\gamma_2 \in (0,1]$, $\kappa \in [0, q-1], L_4 > 0$, and a function $\rho^\Delta : [0,+\infty) \to [0,+\infty)$ with

$$\sup_{\{h\in[0,\tilde{h}],\tau\in(0,\tilde{\tau}]\}} \tau \sum_{k=0}^{\infty} \big(\rho^\Delta(t_k)\big)^{\gamma_2} < \infty$$

such that for any $x^h, y^h \in E^h$,

$$\big(\mathbb{E}[\|Y_{t_k}^{x^h,\Delta} - Y_{t_k}^{y^h,\Delta}\|^2]\big)^{\frac{1}{2}} \leq L_4\|x^h - y^h\|(1 + \|x^h\|^\kappa + \|y^h\|^\kappa)\rho^\Delta(t_k).$$

Below, when there is no confusion, we always assume that $h \in [0, \tilde{h}]$ and $\tau \in (0, \tilde{\tau}]$. And constants $r$ and $q$ in Assumptions 1 (i) and 2 (i), respectively, are always assumed to be large enough to meet the need.

**Assumption 3.** Assume that there exist $\alpha = (\alpha_1, \alpha_2)^\top \in \mathbb{R}_+^2$ and $L_5 > 0$ such that for any $x \in E$,

$$\sup_{t\geq 0} \big(\mathbb{E}[\|X_t^x - Y_t^{x^h,\Delta}\|^2]\big)^{\frac{1}{2}} \leq L_5(1 + \|x\|^{\tilde{r}\vee\tilde{q}})|\Delta^\alpha|,$$

where $\Delta^\alpha := (h^{\alpha_1}, \tau^{\alpha_2})^\top$ and $|\Delta^\alpha| := h^{\alpha_1} + \tau^{\alpha_2}$.

Denote the time-average of the numerical discretization by

$$\frac{1}{k} S_k^{x^h,\Delta} := \frac{1}{k} \sum_{i=0}^{k-1} f(Y_{t_i}^{x^h,\Delta})$$

for $f \in C_{p,\gamma}$ with suitable parameters $p \geq 1$ and $\gamma \in (0,1]$. Then we obtain the strong LLN of the time-average of numerical discretizations.

**Theorem 3.1.** *Let Assumption 1 hold for the time-homogeneous Markov process $\{X_t^x\}_{t\geq 0}$, Assumption 2 hold for the time-homogeneous Markov process $\{Y_{t_k}^{x^h,\Delta}\}_{k\in\mathbb{N}}$, Assumption 3 hold for $\{X_t^x\}_{t\geq 0}$ and $\{Y_{t_k}^{x^h,\Delta}\}_{k\in\mathbb{N}}$, $\gamma \in [\gamma_2, 1]$, and $p \geq 1$ with $p(1+\tilde{q}) + 2(1+\kappa)\gamma \leq q$. Then for any $f \in C_{p,\gamma}$, we have*

$$\lim_{|\Delta|\to 0} \lim_{k\to\infty} \frac{1}{k} S_k^{x^h,\Delta} = \mu(f) \quad a.s. \tag{6}$$

The following CLT characterizes the limit behavior of $\frac{1}{\sqrt{k\tau}}\sum_{i=0}^{k-1}\big(f(Y_{t_i}^{x^h,\Delta})-\mu(f)\big)\tau$, which is called the normalized time-average of the numerical discretization. Here, $f\in C_{p,\gamma}$ with suitable parameters $p\geq 1$ and $\gamma\in(0,1]$. Recall that $v^2=2\mu\big((f-\mu(f))\int_0^\infty(P_tf-\mu(f))\mathrm{d}t\big)$ is given in Proposition 2.1.

**Theorem 3.2.** *Let Assumption 1 hold for the time-homogeneous Markov process $\{X_t^x\}_{t\geq 0}$, Assumption 2 hold for the time-homogeneous Markov process $\{Y_{t_k}^{x^h,\Delta}\}_{k\in\mathbb{N}}$, Assumption 3 hold for $\{X_t^x\}_{t\geq 0}$ and $\{Y_{t_k}^{x^h,\Delta}\}_{k\in\mathbb{N}}$. For any fixed $\lambda\in(0,\alpha_2\gamma)$ with $\gamma\in[\gamma_1\vee\gamma_2,1]$, set $k:=\lceil\tau^{-1-2\lambda}\rceil$ and $h:=\tau^{\alpha_2/\alpha_1}$ when $h\neq 0$. Let $p\geq 1$ satisfy*

$$\tilde{q}^2\Big(3\vee(\frac{1}{\lambda}+1)\Big)\Big(p(\tilde{q}\vee\tilde{r})+\big(3+4(\kappa\vee\beta)\big)\gamma\Big)\leq q\wedge r, \tag{7}$$

*and $\varepsilon\ll\lambda$ with $\sqrt{\varepsilon}-\frac{\varepsilon}{\lambda}-\frac{\varepsilon\sqrt{\varepsilon}}{\lambda}>0$ satisfy*

$$\lim_{\tau\to 0}\tau^{\varepsilon}\sum_{i=\lceil\tau^{-1-\frac{\varepsilon}{4}}\rceil}^{k}(\rho^{\Delta}(t_i))^{\gamma}=0. \tag{8}$$

*Then for any $f\in C_{p,\gamma}$,*

$$\frac{1}{\sqrt{k\tau}}\sum_{i=0}^{k-1}\big(f(Y_{t_i}^{x^h,\Delta})-\mu(f)\big)\tau\xrightarrow{d}\mathscr{N}(0,v^2)\quad \text{as } \tau\to 0. \tag{9}$$

**Remark 1.** (i) If Assumption 1 (i) is replaced by: "for any $r\geq 2$, there exist $\tilde{r}\geq 1$ and $L_1>0$ such that $\sup_{t\geq 0}\mathbb{E}[\|X_t^x\|^r]\leq L_1(1+\|x\|^{\tilde{r}r})$", and Assumption 2 (i) is replaced by: "for any $q\geq 2$, there exist $\tilde{q}\geq 1$ and $L_3>0$ such that $\sup_{k\geq 0}\mathbb{E}[\|Y_{t_k}^{x^h,\Delta}\|^q]\leq L_3(1+\|x^h\|^{\tilde{q}q})$", then the restriction (7) on $p$ is not necessary.

(ii) When $P_{t_k}^{\Delta}$ is exponential mixing, i.e., $\rho^{\Delta}(t_k)=e^{-ct_k}$ with some $c>0$, then (8) is satisfied naturally.

**Remark 2.** Our main results in Theorems 3.1 and 3.2 are proved with a strong convergence condition but for test functionals with lower regularity, i.e., $f\in C_{p,\gamma}$. In general, strong convergence orders are no larger than the weak ones. The numerical LLN and CLT still hold with certain trade-off between the convergence condition of numerical discretizations (i.e., Assumption 3) and the regularity of test functionals. Precisely, if Assumption 3 is replaced by:

$$\sup_{t\geq 0}|\mathbb{E}[f(X_t^x)-f(Y_t^{x^h,\Delta})]|\leq K|\Delta^{\alpha}| \tag{10}$$

for test functionals belonging to some space $\mathfrak{C}$, then Theorems 3.1 and 3.2 still hold for $f\in C_{p,\gamma}\cap\mathfrak{C}$.

The main difference in proofs of this remark and Theorem 3.2 is the estimation of terms $|P_tf(u)-P_t^{\Delta}f(u)|$ and $|\mu(f)-\mu^{\Delta}(f)|$, where $u\in E^h$, and $\mu^{\Delta}$ is the invariant measure of the numerical discretization (see Proposition 4.1). Under the assumption (10), we have that for $u\in E^h$,

$$|P_tf(u)-P_t^{\Delta}f(u)|\leq\sup_{t\geq 0}|\mathbb{E}[f(X_t^u)-f(Y_t^{u,\Delta})]|\leq K|\Delta^{\alpha}|.$$

Hence, by (5), we have that $|\mu(f)-\mu^{\Delta}(f)|$ can be estimated as

$$|\mu(f)-\mu^{\Delta}(f)|\leq|P_{t_k}f(u)-\mu(f)|+|P^{\Delta}_{t_k}f(u)-\mu^{\Delta}(f)|+|P_{t_k}f(u)-P^{\Delta}_{t_k}f(u)|\leq K|\Delta^{\alpha}|,$$

where we used (23), and in the last step we let $k\to\infty$.

**Remark 3.** Based on the LLN and CLT results of numerical discretizations, we provide an analysis on the computational cost of the time-average of numerical discretizations for approximating the ergodic limit to achieve a given accuracy $\epsilon$. For a significance level $\theta\in(0,1)$, let the Gaussian confidence interval $[-\mathcal{I}_\theta,\mathcal{I}_\theta]$ be such that $\mathbb{P}\{|Z|\leq\mathcal{I}_\theta\}=1-\theta$, where $Z$ is a standard Gaussian random variable. Then by the numerical CLT, an asymptotic confidence interval with level $1-\theta$, centered around $\mu(f)$, is $[\mu(f)-\frac{\mathcal{I}_\theta v}{\sqrt{k\tau}},\mu(f)+\frac{\mathcal{I}_\theta v}{\sqrt{k\tau}}]$. This, along with the relation $k=\lceil\tau^{-1-2\lambda}\rceil$ implies that for a given accuracy $\epsilon$, when $\tau$ is set to be the size of $(\frac{\epsilon}{\mathcal{I}_\theta v})^{\frac{1}{\lambda}}$, one arrives at $\mathbb{P}\{|\frac{1}{k}S_k^{x^h,\Delta}-\mu(f)|\leq\epsilon\}=1-\theta$. Hence, for any $\lambda\in(0,\alpha_2\gamma)$, the computational cost of $\frac{1}{k}S_k^{x^h,\Delta}$ is estimated as $h^{-1}k=O(\epsilon^{-\frac{1}{\lambda}(\frac{\alpha_2}{\alpha_1}+1)-2})$ (resp. $k=O(\epsilon^{-\frac{1}{\lambda}-2})$) for $h\neq0$ (resp. $h=0$), where $\alpha=(\alpha_1,\alpha_2)^\top$ is the weak convergence order given in Remark 2 of the numerical discretization.

## 3.2. Applications

The main results in Theorems 3.1 and 3.2 can be applied to numerical discretizations of a large class of autonomous stochastic differential equations. Below, we present the applications to the cases of SODEs, SPDEs, and SFDEs. Denote by $C_b^k:=C_b^k(E;\mathbb{R})$ (resp. $\widetilde{C}_b^k:=\widetilde{C}_b^k(E;\mathbb{R})$) the bounded continuous functions (resp. continuous functions) that are continuously differentiable with bounded derivatives up to order $k$.

### *3.2.1. SODEs*

In this part, we demonstrate the application of our results to the SODE with non-globally Lipschitz continuous drift term, as well as its BEM method. The equation has the following form

$$\mathrm{d}X_t^x=b(X_t^x)\mathrm{d}t+\sigma(X_t^x)\mathrm{d}W(t),\quad X_0^x=x\in\mathbb{R}^d, \tag{11}$$

where $\{W(t),t\geq0\}$ is a $D$-dimensional standard Brownian motion on a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq0},\mathbb{P})$. We suppose that coefficients $b:\mathbb{R}^d\to\mathbb{R}^d$ and $\sigma:\mathbb{R}^d\to\mathbb{R}^{d\times D}$ satisfy the following assumptions.

**(O1).** *Assume that there exists a positive constant* $\tilde{L}_1$ *such that for any* $u_1,u_2\in\mathbb{R}^d$, $\langle u_1-u_2,b(u_1)-b(u_2)\rangle\leq-\tilde{L}_1|u_1-u_2|^2$.

**(O2).** *Assume that there exist positive constants* $\tilde{L}_2,\tilde{L}_3$ *with* $2\tilde{L}_1>\tilde{L}_2$ *such that* $|\sigma(u_1)-\sigma(u_2)|\leq\tilde{L}_2|u_1-u_2|$ *and* $|\sigma(u_1)|\leq\tilde{L}_3$ *for any* $u_1,u_2\in\mathbb{R}^d$.

**(O3).** *Assume that there exist constants* $\tilde{L}_4>0$ *and* $\tilde{\ell}\geq0$ *such that for any* $u_1,u_2\in\mathbb{R}^d$, $|b(u_1)-b(u_2)|\leq\tilde{L}_4(1+|u_1|^{\tilde{\ell}}+|u_2|^{\tilde{\ell}})|u_1-u_2|$.

**(O1)** and **(O3)** refer to the dissipative condition and the locally Lipschitz condition of the coefficient $b$, respectively. It follows from **(O3)** that $b$ is superlinearly growing when $\tilde{\ell}>0$. **(O2)** is the globally Lipschitz condition and the boundedness condition of the coefficient $\sigma$. It is shown in (Jin, 2025, Propositions 2.2) that under **(O1)** and **(O2)**, the exact solution $\{X_t^x\}_{t\geq0}$ of (11) satisfies Assumption 1, in which parameters are taken as $\tilde{r}=1$, $r\geq2$ being arbitrary, $\beta=0$, $\rho(t)=e^{-ct}$ for some $c>0$, and

$\gamma_1 \in (0,1]$ being arbitrary. Thus by virtue of Proposition 2.1, the exact solution $\{X_t^x\}_{t\geq 0}$ of (11) admits a unique invariant measure $\mu$ and fulfills the strong LLN and the CLT.

Noticing that $b$ may be superlinearly growing, we focus on the following BEM method

$$Y_{t_{i+1}}^{x,\Delta} = Y_{t_i}^{x,\Delta} + b(Y_{t_{i+1}}^{x,\Delta})\tau + \sigma(Y_{t_i}^{x,\Delta})\delta W_i, \quad \delta W_i = W(t_{i+1}) - W(t_i), \tag{12}$$

and investigate the strong LLN and the CLT of the corresponding numerical solution.

**Theorem 3.3.** *Let **(O1)**–**(O3)** hold. Then for any $p \geq 1$, $\gamma \in (0,1]$, and $f \in C_{p,\gamma}$, the numerical discretization $\{Y_{t_k}^{x,\Delta}\}_{k\in\mathbb{N}}$ of BEM method (12) satisfies (6). Furthermore, for any fixed $\lambda \in (0, \frac{1}{2}\gamma)$, set $k = \lceil \tau^{-1-2\lambda} \rceil$, then $\{Y_{t_k}^{x,\Delta}\}_{k\in\mathbb{N}}$ satisfies (9).*

**Proof.** Based on Theorems 3.1 and 3.2, it suffices to verify that the numerical discretization $\{Y_{t_k}^{x,\Delta}\}_{k\in\mathbb{N}}$ of the BEM method (12) satisfies Assumption 2, $\{X_t^x\}_{t\geq 0}$ and $\{Y_{t_k}^{x,\Delta}\}_{k\in\mathbb{N}}$ satisfy Assumption 3, and (7)–(8) hold.

*Verification of Assumption 2.* According to (Jin, 2025, Theorem 4.1) and (Liu, Mao and Wu, 2023, Lemma 4.2), we have that for sufficiently small $\Delta \in (0,1]$, Assumption 2 is fulfilled with $\tilde{q} = 1$, any $q \geq 2$, $\kappa = 0$, $\rho^\Delta(t) = e^{-ct}$ for some $c > 0$, and any $\gamma_2 \in (0,1]$.

*Verification of Assumption 3.* It can be found in (Liu, Mao and Wu, 2023, Theorem 4.2) that for sufficiently small $\Delta \in (0,1]$, Assumption 3 is fulfilled with $|\Delta^\alpha| = \tau^{\frac{1}{2}}$.

*Verification of (7) and (8).* It is straightforward to deduce from Remark 1 that (7) and (8) hold.

Thus we complete the proof. □

As we stated in Remark 2, when the weak convergence of the BEM method (12) is used, Theorems 3.1 and 3.2 still hold for some class of test functionals. To simplify the presentation, we use the weak error result given in Chen, Hong and Lu (2023), where the time-independent weak error analysis of the BEM method for the SODE with piecewise continuous arguments is investigated. Denote $\mathcal{D}^{(i)}$ by the Fréchet derivative operator up to order $i$. Assume in addition that derivatives of coefficients $b, \sigma$ satisfy the polynomial growth conditions.

**(O4).** *Assume that coefficients $b, \sigma$ have continuous derivatives up to order 3, and there exist positive constants $\tilde{L}_5, \tilde{\ell}_1$ such that for any $u_1, u_2, v_1, v_2 \in \mathbb{R}^d$,*

$$\begin{aligned}
&\left|\mathcal{D}^{(1)}b(u_1)v_1 - \mathcal{D}^{(1)}b(u_2)v_1\right| \vee \left|\mathcal{D}^{(1)}\sigma(u_1)v_1 - \mathcal{D}^{(1)}\sigma(u_2)v_1\right| \\
\leq\ & \tilde{L}_5(1 + |u_1|^{\tilde{\ell}_1} + |u_2|^{\tilde{\ell}_1})|u_1 - u_2||v_1|, \\
&\left|\mathcal{D}^{(2)}b(u_1)(v_1,v_2) - \mathcal{D}^{(2)}b(u_2)(v_1,v_2)\right| \vee \left|\mathcal{D}^{(2)}\sigma(u_1)(v_1,v_2) - \mathcal{D}^{(2)}\sigma(u_2)(v_1,v_2)\right| \\
\leq\ & \tilde{L}_5(1 + |u_1|^{(\tilde{\ell}_1-1)\vee 0} + |u_2|^{(\tilde{\ell}_1-1)\vee 0})|u_1 - u_2||v_1||v_2|.
\end{aligned}$$

By means of (Chen, Hong and Lu, 2023, Theorem 4.8) for the case without memory, under **(O1)**–**(O4)**, the numerical discretization of the BEM method (12) satisfies $\sup_{t\geq 0}|\mathbb{E}[f(X_t^x) - f(Y_t^{x,\Delta})]| \leq K\tau$ for $f \in C_b^3$. Owing to $C_b^3 \subset C_{2,1}$ and making use of Remark 2, we derive the following corollary.

**Corollary 3.4.** *Let **(O1)**–**(O4)** hold. For any fixed $\lambda \in (0,1)$, set $k = \lceil \tau^{-1-2\lambda} \rceil$. Then for any $f \in C_b^3$, the numerical solution $\{Y_{t_k}^{x,\Delta}\}_{k\in\mathbb{N}}$ of (12) satisfies (9).*

Compared with Theorem 3.3, the upper bound of the parameter $\lambda$ is improved from $\frac{1}{2}\gamma$ to 1 in the result of Corollary 3.4, while the class of test functionals changes from $C_{p,\gamma}$ to $C_b^3$. Note that the solution of the Poisson equation $\mathcal{L}\varphi = f - \mu(f)$ is $\varphi = -\int_0^\infty (P_t f - \mu(f))\mathrm{d}t$, where $\mathcal{L}$ is the generator of (11). When $f \in C_b^3$, properties of the Poisson equation are well-studied, see e.g. Mattingly, Stuart and Tretyakov (2010). It follows from $\varphi\mathcal{L}\varphi = \frac{1}{2}\mathcal{L}\varphi^2 - \frac{1}{2}\|\sigma^\top\nabla\varphi\|^2$ and $\mu(\mathcal{L}\varphi^2) = 0$ that $v^2 = -2\mu(\varphi\mathcal{L}\varphi) = \mu(\|\sigma^\top\nabla\varphi\|^2)$, which is consistent with that of (Jin, 2025, Theorems 3.2 and 3.4).

### *3.2.2. SPDEs*

In this part, we consider the probabilistic limit behaviors for numerical discretizations of SPDEs, by applying the numerical LLN and CLT results to the globally Lipschitz drift term case and the non-globally Lipschitz case. Consider the following SPDE on $E := L^2((0,1);\mathbb{R})$

$$\mathrm{d}X_t^x = AX_t^x\mathrm{d}t + F(X_t^x)\mathrm{d}t + \mathrm{d}W(t), \quad X_0^x = x, \tag{13}$$

where $A : \mathrm{Dom}(A) \subset E \to E$ is the Dirichlet Laplacian with homogeneous Dirichlet boundary conditions, and $\{W(t), t \geq 0\}$ is a generalized $Q$-Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual regularity condition $\|(-A)^{\frac{\beta_1-1}{2}}Q^{\frac{1}{2}}\|_{\mathcal{L}_2} < \infty$ for $\beta_1 \in (0,1]$ with $\|\cdot\|_{\mathcal{L}_2}$ being the Hibert–Schmidt operator norm. Here, $Q$ is a bounded, linear, self-adjoint, and non-negative definite operator, which commutes with $A$. It is known that there is a sequence of increasing real numbers $\{\lambda_i\}_{i\in\mathbb{N}_+}$ and an orthonormal basis $\{e_i\}_{i\in\mathbb{N}_+}$ such that $-Ae_i = \lambda_i e_i$.

I. **Globally Lipschitz continuous case for $F$.** We suppose that $F$ satisfies the following globally Lipschitz condition and the dissipative condition.

**(P1).** *Assume that for any $X, Y \in E$, $\|F(X) - F(Y)\| \leq K\|X - Y\|$ for some constant $K > 0$.*

**(P2).** *Assume that there exists a constant $\lambda_F$ with $\lambda_F < \lambda_1$ such that $\langle X - Y, F(X) - F(Y)\rangle \leq \lambda_F\|X - Y\|^2$ for any $X, Y \in E$, and that $F$ is Gâteaux differentiable with the Gâteaux derivative satisfying $\sup_{Y\in E}\|DF(Y)X\| \leq L_F\|X\|$, $L_F > 0$.*

Under **(P1)–(P2)**, by (Chen et al., 2023, Proposition 2.1) and Proposition 2.1, the mild solution $\{X_t^x\}_{t\geq 0}$ of (13) admits a unique invariant measure $\mu$ and fulfills the strong LLN and the CLT.

Consider the full discretization $\{Y_{t_k}^{x^h,\Delta}\}_{k\in\mathbb{N}}$, whose spatial direction is based on the spectral Galerkin method and temporal direction is the exponential Euler method,

$$Y_{t_{k+1}}^{x^h,\Delta} = S^h(\tau)(Y_{t_k}^{x^h,\Delta} + P^hF(Y_{t_k}^{x^h,\Delta})\tau + \delta W_k^h), \quad Y_0^{x^h,\Delta} = x^h, \tag{14}$$

where $h := \frac{1}{N}$ with $N \in \mathbb{N}_+$, $P^h$ is the spectral Galerkin projection, $\delta W_k^h = P^h(W(t_{k+1}) - W(t_k))$, and $S^h(\tau) := e^{A^h\tau}$ with $A^h := P^hA$. For this numerical discretization, by (Chen et al., 2023, Propositions 2.3–2.5), and Theorems 3.1 and 3.2, we can obtain the following result, whose proof is similar to that of Theorem 3.3 and thus is omitted.

**Theorem 3.5.** *Let **(P1)–(P2)** hold. Then for any $p \geq 1$, $\gamma \in (0,1]$, and $f \in C_{p,\gamma}$, the numerical discretization $\{Y_{t_k}^{x,\Delta}\}_{k\in\mathbb{N}}$ of (14) satisfies (6). Furthermore, for any fixed $\lambda \in (0, \frac{\beta_1}{2}\gamma)$, set $k = \lceil\tau^{-1-2\lambda}\rceil$ and $h = \tau^{\frac{1}{2}}$, then $\{Y_{t_k}^{x,\Delta}\}_{k\in\mathbb{N}}$ satisfies (9).*

II. **Non-globally Lipschitz continuous case for $F$.** In this part, we consider (13) with the super-linear coefficient $F$, for instance the stochastic Allen–Cahn equation (i.e., $F$ in (13) has the form

$F(X)(\zeta) = X(\zeta) - X(\zeta)^3, \zeta \in (0,1)$). By the use of our main results, probabilistic limit behaviors including the strong LLN and the CLT of the full discretization of the stochastic Allen–Cahn equation can be obtained. To this end, we propose the following dissipative condition on $F$.

**(P3).** *Let $\mathfrak{g}$ be a cubic polynomial with $\mathfrak{g}(\xi) = -a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0$, $\xi \in \mathbb{R}$, $a_i \in \mathbb{R}$, $i = 0, 1, 2$, and $a_3 > 0$. Assume that $F : L^6((0,1);\mathbb{R}) \to E$ is the Nemytskii operator defined by $F(X)(\zeta) = \mathfrak{g}(X(\zeta))$, and that $\lambda_F := \sup_{\xi\in\mathbb{R}} \mathfrak{g}'(\xi) < \lambda_1$.*

Similarly to the proof of (Cui, Hong and Sun, 2021, Lemma 2), under **(P3)**, we can obtain that the mild solution $\{X_t^x\}_{t\ge 0}$ of (13) fulfills $\sup_{t\ge0} \mathbb{E}[\|X_t^x\|^r] \le K(1 + \|x\|^{\tilde{r}r})$ for any $r \ge 2$ and some fixed constant $\tilde{r}$. And it follows from the Itô formula that for any $p \ge 2$, $\mathbb{E}[\|X_t^{x_1} - X_t^{x_2}\|^p] \le \mathbb{E}[\|x_1 - x_2\|^p]e^{-p(\lambda_1-\lambda_F)t}$. Thus by Proposition 2.1, under **(P3)**, the mild solution $\{X_t^x\}_{t\ge0}$ of (13) admits a unique invariant measure $\mu$ and fulfills the strong LLN and the CLT for $f \in C_{p,\gamma}$ with any $p \ge 1, \gamma \in (0,1]$.

For the numerical method considered in Cui, Hong and Sun (2021), i.e., the full discretization with spectral Galerkin method in space and BEM method in time,

$$Y_{t_{k+1}}^{x^h,\Delta} = Y_{t_k}^{x^h,\Delta} + A^h Y_{t_{k+1}}^{x^h,\Delta}\tau + P^h F(Y_{t_{k+1}}^{x^h,\Delta})\tau + \delta W_k^h, \quad Y_0^{x^h,\Delta} = x^h, \tag{15}$$

we can also derive $\mathbb{E}[\|Y_{t_{k+1}}^{x_1^h,\Delta} - Y_{t_{k+1}}^{x_2^h,\Delta}\|^p] \le K\mathbb{E}[\|x_1 - x_2\|^p]e^{-p(\lambda_1-\lambda_F)t_{k+1}}$. By part 2 of the supplementary material (Chen et al. (2025)), we have the time-independent strong convergence of the full discretization:

$$\sup_{t\ge0} \|X_t^x - Y_t^{x^h,\Delta}\|_{L^2(\Omega;E)} \le \Upsilon(\|x\|_{C((0,1);\mathbb{R})}, \|x\|_{\dot{E}^{\beta_1}})(\tau^{\frac{\beta_1}{2}} + h^{\beta_1}), \tag{16}$$

where $\Upsilon(a,b)$ is a polynomial with respect to variables $a$ and $b$, and $\dot{E}^{\beta_1}$ is the Sobolev space generated by the fractional power of $-A$ (see e.g. (Kruse, 2014, Section B.2)). We also remark that in the proof of the numerical strong LLN and CLT, Assumption 3 is used to derive Proposition 4.1 and $\lim_{|\Delta|\to0} \frac{\mu^\Delta(H^\Delta)}{\tau} = v^2$ in Proposition 5.3. If we replace Assumption 3 by $\sup_{t\ge0} \left(\mathbb{E}[\|X_t^x - Y_t^{x^h,\Delta}\|^2]\right)^{\frac{1}{2}} \le L_5(1 + \|x\|_{\tilde{E}}^{\tilde{r}\vee\tilde{q}})|\Delta^\alpha|$ with $\sup_{t\ge0}\mathbb{E}[\|X_t^x\|_{\tilde{E}}^{2\tilde{r}\vee2\tilde{q}}] < \infty$, then the numerical strong LLN and CLT still hold, where $\tilde{E}(\subset E)$ is some Banach space. Noting that from part 2 of the supplementary material (Chen et al. (2025)), one has $\sup_{t\ge0}\mathbb{E}[\|X_t^x\|_{C((0,1);\mathbb{R})}^r] + \sup_{t\ge0}\mathbb{E}[\|X_t^x\|_{\dot{E}^{\beta_1}}^r] \le K(x)$ with any $r \ge 2$. Hence, together with (Cui, Hong and Sun, 2021, Lemma 4), and Theorems 3.1 and 3.2, we obtain the following result.

**Theorem 3.6.** *If conditions **(P1)**–**(P2)** are replaced by **(P3)**, the result of Theorem 3.5 still holds.*

By using the weak error of the numerical discretization (15) from (Cui, Hong and Sun, 2021, Theorem 2), we derive the following result.

**Corollary 3.7.** *Let **(P3)** hold. For any fixed $\lambda \in (0, \beta_1)$, set $k = \lceil \tau^{-1-2\lambda} \rceil$. Then for any $f \in \widetilde{C}_b^2$, the numerical discretization $\{Y_{t_k}^{x,\Delta}\}_{k\in\mathbb{N}}$ of (15) satisfies (9).*

### *3.2.3. SFDEs*

The probabilistic limit behaviors of the functional solution of SFDEs have been studied in the literature; see e.g. Bao, Wang and Yuan (2020) and references therein. However, to the best of our knowledge, there is no result on the preservation of probabilistic limit behaviors of numerical discretizations for

SFDEs. Based on the application of our main results, we focus on the probabilistic limit behaviors for the EM method of the SFDE. Let $\delta_0 > 0$ be the delay. Denote by $C([-\delta_0, 0];\ \mathbb{R}^d)$ the space of all continuous functions $\phi(\cdot)$ from $[-\delta_0, 0]$ to $\mathbb{R}^d$ equipped with the norm $\|\phi\| = \sup_{-\delta_0 \le \theta \le 0} |\phi(\theta)|$.

The SFDE on $E = C([-\delta_0, 0]; \mathbb{R}^d)$ has the following form

$$\mathrm{d}X^x(t) = b(X_t^x)\mathrm{d}t + \sigma(X_t^x)\mathrm{d}W(t), \quad X_0^x = x \in E, \tag{17}$$

where $\{W(t),\ t \ge 0\}$ is a $D$-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$, and coefficients $b : E \to \mathbb{R}^d$ and $\sigma : E \to \mathbb{R}^{d\times D}$ are measurable functions. We impose the globally Lipschitz condition on $\sigma$, the dissipative condition and the globally Lipschitz condition on $b$, and the Hölder continuity condition on the initial datum.

**(F1).** *Assume that there are positive constants $\bar{L}_1, \bar{L}_2$, and the probability measure $\nu_1$ on $[-\delta_0, 0]$ such that $|\sigma(\phi_1) - \sigma(\phi_2)|^2 \le \bar{L}_1\big(|\phi_1(0) - \phi_2(0)|^2 + \int_{-\delta_0}^0 |\phi_1(\theta) - \phi_2(\theta)|^2 \mathrm{d}\nu_1(\theta)\big)$ and $|\sigma(\phi_1)| \le \bar{L}_2$ for $\phi_1, \phi_2 \in E$.*

**(F2).** *Assume that there are positive constants $\bar{L}_3, \bar{L}_4, \bar{L}_5$ with $\bar{L}_3 > \bar{L}_1 + \bar{L}_4$ and the probability measure $\nu_2$ on $[-\delta_0, 0]$ such that for any $\phi_1, \phi_2 \in E$,*

$$\big\langle \phi_1(0) - \phi_2(0), b(\phi_1) - b(\phi_2)\big\rangle \le -\bar{L}_3|\phi_1(0) - \phi_2(0)|^2 + \bar{L}_4 \int_{-\delta_0}^0 |\phi_1(\theta) - \phi_2(\theta)|^2 \mathrm{d}\nu_2(\theta),$$

$$|b(\phi_1) - b(\phi_2)|^2 \le \bar{L}_5\Big(|\phi_1(0) - \phi_2(0)|^2 + \int_{-\delta_0}^0 |\phi_1(\theta) - \phi_2(\theta)|^2 \mathrm{d}\nu_2(\theta)\Big).$$

**(F3).** *Assume that there are positive constants $\bar{L}_6$ and $\bar{\ell} \ge 1/2$ such that $|x(\theta_1) - x(\theta_2)| \le \bar{L}_6|\theta_1 - \theta_2|^{\bar{\ell}} \quad \forall\, \theta_1, \theta_2 \in [-\delta_0, 0]$.*

The functional solution $X_t^x : \theta \mapsto X^x(t+\theta)$ of (17) is the $E$-valued random variable for $t \ge 0$. Analogous calculations to ones carried out in proofs of (Bao, Wang and Yuan, 2020, Lemma 3.1) and (Bao, Shao and Yuan, 2023, Lemma 3.1) yield that, under **(F1)** and **(F2)**, the functional solution $\{X_t^x\}_{t\ge0}$ satisfies Assumption 1 with $\tilde{r} = 1$, any $r \ge 2$, $\beta = 0$, $\rho(t) = e^{-ct}$ for some $c > 0$, and any $\gamma_1 \in (0, 1]$. Thus, by Proposition 2.1, the exact functional solution $\{X_t^x\}_{t\ge0}$ admits a unique invariant measure $\mu$ and fulfills the strong LLN and the CLT.

Without loss of generality, we assume that there exists an integer $N \ge \delta_0$ sufficiently large such that $\tau = \frac{\delta_0}{N} \in (0, 1]$. Let $t_k = k\tau$ for $k = -N, -N+1, \dots$ The $E$-valued numerical discretization $\{Y_{t_k}^{x^h,\Delta}\}_{k\in\mathbb{N}}$ based on the EM method associated with (17) is defined by: for $\theta \in [t_j, t_{j+1}],\ j \in \{-N, \dots, -1\}$,

$$Y_{t_k}^{x^h,\Delta}(\theta) = \frac{t_{j+1} - \theta}{\tau} y^{x,\tau}(t_{k+j}) + \frac{\theta - t_j}{\tau} y^{x,\tau}(t_{k+j+1}), \tag{18}$$

where $x^h$ is the linear interpolation of $x(t_{-N}), x(t_{-N+1}), \dots, x(0)$,

$$y^{x,\tau}(t_{k+1}) = y^{x,\tau}(t_k) + b(Y_{t_k}^{x^h,\Delta})\tau + \sigma(Y_{t_k}^{x^h,\Delta})\delta W_k, \quad k \in \mathbb{N}, \tag{19}$$

and $y^{x,\tau}(\theta) = x(\theta)$ for $\theta \in [-\delta_0, 0]$. For the numerical discretization $\{Y_{t_k}^{x^h,\Delta}\}_{k\in\mathbb{N}}$, using the similar technique as the one in the proof of (Bao, Shao and Yuan, 2023, Lemma 4.2), by **(F1)** and **(F2)**, we deduce that for sufficiently small $\tau \in (0, 1]$, Assumption 2 (ii) is satisfied with $\kappa = 0$, $\rho^\Delta(t) = e^{-ct}$ for some $c > 0$, and any $\gamma_2 \in (0, 1]$. Moreover, one has from **(F1)–(F3)** that for any $q \ge 2$,

$$\sup_{k\ge0} \mathbb{E}[\|Y_{t_k}^{x^h,\Delta}\|^q] \le K(1 + \|x\|^q), \tag{20}$$

$$\sup_{t\geq 0}\mathbb{E}[\|X_t^x - Y_t^{x^h,\Delta}\|^2] \leq K(1+\|x\|^2)\tau; \tag{21}$$

see part 3 of the supplementary material (Chen et al. (2025)) for the proofs. Hence, utilizing Theorems 3.1 and 3.2, the strong LLN and the CLT of the numerical discretization (18) are stated as follows.

**Theorem 3.8.** *Let (F1)–(F3) hold. Then for any $p \geq 1$, $\gamma \in (0,1]$, and $f \in C_{p,\gamma}$, the numerical discretization (18) satisfies (6). Furthermore, for any fixed $\lambda \in (0, \frac{1}{2}\gamma)$, set $k = \lceil \tau^{-1-2\lambda} \rceil$, then $\{Y_{t_k}^{x,\Delta}\}_{k\in\mathbb{N}}$ satisfies (9).*

## 4. Proof of Theorem 3.1

This section is devoted to giving the proof of the preservation of the strong LLN of numerical discretizations, based on the existence and uniqueness as well as the convergence of the numerical invariant measure.

**Proposition 4.1.** *Let Assumption 2 hold for the time-homogeneous Markov process $\{Y_{t_k}^{x^h,\Delta}\}_{k\in\mathbb{N}}$. Then for each fixed $\Delta$, $\{Y_{t_k}^{x^h,\Delta}\}_{k\in\mathbb{N}}$ admits a unique numerical invariant measure $\mu^\Delta \in \mathcal{P}(E^h)$. Furthermore, if in addition Assumption 1 holds for the time-homogeneous Markov process $\{X_t^x\}_{t\geq 0}$ and Assumption 3 holds for $\{X_t^x\}_{t\geq 0}$ and $\{Y_{t_k}^{x^h,\Delta}\}_{k\in\mathbb{N}}$, then we have $\mathbb{W}_2(\mu,\mu^\Delta) \leq K|\Delta^\alpha|$, where $|\Delta^\alpha|$ is given in Assumption 3.*

Note that Assumption 2 leads to

$$\mu^\Delta(\|\cdot\|^q) \leq \lim_{t\to\infty}\int_{E^h}\|u\|^q \mathrm{d}\mu_t^{\mathbf{0},\Delta}(u) \leq \sup_{t\geq 0}\mathbb{E}[\|Y_t^{\mathbf{0},\Delta}\|^q] \leq L_3. \tag{22}$$

Similar to the proof of (5), we derive that for $f \in C_{p,\gamma}$ with $p \in [1,q]$ and $\gamma \in (0,1]$,

$$|P_{t_k}^\Delta f(u) - \mu^\Delta(f)| \leq K\|f\|_{p,\gamma}(1+\|u\|^{\frac{p\tilde{q}}{2}+(1+\kappa)\gamma})(\rho^\Delta(t_k))^\gamma, \quad k\in\mathbb{N}. \tag{23}$$

This is the uniformly mixing property of numerical discretizations. Based on (Shirikyan, 2006, Proposition 2.6 and Remark 2.7), we obtain the estimate between $\frac{1}{k}S_k^{x^h,\Delta}$ and $\mu^\Delta(f)$.

**Lemma 4.2.** *Let Assumption 2 hold for the time-homogeneous Markov process $\{Y_{t_k}^{x^h,\Delta}\}_{k\in\mathbb{N}}$, $\gamma \in [\gamma_2, 1]$, and $\varrho \in \mathbb{N}$, $p \geq 1$ satisfy $\varrho\big(\frac{p}{2}(1+\tilde{q}) + (1+\kappa)\gamma\big) \leq q$. Then for any $f \in C_{p,\gamma}$, $x^h \in E^h$, we have*

$$\mathbb{E}\Big[\Big|\frac{1}{k}S_k^{x^h,\Delta} - \mu^\Delta(f)\Big|^{2\varrho}\Big] \leq K\|f\|_{p,\gamma}^{2\varrho}(1+\|x^h\|^{\varrho\tilde{q}(\frac{p}{2}(1+\tilde{q})+(1+\kappa)\gamma)})t_k^{-\varrho} \quad \forall\, k\in\mathbb{N}_+. \tag{24}$$

With these in hand, we present the proof of the strong LLN, i.e., $\frac{1}{k}S_k^{x^h,\Delta} \xrightarrow{a.s.} \mu(f)$ as $k\to\infty$ and $|\Delta| \to 0$.

**Proof of Theorem 3.1.** Applying (4) and Proposition 4.1 leads to

$$|\mu(f) - \mu^\Delta(f)| \leq K\|f\|_{p,\gamma}|\Delta^\alpha|^\gamma, \quad f\in C_{p,\gamma}. \tag{25}$$

Hence it suffices to prove that $\lim_{k\to\infty}\big|\frac{1}{k}S_k^{x^h,\Delta}-\mu^\Delta(f)\big|=0$ a.s. for each fixed step-size $\Delta$ and $f\in C_{p,\gamma}$.

For any $\delta\in(0,\frac{1}{4})$ and $k\in\mathbb{N}$ with $k\ge\lceil 1/\tau\rceil$, define $\mathcal{A}_k^{x^h,\Delta}:=\big\{\omega\in\Omega:\big|\frac{1}{k}S_k^{x^h,\Delta}-\mu^\Delta(f)\big|(\omega)>\|f\|_{p,\gamma}t_k^{-\delta}\tau^{-\frac{1}{4}}\big\}$. Note that the condition $p(1+\tilde{q})+2(1+\kappa)\gamma\le q$ coincides with the one in Lemma 4.2 with $\varrho=2$. By virtue of (24) with $\varrho=2$ one has $\mathbb{E}\big[\big|\frac{1}{k}S_k^{x^h,\Delta}-\mu^\Delta(f)\big|^4\big]\le K\|f\|_{p,\gamma}^4(1+\|x^h\|^{\tilde{q}(p(1+\tilde{q})+2(1+\kappa)\gamma)})t_k^{-2}$. Applying the Chebyshev inequality yields

$$\mathbb{P}(\mathcal{A}_k^{x^h,\Delta})\le\frac{t_k^{4\delta}\tau}{\|f\|_{p,\gamma}^4}\mathbb{E}\Big[\big|\frac{1}{k}S_k^{x^h,\Delta}-\mu^\Delta(f)\big|^4\Big]\le K(1+\|x^h\|^{\tilde{q}(p(1+\tilde{q})+2(1+\kappa)\gamma)})t_k^{2(2\delta-1)}\tau. \tag{26}$$

Then it follows from $2(1-2\delta)>1$ that

$$\sum_{k=\lceil\frac{1}{\tau}\rceil}^{\infty}\mathbb{P}(\mathcal{A}_k^{x^h,\Delta})\le K(1+\|x^h\|^{\tilde{q}(p(1+\tilde{q})+2(1+\kappa)\gamma)}). \tag{27}$$

Define the random variable $\mathcal{K}^{x^h,\Delta}$ by

$$\mathcal{K}^{x^h,\Delta}(\omega):=\inf\big\{j\in\mathbb{N}\text{ with }j\ge\lceil\frac{1}{\tau}\rceil:\big|\frac{1}{k}S_k^{x^h,\Delta}-\mu^\Delta(f)\big|(\omega)\le\|f\|_{p,\gamma}t_k^{-\delta}\tau^{-\frac{1}{4}}\ \forall k\ge j+1\big\}.$$

Inequality (27), along with the Borel–Cantelli lemma implies that $\mathcal{K}^{x^h,\Delta}<\infty$ a.s. and

$$\mathbb{P}\Big\{\omega\in\Omega:\big|\frac{1}{k}S_k^{x^h,\Delta}-\mu^\Delta(f)\big|(\omega)\le\|f\|_{p,\gamma}t_k^{-\delta}\tau^{-\frac{1}{4}}\text{ for all }k\ge\mathcal{K}^{x^h,\Delta}(\omega)+1\Big\}=1. \tag{28}$$

Let $T^{x^h,\Delta}=\tau\mathcal{K}^{x^h,\Delta}$. Noting $2(2\delta-1)<-1$, there exists a constant $l>0$ (e.g., $l=-\frac{1+2(2\delta-1)}{2}>0$) such that $l+2(2\delta-1)<-1$. This, combining (26) implies

$$\begin{aligned}\mathbb{E}[(T^{x^h,\Delta})^l]&=\sum_{j=\lceil\frac{1}{\tau}\rceil}^{\infty}\mathbb{E}\big[(T^{x^h,\Delta})^l\mathbf{1}_{\{\mathcal{K}^{x^h,\Delta}=j\}}\big]=\sum_{j=\lceil\frac{1}{\tau}\rceil}^{\infty}(j\tau)^l\mathbb{P}(\mathcal{K}^{x^h,\Delta}=j)\\&\le\sum_{j=\lceil\frac{1}{\tau}\rceil}^{\infty}(t_j)^l\mathbb{P}(\mathcal{A}_j^{x^h,\Delta})\le K(1+\|x^h\|^{\tilde{q}(p(1+\tilde{q})+2(1+\kappa)\gamma)}).\end{aligned} \tag{29}$$

For $\tilde{\delta}>0$ and $N\in\mathbb{N}_+$, define $\mathcal{D}_N^{x^h}:=\big\{\omega\in\Omega:T^{x^h,\Delta}+1>N^{\tilde{\delta}}\big\}$. When $\tilde{\delta}$ is chosen so that $\tilde{\delta}l>1$, it follows from (29) that $\sum_{N=1}^{\infty}\mathbb{P}(\mathcal{D}_N^{x^h})\le\sum_{N=1}^{\infty}(\frac{1}{N})^{\tilde{\delta}l}\mathbb{E}[(T^{x^h,\Delta}+1)^l]\le K(1+\|x^h\|^{\tilde{q}(p(1+\tilde{q})+2(1+\kappa)\gamma)})$. Using the Borel–Cantelli lemma again yields that there exists a random variable $\mathfrak{N}^{x^h}<\infty$ a.s. such that

$$\mathbb{P}\Big\{\omega\in\Omega:T^{x^h,\Delta}(\omega)+1\le N^{\tilde{\delta}}\text{ for all }N\ge\mathfrak{N}^{x^h}(\omega)\Big\}=1. \tag{30}$$

Combining (28) and (30) yields that for a.s. $\omega\in\Omega$, when $k\ge\frac{(\mathfrak{N}^{x^h}(\omega))^{\tilde{\delta}}}{\tau^{1+\frac{1}{4\delta}}}\ge\frac{T^{x^h,\Delta}(\omega)+1}{\tau}\ge\mathcal{K}^{x^h,\Delta}(\omega)+1$, we have

$$\big|\frac{1}{k}S_k^{x^h,\Delta}-\mu^\Delta(f)\big|(\omega)\le\|f\|_{p,\gamma}t_k^{-\delta}\tau^{-\frac{1}{4}}\le\|f\|_{p,\gamma}(\mathfrak{N}^{x^h}(\omega))^{-\tilde{\delta}\delta},$$

which implies $\lim_{k\to\infty}\frac{1}{k}S_k^{x^h,\Delta}=\mu^\Delta(f)$ a.s. for fixed $\Delta$. This finishes the proof. $\square$

# 5. Proof of Theorem 3.2

In this section, we present the proof of the preservation of the CLT of numerical discretizations, based on the decomposition of the normalized time-average of numerical discretizations. Precisely, we introduce the following decomposition of the normalized time-average $\frac{1}{\sqrt{k\tau}}\sum_{i=0}^{k-1}\big(f(Y_{t_i}^{x^h,\Delta})-\mu(f)\big)\tau$,

$$\frac{1}{\sqrt{k\tau}}\sum_{i=0}^{k-1}\big(f(Y_{t_i}^{x^h,\Delta})-\mu(f)\big)\tau=\frac{1}{\sqrt{k\tau}}\mathscr{M}_k^{x^h,\Delta}+\frac{1}{\sqrt{k\tau}}\mathscr{R}_k^{x^h,\Delta}, \tag{31}$$

where

$$\mathscr{M}_k^{x^h,\Delta}:=\tau\sum_{i=0}^{\infty}\Big(\mathbb{E}\big[f(Y_{t_i}^{x^h,\Delta})|\mathcal{F}_{t_k}\big]-\mu^\Delta(f)\Big)-\tau\sum_{i=0}^{\infty}\Big(\mathbb{E}\big[f(Y_{t_i}^{x^h,\Delta})|\mathcal{F}_0\big]-\mu^\Delta(f)\Big), \tag{32}$$

$$\mathscr{R}_k^{x^h,\Delta}:=-\tau\sum_{i=k}^{\infty}\Big(\mathbb{E}\big[f(Y_{t_i}^{x^h,\Delta})|\mathcal{F}_{t_k}\big]-\mu^\Delta(f)\Big)+\tau\sum_{i=0}^{\infty}\Big(\mathbb{E}\big[f(Y_{t_i}^{x^h,\Delta})|\mathcal{F}_0\big]-\mu^\Delta(f)\Big)+k\tau(\mu^\Delta(f)-\mu(f)).$$

By virtue of the Slutsky theorem, the idea for proving Theorem 3.2 is to prove that $\frac{1}{\sqrt{k\tau}}\mathscr{M}_k^{x^h,\Delta}\xrightarrow{d}\mathcal{N}(0,v^2)$ and $\frac{1}{\sqrt{k\tau}}\mathscr{R}_k^{x^h,\Delta}\xrightarrow{\mathbb{P}}0$.

By fully utilizing the Markov property and the uniformly mixing property of numerical discretizations of the time-homogeneous Markov process, it is shown in the following proposition that $\mathscr{M}_k^{x^h,\Delta}$ is well-defined and is a martingale for $k\in\mathbb{N}$.

**Proposition 5.1.** *Let Assumption 2 hold for the time-homogeneous Markov process $\{Y_{t_k}^{x^h,\Delta}\}_{k\in\mathbb{N}}$, $\gamma\in[\gamma_2,1]$, and $p\geq 1$ satisfy $\frac{p\tilde{q}}{2}+(1+\kappa)\gamma\leq q$. Then for any $x^h\in E^h$, $f\in C_{p,\gamma}$, the sequence $\{\mathscr{M}_k^{x^h,\Delta}\}_{k\in\mathbb{N}}$ is an $\{\mathcal{F}_{t_k}\}_{k\in\mathbb{N}}$-adapted martingale with $\mathscr{M}_0^{x^h,\Delta}=0$. Moreover, we have*

$$|\mathscr{M}_k^{x^h,\Delta}|\leq\tau\Big|\sum_{i=0}^{k-1}\big(f(Y_{t_i}^{x^h,\Delta})-\mu^\Delta(f)\big)\Big|+K\|f\|_{p,\gamma}(1+\|Y_{t_k}^{x^h,\Delta}\|^{\frac{p\tilde{q}}{2}+(1+\kappa)\gamma}+\|x^h\|^{\frac{p\tilde{q}}{2}+(1+\kappa)\gamma}). \tag{33}$$

Define the martingale difference sequence of $\{\mathscr{M}_k^{x^h,\Delta}\}_{k\in\mathbb{N}}$ by

$$\mathcal{Z}_k^{x^h,\Delta}:=\mathscr{M}_k^{x^h,\Delta}-\mathscr{M}_{k-1}^{x^h,\Delta}\quad\forall\,k\in\mathbb{N}_+,\quad\mathcal{Z}_0^{x^h,\Delta}=0.$$

Denote by $\mathbb{E}\big[|\mathcal{Z}_k^{x^h,\Delta}|^2|\mathcal{F}_{t_{k-1}}\big]$ the conditional variance of the martingale difference sequence, whose properties are presented in following propositions.

**Proposition 5.2.** *Let Assumption 2 hold for the time-homogeneous Markov process $\{Y_{t_k}^{x^h,\Delta}\}_{k\in\mathbb{N}}$ and $\gamma\in[\gamma_2,1]$.*

(i) *For any constant $c$ satisfying $c\big(\frac{p\tilde{q}}{2}+(1+\kappa)\gamma\big)\leq q$, it holds that*

$$\sup_{h\in[0,\tilde{h}],\tau\in(0,\tilde{\tau}]}\sup_{k\geq 0}\mathbb{E}[|\mathcal{Z}_k^{x^h,\Delta}|^c]\leq K\|f\|_{p,\gamma}^c(1+\|x^h\|^{c\tilde{q}(\frac{p\tilde{q}}{2}+(1+\kappa)\gamma)}). \tag{34}$$

(ii) *Let $p \geq 1$ satisfy $p\tilde{q} + 2(1+\kappa)\gamma \leq q$. Then $\mathbb{E}[|\mathcal{Z}_{k+1}^{x^h,\Delta}|^2|\mathcal{F}_{t_k}] = H^\Delta(Y_{t_k}^{x^h,\Delta})$ for all $k \in \mathbb{N}$, where $H^\Delta : E^h \to \mathbb{R}$ is defined as*

$$H^\Delta(u) = -\tau^2|f(u) - \mu^\Delta(f)|^2 + \tau^2\mathbb{E}\Big[\Big|\sum_{i=0}^{\infty}\big(P_{t_i}^\Delta f(Y_{t_1}^{u,\Delta}) - \mu^\Delta(f)\big)\Big|^2\Big]$$

$$-\tau^2|\sum_{i=0}^{\infty}\big(P_{t_i}^\Delta f(u) - \mu^\Delta(f)\big)|^2 + 2\tau^2\big(f(u) - \mu^\Delta(f)\big)\sum_{i=0}^{\infty}\big(P_{t_i}^\Delta f(u) - \mu^\Delta(f)\big). \quad (35)$$

(iii) *Let $p \geq 1$ satisfy $2p\tilde{q} + 2(1+2\kappa)\gamma \leq q$. Then $H^\Delta \in C_{2\tilde{p}_\gamma,\gamma}$ and $\|H^\Delta\|_{2\tilde{p}_\gamma,\gamma} \leq K\|f\|_{p,\gamma}^2$, where $\tilde{p}_\gamma := \tilde{q}\big(p\tilde{q} + (2+3\kappa)\gamma\big)$.*

**Proposition 5.3.** *Let Assumption 1 hold for the time-homogeneous Markov process $\{X_t^x\}_{t\geq 0}$, Assumption 2 hold for the time-homogeneous Markov process $\{Y_{t_k}^{x^h,\Delta}\}_{k\in\mathbb{N}}$, Assumption 3 hold for $\{X_t^x\}_{t\geq 0}$ and $\{Y_{t_k}^{x^h,\Delta}\}_{k\in\mathbb{N}}$, $\gamma \in [\gamma_1 \vee \gamma_2, 1]$, and $p \geq 1$ satisfy $p(1+\tilde{q}\vee\tilde{r}) + 2(1+\kappa\vee\beta)\gamma \leq q \wedge r$. Then for any $f \in C_{p,\gamma}$,*

$$\mu^\Delta(H^\Delta) = -\tau^2\mu^\Delta\big(|f - \mu^\Delta(f)|^2\big) + 2\tau\mu^\Delta\Big(\big(f - \mu^\Delta(f)\big)\sum_{i=0}^{\infty}\big(P_{t_i}^\Delta f - \mu^\Delta(f)\big)\tau\Big). \quad (36)$$

*Moreover, $|\mu^\Delta(H^\Delta)| \leq K\tau\|f\|_{p,\gamma}^2$ and $\lim_{|\Delta|\to 0}\frac{\mu^\Delta(H^\Delta)}{\tau} = v^2$.*

The proofs of Propositions 5.1–5.3 are presented in part 1 of supplementary materials (Chen et al. (2025)) to avoid a possible distraction from the presentation of the main results. With these in hand, we give the proof of the preservation of the CLT of numerical discretizations.

**Proof of Theorem 3.2.** *Step 1: Prove that $\frac{1}{\sqrt{k\tau}}\mathscr{M}_k^{x^h,\Delta}$ converges in distribution to $\mathscr{N}(0,v^2)$ as* $|\Delta| \to 0$. Recalling $k = \lceil \tau^{-2\lambda-1}\rceil$, it is equivalent to show that $\tau^\lambda\mathscr{M}_{\lceil\tau^{-2\lambda-1}\rceil}^{x^h,\Delta}$ converges in distribution to $\mathscr{N}(0,v^2)$. To this end, we show that the characteristic function of $\tau^\lambda\mathscr{M}_{\lceil\tau^{-2\lambda-1}\rceil}^{x^h,\Delta}$ satisfies

$$\lim_{|\Delta|\to 0}\mathbb{E}\big[\exp\big(\mathbf{i}\iota\tau^\lambda\mathscr{M}_{\lceil\tau^{-2\lambda-1}\rceil}^{x^h,\Delta}\big)\big] = e^{-\frac{v^2\iota^2}{2}} \quad \forall\, \iota \in \mathbb{R}, \quad (37)$$

where $\mathbf{i}$ is the imaginary unit. Without loss of generality, we assume that $\mu(f) = 0$. Otherwise, we let $\tilde{f} = f - \mu(f)$ and consider $\tilde{f}$ instead of $f$.

A direct calculation gives

$$e^{\frac{v^2\iota^2}{2}}\mathbb{E}\big[\exp\big(\mathbf{i}\iota\tau^\lambda\mathscr{M}_k^{x^h,\Delta}\big)\big] - 1 = \sum_{j=0}^{k-1}\Big(e^{\frac{v^2\iota^2(j+1)}{2k}}\mathbb{E}\big[\exp\big(\mathbf{i}\iota\tau^\lambda\mathscr{M}_{j+1}^{x^h,\Delta}\big)\big] - e^{\frac{v^2\iota^2 j}{2k}}\mathbb{E}\big[\exp\big(\mathbf{i}\iota\tau^\lambda\mathscr{M}_j^{x^h,\Delta}\big)\big]\Big)$$

$$= \sum_{j=0}^{k-1}e^{\frac{v^2\iota^2(j+1)}{2k}}\mathbb{E}\Big[\exp\big(\mathbf{i}\iota\tau^\lambda\mathscr{M}_j^{x^h,\Delta}\big)\big(\exp\big(\mathbf{i}\iota\tau^\lambda\mathcal{Z}_{j+1}^{x^h,\Delta}\big) - 1 + \frac{v^2\iota^2}{2k}\big)\Big]$$

$$+ \sum_{j=0}^{k-1}e^{\frac{v^2\iota^2(j+1)}{2k}}\big(1 - e^{-\frac{v^2\iota^2}{2k}} - \frac{v^2\iota^2}{2k}\big)\mathbb{E}\big[\exp\big(\mathbf{i}\iota\tau^\lambda\mathscr{M}_j^{x^h,\Delta}\big)\big]. \quad (38)$$

Note that for any $\zeta\in\mathbb{R}\setminus\{0\}$, $e^{\mathbf{i}\zeta}=1+\mathbf{i}\zeta-\frac{\zeta^2}{2}-\zeta^2 Q(\zeta)$, where $Q(\zeta)=\zeta^{-2}\int_0^{\zeta}\int_0^{\zeta_2}(e^{\mathbf{i}\zeta_1}-1)\mathrm{d}\zeta_1\mathrm{d}\zeta_2$ satisfies $\sup_{\zeta\in\mathbb{R}}|Q(\zeta)|\le 1, |Q(\zeta)|=O(|\zeta|)$ for $\zeta\ll 1$. This, along with $\mathbb{E}[\mathcal{Z}_{j+1}^{x^h,\Delta}|\mathcal{F}_{t_j}]=0$ implies that

$$\begin{aligned}
&\mathbb{E}\Big[\exp(\mathbf{i}\iota\tau^{\lambda}\mathscr{M}_j^{x^h,\Delta})\big(\exp(\mathbf{i}\iota\tau^{\lambda}\mathcal{Z}_{j+1}^{x^h,\Delta})-1+\frac{\nu^2\iota^2}{2k}\big)\Big]\\
&=\mathbb{E}\Big[\exp(\mathbf{i}\iota\tau^{\lambda}\mathscr{M}_j^{x^h,\Delta})\mathbb{E}\Big[\Big(-\frac{\iota^2\tau^{2\lambda}(\mathcal{Z}_{j+1}^{x^h,\Delta})^2}{2}-\iota^2\tau^{2\lambda}(\mathcal{Z}_{j+1}^{x^h,\Delta})^2 Q_{j+1}+\frac{\nu^2\iota^2}{2k}\Big)\Big|\mathcal{F}_{t_j}\Big]\Big]\\
&=\frac{\iota^2\tau^{2\lambda}}{2}\mathbb{E}\Big[\exp(\mathbf{i}\iota\tau^{\lambda}\mathscr{M}_j^{x^h,\Delta})\big(\frac{\nu^2}{k\tau^{2\lambda}}-(\mathcal{Z}_{j+1}^{x^h,\Delta})^2\big)\Big]-\iota^2\tau^{2\lambda}\mathbb{E}\Big[\exp(\mathbf{i}\iota\tau^{\lambda}\mathscr{M}_j^{x^h,\Delta})(\mathcal{Z}_{j+1}^{x^h,\Delta})^2 Q_{j+1}\Big],
\end{aligned}\tag{39}$$

where $Q_{j+1}:=Q(\iota\tau^{\lambda}\mathcal{Z}_{j+1}^{x^h,\Delta})$. Plugging (39) into (38) yields

$$e^{\frac{\nu^2\iota^2}{2}}\mathbb{E}[\exp(\mathbf{i}\iota\tau^{\lambda}\mathscr{M}_k^{x^h,\Delta})]-1=I_1(k)+I_2(k)+I_3(k),\tag{40}$$

where

$$\begin{aligned}
I_1(k)&:=\sum_{j=0}^{k-1}e^{\frac{\nu^2\iota^2(j+1)}{2k}}\big(1-e^{-\frac{\nu^2\iota^2}{2k}}-\frac{\nu^2\iota^2}{2k}\big)\mathbb{E}[\exp(\mathbf{i}\iota\tau^{\lambda}\mathscr{M}_j^{x^h,\Delta})],\\
I_2(k)&:=-\iota^2\tau^{2\lambda}\sum_{j=0}^{k-1}e^{\frac{\nu^2\iota^2(j+1)}{2k}}\mathbb{E}\Big[\exp(\mathbf{i}\iota\tau^{\lambda}\mathscr{M}_j^{x^h,\Delta})(\mathcal{Z}_{j+1}^{x^h,\Delta})^2 Q_{j+1}\Big],\\
I_3(k)&:=\frac{\iota^2\tau^{2\lambda}}{2}\sum_{j=0}^{k-1}e^{\frac{\nu^2\iota^2(j+1)}{2k}}\mathbb{E}\Big[\exp(\mathbf{i}\iota\tau^{\lambda}\mathscr{M}_j^{x^h,\Delta})\big(\frac{\nu^2}{k\tau^{2\lambda}}-(\mathcal{Z}_{j+1}^{x^h,\Delta})^2\big)\Big].
\end{aligned}$$

**Estimate of the term $I_1(k)$.** Applying the inequality $|1-e^{-\zeta}-\zeta|\le\frac{1}{2}\zeta^2$ for $\zeta>0$ gives

$$\lim_{\tau\to 0}|I_1(k)|\le\lim_{\tau\to 0}e^{\frac{\nu^2\iota^2}{2}}\sum_{j=0}^{k-1}\big|1-e^{-\frac{\nu^2\iota^2}{2k}}-\frac{\nu^2\iota^2}{2k}\big|\le\lim_{\tau\to 0}e^{\frac{\nu^2\iota^2}{2}}\sum_{j=0}^{k-1}\frac{\nu^4\iota^4}{8k^2}\le\lim_{\tau\to 0}\frac{K}{k}=0.$$

**Estimate of the term $I_2(k)$.** It follows from $\sup_{\zeta\in\mathbb{R}}|Q(\zeta)|\le 1$ that for any $\epsilon>0$,

$$I_2(k)\le\iota^2 e^{\frac{\nu^2\iota^2}{2}}\tau^{2\lambda}\sum_{j=0}^{k-1}\mathbb{E}\big[(\mathcal{Z}_{j+1}^{x^h,\Delta})^2|Q_{j+1}|\big]\le I_{2,1}(k)+I_{2,2}(k),\tag{41}$$

where

$$\begin{aligned}
I_{2,1}(k)&:=\iota^2 e^{\frac{\nu^2\iota^2}{2}}\tau^{2\lambda}\sum_{j=0}^{k-1}\mathbb{E}\big[(\mathcal{Z}_{j+1}^{x^h,\Delta})^2\mathbf{1}_{\{|\mathcal{Z}_{j+1}^{x^h,\Delta}|>\epsilon\tau^{-\lambda}\}}\big],\\
I_{2,2}(k)&:=\iota^2 e^{\frac{\nu^2\iota^2}{2}}\tau^{2\lambda}\sum_{j=0}^{k-1}\mathbb{E}\big[(\mathcal{Z}_{j+1}^{x^h,\Delta})^2|Q_{j+1}|\mathbf{1}_{\{|\mathcal{Z}_{j+1}^{x^h,\Delta}|\le\epsilon\tau^{-\lambda}\}}\big].
\end{aligned}$$

Using Propositions 5.2–5.3 leads to that for any $\varepsilon \in (0, \lambda)$,

$$
\begin{aligned}
I_{2,2}(k) &\leq \iota^2 e^{\frac{\nu^2 \iota^2}{2}} \tau^{2\lambda} \sup_{|\zeta| \leq \iota\epsilon} |Q(\zeta)| \sum_{j=0}^{k-1} \mathbb{E}[(\mathcal{Z}_{j+1}^{x^h,\Delta})^2] \\
&\leq \iota^2 e^{\frac{\nu^2 \iota^2}{2}} \sup_{|\zeta| \leq \iota\epsilon} |Q(\zeta)| \Big[\tau^{2\lambda} \sum_{j=0}^{\lceil \tau^{-1-(\lambda-\varepsilon)} \rceil - 1} \mathbb{E}[(\mathcal{Z}_{j+1}^{x^h,\Delta})^2] \\
&\quad + \tau^{2\lambda} \sum_{j=\lceil \tau^{-1-(\lambda-\varepsilon)} \rceil}^{k-1} \big(P_{t_j}^{\Delta} H^{\Delta}(x^h) - \mu^{\Delta}(H^{\Delta})\big) + K\|f\|_{p,\gamma}^2\Big].
\end{aligned} \tag{42}
$$

Combining $\mathbb{E}\big[|\mathscr{M}_{\lceil \tau^{-1-(\lambda-\varepsilon)} \rceil}^{x^h,\Delta}|^2\big] = \sum_{j=0}^{\lceil \tau^{-1-(\lambda-\varepsilon)} \rceil - 1} \mathbb{E}[(\mathcal{Z}_{j+1}^{x^h,\Delta})^2]$ and (33) implies that

$$
\begin{aligned}
&\sum_{j=0}^{\lceil \tau^{-1-(\lambda-\varepsilon)} \rceil - 1} \mathbb{E}[(\mathcal{Z}_{j+1}^{x^h,\Delta})^2] \leq K\tau^{1-(\lambda-\varepsilon)} \sum_{j=0}^{\lceil \tau^{-1-(\lambda-\varepsilon)} \rceil - 1} \mathbb{E}\big[|f(Y_{t_j}^{x^h,\Delta})|^2 + |\mu^{\Delta}(f)|^2\big] \\
&+ K\|f\|_{p,\gamma}^2 \mathbb{E}\Big[1 + \|Y_{t_{\lceil \tau^{-1-(\lambda-\varepsilon)} \rceil}}^{x^h,\Delta}\|^{p\tilde{q}+2(1+\kappa)\gamma} + \|x^h\|^{p\tilde{q}+2(1+\kappa)\gamma}\Big].
\end{aligned}
$$

Since the condition (7) implies that $p\tilde{q} + 2(1+\kappa)\gamma \leq r$, by Assumption 2 (i) we obtain

$$
\sum_{j=0}^{\lceil \tau^{-1-(\lambda-\varepsilon)} \rceil - 1} \mathbb{E}[(\mathcal{Z}_{j+1}^{x^h,\Delta})^2] \leq K\|f\|_{p,\gamma}^2 (1 + \|x^h\|^{\tilde{q}(p\tilde{q}+2(1+\kappa)\gamma)}) \tau^{-2(\lambda-\varepsilon)}. \tag{43}
$$

It follows from (7) and Proposition 5.2 (iii) that $H^{\Delta} \in C_{2\tilde{p}_{\gamma},\gamma}$ with $2\tilde{p}_{\gamma} \leq q$. Making use of (23) yields

$$
\tau^{2\lambda} \sum_{j=\lceil \tau^{-1-(\lambda-\varepsilon)} \rceil}^{k-1} \big(P_{t_j}^{\Delta} H^{\Delta}(x^h) - \mu^{\Delta}(H^{\Delta})\big) \leq K\|f\|_{p,\gamma}^2 (1 + \|x^h\|^{\tilde{q}\tilde{p}_{\gamma}+(1+\kappa)\gamma}) \tau^{2\lambda} \sum_{j=\lceil \tau^{-1-(\lambda-\varepsilon)} \rceil}^{k-1} \big(\rho^{\Delta}(t_j)\big)^{\gamma}. \tag{44}
$$

Inserting (43) and (44) into (42) leads to

$$
\begin{aligned}
I_{2,2}(k) &\leq K\iota^2 e^{\frac{\nu^2 \iota^2}{2}} \sup_{|\zeta| \leq \iota\epsilon} |Q(\zeta)| \|f\|_{p,\gamma}^2 \Big(1 + \|x^h\|^{\tilde{q}(p\tilde{q}+2(1+\kappa)\gamma)} + \|x^h\|^{\tilde{q}\tilde{p}_{\gamma}+(1+\kappa)\gamma}\Big) \\
&\quad \times \Big(\tau^{2\varepsilon} + \tau^{2\lambda} \sum_{j=\lceil \tau^{-1-(\lambda-\varepsilon)} \rceil}^{k-1} \big(\rho^{\Delta}(t_j)\big)^{\gamma} + 1\Big).
\end{aligned} \tag{45}
$$

Note that (8) implies $\lim_{\tau \to 0} \tau^{2\lambda} \sum_{j=\lceil \tau^{-1-(\lambda-\varepsilon)} \rceil}^{k-1} \big(\rho^{\Delta}(t_j)\big)^{\gamma} = 0$. This, along with (45) leads to

$$
\lim_{\epsilon \to 0} \lim_{\tau \to 0} I_{2,2}(k) = 0. \tag{46}
$$

By virtue of the condition (7) there exists a positive constant $c$ such that $c > \frac{2}{\lambda} \vee 4$ and $c\big(\frac{p\tilde{q}}{2} + (1 + \frac{\kappa}{2})\gamma\big) \le q$. Applying the Hölder inequality, the Chebyshev inequality, and (34), we have

$$\begin{aligned}
I_{2,1}(k) &\le \iota^2 e^{\frac{\nu^2\iota^2}{2}} \tau^{2\lambda} \sum_{j=0}^{k-1} \big(\mathbb{E}[|\mathcal{Z}_{j+1}^{x^h,\Delta}|^4]\big)^{\frac{1}{2}} \big(\mathbb{P}\{|\mathcal{Z}_{j+1}^{x^h,\Delta}| > \epsilon\tau^{-\lambda}\}\big)^{\frac{1}{2}} \\
&\le \iota^2 e^{\frac{\nu^2\iota^2}{2}} \frac{1}{\epsilon^{c/2}} \tau^{2\lambda + \frac{c\lambda}{2}} \sum_{j=0}^{k-1} \big(\mathbb{E}[|\mathcal{Z}_{j+1}^{x^h,\Delta}|^4]\big)^{\frac{1}{2}} \big(\mathbb{E}[|\mathcal{Z}_{j+1}^{x^h,\Delta}|^c]\big)^{\frac{1}{2}} \\
&\le K\iota^2 e^{\frac{\nu^2\iota^2}{2}} \frac{1}{\epsilon^{c/2}} \tau^{\frac{c\lambda}{2}-1} \|f\|_{p,\gamma}^{\frac{4+c}{2}} \big(1 + \|x^h\|^{\frac{(4+c)\tilde{q}}{2}\left(\frac{p\tilde{q}}{2}+(1+\kappa)\gamma\right)}\big).
\end{aligned} \tag{47}$$

Plugging (46) and (47) into (41) yields

$$\lim_{\tau\to 0} |I_2(k)| \le \lim_{\epsilon\to 0}\lim_{\tau\to 0} K\iota^2 e^{\frac{\nu^2\iota^2}{2}} \frac{1}{\epsilon^{c/2}} \tau^{\frac{c\lambda}{2}-1} \|f\|_{p,\gamma}^{\frac{4+c}{2}} \big(1+\|x^h\|^{\frac{(4+c)\tilde{q}}{2}\left(\frac{p\tilde{q}}{2}+(1+\kappa)\gamma\right)}\big) + \lim_{\epsilon\to 0}\lim_{\tau\to 0} |I_{2,2}(k)| = 0.$$

**Estimate of the term $I_3(k)$.** To simplify notations, we denote $\tilde{k} = \lceil \tau^{-1-\varepsilon} \rceil$ and $M = \lceil k/\tilde{k} \rceil$. Divide $\{0, 1, \ldots, k-1\}$ into $M$ blocks, and denote

$$\mathbb{B}_i = \{(i-1)\tilde{k}, (i-1)\tilde{k}+1, \ldots, i\tilde{k}-1\} \text{ for } i \in \{1, \ldots, M-1\}, \quad \mathbb{B}_M = \{(M-1)\tilde{k}, \ldots, k-1\}.$$

We rewrite the term $I_3(k)$ as

$$\begin{aligned}
I_3(k) &= \frac{\iota^2\tau^{2\lambda}}{2} \sum_{i=1}^{M} \mathbb{E}\Big[\exp\big(\mathbf{i}\iota\tau^\lambda \mathscr{M}_{(i-1)\tilde{k}}^{x^h,\Delta}\big) \sum_{j\in\mathbb{B}_i} e^{\frac{\nu^2\iota^2(j+1)}{2k}} \Big(\frac{\nu^2}{k\tau^{2\lambda}} - (\mathcal{Z}_{j+1}^{x^h,\Delta})^2\Big)\Big] + \frac{\iota^2\tau^{2\lambda}}{2} \sum_{i=1}^{M} \sum_{j\in\mathbb{B}_i} \\
&\mathbb{E}\Big[\big(\exp(\mathbf{i}\iota\tau^\lambda \mathscr{M}_j^{x^h,\Delta}) - \exp(\mathbf{i}\iota\tau^\lambda \mathscr{M}_{(i-1)\tilde{k}}^{x^h,\Delta})\big) e^{\frac{\nu^2\iota^2(j+1)}{2k}} \Big(\frac{\nu^2}{k\tau^{2\lambda}} - (\mathcal{Z}_{j+1}^{x^h,\Delta})^2\Big)\Big] \le I_{3,1}(k) + I_{3,2}(k),
\end{aligned} \tag{48}$$

where

$$\begin{aligned}
I_{3,1}(k) &:= \frac{\iota^2\tau^{2\lambda}}{2} \sum_{i=1}^{M} \mathbb{E}\Big[\exp\big(\mathbf{i}\iota\tau^\lambda \mathscr{M}_{(i-1)\tilde{k}}^{x^h,\Delta}\big) \sum_{j\in\mathbb{B}_i} e^{\frac{\nu^2\iota^2(j+1)}{2k}} \Big(\frac{\nu^2}{k\tau^{2\lambda}} - (\mathcal{Z}_{j+1}^{x^h,\Delta})^2\Big)\Big], \\
I_{3,2}(k) &:= \frac{\iota^2\tau^{2\lambda}}{2} e^{\frac{\nu^2\iota^2}{2}} \sum_{i=1}^{M} \sum_{j\in\mathbb{B}_i} \mathbb{E}\Big[\Big|\exp\big(\mathbf{i}\iota\tau^\lambda(\mathscr{M}_j^{x^h,\Delta} - \mathscr{M}_{(i-1)\tilde{k}}^{x^h,\Delta})\big) - 1\Big| \Big(\frac{\nu^2}{k\tau^{2\lambda}} + (\mathcal{Z}_{j+1}^{x^h,\Delta})^2\Big)\Big].
\end{aligned}$$

It follows from the property of the conditional expectation and the time-homogeneous Markov property of $\{\mathcal{Z}_k^{x^h,\Delta}\}_{k\in\mathbb{N}}$ that

$$\begin{aligned}
I_{3,1}(k) &\le \frac{\iota^2\tau^{2\lambda}}{2} \sum_{i=1}^{M} \mathbb{E}\Big[\Big|\mathbb{E}\Big[\sum_{j\in\mathbb{B}_i} e^{\frac{\nu^2\iota^2(j+1)}{2k}} \Big(\frac{\nu^2}{k\tau^{2\lambda}} - (\mathcal{Z}_{j+1}^{x^h,\Delta})^2\Big)\Big|\mathcal{F}_{t_{(i-1)\tilde{k}}}\Big]\Big|\Big] \\
&\le \frac{\iota^2\tau^{2\lambda-\varepsilon}}{2} e^{\frac{\nu^2\iota^2}{2}} \sum_{i=1}^{M} \mathbb{E}\Big[\Big|\tau^\varepsilon \sum_{j\in\mathbb{B}_1} \Big|\frac{\nu^2}{k\tau^{2\lambda}} - \mathbb{E}[(\mathcal{Z}_{j+1}^{y^h,\Delta})^2]\Big|\Big|_{y^h = Y_{t_{(i-1)\tilde{k}}}^{x^h,\Delta}}\Big|\Big].
\end{aligned} \tag{49}$$

It is straightforward to see from Proposition 5.2 (ii) that

$$\begin{aligned}
&\tau^{\varepsilon}\sum_{j\in\mathbb{B}_1}\Big|\frac{v^2}{k\tau^{2\lambda}}-\mathbb{E}[(\mathcal{Z}_{j+1}^{y^h,\Delta})^2]\Big|\\
&\le \tau^{\varepsilon}\sum_{j=0}^{\tilde{k}-1}\Big|\frac{v^2}{k\tau^{2\lambda}}-\mu^{\Delta}(H^{\Delta})\Big|+\tau^{\varepsilon}\sum_{j=0}^{\tilde{k}-1}|\mu^{\Delta}(H^{\Delta})-\mathbb{E}[(\mathcal{Z}_{j+1}^{y^h,\Delta})^2]|+\tau^{\varepsilon}\sum_{j=\lceil\tau^{-1-\frac{\varepsilon}{4}}\rceil}^{\tilde{k}-1}|\mu^{\Delta}(H^{\Delta})-\mathbb{E}[(\mathcal{Z}_{j+1}^{y^h,\Delta})^2]|\\
&\le \tau^{\varepsilon+1}\tilde{k}v^2\Big|\frac{1}{k\tau^{2\lambda+1}}-1\Big|+\tau^{\varepsilon+1}\tilde{k}\Big|v^2-\frac{\mu^{\Delta}(H^{\Delta})}{\tau}\Big|+\tau^{\varepsilon-1-\frac{\varepsilon}{4}}|\mu^{\Delta}(H^{\Delta})|\\
&\quad+\tau^{\varepsilon}\sum_{j=0}^{\lceil\tau^{-1-\frac{\varepsilon}{4}}\rceil-1}\mathbb{E}[(\mathcal{Z}_{j+1}^{y^h,\Delta})^2]+\tau^{\varepsilon}\sum_{j=\lceil\tau^{-1-\frac{\varepsilon}{4}}\rceil}^{\tilde{k}-1}|\mu^{\Delta}(H^{\Delta})-P_{t_j}^{\Delta}H^{\Delta}(y^h)|. \qquad (50)
\end{aligned}$$

Using the same techniques as (43) and (44), we derive

$$\sum_{j=0}^{\lceil\tau^{-1-\frac{\varepsilon}{4}}\rceil-1}\mathbb{E}[(\mathcal{Z}_{j+1}^{y^h,\Delta})^2]\le K\|f\|_{p,\gamma}^2(1+\|y^h\|^{\tilde{q}(p\tilde{q}+2(1+\kappa)\gamma)})\tau^{-\frac{\varepsilon}{2}} \qquad (51)$$

and

$$\tau^{\varepsilon}\sum_{j=\lceil\tau^{-1-\frac{\varepsilon}{4}}\rceil}^{\tilde{k}-1}|\mu^{\Delta}(H^{\Delta})-P_{t_j}^{\Delta}H^{\Delta}(y^h)|\le K\|f\|_{p,\gamma}^2(1+\|y^h\|^{\tilde{q}\tilde{p}_{\gamma}+(1+\kappa)\gamma})\tau^{\varepsilon}\sum_{j=\lceil\tau^{-1-\frac{\varepsilon}{4}}\rceil}^{\tilde{k}-1}\big(\rho^{\Delta}(t_j)\big)^{\gamma}. \qquad (52)$$

Inserting (51) and (52) into (50) and using $|\mu^{\Delta}(H^{\Delta})|\le K\tau\|f\|_{p,\gamma}^2$, we arrive at

$$\begin{aligned}
&\tau^{\varepsilon}\sum_{j\in\mathbb{B}_1}\Big|\frac{v^2}{k\tau^{2\lambda}}-\mathbb{E}[(\mathcal{Z}_{j+1}^{y^h,\Delta})^2]\Big|\le \tau^{\varepsilon+1}\tilde{k}v^2\Big|\frac{1}{k\tau^{2\lambda+1}}-1\Big|+\tau^{\varepsilon+1}\tilde{k}\Big|v^2-\frac{\mu^{\Delta}(H^{\Delta})}{\tau}\Big|+\tau^{\frac{3\varepsilon}{4}}K\|f\|_{p,\gamma}^2\\
&+K\|f\|_{p,\gamma}^2(1+\|y^h\|^{\tilde{q}(p\tilde{q}+2(1+\kappa)\gamma)})\tau^{\frac{\varepsilon}{2}}+K\|f\|_{p,\gamma}^2(1+\|y^h\|^{\tilde{q}\tilde{p}_{\gamma}+(1+\kappa)\gamma})\tau^{\varepsilon}\sum_{j=\lceil\tau^{-1-\frac{\varepsilon}{4}}\rceil}^{\tilde{k}-1}\big(\rho^{\Delta}(t_j)\big)^{\gamma}. \qquad (53)
\end{aligned}$$

Since $\tilde{q}(p\tilde{q}+2(1+\kappa)\gamma)\le q$ and $\tilde{q}\tilde{p}_{\gamma}+(1+\kappa)\gamma\le q$, plugging (53) into (49) and applying Assumption 2 (i), we have

$$\begin{aligned}
I_{3,1}(k)\le K\iota^2e^{\frac{v^2\iota^2}{2}}\Big(&v^2\Big|\frac{1}{k\tau^{2\lambda+1}}-1\Big|+\Big|v^2-\frac{\mu^{\Delta}(H^{\Delta})}{\tau}\Big|+\tau^{\frac{3\varepsilon}{4}}\|f\|_{p,\gamma}^2\\
&+\|f\|_{p,\gamma}^2(1+\|x^h\|^{\tilde{q}^2(p\tilde{q}+2(1+\kappa)\gamma)})\tau^{\frac{\varepsilon}{2}}\\
&+\|f\|_{p,\gamma}^2(1+\|x^h\|^{\tilde{q}(q\tilde{p}_{\gamma}+(1+\kappa)\gamma)})\tau^{\varepsilon}\sum_{j=\lceil\tau^{-1-\frac{\varepsilon}{4}}\rceil}^{\tilde{k}-1}\big(\rho^{\Delta}(t_j)\big)^{\gamma}\Big),
\end{aligned}$$

where we used $k=\lceil\tau^{-2\lambda-1}\rceil$, $\tilde{k}=\lceil\tau^{-1-\varepsilon}\rceil$, and $M=\lceil k/\tilde{k}\rceil=O(\tau^{-2\lambda+\varepsilon})$. It follows from (8) that $\lim_{\tau\to 0}\tau^{\varepsilon}\sum_{j=\lceil\tau^{-1-\frac{\varepsilon}{4}}\rceil}^{\tilde{k}-1}\big(\rho^{\Delta}(t_j)\big)^{\gamma}=0$, This, along with $p(1+\tilde{q}\vee\tilde{r})+2(1+\kappa\vee\beta)\gamma\le q\wedge r$ and Proposition 5.3 implies that

$$\lim_{|\Delta|\to 0} I_{3,1}(k)=0. \tag{54}$$

Furthermore, by $|e^{\mathbf{i}u}-1|\le u$ and the Hölder inequality, one has that for any $\varepsilon_1>0$,

$$\begin{aligned}
I_{3,2}(k) &\le \varepsilon_1\iota\frac{\iota^2\tau^{2\lambda}}{2}e^{\frac{\nu^2\iota^2}{2}}\sum_{i=1}^{M}\sum_{j\in\mathbb{B}_i}\big(\frac{\nu^2}{k\tau^{2\lambda}}+\mathbb{E}[(\mathcal{Z}_{j+1}^{x^h,\Delta})^2]\big)\\
&\quad+\iota^2\tau^{2\lambda}e^{\frac{\nu^2\iota^2}{2}}\sum_{i=1}^{M}\sum_{j\in\mathbb{B}_i}\mathbb{E}\Big[\big(\frac{\nu^2}{k\tau^{2\lambda}}+(\mathcal{Z}_{j+1}^{x^h,\Delta})^2\big)\mathbf{1}_{\{\tau^{\lambda}|\mathscr{M}_j^{x^h,\Delta}-\mathscr{M}_{(i-1)\tilde{k}}^{x^h,\Delta}|>\varepsilon_1\}}\Big]\\
&\le K\varepsilon_1\iota^3 e^{\frac{\nu^2\iota^2}{2}}+\varepsilon_1\frac{\iota^3\tau^{2\lambda}}{2}e^{\frac{\nu^2\iota^2}{2}}\sum_{j=0}^{k-1}\mathbb{E}\big[H^{\Delta}(Y_{t_j}^{x^h,\Delta})\big]\\
&\quad+K\iota^2\tau^{2\lambda}e^{\frac{\nu^2\iota^2}{2}}\sum_{i=1}^{M}\sum_{j\in\mathbb{B}_i}\big(\nu^4+\mathbb{E}[(\mathcal{Z}_{j+1}^{x^h,\Delta})^4]\big)^{\frac12}\big(\mathbb{P}\{\tau^{\lambda}|\mathscr{M}_j^{x^h,\Delta}-\mathscr{M}_{(i-1)\tilde{k}}^{x^h,\Delta}|>\varepsilon_1\}\big)^{\frac12}.
\end{aligned}\tag{55}$$

It follows from (23) and the definition of $\mathscr{M}_i^{x^h,\Delta}$ that for any $j\in\mathbb{B}_i\setminus\{(i-1)\tilde{k}\}$,

$$\begin{aligned}
|\mathscr{M}_j^{x^h,\Delta}-\mathscr{M}_{(i-1)\tilde{k}}^{x^h,\Delta}| &\le \tau\Big|\sum_{l=(i-1)\tilde{k}}^{j-1}\big(f(Y_{t_l}^{x^h,\Delta})-\mu^{\Delta}(f)\big)\Big|\\
&\quad+K\|f\|_{p,\gamma}(1+\|Y_{t_j}^{x^h,\Delta}\|^{\frac{p\tilde{q}}{2}+(1+\kappa)\gamma}+\|Y_{t_{(i-1)\tilde{k}}}^{x^h,\Delta}\|^{\frac{p\tilde{q}}{2}+(1+\kappa)\gamma}).
\end{aligned}$$

Let $\varepsilon\ll 1$ such that $\lceil\frac{1+\sqrt{\varepsilon}}{\lambda}\rceil\le\frac{1}{\lambda}+1$. This, along with the condition (7) implies that

$$2\lceil 3\vee\frac{1+\sqrt{\varepsilon}}{\lambda}\rceil\Big(\frac{p\tilde{q}}{2}+(1+\kappa)\gamma\Big)\le q.$$

Combining the Chebyshev inequality and Assumption 2 (i) yields

$$\begin{aligned}
&\mathbb{P}\{\tau^{\lambda}|\mathscr{M}_j^{x^h,\Delta}-\mathscr{M}_{(i-1)\tilde{k}}^{x^h,\Delta}|>\varepsilon_1\}\\
&\le K(\frac{\tau^{\lambda+1}}{\varepsilon_1})^{2\lceil 3\vee\frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}\mathbb{E}\Big[\mathbb{E}\Big[\Big|\sum_{l=0}^{j-(i-1)\tilde{k}-1}\big(f(Y_{t_l}^{y^h,\Delta})-\mu^{\Delta}(f)\big)\Big|^{2\lceil 3\vee\frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}\Big]\Big|_{y^h=Y_{t_{(i-1)\tilde{k}}}^{x^h,\Delta}}\Big]\\
&\quad+K\|f\|_{p,\gamma}^{2\lceil 3\vee\frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}(\frac{\tau^{\lambda}}{\varepsilon_1})^{2\lceil 3\vee\frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}(1+\|x^h\|^{2\tilde{q}\lceil 3\vee\frac{1+\sqrt{\varepsilon}}{\lambda}\rceil\left(\frac{p\tilde{q}}{2}+(1+\kappa)\gamma\right)}).
\end{aligned}\tag{56}$$

Note that the condition (7) leads to that the one on $p$ in Lemma 4.2 holds with $\varrho = \lceil 3 \vee \frac{1+\sqrt{\varepsilon}}{\lambda} \rceil$. By virtue of Lemma 4.2 with $\varrho = \lceil 3 \vee \frac{1+\sqrt{\varepsilon}}{\lambda} \rceil$, we obtain

$$
\begin{aligned}
&\mathbb{E}\Big[\Big| \sum_{l=0}^{j-(i-1)\tilde{k}-1} \big(f(Y_{t_l}^{y^h,\Delta}) - \mu^{\Delta}(f)\big)\Big|^{2\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}\Big] \\
&\leq K\|f\|_{p,\gamma}^{2\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}\big(j-(i-1)\tilde{k}\big)^{2\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}\big(1+\|y^h\|^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil \tilde{q}(\frac{p}{2}(1+\tilde{q})+(1+\kappa)\gamma)}\big)t_{j-(i-1)\tilde{k}}^{-\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}.
\end{aligned}
$$

This, along with (56), $\tilde{q}\lceil 3 \vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil \left(\frac{p}{2}(1+\frac{\tilde{q}}{q}) + (1+\kappa)\gamma\right) \leq q$, and Assumption 2 (i) implies that

$$
\begin{aligned}
&\sum_{j\in\mathbb{B}_i} \big(\mathbb{P}\{\tau^{\lambda}|\mathscr{M}_j^{x^h,\Delta} - \mathscr{M}_{(i-1)\tilde{k}}^{x^h,\Delta}| > \varepsilon_1\}\big)^{\frac{1}{2}} \\
&\leq K\|f\|_{p,\gamma}^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}\sum_{j\in\mathbb{B}_i}\Big(\frac{\tau^{\lambda+1}(j-(i-1)\tilde{k})}{\varepsilon_1}\Big)^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}\big(\mathbb{E}[\|Y_{t_{(i-1)\tilde{k}}}^{x^h,\Delta}\|^{\tilde{q}\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil(\frac{p}{2}(1+\tilde{q})+(1+\kappa)\gamma)}] \\
&\quad +1\big)^{\frac{1}{2}}t_{j-(i-1)\tilde{k}}^{-\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil/2} + K\|f\|_{p,\gamma}^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}\Big(\frac{\tau^{\lambda}}{\varepsilon_1}\Big)^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}\big(1+\|x^h\|^{\tilde{q}\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil\left(\frac{p\tilde{q}}{2}+(1+\kappa)\gamma\right)}\big) \\
&\leq K\|f\|_{p,\gamma}^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}\big(1+\|x^h\|^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil \frac{\tilde{q}^2}{2}(\frac{p}{2}(1+\tilde{q})+(1+\kappa)\gamma)}\big)\sum_{j\in\mathbb{B}_1}\Big(\frac{\tau^{\lambda+1}j}{\varepsilon_1}\Big)^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}t_j^{-\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil/2} \\
&\quad + K\|f\|_{p,\gamma}^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}\Big(\frac{\tau^{\lambda}}{\varepsilon_1}\Big)^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}\big(1+\|x^h\|^{\tilde{q}\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil\left(\frac{p\tilde{q}}{2}+(1+\kappa)\gamma\right)}\big) \\
&\leq K\|f\|_{p,\gamma}^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}\big(1+\|x^h\|^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil \tilde{q}^2\left(\frac{p\tilde{q}}{2}+(1+\kappa)\gamma\right)}\big)\Big(\Big(\frac{\tau^{\lambda}}{\varepsilon_1}\Big)^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}\sum_{j=0}^{\lceil \tau^{-1}\rceil -1} t_j^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil/2} \\
&\quad + \Big(\frac{\tau^{\lambda-\varepsilon}}{\varepsilon_1}\Big)^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}\sum_{j=\lceil \tau^{-1}\rceil}^{\tilde{k}-1}\Big(\frac{j}{\tau^{-1-\varepsilon}}\Big)^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}t_j^{-\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil/2} + \Big(\frac{\tau^{\lambda}}{\varepsilon_1}\Big)^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}\Big) \\
&\leq \frac{1}{\varepsilon_1^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}}K\|f\|_{p,\gamma}^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}\big(1+\|x^h\|^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil \tilde{q}^2\left(\frac{p\tilde{q}}{2}+(1+\kappa)\gamma\right)}\big)\big(1+\tau^{\frac{(\lambda-\varepsilon)(1+\sqrt{\varepsilon})}{\lambda}-1}+\tau\big).
\end{aligned}
$$

Inserting the above inequality into (55) leads to

$$
\begin{aligned}
I_{3,2}(k) \leq{}& K\varepsilon_1\iota^3 e^{\frac{v^2\iota^2}{2}} + \varepsilon_1\frac{\iota^3\tau^{2\lambda}}{2}e^{\frac{v^2\iota^2}{2}}\sum_{j=0}^{k-1}\mathbb{E}\big[H^{\Delta}(Y_{t_j}^{x^h,\Delta})\big] \\
&+ K\iota^2 e^{\frac{v^2\iota^2}{2}}\big(v^4+\sup_{n\geq 0}\mathbb{E}[(\mathcal{Z}_n^{x^h,\Delta})^4]\big)^{\frac{1}{2}}\frac{1}{\varepsilon_1^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil}}\|f\|_{p,\gamma}^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil} \\
&\times \big(1+\|x^h\|^{\lceil 3\vee \frac{1+\sqrt{\varepsilon}}{\lambda}\rceil \tilde{q}^2\left(\frac{p\tilde{q}}{2}+(1+\kappa)\gamma\right)}\big)\big(1+\tau^{\sqrt{\varepsilon}-\frac{\varepsilon}{\lambda}-\frac{\varepsilon\sqrt{\varepsilon}}{\lambda}}+\tau\big)M\tau^{2\lambda}.
\end{aligned}
$$

Recalling $M = O(\tau^{-2\lambda+\varepsilon})$, letting $\varepsilon \in (0,1)$ satisfy $\sqrt{\varepsilon} - \frac{\varepsilon}{\lambda} - \frac{\varepsilon\sqrt{\varepsilon}}{\lambda} > 0$, and using (34) and (42)–(44) yield

$$\lim_{\varepsilon_1 \to 0} \lim_{\tau \to 0} I_{3,2}(k) = 0. \tag{57}$$

We conclude from (48), (54), and (57) that $\lim_{|\Delta|\to 0} I_3(k) = 0$. The desired argument (37) follows from estimates of terms $I_i, i = 1, 2, 3$.

*Step 2: Prove that* $\frac{1}{\sqrt{k\tau}} \mathscr{R}_k^{x^h,\Delta}$ *converges in probability to* 0. Recalling $k = \lceil \tau^{-2\lambda-1} \rceil$, it is equivalent to show that $\tau^\lambda \mathscr{R}^{x^h,\Delta}_{\lceil \tau^{-2\lambda-1} \rceil}$ converges in probability to 0. Let $\mathscr{R}_k^{x^h,\Delta} = \mathscr{R}_{k,1}^{x^h,\Delta} + \mathscr{R}_{k,2}^{x^h,\Delta}$, where

$$\mathscr{R}_{k,1}^{x^h,\Delta} := -\tau \sum_{i=k}^{\infty} \Big( \mathbb{E}\big[f(Y_{t_i}^{x^h,\Delta}) | \mathcal{F}_{t_k}\big] - \mu^\Delta(f) \Big) + \tau \sum_{i=0}^{\infty} \Big( \mathbb{E}\big[f(Y_{t_i}^{x^h,\Delta}) | \mathcal{F}_0\big] - \mu^\Delta(f) \Big)$$

and $\mathscr{R}_{k,2}^{x^h,\Delta} := k\tau(\mu^\Delta(f) - \mu(f))$. By (23) we deduce

$$\begin{aligned} |\mathscr{R}_{k,1}^{x^h,\Delta}| &\le \tau \sum_{i=0}^{\infty} \big| P_{t_i}^\Delta f(Y_{t_k}^{x^h,\Delta}) - \mu^\Delta(f) \big| + \tau \sum_{i=0}^{\infty} \big| P_{t_i}^\Delta f(x^h) - \mu^\Delta(f) \big| \\ &\le K \|f\|_{p,\gamma} \big(1 + \|Y_{t_k}^{x^h,\Delta}\|^{\frac{p\tilde{q}}{2} + (1+\kappa)\gamma} + \|x^h\|^{\frac{p\tilde{q}}{2} + (1+\kappa)\gamma}\big). \end{aligned}$$

Taking expectation on both sides of $|\mathscr{R}_{k,1}^{x^h,\Delta}|$ and using Assumption 2 (i), we derive $\mathbb{E}[|\mathscr{R}_{k,1}^{x^h,\Delta}|] \le K\|f\|_{p,\gamma}(1 + \|x^h\|^{\tilde{q}(\frac{p\tilde{q}}{2} + (1+\kappa)\gamma)})$. This, along with $\frac{p\tilde{q}}{2} + (1+\kappa)\gamma \le q$ and Assumption 2 (i) leads to that

$$\lim_{|\Delta| \to 0} \frac{1}{\sqrt{k\tau}} \mathbb{E}[|\mathscr{R}_{k,1}^{x^h,\Delta}|] \le \lim_{|\Delta|\to 0} K\|f\|_{p,\gamma}(1 + \|x^h\|^{\tilde{q}(\frac{p\tilde{q}}{2} + (1+\kappa)\gamma)})\tau^\lambda = 0. \tag{58}$$

It follows from (25) that $\frac{1}{\sqrt{k\tau}} |\mathscr{R}_{k,2}^{x^h,\Delta}| = \sqrt{k\tau} \big|\mu^\Delta(f) - \mu(f)\big| \le K\|f\|_{p,\gamma}\sqrt{k\tau}(h^{\alpha_1\gamma} + \tau^{\alpha_2\gamma})$. Since $h = \tau^{\alpha_2/\alpha_1}$ when $h \ne 0$, one has $K\|f\|_{p,\gamma}\sqrt{k\tau}(h^{\alpha_1\gamma} + \tau^{\alpha_2\gamma}) = K\|f\|_{p,\gamma}\tau^{\alpha_2\gamma - \lambda}$, which tends to 0 as $\tau \to 0$ by letting $\lambda \in (0, \alpha_2\gamma)$.

Combining *Steps 1–2* finishes the proof. □

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# Supplementary Material

**Supplement to "On numerical discretizations that preserve probabilistic limit behaviors for time-homogeneous Markov processes"** (DOI: 10.3150/24-BEJ1841SUPP; .pdf). The supplementary material Chen et al. (2025) contains the proofs of Propositions 4.1 and 5.1–5.3, (16), (20), and (21).

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