Supplement to "On numerical discretizations that preserve probabilistic limit behaviors for time-homogeneous Markov processes"

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1. Proofs of Propositions 4.1, 5.1–5.3

This section is devoted to proofs of Propositions 4.1, 5.1–5.3.

Proof of Proposition 4.1. (i) We first prove the existence and uniqueness of the numerical invariant measure in the following three steps.

Step 1: Prove that for each fixed $x^h \in E^h$, the family $\{\mu_{t_k}^{x^h,\Delta}\}_{k=0}^{\infty}$ is tight, and the limit is unique in $(\mathcal{P}(E^h), \mathbb{W}_1)$. By virtue of (Villani, 2009, Lemma 6.14), it suffices to prove that $\{\mu_{t_k}^{x^h,\Delta}\}_{k=0}^{\infty}$ is a Cauchy sequence in the Polish space $(\mathcal{P}(E^h), \mathbb{W}_1)$. In fact, for any $k, i \in \mathbb{N}$, applying (Villani, 2009, Remark 6.5) leads to

$$\mathbb{W}_1(\mu_{t_{k+i}}^{x^h,\Delta}, \mu_{t_k}^{x^h,\Delta}) = \sup_{\|\Psi\|_{Lip} \vee \|\Psi\|_\infty \le 1} \Big| \int_{E^h} \Psi(u) \mathrm{d}\mu_{t_{k+i}}^{x^h,\Delta}(u) - \int_{E^h} \Psi(u) \mathrm{d}\mu_{t_k}^{x^h,\Delta}(u) \Big|,$$

where $\Psi : E^h \to \mathbb{R}$ is bounded and Lipschitz continuous, and norms $\|\cdot\|_{Lip}$ and $\|\cdot\|_\infty$ are defined by $\|\Psi\|_{Lip} := \sup_{\substack{u_1,u_2 \in E^h \\ u_1 \neq u_2}} \frac{|\Psi(u_1)-\Psi(u_2)|}{\|u_1-u_2\|}$ and $\|\Psi\|_\infty := \sup_{u \in E^h} |\Psi(u)|$, respectively. By the definition of $\mu_\cdot^{x^h,\Delta}$, the time-homogeneous Markov property of $\{Y_{t_k}^{x^h,\Delta}\}_{k=0}^{\infty}$, and Assumption 2, we deduce that

$$\begin{aligned}
\mathbb{W}_1(\mu_{t_{k+i}}^{x^h,\Delta}, \mu_{t_k}^{x^h,\Delta}) &= \sup_{\|\Psi\|_{Lip} \vee \|\Psi\|_\infty \le 1} \Big| \mathbb{E}\Big[\mathbb{E}\big[\Psi(Y_{t_k}^{y^h,\Delta}) - \Psi(Y_{t_k}^{x^h,\Delta})\big]\Big|_{y^h = Y_{t_i}^{x^h,\Delta}} \Big] \Big| \\
&\le K\mathbb{E}\Big[\|Y_{t_i}^{x^h,\Delta} - x^h\| \big(1 + \|Y_{t_i}^{x^h,\Delta}\|^\kappa + \|x^h\|^\kappa\big) \Big] \rho^\Delta(t_k) \\
&\le K(1 + \|x^h\|^{\tilde{q}(1+\kappa)}) \rho^\Delta(t_k),
\end{aligned}$$

which together with $\lim_{k\to\infty} \rho^\Delta(t_k) = 0$ finishes the proof of *Step 1*.

Step 2: Prove that for each $x^h \in E^h$, $\{Y_{t_k}^{x^h,\Delta}\}_{k\in\mathbb{N}}$ admits a unique invariant measure which is denoted by $\mu^{x^h,\Delta}$. The uniqueness of the invariant measure is proved in *Step 1*. For the proof of the existence, based on the Krylov–Bogoliubov theorem (Da Prato and Zabczyk, 1996, Theorem 3.1.1), it suffices to show that $P^{\Delta}_{t_k}$ is Feller, i.e., the mapping $x^h \to P^{\Delta}_{t_k}\Phi(x^h) = \mathbb{E}[\Phi(Y^{x^h,\Delta}_{t_k})]$ is bounded and continuous for any $\Phi \in C_b$. This can be derived by the use of Assumption 2 (ii).

Step 3: Prove that for any $x^h \neq y^h$, $\mu^{x^h,\Delta} = \mu^{y^h,\Delta}(=: \mu^{\Delta})$. This can be obtained by

$$\begin{aligned}\mathbb{W}_1(\mu^{x^h,\Delta}, \mu^{y^h,\Delta}) &\le \mathbb{W}_1(\mu^{x^h,\Delta}, \mu^{x^h,\Delta}_{t_k}) + \mathbb{W}_1(\mu^{x^h,\Delta}_{t_k}, \mu^{y^h,\Delta}_{t_k}) + \mathbb{W}_1(\mu^{y^h,\Delta}_{t_k}, \mu^{y^h,\Delta})\\ &\le \|x^h - y^h\|(1 + \|x^h\|^{\kappa} + \|y^h\|^{\kappa})\rho^{\Delta}(t_k) \to 0 \text{ as } t_k \to \infty.\end{aligned}$$

(ii) We show the convergence of the numerical invariant measure. It follows from $(P_t)^*\mu = \mu$ and Assumption 1 that

$$\mathbb{W}_2(\mu^x_t, \mu) \le \mathbb{E}\Big[\big(\mathbb{E}[\|X^x_t - X^y_t\|^2]\big)^{\frac{1}{2}}\big|_{y=\xi}\Big] \le K(1 + \|x\|^{1+\beta})\rho(t).$$

This, along with Assumptions 1–3 implies that

$$\begin{aligned}\mathbb{W}_2(\mu, \mu^{\Delta}) &\le \mathbb{W}_2(\mu, \mu^{\mathbf{0}}_{t_k}) + \mathbb{W}_2(\mu^{\mathbf{0}}_{t_k}, \mu^{\mathbf{0},\Delta}_{t_k}) + \mathbb{W}_2(\mu^{\mathbf{0},\Delta}_{t_k}, \mu^{\Delta})\\ &\le K(\rho(t_k) + |\Delta^{\alpha}| + \rho^{\Delta}(t_k)).\end{aligned} \tag{S1}$$

Letting $t_k \to \infty$ we finish the proof. □

Proof of Proposition 5.1. From the definition of $\mathscr{M}^{x^h,\Delta}_k$ (see (32)), we have

$$\begin{aligned}\mathscr{M}^{x^h,\Delta}_k = {}& \tau\sum_{i=0}^{k-1}\big(f(Y^{x^h,\Delta}_{t_i}) - \mu^{\Delta}(f)\big) + \tau\sum_{i=0}^{\infty}\big(P^{\Delta}_{t_i}f(Y^{x^h,\Delta}_{t_k}) - \mu^{\Delta}(f)\big)\\ &- \tau\sum_{i=0}^{\infty}\big(P^{\Delta}_{t_i}f(x^h) - \mu^{\Delta}(f)\big).\end{aligned} \tag{S2}$$

It follows from (23) that

$$\begin{aligned}|\mathscr{M}^{x^h,\Delta}_k| \le {}& \tau\Big|\sum_{i=0}^{k-1}\big(f(Y^{x^h,\Delta}_{t_i}) - \mu^{\Delta}(f)\big)\Big| + K\|f\|_{p,\gamma}(1 + \|Y^{x^h,\Delta}_{t_k}\|^{\frac{p\tilde{q}}{2}+(1+\kappa)\gamma})\times\\ &\tau\sum_{i=0}^{\infty}(\rho^{\Delta}(t_i))^{\gamma} + K\|f\|_{p,\gamma}(1 + \|x^h\|^{\frac{p\tilde{q}}{2}+(1+\kappa)\gamma})\tau\sum_{i=0}^{\infty}(\rho^{\Delta}(t_i))^{\gamma}.\end{aligned}$$

Using the conditions $\gamma \in [\gamma_2, 1]$ and $\frac{p\tilde{q}}{2} + (1+\kappa)\gamma \le q$, and Assumption 2 leads to

$$\mathbb{E}[|\mathscr{M}^{x^h,\Delta}_k|] \le K\|f\|_{p,\gamma}k(1 + \|x^h\|^{\tilde{q}(\frac{p\tilde{q}}{2}+(1+\kappa)\gamma)}) < \infty.$$

In addition, by (S2), we have

$$\begin{aligned}\mathscr{M}^{x^h,\Delta}_k - \mathscr{M}^{x^h,\Delta}_{k-1} = {}& \tau\big(f(Y^{x^h,\Delta}_{t_{k-1}}) - \mu^{\Delta}(f)\big) + \tau\sum_{i=0}^{\infty}\big(P^{\Delta}_{t_i}f(Y^{x^h,\Delta}_{t_k}) - \mu^{\Delta}(f)\big)\\ &- \tau\sum_{i=0}^{\infty}\big(P^{\Delta}_{t_i}f(Y^{x^h,\Delta}_{t_{k-1}}) - \mu^{\Delta}(f)\big),\end{aligned} \tag{S3}$$

which gives

$$\mathbb{E}\Big[(\mathscr{M}_{k+j}^{x^h,\Delta} - \mathscr{M}_k^{x^h,\Delta})|\mathcal{F}_{t_k}\Big] = \tau \sum_{i=0}^{j-1} \big(P_{t_i}^\Delta f(Y_{t_k}^{x^h,\Delta}) - \mu^\Delta(f)\big)$$

$$+\tau P_{t_j}^\Delta \sum_{i=0}^{\infty} \big(P_{t_i}^\Delta f - \mu^\Delta(f)\big)(Y_{t_k}^{x^h,\Delta}) - \tau \sum_{i=0}^{\infty} \big(P_{t_i}^\Delta f(Y_{t_k}^{x^h,\Delta}) - \mu^\Delta(f)\big). \tag{S4}$$

Applying the dominated convergence theorem yields

$$\tau P_{t_j}^\Delta \sum_{i=0}^{\infty} \big(P_{t_i}^\Delta f - \mu^\Delta(f)\big)(Y_{t_k}^{x^h,\Delta}) = \tau \sum_{i=0}^{\infty} \big(P_{t_{i+j}}^\Delta f - \mu^\Delta(f)\big)(Y_{t_k}^{x^h,\Delta})$$

$$= \tau \sum_{i=j}^{\infty} \big(P_{t_i}^\Delta f(Y_{t_k}^{x^h,\Delta}) - \mu^\Delta(f)\big).$$

Inserting the above equality into (S4) leads to $\mathbb{E}\big[(\mathscr{M}_{k+j}^{x^h,\Delta} - \mathscr{M}_k^{x^h,\Delta})|\mathcal{F}_{t_k}\big] = 0$. This finishes the proof. □

Proof of Proposition 5.2. (i) It follows from (22), (23), (S3), Assumption 2, and the condition $\gamma \in [\gamma_2, 1]$ that

$$|\mathcal{Z}_k^{x^h,\Delta}| \le \tau\big|f(Y_{t_{k-1}}^{x^h,\Delta}) - \mu^\Delta(f)\big| + \tau \sum_{i=0}^{\infty} \big|P_{t_i}^\Delta f(Y_{t_k}^{x^h,\Delta}) - \mu^\Delta(f)\big|$$

$$+\tau \sum_{i=0}^{\infty} \big|P_{t_i}^\Delta f(Y_{t_{k-1}}^{x^h,\Delta}) - \mu^\Delta(f)\big|$$

$$\le K\|f\|_{p,\gamma}\tau\big(1 + \|Y_{t_k}^{x^h,\Delta}\|^{\frac{p}{2}}\big) + K\|f\|_{p,\gamma}\Big(1 + \|Y_{t_k}^{x^h,\Delta}\|^{\frac{p\tilde{q}}{2}+(1+\kappa)\gamma}$$

$$+ \|Y_{t_{k-1}}^{x^h,\Delta}\|^{\frac{p\tilde{q}}{2}+(1+\kappa)\gamma}\Big). \tag{S5}$$

Taking any constant c satisfying $c\big(\frac{p\tilde{q}}{2} + (1+\kappa)\gamma\big) \le q$ and using Assumption 2 (i) finish the proof of (i).

(ii) The condition $p\tilde{q} + 2(1+\kappa)\gamma \le q$ coincides with the one in (i) with $c = 2$, thus $\mathbb{E}\big[|\mathcal{Z}_{l+1}^{x^h,\Delta}|^2|\mathcal{F}_{t_l}\big]$ is well-defined. It follows from (S3) that

$$\mathbb{E}\big[|\mathcal{Z}_{l+1}^{x^h,\Delta}|^2|\mathcal{F}_{t_l}\big] = \mathbb{E}\Big[\Big|\tau\big(f(y^h) - \mu^\Delta(f)\big) + \tau \sum_{i=0}^{\infty} \big(P_{t_i}^\Delta f(Y_{t_1}^{y^h,\Delta}) - \mu^\Delta(f)\big)$$

$$- \tau \sum_{i=0}^{\infty} \big(P_{t_i}^\Delta f(y^h) - \mu^\Delta(f)\big)\Big|^2\Big]\Big|_{y^h = Y_{t_l}^{x^h,\Delta}} =: G^\Delta(Y_{t_l}^{x^h,\Delta}). \tag{S6}$$

For any $u \in E^h$,

$$G^\Delta(u) = \tau^2|f(u) - \mu^\Delta(f)|^2 + \tau^2\Big|\sum_{i=0}^{\infty} \big(P_{t_i}^\Delta f(u) - \mu^\Delta(f)\big)\Big|^2$$

$$+\tau^2\mathbb{E}\Big[\Big|\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta}f(Y_{t_1}^{u,\Delta})-\mu^{\Delta}(f)\big)\Big|^2\Big]$$

$$+2\tau^2\big(f(u)-\mu^{\Delta}(f)\big)P_{t_1}^{\Delta}\Big(\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta}f-\mu^{\Delta}(f)\big)\Big)(u)$$

$$-2\tau^2\big(f(u)-\mu^{\Delta}(f)\big)\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta}f(u)-\mu^{\Delta}(f)\big)$$

$$-2\tau^2\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta}f(u)-\mu^{\Delta}(f)\big)P_{t_1}^{\Delta}\Big(\sum_{j=0}^{\infty}\big(P_{t_j}^{\Delta}f-\mu^{\Delta}(f)\big)\Big)(u). \tag{S7}$$

Applying the dominated convergence theorem yields

$$\big(f(u)-\mu^{\Delta}(f)\big)P_{t_1}^{\Delta}\Big(\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta}f-\mu^{\Delta}(f)\big)\Big)(u)$$
$$=\big(f(u)-\mu^{\Delta}(f)\big)\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta}f(u)-\mu^{\Delta}(f)\big)-|f(u)-\mu^{\Delta}(f)|^2. \tag{S8}$$

Similarly,

$$\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta}f(u)-\mu^{\Delta}(f)\big)P_{t_1}^{\Delta}\Big(\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta}f-\mu^{\Delta}(f)\big)\Big)(u)$$
$$=\Big|\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta}f(u)-\mu^{\Delta}(f)\big)\Big|^2-\big(f(u)-\mu^{\Delta}(f)\big)\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta}f(u)-\mu^{\Delta}(f)\big). \tag{S9}$$

Inserting (S8) and (S9) into (S7) implies $G^{\Delta}=H^{\Delta}$, which finishes the proof of (ii).

(iii) For any $u\in E^h$, using (22), (23), the conditions $\gamma\in[\gamma_2,1]$ and $p\tilde{q}+2(1+\kappa)\gamma\le q$, and Assumption 2, we deduce

$$|H^{\Delta}(u)|$$
$$\le K\|f\|_{p,\gamma}^2\tau^2(1+\|u\|^p)+K\|f\|_{p,\gamma}^2\big(1+\mathbb{E}[\|Y_{t_1}^{u,\Delta}\|^{p\tilde{q}+2(1+\kappa)\gamma}]\big)\Big(\tau\sum_{i=0}^{\infty}\big(\rho^{\Delta}(t_i)\big)^{\gamma}\Big)^2$$
$$+K\|f\|_{p,\gamma}^2\big(1+\|u\|^{p\tilde{q}+2(1+\kappa)\gamma}\big)\Big(\tau\sum_{i=0}^{\infty}\big(\rho^{\Delta}(t_i)\big)^{\gamma}\Big)^2$$
$$+K\|f\|_{p,\gamma}^2\tau(1+\|u\|^{\frac{p}{2}})\big(1+\|u\|^{\frac{p}{2}(1+\tilde{q})+(1+\kappa)\gamma}\big)\Big(\tau\sum_{i=0}^{\infty}\big(\rho^{\Delta}(t_i)\big)^{\gamma}\Big)$$
$$\le K\|f\|_{p,\gamma}^2(1+\|u\|^{\tilde{q}(p\tilde{q}+2(1+\kappa)\gamma)})\le K\|f\|_{p,\gamma}^2(1+\|u\|^{\tilde{p}_{\gamma}}),$$

which implies

$$\sup_{h\in[0,\tilde{h}],\tau\in(0,\tilde{\tau}]}\sup_{u\in E^h}\frac{|H^{\Delta}(u)|}{1+\|u\|^{\tilde{p}_{\gamma}}}\le K\|f\|_{p,\gamma}^2. \tag{S10}$$

By the definition of H^{Δ}, we deduce that for any $u_1, u_2 \in E^h$,

$$H^{\Delta}(u_1) - H^{\Delta}(u_2) = \sum_{l=1}^{4} I_l(u_1, u_2), \tag{S11}$$

where

$$\begin{aligned}
I_1(u_1,u_2) &:= \tau^2\big(|f(u_2) - \mu^{\Delta}(f)|^2 - |f(u_1) - \mu^{\Delta}(f)|^2\big),\\
I_2(u_1,u_2) &:= \tau^2\mathbb{E}\Big[\Big|\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta} f(Y_{t_1}^{u_1,\Delta}) - \mu^{\Delta}(f)\big)\Big|^2\Big] - \tau^2\mathbb{E}\Big[\Big|\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta} f(Y_{t_1}^{u_2,\Delta}) - \mu^{\Delta}(f)\big)\Big|^2\Big],\\
I_3(u_1,u_2) &:= -\tau^2\Big|\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta} f(u_1) - \mu^{\Delta}(f)\big)\Big|^2 + \tau^2\Big|\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta} f(u_2) - \mu^{\Delta}(f)\big)\Big|^2,\\
I_4(u_1,u_2) &:= 2\tau^2\big(f(u_1) - \mu^{\Delta}(f)\big)\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta} f(u_1) - \mu^{\Delta}(f)\big)\\
&\qquad - 2\tau^2\big(f(u_2) - \mu^{\Delta}(f)\big)\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta} f(u_2) - \mu^{\Delta}(f)\big).
\end{aligned}$$

By the definition of $C_{p,\gamma}$ and (22), we have

$$\begin{aligned}
|I_1(u_1,u_2)| &= \tau^2|f(u_1) + f(u_2) - 2\mu^{\Delta}(f)||f(u_1) - f(u_2)|\\
&\le K\|f\|_{p,\gamma}^2\tau^2(1 \wedge \|u_1 - u_2\|^{\gamma})(1 + \|u_1\|^{2p} + \|u_2\|^{2p})^{\frac{1}{2}}.
\end{aligned} \tag{S12}$$

It follows from (23) that

$$\begin{aligned}
|I_3(u_1,u_2)| &= \tau^2\Big|\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta} f(u_1) - \mu^{\Delta}(f)\big)\\
&\qquad + \sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta} f(u_2) - \mu^{\Delta}(f)\big)\Big|\Big|\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta} f(u_2) - P_{t_i}^{\Delta} f(u_1)\big)\Big|\\
&\le K\|f\|_{p,\gamma}^2(1 + \|u_1\|^{\frac{p\tilde{q}}{2}+(1+\kappa)\gamma} + \|u_2\|^{\frac{p\tilde{q}}{2}+(1+\kappa)\gamma})(\tau\sum_{i=0}^{\infty}(\rho^{\Delta}(t_i))^{\gamma})^2\\
&\qquad \times \|u_1 - u_2\|^{\gamma}(1 + \|u_1\|^{\frac{p\tilde{q}}{2}+\gamma\kappa} + \|u_2\|^{\frac{p\tilde{q}}{2}+\gamma\kappa})\\
&\le K\|f\|_{p,\gamma}^2\|u_1 - u_2\|^{\gamma}(1 + \|u_1\|^{p\tilde{q}+(1+2\kappa)\gamma} + \|u_2\|^{p\tilde{q}+(1+2\kappa)\gamma}),
\end{aligned} \tag{S13}$$

where we used

$$\begin{aligned}
&|P_{t_k}^{\Delta} f(u_1) - P_{t_k}^{\Delta} f(u_2)| = |\mu_{t_k}^{u_1,\Delta}(f) - \mu_{t_k}^{u_2,\Delta}(f)|\\
&\le K\|f\|_{p,\gamma}\|u_1 - u_2\|^{\gamma}(1 + \|u_1\|^{\frac{p\tilde{q}}{2}+\gamma\kappa} + \|u_2\|^{\frac{p\tilde{q}}{2}+\gamma\kappa})(\rho^{\Delta}(t_k))^{\gamma},\ u_1, u_2 \in E^h.
\end{aligned}$$

Noting $I_2(u_1,u_2)=-\mathbb{E}\big[I_3(Y_{t_1}^{u_1,\Delta},Y_{t_1}^{u_2,\Delta})\big]$, by (S13), $2p\tilde{q}+2(1+2\kappa)\gamma\le q$, the Hölder inequality, and Assumption 2, we obtain

$$\begin{aligned}
&|I_2(u_1,u_2)|\\
&\le K\|f\|_{p,\gamma}^2\mathbb{E}\Big[\|Y_{t_1}^{u_1,\Delta}-Y_{t_1}^{u_2,\Delta}\|^{\gamma}(1+\|Y_{t_1}^{u_1,\Delta}\|^{p\tilde{q}+(1+2\kappa)\gamma}+\|Y_{t_1}^{u_2,\Delta}\|^{p\tilde{q}+(1+2\kappa)\gamma})\Big]\\
&\le K\|f\|_{p,\gamma}^2\|u_1-u_2\|^{\gamma}\big(1+\|u_1\|^{\tilde{q}(p\tilde{q}+(1+2\kappa)\gamma)+\gamma\kappa}+\|u_2\|^{\tilde{q}(p\tilde{q}+(1+2\kappa)\gamma)+\gamma\kappa}\big).
\end{aligned}\tag{S14}$$

The similar arguments as those of I_1 and I_3 lead to

$$\begin{aligned}
|I_4(u_1,u_2)|\le{}& 2\tau^2|f(u_1)-f(u_2)||\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta}f(u_1)-\mu^{\Delta}(f)\big)|\\
&+2\tau^2|f(u_2)-\mu^{\Delta}(f)||\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta}f(u_1)-P_{t_i}^{\Delta}f(u_2)\big)|\\
\le{}& K\|f\|_{p,\gamma}^2\tau\|u_1-u_2\|^{\gamma}(1+\|u_1\|^{\frac{p}{2}(1+\tilde{q})+(1+\kappa)\gamma}+\|u_2\|^{\frac{p}{2}(1+\tilde{q})+(1+\kappa)\gamma}).
\end{aligned}\tag{S15}$$

According to $\tilde{q}\ge 1$ and $p\ge 1$, one has $p\tilde{q}^2\ge\frac{p}{2}(1+\tilde{q})$, which implies

$$\tilde{p}_{\gamma}:=\tilde{q}\big(p\tilde{q}+(2+3\kappa)\gamma\big)\ge\frac{p}{2}(1+\tilde{q})+(1+\kappa)\gamma.$$

Plugging (S12)–(S15) into (S11) deduces

$$\sup_{h\in[0,\tilde{h}],\tau\in(0,\tilde{\tau}]}\ \sup_{\substack{u_1,u_2\in E^h\\ u_1\neq u_2}}\frac{|H^{\Delta}(u_1)-H^{\Delta}(u_2)|}{\|u_1-u_2\|^{\gamma}(1+\|u_1\|^{2\tilde{p}_{\gamma}}+\|u_2\|^{2\tilde{p}_{\gamma}})^{\frac12}}\le K\|f\|_{p,\gamma}^2.$$

This, along with (S10) implies that

$$\begin{aligned}
&\sup_{h\in[0,\tilde{h}],\tau\in(0,\tilde{\tau}]}\ \sup_{\substack{u_1,u_2\in E^h\\ u_1\neq u_2}}\frac{|H^{\Delta}(u_1)-H^{\Delta}(u_2)|}{(1\wedge\|u_1-u_2\|^{\gamma})(1+\|u_1\|^{2\tilde{p}_{\gamma}}+\|u_2\|^{2\tilde{p}_{\gamma}})^{\frac12}}\\
&\le K\|f\|_{p,\gamma}^2+\sup_{h\in[0,\tilde{h}],\tau\in(0,\tilde{\tau}]}\ \sup_{\substack{u_1,u_2\in E^h\\ u_1\neq u_2}}\frac{|H^{\Delta}(u_1)-H^{\Delta}(u_2)|\mathbf{1}_{\{\|u_1-u_2\|>1\}}}{(1+\|u_1\|^{2\tilde{p}_{\gamma}}+\|u_2\|^{2\tilde{p}_{\gamma}})^{\frac12}}\\
&\le K\|f\|_{p,\gamma}^2+\sup_{h\in[0,\tilde{h}],\tau\in(0,\tilde{\tau}]}\ \sup_{\substack{u_1,u_2\in E^h\\ u_1\neq u_2}}\frac{|H^{\Delta}(u_1)|+|H^{\Delta}(u_2)|}{(1+\|u_1\|^{2\tilde{p}_{\gamma}}+\|u_2\|^{2\tilde{p}_{\gamma}})^{\frac12}}\le\ K\|f\|_{p,\gamma}^2.
\end{aligned}$$

The proof is completed. □

Proof of Proposition 5.3. *Step 1: Proof of* (36). The condition of p in Proposition 5.3 implies that the one of Proposition 5.2 (ii) holds and then H^{Δ} is well-defined. Moreover, H^{Δ} can be rewritten as

$$H^{\Delta}(u)=-\tau^2|f(u)-\mu^{\Delta}(f)|^2+\tau^2P_{t_1}^{\Delta}\Big|\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta}f-\mu^{\Delta}(f)\big)\Big|^2(u)-\tau^2\Big|\sum_{i=0}^{\infty}\big(P_{t_i}^{\Delta}f(u)$$

$$-\mu^{\Delta}(f)\Big|^2+2\tau^2\big(f(u)-\mu^{\Delta}(f)\big)\sum_{i=0}^{\infty}\big(P^{\Delta}_{t_i}f(u)-\mu^{\Delta}(f)\big),$$

which together with $(P^{\Delta}_{t_1})^*\mu^{\Delta}=\mu^{\Delta}$ gives

$$\mu^{\Delta}(H^{\Delta})=-\tau^2\mu^{\Delta}\big(|f-\mu^{\Delta}(f)|^2\big)+2\tau\mu^{\Delta}\Big(\big(f-\mu^{\Delta}(f)\big)\sum_{i=0}^{\infty}\big(P^{\Delta}_{t_i}f-\mu^{\Delta}(f)\big)\tau\Big).$$

Here, $(P^{\Delta}_t)^*$ is the transpose operator of P^{Δ}_t; see (Da Prato, 2006, Section 5.2).

Step 2: Proof of $|\mu^{\Delta}(H^{\Delta})|\le K\tau\|f\|^2_{p,\gamma}$. According to the Young inequality, we arrive at

$$|\mu^{\Delta}(H^{\Delta})|\le 2\tau\mu^{\Delta}\big(|f-\mu^{\Delta}(f)|^2\big)+\tau\mu^{\Delta}\Big(\big(\sum_{i=0}^{\infty}\big(P^{\Delta}_{t_i}f-\mu^{\Delta}(f)\big)\big)^2\tau^2\Big). \tag{S16}$$

The fact $f\in C_{p,\gamma}$ implies that $f^2\in C_{2p,\gamma}$ and $\|f^2\|_{2p,\gamma}\le K\|f\|^2_{p,\gamma}$. Together with (22), we derive

$$\begin{aligned}&\mu^{\Delta}(|f-\mu^{\Delta}(f)|^2)=\mu^{\Delta}(f^2)-|\mu^{\Delta}(f)|^2\\ &\le K\|f\|^2_{p,\gamma}\mu^{\Delta}(1+\|\cdot\|^p)+K\big(\|f\|_{p,\gamma}\mu^{\Delta}(1+\|\cdot\|^{\frac{p}{2}})\big)^2\le K\|f\|^2_{p,\gamma}.\end{aligned} \tag{S17}$$

It follows from (23), $p\tilde q+2(1+\kappa)\gamma\le q$, and (22) that

$$\begin{aligned}&\mu^{\Delta}\Big(\big(\sum_{i=0}^{\infty}\big(P^{\Delta}_{t_i}f-\mu^{\Delta}(f)\big)\big)^2\tau^2\Big)\\ &\le K\|f\|^2_{p,\gamma}\mu^{\Delta}(1+\|\cdot\|^{p\tilde q+2(1+\kappa)\gamma})\le K\|f\|^2_{p,\gamma}.\end{aligned} \tag{S18}$$

Inserting (S17) and (S18) into (S16) yields $|\mu^{\Delta}(H^{\Delta})|\le K\tau\|f\|^2_{p,\gamma}$.

Step 3: Proof of $\lim_{|\Delta|\to 0}\frac{\mu^{\Delta}(H^{\Delta})}{\tau}=\nu^2$. Without loss of generality, below we assume that $\mu(f)=0$. Otherwise, we let $\tilde f:=f-\mu(f)$ and consider $\tilde f$ instead of f. The condition on p in Proposition 5.3 leads to $\frac{p}{2}(1+\tilde q\vee\tilde r)+(1+\beta\vee\kappa)\gamma\le r$, which along with $f\in C_{p,\gamma}$, (5), and (23) implies that

$$\mu\Big(f\sum_{i=0}^{\infty}\big(P^{\Delta}_{t_i}f-\mu^{\Delta}(f)\big)\tau\mathbf{1}_{\{E^h\}}\Big)<\infty\quad\text{and}\quad\mu\Big(f\int_0^{\infty}P_tf\,\mathrm{d}t\Big)<\infty.$$

Together with (36) we arrive at

$$\begin{aligned}&\Big|\frac{\mu^{\Delta}(H^{\Delta})}{\tau}-\nu^2\Big|\\ &\le\tau\mu^{\Delta}\big(|f-\mu^{\Delta}(f)|^2\big)+2|\mu^{\Delta}(f)|\mu^{\Delta}\Big(\sum_{i=0}^{\infty}\big|P^{\Delta}_{t_i}f-\mu^{\Delta}(f)\big|\tau\Big)\\ &\quad+2\Big|\mu^{\Delta}\Big(f\sum_{i=0}^{\infty}\big(P^{\Delta}_{t_i}f-\mu^{\Delta}(f)\big)\tau\mathbf{1}_{\{E^h\}}\Big)-\mu\Big(f\sum_{i=0}^{\infty}\big(P^{\Delta}_{t_i}f-\mu^{\Delta}(f)\big)\tau\mathbf{1}_{\{E^h\}}\Big)\Big|\\ &\quad+2\Big|\mu\Big(f\sum_{i=0}^{\infty}\big(P^{\Delta}_{t_i}f-\mu^{\Delta}(f)\big)\tau\mathbf{1}_{\{E^h\}}\Big)-\mu\Big(f\int_0^{\infty}P_tf\,\mathrm{d}t\Big)\Big|=:\sum_{i=1}^{4}II_i.\end{aligned}$$

Due to (S17), we have $II_1 \to 0$ as $|\Delta| \to 0$. By (22), (23), and (25), we arrive at

$$\begin{aligned} II_2 &\le K|\mu^\Delta(f)|\mu^\Delta\Big(\|f\|_{p,\gamma}(1+\|\cdot\|^{\frac{p\tilde{q}}{2}+(1+\kappa)\gamma})\sum_{i=0}^{\infty}\big(\rho^\Delta(t_i)\big)^\gamma\tau\Big) \\ &\le K\|f\|_{p,\gamma}|\mu^\Delta(f)-\mu(f)| \le\; K\|f\|^2_{p,\gamma}|\Delta^\alpha|^\gamma, \end{aligned} \tag{S19}$$

where we used $\mu(f)=0$. Denote

$$F^\Delta := f\sum_{i=0}^{\infty}\big(P^\Delta_{t_i}f-\mu^\Delta(f)\big)\tau\mathbf{1}_{\{E^h\}}.$$

Using the similar techniques as the proof of Proposition 5.2, we obtain

$$F^\Delta \in C_{p(1+\tilde{q})+2(1+\kappa)\gamma,\gamma} \quad \text{and} \quad \sup_{h\in[0,\tilde{h}],\tau\in(0,\tilde{\tau}]}\|F^\Delta\|_{p(1+\tilde{q})+2(1+\kappa)\gamma,\gamma} \le K\|f\|^2_{p,\gamma}.$$

Then by $p(1+\tilde{q})+2(1+\kappa)\gamma \le r\wedge q$ and (25), we deduce

$$II_3 \le K\|F^\Delta\|_{p(1+\tilde{q})+2(1+\kappa)\gamma,\gamma}|\Delta^\alpha|^\gamma \le K\|f\|^2_{p,\gamma}|\Delta^\alpha|^\gamma. \tag{S20}$$

For the term II_4, we have

$$\begin{aligned} II_4 \le\; & 2\mu\Big(\mathbf{1}_{\{E^h\}}\int_0^\infty \big|f\big(P^\Delta_t f-P_t f\big)+f\big(\mu(f)-\mu^\Delta(f)\big)\big|\mathrm{d}t\Big) \\ &+2\mu\Big(f\int_0^\infty P_t f\mathrm{d}t\mathbf{1}_{\{E\setminus E^h\}}\Big), \end{aligned}$$

where $E\setminus E^h$ represents the complementary set of E^h and we used $\mu(f)=0$. It follows from (4) and Assumptions 2–3 that for any $u\in E^h$,

$$\begin{aligned} |P_t f(u)-P^\Delta_t f(u)| &\le \|f\|_{p,\gamma}\big(1+\mathbb{E}[\|X^u_t\|^p]+\mathbb{E}[\|Y^u_t\|^p]\big)^{\frac{1}{2}}\big(\mathbb{W}_2(\mu^u_t,\mu^{u,\Delta}_t)\big)^\gamma \\ &\le K\|f\|_{p,\gamma}(1+\|u\|^{(\frac{p}{2}+\gamma)(\tilde{r}\vee\tilde{q})})|\Delta^\alpha|^\gamma. \end{aligned} \tag{S21}$$

This, along with (25) implies that

$$\begin{aligned} &\mathbf{1}_{\{u\in E^h\}}\big|f(u)\big(P^\Delta_t f(u)-P_t f(u)\big)+f(u)\big(\mu(f)-\mu^\Delta(f)\big)\big| \\ &\le \mathbf{1}_{\{u\in E^h\}}|f(u)|\Big(\big|P^\Delta_t f(u)-P_t f(u)\big|+K\|f\|_{p,\gamma}|\Delta^\alpha|^\gamma\Big) \\ &\le K\|f\|^2_{p,\gamma}(1+\|u\|^{(\frac{p}{2}+\gamma)(\tilde{r}\vee\tilde{q})+\frac{p}{2}}\mathbf{1}_{\{u\in E^h\}})|\Delta^\alpha|^\gamma. \end{aligned}$$

Applying the Fubini theorem and the Fatou lemma leads to

$$\begin{aligned} &\overline{\lim_{|\Delta|\to 0}}\,\mu\Big(\mathbf{1}_{\{E^h\}}\int_0^\infty \big|f\big(P^\Delta_t f-P_t f\big)-f\big(\mu^\Delta(f)-\mu(f)\big)\big|\mathrm{d}t\Big) \\ &\le \int_0^\infty \mu\Big(\overline{\lim_{|\Delta|\to 0}}\,\mathbf{1}_{\{E^h\}}\big|f\big(P^\Delta_t f-P_t f\big)+f\big(\mu(f)-\mu^\Delta(f)\big)\big|\Big)\mathrm{d}t=0, \end{aligned}$$

which together with $\lim_{|\Delta|\to 0}\mu\big(f\int_0^\infty P_t f\mathrm{d}t\mathbf{1}_{\{E\setminus E^h\}}\big)=0$ gives $II_4\to 0$ as $|\Delta|\to 0$. Combining estimates of terms $II_i, i=1,2,3,4$ completes the proof. □

2. Proof of (16)

Proof. By the Hölder continuity $\|X_t^x - X^x_{\lfloor \frac{t}{\tau} \rfloor \tau}\|_{L^2(\Omega;E)} \le \Upsilon(\|x\|_{C((0,1);\mathbb{R})}, \|x\|_{\dot{E}^{\beta_1}})(t - \lfloor \frac{t}{\tau} \rfloor \tau)^{\frac{\beta_1}{2}} \le \Upsilon(\|x\|_{C((0,1);\mathbb{R})}, \|x\|_{\dot{E}^{\beta_1}})\tau^{\frac{\beta_1}{2}}$ (see e.g. (Cui, Hong and Sun, 2021, Section 2.2)) and the definition of $\{Y_t^{x^h,\Delta}\}_{t\ge 0}$, it suffices to prove that $\sup_{k\ge 0}\|X^x_{t_k} - Y^{x^h,\Delta}_{t_k}\|_{L^2(\Omega;E)} \le \Upsilon(\|x\|_{C((0,1);\mathbb{R})}, \|x\|_{\dot{E}^{\beta_1}})(\tau^{\frac{\beta_1}{2}} + N^{-\beta_1})$ with some polynomial Υ. Introduce the auxiliary process

$$\widetilde{Y}^{x^h,\Delta}_{t_{k+1}} = \widetilde{Y}^{x^h,\Delta}_{t_k} + A^h \widetilde{Y}^{x^h,\Delta}_{t_{k+1}}\tau + P^h F(X^x_{t_{k+1}})\tau + \delta W^h_k. \tag{S22}$$

We can rewritten (15) and (S22) as

$$\begin{aligned} Y^{x^h,\Delta}_{t_k} &= S^k_{h,\tau}x^h + \tau\sum_{i=0}^{k-1} S^{k-i}_{h,\tau}P^hF(Y^{x^h,\Delta}_{t_{i+1}}) + W^k_{A^h}, \\ \widetilde{Y}^{x^h,\Delta}_{t_k} &= S^k_{h,\tau}x^h + \tau\sum_{i=0}^{k-1} S^{k-i}_{h,\tau}P^hF(X^x_{t_{i+1}}) + W^k_{A^h}, \end{aligned}$$

where $S_{h,\tau} := (\mathrm{Id} - A^h\tau)^{-1}$, $W^k_{A^h} := \sum_{i=0}^{k-1} S^{k-i}_{h,\tau}\delta W^h_i$.

We split the error as $\|X^x_{t_k} - Y^{x^h,\Delta}_{t_k}\|_{L^2(\Omega;E)} \le \|X^x_{t_k} - P^hX^x_{t_k}\|_{L^2(\Omega;E)} + \|P^hX^x_{t_k} - \widetilde{Y}^{x^h,\Delta}_{t_k}\|_{L^2(\Omega;E)} + \|\widetilde{Y}^{x^h,\Delta}_{t_k} - Y^{x^h,\Delta}_{t_k}\|_{L^2(\Omega;E)}$. The first term on the right-hand side of the above inequality is estimated as

$$\|X^x_{t_k} - P^hX^x_{t_k}\|_{L^2(\Omega;E)} \le \|(-A)^{-\frac{\beta_1}{2}}(\mathrm{Id} - P^h)\|_{\mathcal{L}(E)}\|(-A)^{\frac{\beta_1}{2}}X^x_{t_k}\|_{L^2(\Omega;E)} \le \lambda_N^{-\frac{\beta_1}{2}}\|X^x_{t_k}\|_{L^2(\Omega;\dot{E}^{\beta_1})},$$

where $\lambda_N \sim N^2$ is the N-th eigenvalue of $-A$, and $\dot{E}^{\beta_1}$ is the Sobolev space generated by the fractional power of $-A$ (see e.g. (Kruse, 2014, Section B.2)). We claim that $\sup_{t\ge 0}\|X^x_t\|_{L^r(\Omega;\dot{E}^{\beta_1})} \le \Upsilon(\|x\|_{C((0,1);\mathbb{R})}, \|x\|_{\dot{E}^{\beta_1}})$ $\forall r \ge 2$ with some polynomial Υ. In fact, decompose the exact solution as $X^x_t = Y^0_t + Z^x_t$, where

$$\begin{aligned} \mathrm{d}Y^0_t &= AY^0_t\mathrm{d}t + F(Y^0_t + Z^x_t)\mathrm{d}t, \quad Y^0_0 = 0, \\ \mathrm{d}Z^x_t &= AZ^x_t\mathrm{d}t + \mathrm{d}W(t), \quad Z^x_0 = x. \end{aligned}$$

Then by the similar argument to the proof of (Cui, Hong and Sun, 2021, Lemma 2, Corollary 2), we derive that $\sup_{t\ge 0}\|Y^0_t\|_{L^r(\Omega;C((0,1);\mathbb{R}))} + \sup_{t\ge 0}\|Z^x_t\|_{L^r(\Omega;C((0,1);\mathbb{R}))} \le K(1 + \|x\|^{q_1}_{C((0,1);\mathbb{R})})$ with some $q_1 > 0$. As a result, by means of (Chen et al., 2023, Eq. (2.1)), we obtain

$$\begin{aligned} &\|Y^0_t\|_{L^r(\Omega;\dot{E}^{\beta_1})} \le \left\|\int_0^t (-A)^{\frac{\beta_1}{2}}S(t-s)F(Y^0_s + Z^x_s)\mathrm{d}s\right\|_{L^r(\Omega;E)} \\ &\le K\int_0^t (t-s)^{-\frac{\beta_1}{2}}e^{-\frac{\lambda_1}{2}(t-s)}\mathrm{d}s \sup_{s\ge 0}(\|Y^0_s\|^3_{L^{3r}(\Omega;C((0,1);\mathbb{R}))} + \|Z^x_s\|^3_{L^{3r}(\Omega;C((0,1);\mathbb{R}))}) \\ &\le K(1 + \|x\|^{3q_1}_{C((0,1);\mathbb{R})}). \end{aligned}$$

By the Burkholder–Davis–Gundy inequality and (Chen et al., 2023, Eq. (2.3)), we have

$$\sup_{t\geq 0}\|Z_t^x\|^r_{L^r(\Omega;\dot{E}^{\beta_1})}\leq K\|x\|^r_{\dot{E}^{\beta_1}}+K\sup_{t\geq 0}\Big(\sum_{j=1}^{\infty}\int_0^t\|(-A)^{\frac{\beta_1}{2}}S(t-s)Q^{\frac{1}{2}}e_j\|^2\mathrm{d}s\Big)^{\frac{r}{2}}$$

$$=K\|x\|^r_{\dot{E}^{\beta_1}}+K\sup_{t\geq 0}\Big(\sum_{j=1}^{\infty}\int_0^t\|(-A)^{\frac{1}{2}}S(t-s)(-A)^{\frac{\beta_1-1}{2}}Q^{\frac{1}{2}}e_j\|^2\mathrm{d}s\Big)^{\frac{r}{2}}\leq K(1+\|x\|^r_{\dot{E}^{\beta_1}}).$$

Combining the above two inequalities yields that

$$\sup_{t\geq 0}\|X_t^x\|_{L^r(\Omega;\dot{E}^{\beta_1})}\leq \sup_{t\geq 0}\|Y_t^0\|_{L^r(\Omega;\dot{E}^{\beta_1})}+\sup_{t\geq 0}\|Z_t^x\|_{L^r(\Omega;\dot{E}^{\beta_1})}\leq K(1+\|x\|^{3q_1}_{C((0,1);\mathbb{R})}+\|x\|_{\dot{E}^{\beta_1}}).$$

Hence, we have $\|X_{t_k}^x-P^hX_{t_k}^x\|_{L^2(\Omega;E)}\leq \Upsilon(\|x\|_{C((0,1);\mathbb{R})},\|x\|_{\dot{E}^{\beta_1}})\lambda_N^{-\frac{\beta_1}{2}}$.

For the term $\|P^hX_{t_k}^x-\widetilde{Y}_{t_k}^{x^h,\Delta}\|_{L^2(\Omega;E)}$, we have

$$\begin{aligned}\|P^hX_{t_k}^x-\widetilde{Y}_{t_k}^{x^h,\Delta}\|_{L^2(\Omega;E)}\leq\ &\|(e^{A^ht_k}-S_{h,\tau}^k)x^h\|_{L^2(\Omega;E)}\\ &+\Big\|\int_0^{t_k}e^{A^h(t_k-s)}P^hF(X_s^x)\mathrm{d}s-\tau\sum_{i=0}^{k-1}S_{h,\tau}^{k-i}P^hF(X_{t_{i+1}}^x)\Big\|_{L^2(\Omega;E)}\\ &+\Big\|\int_0^{t_k}e^{A^h(t_k-s)}\mathrm{d}W(s)-\sum_{i=0}^{k-1}S_{h,\tau}^{k-i}\delta W_i^h\Big\|_{L^2(\Omega;E)}=:\ JJ_1+JJ_2+JJ_3.\end{aligned}$$

The term JJ_1 can be estimated as $JJ_1\leq K\tau^{\frac{\beta_1}{2}}$ (see e.g. (Kruse, 2014, Lemma 3.12)).

For the term JJ_2, noting that similar to the proof of (Cui, Hong and Sun, 2021, Lemma 2), it can be shown that

$$\sup_{s\geq 0}\|F(X_s^x)\|_{L^2(\Omega;E)}\leq K\sup_{s\geq 0}\|X_s^x\|_{L^4(\Omega;E)}\sup_{s\geq 0}(1+\|X_s^x\|^2_{L^8(\Omega;C((0,1);\mathbb{R}))})\leq K(1+\|x\|^{6q_1}_{C((0,1);\mathbb{R})}),$$

which combining the Hölder continuity $\|X_s^x-X^x_{\lceil\frac{s}{\tau}\rceil\tau}\|_{L^4(\Omega;E)}\leq\Upsilon(\|x\|_{C((0,1);\mathbb{R})},\|x\|_{\dot{E}^{\beta_1}})(\lceil\frac{s}{\tau}\rceil\tau-s)^{\frac{\beta_1}{2}}$ (see e.g. (Cui, Hong and Sun, 2021, Section 2.2)) and $\|(-A)^{-\frac{\beta_1}{2}}(\mathrm{Id}-e^{A^h(s-\lfloor\frac{s}{\tau}\rfloor\tau)})\|_{\mathcal{L}(E)}\leq K(s-\lfloor\frac{s}{\tau}\rfloor\tau)^{\frac{\beta_1}{2}}$ (see e.g. (Kruse, 2014, Lemma B.9)) leads to

$$\begin{aligned}JJ_2\leq&\int_0^{t_k}e^{-\lambda_1(t_k-s)}\|F(X_s^x)-F(X^x_{\lceil\frac{s}{\tau}\rceil\tau})\|_{L^2(\Omega;E)}\mathrm{d}s\\ &+\int_0^{t_k}(t_k-s)^{-\frac{\beta_1}{2}}e^{-\frac{\lambda_1}{2}(t_k-s)}\|(-A)^{-\frac{\beta_1}{2}}(\mathrm{Id}-e^{A^h(s-\lfloor\frac{s}{\tau}\rfloor\tau)})\|_{\mathcal{L}(E)}\|F(X^x_{\lceil\frac{s}{\tau}\rceil\tau})\|_{L^2(\Omega;E)}\mathrm{d}s\\ &+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\|(e^{A^h(t_k-t_i)}-S_{h,\tau}^{k-i})P^hF(X^x_{\lceil\frac{s}{\tau}\rceil\tau})\|_{L^2(\Omega;E)}\mathrm{d}s\\ \leq&\ \Upsilon(\|x\|_{C((0,1);\mathbb{R})},\|x\|_{\dot{E}^{\beta_1}})\tau^{\frac{\beta_1}{2}}+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\|(e^{A^h(t_k-t_i)}-S_{h,\tau}^{k-i})P^hF(X^x_{\lceil\frac{s}{\tau}\rceil\tau})\|_{L^2(\Omega;E)}\mathrm{d}s.\end{aligned}$$

Note that the inequalities $\ln(1+\zeta) \geq \frac{1}{2}\zeta$ and $\zeta - \ln(1+\zeta) \leq K\zeta^2$ for $\zeta \in (0,1]$ give

$$\begin{aligned}
&\|(e^{A^h t_i} - S^i_{h,\tau})x\|^2 \\
&= \sum_{\lambda_j\tau\leq 1} e^{-2i\ln(1+\tau\lambda_j)}(1-e^{-i(\tau\lambda_j - \ln(1+\tau\lambda_j))})^2 x_j^2 \\
&\quad + \sum_{\lambda_j\tau>1} \frac{1}{t_i}\Big[(\lambda_j i\tau)^{\frac{1}{2}}(e^{-\lambda_j t_i} - (1+\tau\lambda_j)^{-i})\Big]^2 \lambda_j^{-1} x_j^2 \\
&\leq K \sum_{\lambda_j\tau\leq 1} e^{-t_i\lambda_j}(i\tau^2\lambda_j^2)^2 x_j^2 \\
&\quad + K\tau \sum_{\lambda_j\tau>1} \frac{1}{t_i}(1+\tau\lambda_1)^{-i}\Big[(\lambda_j i\tau)^{\frac{1}{2}}(e^{-\lambda_j t_i}(1+\tau\lambda_j)^{\frac{i}{2}} - (1+\tau\lambda_j)^{-\frac{i}{2}})\Big]^2 x_j^2.
\end{aligned}$$

Based on $e^{-\zeta^2} \leq K\zeta^{-s}$ for $s>0, \zeta>0$, we obtain $e^{-\frac{1}{2}t_i\lambda_j} \leq K(t_i\lambda_j)^{-3}$. And due to $\zeta^{\frac{1}{2}}e^{-\frac{1}{2}\zeta} \leq K$, $1+\zeta \leq e^{\zeta}$, and $(1+\zeta)^i \geq 1+i\zeta, i\geq 1$ for $\zeta>0$, we derive

$$(\lambda_j i\tau)^{\frac{1}{2}}e^{-\lambda_j t_i}(1+\tau\lambda_j)^{\frac{i}{2}} + (\lambda_j i\tau)^{\frac{1}{2}}(1+\tau\lambda_j)^{-\frac{i}{2}} \leq Ke^{-\frac{1}{2}\lambda_j t_i}(1+\tau\lambda_j)^{\frac{i}{2}} + \Big(\frac{\lambda_j i\tau}{1+i\tau\lambda_j}\Big)^{\frac{1}{2}} \leq K.$$

Thus,

$$\begin{aligned}
\|(e^{A^h t_i} - S^i_{h,\tau})x\|^2 &\leq K\sum_{\lambda_j\tau\leq 1} e^{-\frac{1}{2}t_i\lambda_1}(t_i\lambda_j)^{-3}t_i^2\tau^2\lambda_j^4 x_j^2 + K\tau t_i^{-1}(1+\tau\lambda_1)^{-i}\sum_{\lambda_j\tau>1} x_j^2 \\
&\leq K\tau t_i^{-1}(e^{-\frac{1}{2}t_i\lambda_1} \vee (1+\tau\lambda_1)^{-i})\|x\|^2.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
JJ_2 &\leq K(x)\tau^{\frac{\beta_1}{2}} + K\tau^{\frac{1}{2}}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}(e^{-\frac{\lambda_1}{4}(t_k-t_i)} \vee (1+\tau\lambda_1)^{-\frac{1}{2}(k-i)})t_{k-i}^{-\frac{1}{2}}\|F(X^x_{\lceil\frac{s}{\tau}\rceil\tau})\|_{L^2(\Omega;E)}\mathrm{d}s \\
&\leq \Upsilon(\|x\|_{C((0,1);\mathbb{R})}, \|x\|_{\dot{E}^{\beta_1}})\tau^{\frac{\beta_1}{2}}.
\end{aligned}$$

For the term JJ_3,

$$\begin{aligned}
JJ_3 &\leq \Big\|\int_0^{t_k} e^{A^h(t_k-s)}(\mathrm{Id} - e^{A^h(s-\lfloor\frac{s}{\tau}\rfloor\tau)})\mathrm{d}W(s)\Big\|_{L^2(\Omega;E)} \\
&\quad + \Big\|\int_0^{t_k}(e^{A^h(t_k-\lfloor\frac{s}{\tau}\rfloor\tau)} - S^{k-\lfloor\frac{s}{\tau}\rfloor}_{h,\tau})\mathrm{d}W(s)\Big\|_{L^2(\Omega;E)} =: JJ_{3,1} + JJ_{3,2}.
\end{aligned}$$

It follows from the Itô isometry and (Kruse, 2014, Lemma B.9) that the term $JJ_{3,1}$ can be estimated as

$$(JJ_{3,1})^2 = \int_0^{t_k}\|(-A)^{\frac{1}{2}}e^{A^h(t_k-s)}(-A)^{-\frac{\beta_1}{2}}(\mathrm{Id} - e^{A^h(s-\lfloor\frac{s}{\tau}\rfloor\tau)})(-A)^{\frac{\beta_1-1}{2}}Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2}\mathrm{d}s \leq K\tau^{\beta_1}.$$

Applying the Itô isometry again, we derive

$$(JJ_{3,2})^2 = \int_0^{t_k} \sum_{j=0}^{\infty} \|(-A)^{-\frac{\beta_1-1}{2}} (e^{A^h(t_k-\lfloor \frac{s}{\tau} \rfloor \tau)} - S_{h,\tau}^{k-\lfloor \frac{s}{\tau} \rfloor})(-A)^{\frac{\beta_1-1}{2}} Q^{\frac{1}{2}} e_j\|^2 \mathrm{d}s$$

$$\leq K\int_0^{t_k} \Big[\sum_{\lambda_j\tau\leq 1} \lambda_j^{-\beta_1+1} e^{-2(k-\lfloor \frac{s}{\tau} \rfloor)\ln(1+\tau\lambda_j)} (1-e^{-(k-\lfloor \frac{s}{\tau} \rfloor)(\tau\lambda_j-\ln(1+\tau\lambda_j))})^2 \|(-A)^{\frac{\beta_1-1}{2}} Q^{\frac{1}{2}} e_j\|^2$$

$$+ \sum_{\lambda_j\tau>1} 2\lambda_j^{-\beta_1}\lambda_j \Big((1+\tau\lambda_j)^{-2(k-\lfloor \frac{s}{\tau} \rfloor)} + e^{-2(t_k-\lfloor \frac{s}{\tau} \rfloor\tau)\lambda_j} \Big) \|(-A)^{\frac{\beta_1-1}{2}} Q^{\frac{1}{2}} e_j\|^2 \Big] \mathrm{d}s$$

$$\leq K\int_0^{t_k} \Big[\sum_{\lambda_j\tau\leq 1} \lambda_j^{1-\beta_1} e^{-\frac{1}{2}(k-\lfloor \frac{s}{\tau} \rfloor)\tau\lambda_j} ((t_k-\lfloor \frac{s}{\tau} \rfloor\tau)\lambda_j)^{-2} (k-\lfloor \frac{s}{\tau} \rfloor)^2 (\tau\lambda_j)^4 \|(-A)^{\frac{\beta_1-1}{2}} Q^{\frac{1}{2}} e_j\|^2$$

$$+ \sum_{\lambda_j\tau>1} \tau^{\beta_1}\lambda_j (e^{-2(k-\lfloor \frac{s}{\tau} \rfloor)\ln(1+\tau\lambda_j)} + e^{-2(t_k-\lfloor \frac{s}{\tau} \rfloor\tau)\lambda_j}) \|(-A)^{\frac{\beta_1-1}{2}} Q^{\frac{1}{2}} e_j\|^2 \Big] \mathrm{d}s \leq K\tau^{\beta_1},$$

where in the last step we used

$$\lambda_j^{-\beta_1}((t_k-\lfloor \frac{s}{\tau} \rfloor\tau)\lambda_j)^{-2}(k-\lfloor \frac{s}{\tau} \rfloor)^2(\tau\lambda_j)^4 \leq \lambda_j^{-\beta_1}\tau^2\lambda_j^2 \leq (\lambda_j\tau)^{2-\beta_1}\tau^{\beta_1} \leq \tau^{\beta_1}$$

for $\lambda_j\tau \leq 1$ and $\sup_{k,j}\int_0^{t_k} \lambda_j (e^{-\zeta(t_k-\lfloor \frac{s}{\tau} \rfloor\tau)\frac{\ln(1+\tau\lambda_j)}{\tau}} + e^{-\zeta(t_k-\lfloor \frac{s}{\tau} \rfloor\tau)\lambda_j})\mathrm{d}s \leq K$ for $\zeta > 0$. Moreover, by using the Burkholder–Davis–Gundy inequality (see e.g. (Kruse, 2014, Proposition 2.12)), one can derive that for $\tilde{q}_0 \geq 2$, $\|P^h X_{t_k}^x - \widetilde{Y}_{t_k}^{x^h,\Delta}\|_{L^{\tilde{q}_0}(\Omega;E)} \leq \Upsilon(\|x\|_{C((0,1);\mathbb{R})}, \|x\|_{\dot{E}^{\beta_1}})\tau^{\frac{\beta_1}{2}}$.

For the term $\|\widetilde{Y}_{t_k}^{x^h,\Delta} - Y_{t_k}^{x^h,\Delta}\|_{L^2(\Omega;E)}$, denoting $\tilde{e}_k := \widetilde{Y}_{t_k}^{x^h,\Delta} - Y_{t_k}^{x^h,\Delta}$, we have

$$\tilde{e}_k - \tilde{e}_{k-1} = A^h\tilde{e}_k\tau + \tau P^h(F(X_{t_k}^x) - F(Y_{t_k}^{x^h,\Delta})).$$

Hence, by taking $\langle\cdot, \tilde{e}_k\rangle$ on both sides, we obtain

$$\frac{1}{2}(\|\tilde{e}_k\|^2 - \|\tilde{e}_{k-1}\|^2) \leq \tau\langle A^h\tilde{e}_k, \tilde{e}_k\rangle + \tau\langle P^h(F(X_{t_k}^x) - F(Y_{t_k}^{x^h,\Delta})), \tilde{e}_k\rangle$$
$$\leq -\frac{\lambda_1-\lambda_F}{2}\|\tilde{e}_k\|^2\tau + \frac{\tau}{2(\lambda_1-\lambda_F)}\|F(X_{t_k}^x) - F(\widetilde{Y}_{t_k}^{x^h,\Delta})\|^2,$$

which gives

$$e^{\epsilon t_k}\|\tilde{e}_k\|^2 \leq \sum_{i=0}^{k-1} e^{\epsilon t_i}\|\tilde{e}_i\|^2\Big(\frac{e^{\epsilon\tau}}{1+(\lambda_1-\lambda_F)\tau} - 1\Big)$$
$$+ \frac{1}{1+(\lambda_1-\lambda_F)\tau}\sum_{i=0}^{k-1} e^{\epsilon t_{i+1}}\frac{\tau}{(\lambda_1-\lambda_F)}\|F(X_{t_{i+1}}^x) - F(\widetilde{Y}_{t_{i+1}}^{x^h,\Delta})\|^2.$$

Take $\epsilon \ll 1$ so that $\frac{e^{\epsilon\tau}}{1+(\lambda_1-\lambda_F)\tau} \leq 1$. Similar to the proof of (Cui, Hong and Sun, 2021, Lemmas 2 and 4), it can be shown that $\sup_{k\geq 0}\mathbb{E}[\|X_{t_k}^x\|_{C((0,1);\mathbb{R})}^8 + \|\widetilde{Y}_{t_k}^{x^h,\Delta}\|_{C((0,1);\mathbb{R})}^8] \leq \Upsilon(\|x\|_{C((0,1);\mathbb{R})}, \|x\|_{\dot{E}^{\beta_1}})$, which implies

$$
\begin{aligned}
&\sup_{k\geq 0}\mathbb{E}[\|F(X_{t_k}^x)-F(\widetilde{Y}_{t_k}^{x^h,\Delta})\|^2]\\
&\leq K\sup_{k\geq 0}\|X_{t_k}^x-\widetilde{Y}_{t_k}^{x^h,\Delta}\|^2_{L^4(\Omega;E)}\sup_{k\geq 0}(\mathbb{E}[1+\|X_{t_k}^x\|^8_{C((0,1);\mathbb{R})}+\|\widetilde{Y}_{t_k}^{x^h,\Delta}\|^8_{C((0,1);\mathbb{R})})])^{\frac{1}{2}}\\
&\leq \Upsilon(\|x\|_{C((0,1);\mathbb{R})},\|x\|_{\dot{E}^{\beta_1}})(\tau^{\beta_1}+N^{-2\beta_1}).
\end{aligned}
$$

Hence, we obtain $\mathbb{E}[\|\tilde{e}_k\|^2]\leq \Upsilon(\|x\|_{C((0,1);\mathbb{R})},\|x\|_{\dot{E}^{\beta_1}})(\tau^{\beta_1}+N^{-2\beta_1})$, which finishes the proof. □

3. Proofs of (20) and (21)

In the section, we are devoted to proving (20) and (21).

Proof of (20). To simplify notations we introduce the following abbreviations

$$
y(t_k)=y^{x,\tau}(t_k),\quad Y_{t_k}=Y_{t_k}^{x^h,\Delta},\quad b_k=b(Y_{t_k}^{x^h,\Delta}),\quad \sigma_k=\sigma(Y_{t_k}^{x^h,\Delta}).
$$

It follows from (18), $y(t_k)=Y_{t_k}(0)$, and (H1)–(H2) that for any $\epsilon\in(0,1)$,

$$
\begin{aligned}
&|y(t_{k+1})|^2=|y(t_k)|^2+|b_k|^2\tau^2+|\sigma_k\delta W_k|^2+2\langle y(t_k),b_k\rangle\tau+\mathcal{M}_k\\
&\leq |y(t_k)|^2+(1+\frac{1}{\epsilon})\tau^2|b(\mathbf{0})|^2+(1+\epsilon)\tau^2|b_k-b(\mathbf{0})|^2+\bar{L}_2^2|\delta W_k|^2+2\tau\langle y(t_k),b(\mathbf{0})\rangle\\
&\quad -2\bar{L}_3\tau|y(t_k)|^2+2\bar{L}_4\tau\int_{-\delta_0}^0|Y_{t_k}(\theta)|^2\mathrm{d}\nu_2(\theta)+\mathcal{M}_k\\
&\leq |y(t_k)|^2+K(1+\frac{1}{\epsilon})\tau+\bar{L}_2^2|\delta W_k|^2-\Big(2\bar{L}_3-(1+\epsilon)\tau\bar{L}_5-\epsilon\Big)\tau|y(t_k)|^2\\
&\quad +\Big(2\bar{L}_4+(1+\epsilon)\tau\bar{L}_5\Big)\tau\int_{-\delta_0}^0|Y_{t_k}(\theta)|^2\mathrm{d}\nu_2(\theta)+\mathcal{M}_k,
\end{aligned}
$$

where the martingale $\{\mathcal{M}_k\}_{k=0}^{\infty}$ is defined by $\mathcal{M}_k:=2\langle y(t_k),\sigma_k\delta W_k\rangle+2\langle b_k,\sigma_k\delta W_k\rangle\tau$. Then for any $q_0\in\mathbb{N}$ and $\tilde{\varepsilon}>0$,

$$
\begin{aligned}
e^{\tilde{\varepsilon}t_{k+1}}|y(t_{k+1})|^{2q_0}&=\sum_{i=0}^{k}\big(e^{\tilde{\varepsilon}t_{i+1}}|y(t_{i+1})|^{2q_0}-e^{\tilde{\varepsilon}t_i}|y(t_i)|^{2q_0}\big)+|x(0)|^{2q_0}\\
&\leq |x(0)|^{2q_0}+\sum_{i=0}^{k}(1-e^{-\tilde{\varepsilon}\tau})e^{\tilde{\varepsilon}t_{i+1}}|y(t_i)|^{2q_0}+\sum_{l=1}^{q_0}\sum_{i=0}^{k}C_{q_0}^l I_{l,i},
\end{aligned}
\tag{S23}
$$

where positive constants $C_{q_0}^l$ are binomial coefficients, and

$$
\begin{aligned}
I_{l,i}:=e^{\tilde{\varepsilon}t_{i+1}}|y(t_i)|^{2(q_0-l)}&\Big(K(1+\frac{1}{\epsilon})\tau+\bar{L}_2^2|\delta W_i|^2-\big(2\bar{L}_3-(1+\epsilon)\tau\bar{L}_5-\epsilon\big)\tau|y(t_i)|^2\\
&+\big(2\bar{L}_4+(1+\epsilon)\tau\bar{L}_5\big)\tau\int_{-\delta_0}^0|Y_{t_i}(\theta)|^2\mathrm{d}\nu_2(\theta)+\mathcal{M}_i\Big)^l.
\end{aligned}
$$

Applying the Young inequality and the Hölder inequality yields

$$I_{1,i} \le \frac{e^{\tilde{\varepsilon}t_{i+1}}K}{(\epsilon\tau)^{q_0-1}}\big((1+\frac{1}{\epsilon})^{q_0}\tau^{q_0}+\bar{L}_2^{2q_0}|\delta W_i|^{2q_0}\big)-\big(2\bar{L}_3-(1+\epsilon)\tau\bar{L}_5-3\epsilon\big)\tau e^{\tilde{\varepsilon}t_{i+1}}|y(t_i)|^{2q_0}$$
$$+\big(2\bar{L}_4+(1+\epsilon)\tau\bar{L}_5\big)\tau e^{\tilde{\varepsilon}t_{i+1}}\Big(\frac{q_0-1}{q_0}|y(t_i)|^{2q_0}+\frac{1}{q_0}\int_{-\delta_0}^{0}|Y_{t_i}(\theta)|^{2q_0}\mathrm{d}\nu_2(\theta)\Big)$$
$$+e^{\tilde{\varepsilon}t_{i+1}}|y(t_i)|^{2(q_0-1)}\mathcal{M}_i,$$

and

$$|I_{l,i}| \le K6^{l-1}e^{\tilde{\varepsilon}t_{i+1}}|y(t_i)|^{2(q_0-l)}\Big((1+\frac{1}{\epsilon})^{l}\tau^{l}+\bar{L}_2^{2l}|\delta W_i|^{2l}+\tau^{l}|y(t_i)|^{2l}$$
$$+\tau^{l}\int_{-\delta_0}^{0}|Y_{t_i}(\theta)|^{2l}\mathrm{d}\nu_2(\theta)+|y(t_i)|^{l}|\sigma_i\delta W_i|^{l}+\tau^{l}|b_i|^{l}|\sigma_i\delta W_i|^{l}\Big)$$
$$\le 3\epsilon\tau e^{\tilde{\varepsilon}t_{i+1}}|y(t_i)|^{2q_0}+\frac{Ke^{\tilde{\varepsilon}t_{i+1}}}{(\epsilon\tau)^{\frac{q_0-l}{l}}}\big((1+\frac{1}{\epsilon})^{q_0}\tau^{q_0}+\bar{L}_2^{2q_0}|\delta W_i|^{2q_0}\big)+K\tau^{l}e^{\tilde{\varepsilon}t_{i+1}}|y(t_i)|^{2q_0}$$
$$+K\tau^{l}\int_{-\delta_0}^{0}|Y_{t_i}(\theta)|^{2q_0}\mathrm{d}\nu_2(\theta)+\frac{Ke^{\tilde{\varepsilon}t_{i+1}}}{(\epsilon\tau)^{\frac{2q_0-l}{l}}}|\delta W_i|^{2q_0}+K\tau^{l}|b_i|^{q_0}|\delta W_i|^{q_0}$$

for the integer $l \in [2, q_0]$. Inserting the above inequalities into (S23) and using (H2), we obtain

$$e^{\tilde{\varepsilon}t_{k+1}}|y(t_{k+1})|^{2q_0}$$
$$\le |x(0)|^{2q_0}+\sum_{i=0}^{k}(1-e^{-\tilde{\varepsilon}\tau})e^{\tilde{\varepsilon}t_{i+1}}|y(t_i)|^{2q_0}+K_\epsilon\tau\sum_{i=0}^{k}e^{\tilde{\varepsilon}t_{i+1}}+K_\epsilon\sum_{i=0}^{k}\frac{|\delta W_i|^{2q_0}}{\tau^{q_0-1}}e^{\tilde{\varepsilon}t_{i+1}}$$
$$+K\tau^2\sum_{i=0}^{k}\mathscr{E}_i-\Big(q_0\big(2\bar{L}_3-(1+\epsilon)\tau\bar{L}_5\big)-3\epsilon\cdot 2^{q_0}\Big)\tau\sum_{i=0}^{k}e^{\tilde{\varepsilon}t_{i+1}}|y(t_i)|^{2q_0}$$
$$+q_0\big(2\bar{L}_4+(1+\epsilon)\tau\bar{L}_5\big)\tau\sum_{i=0}^{k}e^{\tilde{\varepsilon}t_{i+1}}\Big(\frac{q_0-1}{q_0}|y(t_i)|^{2q_0}+\frac{1}{q_0}\int_{-\delta_0}^{0}|Y_{t_i}(\theta)|^{2q_0}\mathrm{d}\nu_2(\theta)\Big)$$
$$+q_0\sum_{i=0}^{k}e^{\tilde{\varepsilon}t_{i+1}}|y(t_i)|^{2(q_0-1)}\mathcal{M}_i, \tag{S24}$$

where $\mathscr{E}_i := e^{\tilde{\varepsilon}t_{i+1}}|y(t_i)|^{2q_0}+e^{\tilde{\varepsilon}t_{i+1}}\int_{-\delta_0}^{0}|Y_{t_i}(\theta)|^{2q_0}\mathrm{d}\nu_2(\theta)+e^{\tilde{\varepsilon}t_{i+1}}|\delta W_i|^{2q_0}$. It follows from (19) and the convex property of $|\cdot|^{2q_0}$ that

$$\sum_{i=0}^{k}e^{\tilde{\varepsilon}t_{i+1}}\int_{-\delta_0}^{0}|Y_{t_i}(\theta)|^{2q_0}\mathrm{d}\nu_2(\theta)$$
$$=\sum_{j=-N}^{-1}\sum_{i=0}^{k}e^{\tilde{\varepsilon}t_{i+1}}\int_{t_j}^{t_{j+1}}\Big|\frac{t_{j+1}-\theta}{\tau}y(t_{i+j})+\frac{\theta-t_j}{\tau}y(t_{i+j+1})\Big|^{2q_0}\mathrm{d}\nu_2(\theta)$$

$$
\begin{aligned}
&\le e^{\tilde{\varepsilon}\delta_0}\sum_{j=-N}^{-1}\sum_{l=-N}^{k}\int_{t_j}^{t_{j+1}}\frac{t_{j+1}-\theta}{\tau}\mathrm{d}\nu_2(\theta)e^{\tilde{\varepsilon}t_{l+1}}|y(t_l)|^{2q_0}\\
&\quad+e^{\tilde{\varepsilon}\delta_0}\sum_{j=-N}^{-1}\sum_{l=-N}^{k}\int_{t_j}^{t_{j+1}}\frac{\theta-t_j}{\tau}\mathrm{d}\nu_2(\theta)e^{\tilde{\varepsilon}t_{l+1}}|y(t_l)|^{2q_0}\\
&\le Ne^{\tilde{\varepsilon}\delta_0}\|x\|^{2q_0}+e^{\tilde{\varepsilon}\delta_0}\sum_{i=0}^{k}e^{\tilde{\varepsilon}t_{i+1}}|y(t_i)|^{2q_0}.
\end{aligned}
\tag{S25}
$$

Plugging (S25) into (S24) and using $1-e^{-\tilde{\varepsilon}\tau}\le\tilde{\varepsilon}\tau$ leads to

$$
\begin{aligned}
&e^{\tilde{\varepsilon}t_{k+1}}|y(t_{k+1})|^{2q_0}\\
&\le Ke^{\tilde{\varepsilon}\delta_0}\|x\|^{2q_0}+K_\epsilon\tau\sum_{i=0}^{k}e^{\tilde{\varepsilon}t_{i+1}}+K_\epsilon\sum_{i=0}^{k}\frac{|\delta W_i|^{2q_0}}{\tau^{q_0-1}}e^{\tilde{\varepsilon}t_{i+1}}+K\tau^2\sum_{i=0}^{k}e^{\tilde{\varepsilon}t_{i+1}}|\delta W_i|^{2q_0}\\
&\quad-\Big(q_0\big(2\bar{L}_3-(1+\epsilon)\tau\bar{L}_5\big)-3\epsilon\cdot 2^{q_0}-\tilde{\varepsilon}-K\tau-q_0\big(2\bar{L}_4+(1+\epsilon)\tau\bar{L}_5\big)e^{\tilde{\varepsilon}\delta_0}\Big)\tau\\
&\quad\times\sum_{i=0}^{k}e^{\tilde{\varepsilon}t_{i+1}}|y(t_i)|^{2q_0}+q_0\sum_{i=0}^{k}e^{\tilde{\varepsilon}t_{i+1}}|y(t_i)|^{2(q_0-1)}\mathcal{M}_i.
\end{aligned}
\tag{S26}
$$

Owing to $\bar{L}_3\ge\bar{L}_1+\bar{L}_4$, choose $\tilde{\tau}\in(0,1]$ sufficiently small such that

$$
q_0\big(2\bar{L}_3-\tilde{\tau}\bar{L}_5\big)-K\tilde{\tau}-q_0\big(2\bar{L}_4+\tilde{\tau}\bar{L}_5\big)>0.
$$

Let $\tilde{\varepsilon}_0,\epsilon_0\in(0,1]$ be sufficiently small such that

$$
q_0\big(2\bar{L}_3-(1+\epsilon_0)\tilde{\tau}\bar{L}_5\big)-3\epsilon_0\cdot 2^{q_0}-\tilde{\varepsilon}_0-K\tilde{\tau}-q_0\big(2\bar{L}_4+(1+\epsilon_0)\tilde{\tau}\bar{L}_5\big)e^{\tilde{\varepsilon}_0\delta_0}\ge 0.
$$

This, along with (S26) implies that for any $\tau\in(0,\tilde{\tau}]$,

$$
\begin{aligned}
e^{\tilde{\varepsilon}_0t_{k+1}}|y(t_{k+1})|^{2q_0}\le{}&Ke^{\tilde{\varepsilon}_0\delta_0}\|x\|^{2q_0}+K_{\epsilon_0}\tau\sum_{i=0}^{k}e^{\tilde{\varepsilon}_0t_{i+1}}+K_{\epsilon_0}\sum_{i=0}^{k}\frac{|\delta W_i|^{2q_0}}{\tau^{q_0-1}}e^{\tilde{\varepsilon}_0t_{i+1}}\\
&+K\tau^2\sum_{i=0}^{k}e^{\tilde{\varepsilon}_0t_{i+1}}|\delta W_i|^{2q_0}+q_0\sum_{i=0}^{k}e^{\tilde{\varepsilon}_0t_{i+1}}|y(t_i)|^{2(q_0-1)}\mathcal{M}_i.
\end{aligned}
\tag{S27}
$$

Taking expectations in the above inequality we deduce

$$
e^{\tilde{\varepsilon}_0t_{k+1}}\mathbb{E}[|y(t_{k+1})|^{2q_0}]\le Ke^{\tilde{\varepsilon}_0\delta_0}\|x\|^{2q_0}+K_{\epsilon_0}e^{\tilde{\varepsilon}_0t_{k+1}},
$$

which implies

$$
\sup_{k\ge 0}\mathbb{E}[|y(t_k)|^{2q_0}]\le K(1+\|x\|^{2q_0}).
\tag{S28}
$$

By virtue of (S27) and (S28) we conclude that for any $\tau\in(0,\tilde{\tau}]$,

$$
e^{\tilde{\varepsilon}_0(t_{k+1}-\delta_0)}\mathbb{E}\Big[\sup_{(k-N)\vee 0\le l\le k}|y(t_{l+1})|^{2q_0}\Big]\le\ \mathbb{E}\Big[\sup_{(k-N)\vee 0\le l\le k}e^{\tilde{\varepsilon}_0t_{l+1}}|y(t_{l+1})|^{2q_0}\Big]
$$

$$
\begin{aligned}
&\leq Ke^{\tilde{\varepsilon}_0\delta_0}\|x\|^{2q_0}+K_{\epsilon_0}\tau\sum_{i=0}^{k}e^{\tilde{\varepsilon}_0 t_{i+1}}\\
&\quad+q_0\mathbb{E}\Big[\sup_{(k-N)\vee 0\leq l\leq k}\sum_{i=(k-N)\vee 0}^{l}e^{\tilde{\varepsilon}_0 t_{i+1}}|y(t_i)|^{2(q_0-1)}\mathcal{M}_i\Big]. \qquad \text{(S29)}
\end{aligned}
$$

Applying the Burkholder–Davis–Gundy inequality, the condition (H1), and the Young inequality yields

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{(k-N)\vee 0\leq l\leq k}\sum_{i=(k-N)\vee 0}^{l}e^{\tilde{\varepsilon}_0 t_{i+1}}|y(t_i)|^{2(q_0-1)}\mathcal{M}_i\Big]\\
&\leq K\mathbb{E}\Big[\Big(\sum_{i=(k-N)\vee 0}^{k}e^{2\tilde{\varepsilon}_0 t_{i+1}}|y(t_i)|^{4q_0-2}|\sigma_i|^2\tau\Big)^{\frac{1}{2}}\Big]\\
&\quad+K\mathbb{E}\Big[\Big(\sum_{i=(k-N)\vee 0}^{k}e^{2\tilde{\varepsilon}_0 t_{i+1}}|y(t_i)|^{4q_0-4}|b_i|^2|\sigma_i|^2\tau^{\frac{3}{2}}\Big)^{\frac{1}{2}}\Big]\\
&\leq\frac{1}{4q_0}\mathbb{E}\Big[e^{\tilde{\varepsilon}_0\tau}\Big(e^{\tilde{\varepsilon}_0 t_{(k-N)\vee 0}}|y(t_{(k-N)\vee 0})|^{2q_0}+\sup_{(k-N)\vee 0\leq i\leq k}e^{\tilde{\varepsilon}_0 t_{i+1}}|y(t_{i+1})|^{2q_0}\Big)\Big]\\
&\quad+Ke^{\tilde{\varepsilon}_0 t_{k+1}}+K\mathbb{E}\Big[\sum_{i=(k-N)\vee 0}^{k}|b_i|^{q_0}\tau^{\frac{q_0}{4}+1}e^{\tilde{\varepsilon}_0 t_{i+1}}\Big]\\
&\leq\frac{1}{2q_0}\mathbb{E}\Big[\sup_{(k-N)\vee 0\leq i\leq k}e^{\tilde{\varepsilon}_0 t_{i+1}}|y(t_{i+1})|^{2q_0}\Big]+K(1+\|x\|^{2q_0})e^{\tilde{\varepsilon}_0 t_{k+1}},
\end{aligned}
$$

where in the last step we used $e^{\zeta}\leq 1+2\zeta\leq 2$ for $\zeta\in(0,\frac{1}{2}]$, the hypothesis (H2), and (S28). Therefore, $\mathbb{E}\big[\sup_{(k-N)\vee 0\leq l\leq k}|y(t_{l+1})|^{2q_0}\big]\leq K(1+\|x\|^{2q_0})$. Combining the definition of $\{Y^{x^h,\Delta}_{t_k}\}_{k\in\mathbb{N}}$ (see (18)) implies (20). $\square$

Proof of (21). Introduce the continuous version of $\{y^{x,\tau}(t_k)\}_{k\in\mathbb{N}}$ as

$$
y^{x,\tau}(t)=y^{x,\tau}(t_k)+b(Y^{x^h,\Delta}_{t_k})(t-t_k)+\sigma(Y^{x^h,\Delta}_{t_k})\big(W(t)-W(t_k)\big)\quad\forall\, t\in[t_k,t_{k+1}). \qquad \text{(S30)}
$$

Recalling $Y^{x^h,\Delta}_t=\sum_{k=0}^{\infty}Y^{x^h,\Delta}_{t_k}\mathbf{1}_{[t_k,t_{k+1})}(t)$, it follows from (17) and (S30) that

$$
X^x(t)-y^{x,\tau}(t)=\int_0^t(b(X^x_s)-b(Y^{x^h,\Delta}_s))\mathrm{d}s+\int_0^t(\sigma(X^x_s)-\sigma(Y^{x^h,\Delta}_s))\mathrm{d}W_s.
$$

By the Itô formula and the Young inequality we deduce that for any $\epsilon\in(0,1)$,

$$
\begin{aligned}
&e^{\epsilon t}|X^x(t)-y^{x,\tau}(t)|^2\\
&=\int_0^t e^{\epsilon s}\Big(\epsilon|X^x(s)-y^{x,\tau}(s)|^2+2\langle X^x(s)-y^{x,\tau}(s),b(X^x_s)-b(Y^{x^h,\Delta}_s)\rangle\\
&\quad+|\sigma(X^x_s)-\sigma(Y^{x^h,\Delta}_s)|^2\Big)\mathrm{d}s+\mathcal{N}_t
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_0^t e^{\epsilon s}\Big(\epsilon|X^x(s)-y^{x,\tau}(s)|^2+2\langle X^x(s)-y^{x,\tau}(s), b(X_s^x)-b(y_s^{x,\tau})\rangle \\
&\quad +(1+\epsilon)|\sigma(X_s^x)-\sigma(y_s^{x,\tau})|^2\Big)\mathrm{d}s+\int_0^t \epsilon e^{\epsilon s}|X^x(s)-y^{x,\tau}(s)|^2\mathrm{d}s+\mathcal{E}_t+\mathcal{N}_t,
\end{aligned}
$$

where $\mathcal{N}_t$ is a martingale defined by

$$
\mathcal{N}_t=2\int_0^t e^{\epsilon s}\langle X^x(s)-y^{x,\tau}(s), (\sigma(X_s^x)-\sigma(Y_s^{x^h,\Delta}))\mathrm{d}W_s\rangle,
$$

the E-valued stochastic process $\{y_t^{x,\tau}\}_{t\ge 0}$ is defined by $y_t^{x,\tau}(\theta)=y^{x,\tau}(t+\theta)$, $\theta\in[-\delta_0,0]$, and $\mathcal{E}_t := \int_0^t e^{\epsilon s}\Big(\frac{1}{\epsilon}|b(y_s^{x,\tau})-b(Y_s^{x^h,\Delta})|^2+(1+\frac{1}{\epsilon})|\sigma(y_s^{x,\tau})-\sigma(Y_s^{x^h,\Delta})|^2\Big)\mathrm{d}s$. Using (H1)–(H2) yields

$$
\begin{aligned}
&e^{\epsilon t}|X^x(t)-y^{x,\tau}(t)|^2 \\
&\le \int_0^t\Big(-\big(2\bar{L}_3-2\epsilon-(1+\epsilon)\bar{L}_1\big)e^{\epsilon s}|X^x(s)-y^{x,\tau}(s)|^2+2\bar{L}_4\int_{-\delta_0}^0 e^{\epsilon s}|X_s^x(r)-y_s^{x,\tau}(r)|^2\mathrm{d}\nu_2(r) \\
&\quad +(1+\epsilon)\bar{L}_1\int_{-\delta_0}^0 e^{\epsilon s}|X_s^x(r)-y_s^{x,\tau}(r)|^2\mathrm{d}\nu_1(r)\Big)\mathrm{d}s+\mathcal{E}_t+\mathcal{N}_t \\
&\le -\Big(2\bar{L}_3-2\epsilon-(1+\epsilon)\bar{L}_1-(2\bar{L}_4+(1+\epsilon)\bar{L}_1)e^{\epsilon\delta_0}\Big)\int_0^t e^{\epsilon s}|X^x(s)-y^{x,\tau}(s)|^2\mathrm{d}s+\mathcal{E}_t+\mathcal{N}_t,
\end{aligned}
$$

where we used

$$
\begin{aligned}
&\int_0^t\int_{-\delta_0}^0 e^{\epsilon s}|X_s^x(r)-y_s^{x,\tau}(r)|^2\mathrm{d}\nu_i(r)\mathrm{d}s \\
&\le e^{\epsilon\delta_0}\int_{-\delta_0}^0 e^{\epsilon s}|X^x(s)-y^{x,\tau}(s)|^2\mathrm{d}s+e^{\epsilon\delta_0}\int_0^t e^{\epsilon s}|X^x(s)-y^{x,\tau}(s)|^2\mathrm{d}s \\
&= e^{\epsilon\delta_0}\int_0^t e^{\epsilon s}|X^x(s)-y^{x,\tau}(s)|^2\mathrm{d}s \quad \text{for } i=1,2.
\end{aligned}
$$

In view of $\bar{L}_3>\bar{L}_1+\bar{L}_4$, choose $\epsilon\in(0,1)$ sufficiently small such that

$$
2\bar{L}_3-2\epsilon-(1+\epsilon)\bar{L}_1-(2\bar{L}_4+(1+\epsilon)\bar{L}_1)e^{\epsilon\delta_0}\ge 0.
$$

This implies

$$
e^{\epsilon t}|X^x(t)-y^{x,\tau}(t)|^2\le\mathcal{E}_t+\mathcal{N}_t. \tag{S31}
$$

Making use of (H1)–(H3), similar to proofs of (Mao, 2003, Lemma 3.3) and (Li, Mao and Song, 2024, Lemma 3.6), one has

$$
\sup_{t\ge 0}\mathbb{E}[\|y_t^{x,\tau}-Y_t^{x^h,\Delta}\|^2]\le K(1+\|x\|^2)\tau. \tag{S32}
$$

This, along with (S31) and (H1) implies that

$$
\mathbb{E}[|X^x(t)-y^{x,\tau}(t)|^2]\le e^{-\epsilon t}\mathbb{E}[\mathcal{E}_t]\le K\int_0^t e^{\epsilon(s-t)}\sup_{\theta\in[-\delta_0,0]}\mathbb{E}[|y_s^{x,\tau}(\theta)-Y_s^{x^h,\Delta}(\theta)|^2]\mathrm{d}s
$$

$$\leq K(1+\|x\|^2)\tau. \tag{S33}$$

Furthermore, according to (S31), and using the Burkholder–Davis–Gundy inequality, we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{(t-\delta_0)\vee 0\leq u\leq t} e^{\epsilon u}|X^x(u)-y^{x,\tau}(u)|^2\Big] \leq \mathbb{E}[\mathcal{E}_t]+\mathbb{E}\Big[\sup_{(t-\delta_0)\vee 0\leq u\leq t}\mathcal{N}_u\Big]\\
\leq& K(1+\|x\|^2)e^{\epsilon t}\tau+K\mathbb{E}\Big[\Big(\sup_{(t-\delta_0)\vee 0\leq u\leq t} e^{\epsilon u}|X^x(u)-y^{x,\tau}(u)|^2\Big)^{\frac{1}{2}}\\
&\times\Big(\int_{(t-\delta_0)\vee 0}^{t} e^{\epsilon s}|\sigma(X_s^x)-\sigma(Y_s^{x^h,\Delta})|^2\mathrm{d}s\Big)^{\frac{1}{2}}\Big].
\end{aligned}$$

Applying the Young inequality, and then by (H1) and (S32) we arrive at

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{(t-\delta_0)\vee 0\leq u\leq t} e^{\epsilon u}|X^x(u)-y^{x,\tau}(u)|^2\Big]\\
\leq& K(1+\|x\|^2)e^{\epsilon t}\tau+\frac{1}{2}\mathbb{E}\Big[\sup_{(t-\delta_0)\vee 0\leq u\leq t} e^{\epsilon u}|X^x(u)-y^{x,\tau}(u)|^2\Big]\\
&+K\int_{(t-\delta_0)\vee 0}^{t} e^{\epsilon s}\mathbb{E}\big[|X^x(s)-Y^{x^h,\Delta}(s)|^2\big]\mathrm{d}s\\
&+\int_{(t-\delta_0)\vee 0}^{t} e^{\epsilon s}\int_{-\delta_0}^{0}\mathbb{E}\big[|X_s^x(r)-Y_s^{x^h,\Delta}(r)|^2\big]\mathrm{d}\nu_1(r)\mathrm{d}s.
\end{aligned}$$

Making use of (S32) and (S33) leads to

$$\begin{aligned}
&\frac{1}{2}e^{\epsilon(t-\delta_0)\vee 0}\mathbb{E}\Big[\sup_{(t-\delta_0)\vee 0\leq u\leq t}|X^x(u)-y^{x,\tau}(u)|^2\Big]\\
\leq& \frac{1}{2}\mathbb{E}\Big[\sup_{(t-\delta_0)\vee 0\leq u\leq t} e^{\epsilon u}|X^x(u)-y^{x,\tau}(u)|^2\Big]\leq K(1+\|x\|^2)e^{\epsilon t}\tau,
\end{aligned}$$

which implies $\mathbb{E}\Big[\sup_{(t-\tau)\vee 0\leq u\leq t}|X^x(u)-y^{x,\tau}(u)|^2\Big]\leq K(1+\|x\|^2)\tau$. Combining (S32) finishes the proof of the desired assertion (21). □

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