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Fully discrete schemes and L^p-strong convergence orders for the SPDE driven by Lévy noise

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Stochastic partial differential equations (SPDEs) driven by Lévy noise naturally arise in physical systems exhibiting abrupt or discontinuous random effects, whose temporal Hölder continuity in L^p sense is known to be at most $\frac{1}{p}$ resulting from the Burkholder–Davis–Gundy inequality. This poses a challenge in the numerical analysis for achieving the uniform L^p-strong convergence orders of numerical schemes for $p \geq 2$. In this paper, we develop a discretization framework for constructing fully discrete schemes tailored to different conditions of Lévy measures, whose spatial direction is based on the spectral Galerkin method and temporal direction employs the Euler-type method. For the case of finite Lévy measures, a jump-adapted time discretization is utilized for the equation that may involve multiplicative noise; while for the case that Lévy measures can be infinite, we introduce an approach based on the quantitative John–Nirenberg inequality for SPDEs driven by additive Lévy noise. We prove that proposed schemes converge in L^p sense with orders nearly $1/2$ in both space and time for all $p \geq 2$, which contributes novel results in the numerical analysis of the SPDE driven by Lévy noise. Numerical experiments are presented to illustrate our theoretical findings.

1. Introduction

Stochastic systems driven by Lévy noise arise naturally in modeling physical phenomena subject to random discontinuities or abrupt energy transfers, such as sudden phase transitions, intermittent turbulent bursts, and stochastic resonance phenomena; see e.g. [1–3] and references therein. In numerical studies, the L^p-strong convergence analysis of numerical methods has received much attention. Specifically, for $p \geq 1$, if there exist $C > 0$ independent of the time step size Δt and $\gamma > 0$ independent of p such that

$$\left(\mathbb{E}\left[\|X(T) - X_M\|^p\right]\right)^{1/p} \leq C\Delta t^{\gamma}, \quad T = M\Delta t > 0,$$

then the numerical approximation $\{X_m\}_{0\leq m\leq M}$ is said to converge strongly in the L^p sense with order γ. The L^p-strong convergence order provides a quantitative measure of the accuracy for approximating sample paths of the underlying solution process. Compared with the case of Gaussian noise, however, the L^p-strong convergence analysis for stochastic differential equations driven by Lévy

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noise presents distinctions due to the presence of jumps (see e.g. [4,5]). For instance, an extra term appears in the Burkholder–Davis–Gundy inequality of the stochastic integral with respect to the compensated Poisson random measure. As a result, the temporal Hölder continuity in L^p sense of the exact solution to a stochastic differential equation with Lévy noise does not exceed $1/p$, resulting in the deterioration of the L^p-strong convergence order γ of a numerical scheme for large p if one uses the temporal Hölder continuity directly. A natural question arises:

Q: *Can one obtain the L^p-strong convergence order γ of a numerical scheme for the stochastic differential equation with Lévy noise that does not depend on p?*

Some progress has been made recently on this question for the case of the stochastic ordinary differential equation (SODE) with Lévy noise: authors in [6] studied the L^p-strong convergence order of the Euler–Maruyama scheme for a multidimensional SODE with irregular Hölder drift, driven by additive Lévy process with exponent $\alpha \in (0, 2]$, and the obtained convergence order does not depend on p. Authors in [7] investigated the L^p-strong approximation of solutions to singular stochastic kinetic equations driven by additive α-stable processes by using an Euler-type scheme, and obtained a convergence rate aligning with the results in [6]. Authors in [8] presented an adaptive approximation scheme for jump-diffusion SODEs with discontinuous drift and (possibly) degenerate diffusion in the finite Lévy measure case, and obtained the strong convergence rate 1 in L^p for $p \in [1, \infty)$. In addition, authors in [9] investigated the strong convergence orders of a class of adaptive numerical methods for the jump-diffusion SODEs with non-globally Lipschitz continuous coefficients. For the case of SPDEs, to the best of our knowledge, this question is still open so far. The aim of this paper is to provide a positive answer for fully discrete schemes of the SPDE driven by Lévy noise.

We consider the following SPDE driven by Gaussian noise and Poisson random measure:

$$\begin{cases} dX(t) = AX(t)dt + F(X(t))dt + dW(t) + \int_\chi G(X(t), z)\tilde{N}(dz, dt), \ t \in (0, T], \\ X(0) = x_0, \end{cases} \tag{1}$$

where $T > 0$ and $A := \Delta : \mathrm{Dom}(A) \subset H \to H$ is the Laplacian operator with homogeneous Dirichlet boundary condition. Here $H := L^2(0, 1)$ endowed with the usual inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and $\chi := H \backslash \{0\}$ is the mark set. The process $\{W(t)\}_{t\in[0,T]}$ is an H-valued Gaussian space-time white noise. Let $\tilde{N}(dz, dt) := N(dz, dt) - \nu(dz)dt$ stand for a compensated Poisson random measure, where $\{N(\cdot, t)\}_{t\in[0,T]}$ is the Poisson random measure and ν is a Lévy measure on Borel σ-algebra $\mathcal{B}(\chi)$ satisfying $\nu(\{0\}) = 0$ and $\int_\chi \min\{1, \|z\|^2\}\nu(dz) < \infty$. Here $\{W(t)\}_{t\in[0,T]}$ and $\{\tilde{N}(\cdot, t)\}_{t\in[0,T]}$ are supposed to be independent. Define a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ generated by $\{W(t), \tilde{N}(\cdot, t)\}_{t\in[0,T]}$ that satisfies usual conditions. Precise assumptions on coefficients F and G will be given in Section 2.1. Equations of type (1) are widely studied in modeling physics phenomena, such as phenomena in fluid mechanics [2], neural model with random inputs [10] and so on. The theoretical analysis of such equations has been well developed, including the existence and uniqueness of solutions, as well as long-term dynamical behaviors, e.g., [11–13]. For the numerical studies, authors of [14] investigated the structure preservation and convergence analysis of a class of fully discrete finite difference schemes, and obtained the L^2-strong convergence orders that are nearly $1/2$ in space and $1/4$ in time. Authors of [15,16] also investigated the L^2-strong convergence order of the fully discrete schemes for (1), using the finite element method in space and Euler-type methods in time. In [17,18], authors studied the accuracy of spacial and temporal approximations by considering spectral method or finite elements method, and the explicit or implicit Euler method, respectively, and established the L^p-strong convergence result for $1 \le p \le 2$.

In this paper, we apply the spectral Galerkin method to approximate (1) in space, and further use the Euler-type methods in time to obtain the fully discrete schemes. The corresponding mapping for the one-step approximation is given as follows:

$$\begin{aligned} \Phi(X_i^N, \Delta t_{i+1}) := E(\Delta t_{i+1})X_i^N + \int_{t_i}^{t_{i+1}} E(t_{i+1} - s)P_N F(X_i^N)ds + \int_{t_i}^{t_{i+1}} E(t_{i+1} - s)P_N dW(s) \\ + \int_{t_i}^{t_{i+1}} \int_\chi E(\Delta t_{i+1})P_N G(X_i^N, z)\tilde{N}(dz, ds), \quad i = 0, 1, \dots, n_T - 1, \end{aligned} \tag{2}$$

where $\{E(t)\}_{t\ge 0}$ is the semigroup generated by A, P_N is the projection operator, n_T is the subscript corresponding to T (i.e., $t_{n_T} = T$), and we mention that the time step size $\Delta t_{i+1} := t_{i+1} - t_i$ may be path-dependent; see Section 2 for further details. The main contribution of our work is that, we show for the first time that proposed schemes converge in L^p sense with orders nearly $1/2$ in both space and time for all $p \ge 2$. Indeed, the L^p-strong convergence analysis in time is more technical compared with that in space. This is related to the aforementioned issue yielding poor order for large p. To overcome this order barrier, we develop discretization frameworks tailored to Poisson-driven stochastic systems, classified by different conditions on the Lévy measure $\nu(\chi)$.

We begin with the case $\nu(\chi) < \infty$, i.e., the system exhibits finitely many jumps almost surely, where the jump-adapted discretization strategy emerges as an effective approach. This approach involves superimposing finitely many random jump times onto a deterministic equidistant grid, thereby capitalizing on the system's continuous evolution between successive grids. It enables a crucial reformulation of the stochastic integral with respect to the compensated Poisson random measure in (2) as a more tractable integral against the intensity measure, which has been studied for numerical approximations of SODEs with Lévy noise (see e.g. [8,19–22]). For the considered SPDE (1), applying the jump-adapted strategy to the one-step mapping (2) yields the following form of discretization:

$$\begin{aligned} &X_{(i+1)-}^N = \Phi(X_i^N, \Delta t_{i+1}), \text{ with } \int_{t_i}^{t_{i+1}}\int_\chi Z_N(X_i^N, z)\tilde{N}(dz, ds) = -\int_{t_i}^{t_{i+1}}\int_\chi Z_N(X_i^N, z)\nu(dz)ds, \\ &X_{i+1}^N = X_{(i+1)-}^N + G_N(X_{(i+1)-}^N, p_{t_{i+1}})\mathbf{1}_{\{p_{t_{i+1}} \neq 0\}}. \end{aligned} \tag{3}$$

Here $\{p_{t_i}\}_{i=0,1,\ldots,n_T}$ is an $\mathcal{F}_{t_i}$-adapted Poisson point process with the intensity measure $\nu(dz)dt$. However, rigorous numerical analysis of such schemes to SPDEs introduces several new challenges distinct from the SODE case. One challenge involves the interaction between unbounded linear operators and nonlinear terms which complicates the analysis of temporal regularity. Another challenge arises from the infinite-dimensional nature of stochastic perturbations, in particular for the multiplicative noise case, where the coupling between the stochastic forcing and the nonlinear dynamics introduces additional analytical difficulties for controlling the discretization error. To this end, we first make full use of the smoothing effect and properties of the semigroup generated by the unbounded operator, and establish the one-half temporal Hölder regularity of numerical solution in negative Sobolev spaces. Then we introduce a technique that decouples the interaction between noise and nonlinear dynamics via chaos expansions specifically adapted to the Poisson jumps and Poisson distributions. Combining infinite-dimensional stochastic analysis of stochastic convolutions, we establish the uniform pth moment regularity estimates of numerical solutions in fractional power Sobolev spaces, and further prove that the global discretization error of numerical solution achieves order nearly $\frac{1}{2}$ in both spatial and temporal directions in L^p sense for all $p \geq 2$.

For the case $\nu(\chi) \leq \infty$, there may exist infinitely many jumps that make the jump-adapted strategy invalid. Meanwhile, the accumulation of high-frequency jump noise deeply affects temporal regularity in the stochastic convolution driven by the compensated Poisson random measure. These effects make the analysis of the L^p-strong convergence order particularly delicate. To the best of our knowledge, there has been no result on L^p-strong convergence order of numerical schemes for SPDEs (1) with case $\nu(\chi) = \infty$ and $p > 2$. To move in this direction, we focus our attention on the additive noise case, and consider a fully discrete scheme: $X_{i+1}^N = \Phi(X_i^N, \Delta t)$ with constant step size Δt. We introduce a technique based on nested conditional moment estimates, leveraging a quantitative version of the John–Nirenberg inequality (see [23]). The idea is to bound the L^p-norm of the stochastic convolution by moment estimates of the nested conditional "L^1-norms". This requires careful *a priori* estimates, including the nested conditional "L^2-norms" of both the stochastic convolution and the solution of the perturbed stochastic equation. By further presenting regularity estimates of the numerical solution, we finally establish the L^p-strong convergence order of the scheme in the case of $\nu(\chi) \leq \infty$ with additive noise. The treatment of the multiplicative noise case involves additional challenges and is left for future investigation.

The rest of the paper is organized as follows. The preliminaries, including notations, assumptions, and the well-posedness of the exact solution are given in Section 2.1. Section 2.2 is devoted to the introduction of the fully discrete schemes and their L^p-strong convergence orders. In Section 3, we present the L^p-strong convergence analysis of the fully discrete scheme for (1) in the case of $\nu(\chi) < \infty$. In Section 4, we present the corresponding L^p-strong convergence analysis in the case of $\nu(\chi) \leq \infty$. Numerical experiments are given in Section 5.

2. Fully discrete schemes and main results

In this section, we first give some preliminaries, including notation, assumptions, some useful inequalities, and the well-posedness of the exact solution. Second, we present the fully discrete schemes of (1) for two different cases of Lévy measures, and respectively show their L^p-strong convergence orders.

2.1. Preliminaries

Throughout this paper, we let $C > 0$ be a generic constant that may vary from one place to another. More specific constants which depend on certain parameters a, b are numbered as $C_{a,b}$. We use $\mathbb{N}_+$ to denote the set of positive natural numbers. Set $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. The space of bounded linear operators on H is denoted by $\mathcal{L}(H)$, which endowed with the operator norm $\|\cdot\|_{\mathcal{L}(H)}$. The subspace $\mathcal{L}_2(H) \subset \mathcal{L}(H)$ denotes the set of Hilbert–Schmidt operators on H, with norm denoted by $\|\cdot\|_{\mathcal{L}_2(H)}$. Let $\dot{H}^s$, $s \in \mathbb{R}$ be the Sobolev space generated by the fractional power of $-A$, endowed with the inner product $\langle u, v\rangle_{\dot{H}^s} := \langle (-A)^{\frac{s}{2}} u, (-A)^{\frac{s}{2}} v\rangle$ and the norm $\|u\|_{\dot{H}^s} := \sqrt{\langle u, u\rangle_{\dot{H}^s}}$ for $u, v \in \dot{H}^s$. We use the notation $L^p(\Omega, \dot{H}^s)$ to denote the space of random variables u satisfying $\mathbb{E}[\|u\|_{\dot{H}^s}^p] < \infty$ for $p > 0$.

It is known that there is a sequence of real numbers $\lambda_i = \pi^2 i^2, i \in \mathbb{N}_+$ and an orthonormal basis $\{e_i\}_{i\in\mathbb{N}_+}$ of H with $e_i(x) := \sqrt{2}\sin(i\pi x)$, $x \in [0, 1]$, such that $-Ae_i = \lambda_i e_i$. In addition, operator A generates the semigroup $\{E(t) := e^{tA}, t \geq 0\}$ on H. For all $t \geq 0$, $E(t)$ is a bounded self-adjoint linear operator on H, with $\|E(t)\|_{\mathcal{L}(H)} \leq e^{-t\lambda_1} \leq 1$. It is clear that, for all $t \geq 0$,

$$\int_0^t s^{-\alpha} \|E(s)\|_{\mathcal{L}_2(H)}^2 ds < \infty \qquad \text{if and only if} \quad \alpha \in [0, \frac{1}{2}).$$

Moreover, the semigroup admits the following properties:

$$\|(-A)^{\gamma} E(t)\|_{\mathcal{L}(H)} \leq C t^{-\gamma} e^{-\frac{\lambda_1}{2} t}, \quad t \in (0, T],\ \gamma \geq 0, \tag{4}$$

$$\|(-A)^{-\rho}(E(t) - E(s))\|_{\mathcal{L}(H)} \leq C(t - s)^{\rho}, \quad 0 \leq s < t \leq T,\ \rho \in [0, 1]. \tag{5}$$

Here we impose some assumptions on coefficients and the initial value of (1), which will be used throughout this paper.

Assumption 1 (Nonlinearity). The measurable mapping $F : H \to H$ satisfies that $\|F(x)\| \leq C(1 + \|x\|)$, $x \in H$ with some constant $C > 0$. In addition, F is differentiable and there are constants $L_F, C > 0$ such that

$$\sup_{x\in H} \|DF(x)y\| \leq L_F \|y\|, \quad y \in H,$$

$$\|(-A)^{-\frac{1+\delta}{4}} DF(x)y\| \leq C(1 + \|x\|_{\dot{H}^{\frac{1-\delta}{2}}})\|y\|_{\dot{H}^{-\frac{1-\delta}{2}}}, \quad x \in \dot{H}^{\frac{1-\delta}{2}}, y \in \dot{H}^{-\frac{1-\delta}{2}},$$

where $\delta \in (0, 1)$.

Remark 1. (i) The assumption $\sup_{x\in H}\|DF(x)y\|\le L_F\|y\|$, combining the mean value theorem implies the globally Lipschitz condition on F, namely, for some $\theta\in(0,1)$,

$$\|F(x)-F(y)\|\le\|DF(y+\theta(x-y))(x-y)\|\le L_F\|x-y\|,\quad x,y\in H. \tag{6}$$

(ii) The conditions in Assumption 1 are satisfied if F is the Nemytskii operator defined by $F(u)(x):=f(u(x))$, $x\in[0,1]$, $u\in H$, where $f:\mathbb{R}\to\mathbb{R}$ is continuously differentiable with bounded first order derivative. Indeed, the mapping $F: H\to H$ is then differentiable, and for all $u,h\in H$ and $x\in[0,1]$, we have $[DF(u)h](x)=f'(u(x))h(x)$. This implies $\sup_{u\in H}\|DF(u)h\|\le L_F\|h\|, h\in H$ with some constant $L_F>0$. In addition, for all $u_1\in\dot H^{\frac{1-\delta}{2}}$ and $u_2\in\dot H^{-\frac{1-\delta}{2}}$ with $\delta\in(0,1)$, we obtain

$$\begin{aligned}\left\|(-A)^{-\frac{1+\delta}{4}}DF(u_1)u_2\right\| &= \sup_{h\in H\setminus\{0\}}\frac{\langle(-A)^{-\frac{1+\delta}{4}}DF(u_1)u_2,h\rangle}{\|h\|}\\ &\le \sup_{h\in H\setminus\{0\}}\frac{\|(-A)^{-\frac{1-\delta}{4}}u_2\|\|(-A)^{\frac{1-\delta}{4}}DF(u_1)(-A)^{-\frac{1+\delta}{4}}h\|}{\|h\|}\\ &\le C(1+\|(-A)^{\frac{1-\delta}{4}}u_1\|)\|(-A)^{-\frac{1-\delta}{4}}u_2\|,\end{aligned}$$

where we used [24, Lemma 4.4] in the last step.

Assumption 2 (Jump coefficient)**.** The coefficient $G: H\times\chi\to H$ is defined as $G(x,z)=g_1(z)x+g(z)$, $x\in H$, $z\in\chi$ with mapping $g_1:\chi\to\mathbb{R}$ being bounded by constant $b>0$ and with some mapping $g:\chi\to H$. In addition, for each $p>2$, there exists a constant $C_p>0$ such that

$$\int_\chi\left(|g_1(z)|^2\vee|g_1(z)|^p\right)\nu(dz)\le C_p,\quad \int_\chi\left(\|(-A)^{\frac14}g(z)\|^2\vee\|(-A)^{\frac14}g(z)\|^p\right)\nu(dz)\le C_p. \tag{7}$$

Remark 2. We give an example of g that satisfies Assumption 2: $g(z):=(1\wedge\|z\|^2)\sum_{j\in\mathbb{N}_+}j^{-1-\epsilon}e_j$, where $\{e_j\}_{j\in\mathbb{N}_+}$ is the complete orthonormal basis of H. We verify the second inequality in (7) as follows. Note that for all $p\ge2$, we have

$$\int_\chi\|(-A)^{\frac14}g(z)\|^p\nu(dz)=\int_\chi\left(\sum_{j\in\mathbb{N}_+}\lambda_j^{1/2}|\langle g(z),e_j\rangle|^2\right)^{\frac p2}\nu(dz).$$

For each $j\in\mathbb{N}_+$, $|\langle g(z),e_j\rangle|\le j^{-1-\epsilon}(1\wedge\|z\|^2)$. Thus we have

$$\sum_{j\in\mathbb{N}_+}\lambda_j^{1/2}|\langle g(z),e_j\rangle|^2\le\pi\left(1\wedge\|z\|^2\right)^2\sum_{j\in\mathbb{N}_+}j^{-1-2\epsilon}\le C\pi\left(1\wedge\|z\|^2\right)^2,$$

and it follows that

$$\int_\chi\|(-A)^{\frac14}g(z)\|^p\nu(dz)\le C_p\int_\chi\left(1\wedge\|z\|^2\right)^p\nu(dz)\le C_p\int_\chi(1\wedge\|z\|^2)\nu(dz)\le C_p.$$

Assumption 3 (Initial value)**.** Let x_0 be an $\mathcal{F}_0$-measurable random variable, and $x_0\in L^p(\Omega,\dot H^{\frac{1-\delta}{2}})$ for all $p\ge2$ and $\delta\in(0,1]$.

We introduce the maximal inequality for the Lévy-type stochastic convolution as follows, see e.g. [25, Proposition 3.3] for details.

Lemma 1. *Let $\beta:[0,T]\times\chi\to H$ be a predictable process such that the expectation on the right hand side of the inequality below is finite. Then for all $p\ge2$, there exists a constant $C_{p,T}>0$ such that*

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]}\Big\|\int_0^t\int_\chi E(t-s)\beta(s,z)\tilde N(dz,ds)\Big\|^p\Big]\\ &\le C_{p,T}\mathbb{E}\Big[\int_0^T\Big\{\Big(\int_\chi\|\beta(s,z)\|^2\nu(dz)\Big)^{p/2}+\int_\chi\|\beta(s,z)\|^p\nu(dz)\Big\}ds\Big].\end{aligned}$$

Under assumptions given above, the well-posedness and regularity estimate of the mild solution of (1) can be obtained, which are stated in the following proposition. The proof of the well-posedness is similar to that of [11, Theorem 2.1], and the proof of the strong Markov property is similar to that of [11, Lemma 5.2], which are omitted here. For the proof of the regularity of the exact solution, we present it in the supplementary material [26].

Proposition 1. *Let Assumptions 1–3 hold. Then for each $T>0$, there exists a unique mild solution $\{X(t)\}_{t\in[0,T]}$ of (1)*

$$X(t)=E(t)x_0+\int_0^tE(t-s)F(X(s))ds+\int_0^tE(t-s)dW(s)+\int_0^t\int_\chi E(t-s)G(X(s),z)\tilde N(dz,ds),\quad t\in(0,T]$$

with initial value x_0, satisfying that for any $p\ge2$, $\mathbb{E}[\sup_{t\in[0,T]}\|X(t)\|^p]<\infty$. The process $\{X(t)\}_{t\in[0,T]}$ has a càdlàg modification and is time-homogeneous strong Markovian. Moreover, for all $p\ge2$ and all $\alpha\in[0,\frac12)$, there exists a constant $C_{p,T,\alpha}>0$ such that $\sup_{t\in[0,T]}\|X(t)\|_{L^p(\Omega,\dot H^\alpha)}\le C_{p,T,\alpha}$.

2.2. Fully discrete schemes and L^p-strong convergence orders

In this subsection, we first apply the spectral Galerkin method in the spatial direction and the Euler-type method in the temporal direction to obtain the fully discrete schemes for (1). Then we present the L^p-strong convergence orders of proposed schemes in both space and time.

For fixed $N \in \mathbb{N}_+$, let P_N be the orthogonal projection operator from H onto $H_N := \text{span}\{e_1, e_2, \dots, e_N\}$. Applying the spectral Galerkin method in the spatial direction to (1) yields

$$dX^N(t) = AX^N(t)dt + F_N(X^N(t))dt + P_N dW(t) + \int_\chi G_N(X^N(t), z)\tilde{N}(dz, dt), \quad t \in (0, T] \tag{8}$$

with initial value $X^N(0) = P_N x_0$, where $F_N := P_N F$ and $G_N := P_N G = g_1 P_N + P_N g$. Further, in the temporal direction, we use the Euler-type method to obtain the one-step approximation:

$$\begin{aligned}\Phi(X_i^N, \Delta t_{i+1}) = E(\Delta t_{i+1})X_i^N &+ \int_{t_i}^{t_{i+1}} E(t_{i+1} - s)F_N(X_i^N)ds + \int_{t_i}^{t_{i+1}} E(t_{i+1} - s)P_N dW(s) \\ &+ \int_{t_i}^{t_{i+1}}\int_\chi E(\Delta t_{i+1})G_N(X_i^N, z)\tilde{N}(dz, ds), \quad i = 0, 1, \dots, n_T - 1,\end{aligned} \tag{9}$$

where $\Delta t_{i+1} = t_{i+1} - t_i$ with $\{t_i\}_{i=0,1,\dots,n_T}$ being time grid points. With the one-step approximation at hand, below, we present the fully discrete schemes for two different cases of Lévy measures: $\nu(\chi) < \infty$ and $\nu(\chi) \le \infty$. We remark that the choices of time step sizes $\{\Delta t_i\}_{i=1,\dots,n_T}$ are different in these two cases.

(i) Case of $\nu(\chi) < \infty$. In this case, since $\mathbb{E}[N(\chi, [0, T])] = T\nu(\chi) < \infty$, there are a.s. finitely many jumps. The corresponding non-decreasing jump times are denoted by $\{\sigma_i(\omega), i = 1, 2, \dots, n_J(\omega)\}$, where the random variable n_J denotes the number of jumps. We introduce the jump-adapted time partition by a superposition of finitely many jump times to a deterministic equidistant grid. More precisely, we first give a deterministic partition of the interval $[0, T]$:

$$\mathcal{T}^0 = \{0 = t_0^0 < t_1^0 < \dots < t_M^0 = T\},$$

where $t_i^0 := i\Delta t, i = 0, 1, \dots, M$ with $\Delta t > 0$ being a constant step size. For each sample point, we then merge $\mathcal{T}^0$ and the partition from jump times $\mathcal{T}^1 := \{0 \le \sigma_1 < \sigma_2 < \dots < \sigma_{n_J} \le T\}$ to form a new jump-adapted partition

$$\mathcal{T} = \{0 = t_0 < t_1 < \dots < t_{n_T} = T\},$$

where n_T denotes the subscript corresponding to T and $n_J + M \ge n_T$. Note that in the new partition $\mathcal{T}$, step size $\Delta t_{i+1} = t_{i+1} - t_i$ is path-dependent and non-uniform, with the maximal step size bounded by Δt.

We propose the jump-adapted fully discrete scheme as follows:

$$X_{(i+1)-}^N = \Phi(X_i^N, \Delta t_{i+1}), \quad X_{i+1}^N = X_{(i+1)-}^N + G_N(X_{(i+1)-}^N, p_{t_{i+1}})\mathbf{1}_{\{p_{t_{i+1}} \neq 0\}}, \tag{10}$$

with $\Phi(X_i^N, \Delta t_{i+1})$ being given by (9). Taking advantage of the jump-adapted partition, we have the relation

$$\int_{t_i}^{t_{i+1}}\int_\chi E(\Delta t_{i+1})G_N(X_i^N, z)\tilde{N}(dz, dt) = -\int_{t_i}^{t_{i+1}}\int_\chi E(\Delta t_{i+1})G_N(X_i^N, z)\nu(dz)dt, \quad a.s., \tag{11}$$

for $i = 0, 1, \dots, n_T - 1$. Hence (10) is equivalent to

$$\begin{cases} X_{(i+1)-}^N = E(\Delta t_{i+1})X_i^N + \int_{t_i}^{t_{i+1}} E(t_{i+1} - s)F_N(X_i^N)ds + \int_{t_i}^{t_{i+1}} E(t_{i+1} - s)P_N dW(s) \\ \qquad\qquad - \int_{t_i}^{t_{i+1}}\int_\chi E(\Delta t_{i+1})G_N(X_i^N, z)\nu(dz)ds, \\ X_{i+1}^N = X_{(i+1)-}^N + G_N(X_{(i+1)-}^N, p_{t_{i+1}})\mathbf{1}_{\{p_{t_{i+1}} \neq 0\}}, \quad i = 0, 1, \dots, n_T - 1. \end{cases} \tag{12}$$

We now give the L^p-strong convergence result of (12), whose proof is presented in Section 3.

Theorem 1. *Let Assumptions 1–3 hold, and let $N \in \mathbb{N}_+$ and $p \ge 2$. Then for any $\delta \in (0, 1)$, there exists a constant $C_{p,T,\delta} > 0$ independent of $N, \Delta t$ such that*

$$\|X(T) - X_{n_T}^N\|_{L^p(\Omega, H)} \le C_{p,T,\delta}\big(N^{-\frac{1-\delta}{2}} + (\Delta t)^{\frac{1-\delta}{2}}\big).$$

As a corollary of the pth moment strong convergence, we obtain the following almost sure convergence. The proof is standard, which follows from Theorem 1, together with the Chebyshev inequality and the Borel–Cantelli lemma; see e.g., [27, Corollary 3.19], [28, Theorem 4.2] for a similar proof.

Corollary 1. *Let conditions of Theorem 1 hold. For any $\delta_1 \in (\delta, 1)$ with δ given in Theorem 1, there exists an a.s. finite random variable $C_{p,T,\delta,\delta_1} > 0$ such that*

$$\Big\|X(T, \omega) - X_{n_T}^N(\omega)\Big\| \le C_{p,T,\delta,\delta_1}(\omega)\big(N^{-\frac{1-\delta_1}{2}} + (\Delta t)^{\frac{1-\delta_1}{2}}\big).$$

(ii) Case of $\nu(\chi) \le \infty$. In this case, there may exist infinitely many jumps that make the jump-adapted strategy invalid. In order to obtain the L^p-strong convergence order, we focus on the additive noise case and consider the fully discrete scheme on the deterministic partition $\mathcal{T}^0$ with constant step size Δt. For simplicity, we write t_i instead of time grid point $t_i^0, i = 0, 1, \dots, M$ and $n_T = M$. Then the fully discrete scheme based on the one-step approximation (9) is given as

$$X_{k+1}^N = \Phi(X_k^N, \Delta t),\ k = 0, 1, \dots, n_T - 1, \quad X_0^N = P_N x_0. \tag{13}$$

To obtain the L^p-strong convergence order of this scheme, we present the following assumption on the jump coefficient.

Assumption 4. Let mapping $G : \chi \to H$ satisfy that for any $\delta \in (0, 1)$ and $p > 2$,

$$\int_\chi \big(\|(-A)^{\frac{1-\delta}{2}} G(z)\|^2 \vee \|(-A)^{\frac{1-\delta}{2}} G(z)\|^p\big)\nu(dz) \le C_{p,\delta}. \tag{14}$$

Similar to Remark 2, when $G(z) = (1 \wedge \|z\|^2) \sum_{j\in\mathbb{N}_+} j^{-2-\epsilon} e_j$, Assumption 4 is satisfied. We now give the L^p-strong convergence result of the scheme (13) for the case of the additive Poisson noise as follows, and its proof is presented in Section 4.

Theorem 2. *Let Assumptions 1, 3 and 4 hold, and let $N \in \mathbb{N}_+$ and $p \ge 2$. Then for any $\delta \in (0, 1)$, there exists a constant $C_{p,T,\delta} > 0$ independent of $N, \Delta t$ such that*

$$\Big\|X(T) - X_{n_T}^N\Big\|_{L^p(\Omega,H)} \le C_{p,T,\delta}\big(N^{-\frac{1-\delta}{2}} + (\Delta t)^{\frac{1-\delta}{2}}\big).$$

Corollary 2. *Let conditions of Theorem 2 hold. For any $\delta_2 \in (\delta, 1)$ with δ given in Theorem 2, there exists an a.s. finite random variable $C_{p,T,\delta,\delta_2} > 0$ such that*

$$\Big\|X(T, \omega) - X_{n_T}^N(\omega)\Big\| \le C_{p,T,\delta,\delta_2}(\omega)\big(N^{-\frac{1-\delta_2}{2}} + (\Delta t)^{\frac{1-\delta_2}{2}}\big).$$

3. Proof of Theorem 1

In this section, we present the L^p-strong convergence analysis of fully discrete scheme (12) for (1) driven by the linear multiplicative Poisson noise with finite Lévy measure. Note that the stochastic integral with respect to the compensated Poisson random measure in (12) can be transformed into the integral with respect to the intensity measure, as shown in (11). This transformation is crucial to overcome the order barrier caused by the pth moment estimates of the stochastic integral with respect to the compensated Poisson random measure.

Proof. (**Proof of** Theorem 1) The proof is split into two steps, based on the error between the solutions of (1) and the semi-discrete scheme (8), and the error between the solutions of (8) and the fully discrete scheme (12).

Step 1. *Show that for all $p \ge 2$ and $\delta \in (0, 1)$, there exists a constant $C_{p,T,\delta} > 0$* independent of N such that $\|X(T) - X^N(T)\|_{L^p(\Omega,H)} \le C_{p,T,\delta} N^{-\frac{1-\delta}{2}}$.

It follows from (1) and (8) that, for all $N \in \mathbb{N}_+$,

$$\begin{aligned} X(T) - X^N(T) =& E(T)(\mathrm{I} - P_N)x_0 + \int_0^T E(T-s)(\mathrm{I} - P_N)F(X(s))ds \\ &+ \int_0^T E(T-s)P_N\big(F(X(s)) - F(X^N(s))\big)ds + (\mathrm{I} - P_N)\int_0^T E(T-s)dW(s) \\ &+ \int_0^T \int_\chi E(T-s)(\mathrm{I} - P_N)G(X(s), z)\tilde{N}(dz, ds) \\ &+ \int_0^T \int_\chi E(T-s)P_N\big(G(X(s), z) - G(X^N(s), z)\big)\tilde{N}(dz, ds), \end{aligned}$$

where $\mathrm{I} \colon H \to H$ is an identical mapping. Denote the stochastic convolution $\mathcal{W}(t) := \int_0^t E(t-s)dW(s)$. Owing to properties of projection operator P_N, assumptions on coefficients F and G, Lemma 1, and Proposition 1, we obtain

$$\begin{aligned} &\mathbb{E}\big[\|X(T) - X^N(T)\|^p\big] \\ \le& C_p \lambda_N^{-\frac{1-\delta}{4}p}\|x_0\|^p_{L^p(\Omega,\dot H^{\frac{1-\delta}{2}})} + C_p\lambda_N^{-\frac{1-\delta}{4}p}\Big(\int_0^T (T-s)^{-\frac{1-\delta}{4}}ds\Big)^p \sup_{s\in[0,T]} \mathbb{E}[\|F(X(s))\|^p] \\ &+ C_{p,T}\int_0^T \mathbb{E}\Big[\Big\|X(s) - X^N(s)\Big\|^p\Big]ds + C_p\lambda_N^{-\frac{1-\delta}{4}p}\|\mathcal{W}(T)\|^p_{L^p(\Omega,\dot H^{\frac{1-\delta}{2}})} \\ &+ C_{p,T}\lambda_N^{-\frac{1-\delta}{4}p}\mathbb{E}\Big[\int_0^T\Big\{\int_\chi\Big\|(-A)^{\frac{1-\delta}{4}}G(X(s), z)\Big\|^p\nu(dz) + \Big(\int_\chi\Big\|(-A)^{\frac{1-\delta}{4}}G(X(s), z)\Big\|^2\nu(dz)\Big)^{\frac{p}{2}}\Big\}ds\Big] \\ \le& C_{p,T,\delta}\lambda_N^{-\frac{1-\delta}{4}p} + C_{p,T}\int_0^T \mathbb{E}\Big[\Big\|X(s) - X^N(s)\Big\|^p\Big]ds + C_{p,T}\lambda_N^{-\frac{1-\delta}{4}p}\sup_{s\in[0,T]}\Big\|X(s)\Big\|^p_{L^p(\Omega,\dot H^{\frac{1-\delta}{2}})}. \end{aligned}$$

Applying the Grönwall inequality finishes the proof of **Step 1**.

Step 2. *Show that for all $p \ge 2$ and $\delta \in (0,1)$, there exists a constant $C_{p,T,\delta} > 0$ independent of Δt and N such that $\|X^N(T) - X^N_{n_T}\|_{L^p(\Omega,H)} \le C_{p,T,\delta}(\Delta t)^{\frac{1-\delta}{2}}$.*

On the partition $\mathcal{T}$, the process $\{X^N(t)\}_{t\in[0,T]}$ can be rewritten as

$$\begin{cases} X^N(t_{i+1}-) = E(\Delta t_{i+1})X^N(t_i) + \int_{t_i}^{t_{i+1}} E(t_{i+1}-s)F_N(X^N(s))ds + \int_{t_i}^{t_{i+1}} E(t_{i+1}-s)P_N dW(s) \\ \qquad\qquad - \int_{t_i}^{t_{i+1}}\int_\chi E(t_{i+1}-s)G_N(X^N(s),z)\nu(dz)ds, \\ X^N(t_{i+1}) = X^N(t_{i+1}-) + G_N(X^N(t_{i+1}-), p_{t_{i+1}})\mathbf{1}_{\{p_{t_{i+1}}\neq 0\}}, \quad i = 0,1,\dots,n_T-1. \end{cases} \tag{15}$$

On the time interval $[t_i, t_{i+1})$, it follows from (12) and (15) that

$$\begin{aligned} & X^N_{(i+1)-} - X^N(t_{i+1}-) \\ &= E(\Delta t_{i+1})(X^N_i - X^N(t_i)) + \int_{t_i}^{t_{i+1}} E(t_{i+1}-s)\big(F_N(X^N_i) - F_N(X^N(s))\big)ds \\ &\quad - \int_{t_i}^{t_{i+1}}\int_\chi \big(E(t_{i+1}-t_i)G_N(X^N_i, z) - E(t_{i+1}-s)G_N(X^N(s),z)\big)\nu(dz)ds \end{aligned}$$

and

$$\begin{aligned} & X^N_{i+1} - X^N(t_{i+1}) \\ &= X^N_{(i+1)-} - X^N(t_{i+1}-) + \big(G_N(X^N_{(i+1)-}, p_{t_{i+1}}) - G_N(X^N(t_{i+1}-), p_{t_{i+1}})\big)\mathbf{1}_{\{p_{t_{i+1}}\neq 0\}}. \end{aligned}$$

Denote $e_i := X^N_i - X^N(t_i)$ and $e_{i-} := X^N_{i-} - X^N(t_i-)$. Then it holds that

$$e_{i+1} = \big(\mathrm{I} + g_1(p_{t_{i+1}})\mathbf{1}_{\{p_{t_{i+1}}\neq 0\}}\big)e_{(i+1)-} =: \mathcal{G}_{i+1}e_{(i+1)-}. \tag{16}$$

Using the definition of G_N gives that for $i = 0,1,\dots,n_T-1$,

$$e_{(i+1)-} = E(\Delta t_{i+1})\Gamma_i\mathcal{G}_i e_{i-} + \Xi(t_{i+1}) + \sum_{j=1}^{3}\mathcal{R}_j(t_{i+1}), \tag{17}$$

where $\Gamma_i := \mathrm{I} - \Delta t_{i+1}\int_\chi g_1(z)\nu(dz)$, $\Xi(t_{i+1}) := \int_{t_i}^{t_{i+1}} E(t_{i+1}-s)\big(F_N(X^N_i) - F_N(X^N(t_i))\big)ds$, and

$$\begin{aligned} \mathcal{R}_1(t_{i+1}) &:= \int_{t_i}^{t_{i+1}} E(t_{i+1}-s)\big(F_N(X^N(t_i)) - F_N(X^N(s))\big)ds, \\ \mathcal{R}_2(t_{i+1}) &:= \int_{t_i}^{t_{i+1}}\int_\chi E(t_{i+1}-t_i)\big(G_N(X^N(s),z) - G_N(X^N(t_i),z)\big)\nu(dz)ds, \\ \mathcal{R}_3(t_{i+1}) &:= \int_{t_i}^{t_{i+1}}\int_\chi \big(E(t_{i+1}-s) - E(t_{i+1}-t_i)\big)G_N(X^N(s),z)\nu(dz)ds. \end{aligned}$$

By iteratively applying the relation between $e_{(i+1)-}$ and e_{i-} (i.e., (17)), the local error is accumulated into the global one, namely, we have

$$e_{(i+1)-} = \sum_{k=0}^{i}\Big\{\prod_{l=k+1}^{i} E(\Delta t_{l+1})\Gamma_l\mathcal{G}_l\Big\}\Big(\Xi(t_{k+1}) + \sum_{j=1}^{3}\mathcal{R}_j(t_{k+1})\Big),$$

where we used $e_0 = 0$ and set $\prod_{i+1}^{i} := 1$. Taking $\|\cdot\|_{L^p(\Omega,H)}$-norm yields

$$\begin{aligned} \|e_{n_T-}\|_{L^p(\Omega,H)} &\le \Big\|\sum_{k=0}^{n_T-1}\Big\{\prod_{l=k+1}^{n_T-1} E(\Delta t_{l+1})\Gamma_l\mathcal{G}_l\Big\}\Xi(t_{k+1})\Big\|_{L^p(\Omega,H)} \\ &+ \Big\|\sum_{k=0}^{n_T-1}\Big\{\prod_{l=k+1}^{n_T-1} E(\Delta t_{l+1})\Gamma_l\mathcal{G}_l\Big\}\mathcal{R}_1(t_{k+1})\Big\|_{L^p(\Omega,H)} + \Big\|\sum_{k=0}^{n_T-1}\Big\{\prod_{l=k+1}^{n_T-1} E(\Delta t_{l+1})\Gamma_l\mathcal{G}_l\Big\}\mathcal{R}_2(t_{k+1})\Big\|_{L^p(\Omega,H)} \\ &+ \Big\|\sum_{k=0}^{n_T-1}\Big\{\prod_{l=k+1}^{n_T-1} E(\Delta t_{l+1})\Gamma_l\mathcal{G}_l\Big\}\mathcal{R}_3(t_{k+1})\Big\|_{L^p(\Omega,H)} =: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4. \end{aligned}$$

To proceed, we need to establish the pth moment estimates on terms $\mathcal{J}_1,\dots,\mathcal{J}_4$ respectively, which involve the multiplicative noise or the jump process.

Let $\Delta t_0 := 0$, $\Delta t_{n_T+1} := 0$, and $\lfloor s\rfloor := \min\{t_{n-1}, n \in \{1,\dots,n_T+1\} : \sum_{i=0}^{n}\Delta t_i > s$ or $\sum_{i=0}^{n-1}\Delta t_i = s\}$ for any $s\in[0,T]$, and use $\ell(\lfloor s\rfloor)$ to denote the subscript corresponding to time $\lfloor s\rfloor$ (i.e., $t_{\ell(\lfloor s\rfloor)} = \lfloor s\rfloor$). For the term $\mathcal{J}_1$, we have

$$\mathcal{J}_1 = \Big\|\int_0^T E(T-s)\Big\{\prod_{l=\ell(\lfloor s\rfloor)+1}^{n_T-1}\Gamma_l\mathcal{G}_l\Big\}\big(F_N(X^N_{\ell(\lfloor s\rfloor)}) - F_N(X^N(\lfloor s\rfloor))\big)ds\Big\|_{L^p(\Omega,H)}$$

$$\leq \int_0^T \Bigg(\mathbb{E}\Big[\mathbb{E}\Big[\Big\|\prod_{l=\ell(\lfloor s\rfloor)+1}^{n_T-1}\Gamma_l\mathcal{G}_l\Big\|_{\mathcal{L}(H)}^p\Big\|F_N(X^N_{\ell(\lfloor s\rfloor)})-F_N(X^N(\lfloor s\rfloor))\Big\|^p\Big|\mathcal{F}_{\lfloor s\rfloor}\Big]\Big]\Bigg)^{\frac{1}{p}}ds.$$

Note that random variables Γ_l and $\mathcal{G}_l$ for $l\geq \ell(\lfloor s\rfloor)+1$ are independent of the σ-algebra $\mathcal{F}_{\lfloor s\rfloor}$. Together with the Lipschitz condition (6) of F and (16), we derive

$$\mathcal{J}_1 \leq \int_0^T \Big\|\prod_{l=\ell(\lfloor s\rfloor)+1}^{n_T-1}\Gamma_l\mathcal{G}_l\Big\|_{L^p(\Omega,\mathcal{L}(H))}\Big\|F_N(X^N_{\ell(\lfloor s\rfloor)})-F_N(X^N(\lfloor s\rfloor))\Big\|_{L^p(\Omega,H)}ds$$

$$\leq L_F\Bigg(\mathbb{E}\Big[\sup_{k\in\{0,1,\dots,n_T-1\}}\prod_{l=k+1}^{n_T-1}\|\Gamma_l\mathcal{G}_l\|^p_{\mathcal{L}(H)}\Big]\Bigg)^{\frac{1}{p}}\int_0^T\Big\|\mathcal{G}_{\ell(\lfloor s\rfloor)}e_{\ell(\lfloor s\rfloor)-}\Big\|_{L^p(\Omega,H)}ds.$$

According to Assumption 2 on coefficient G, it holds that

$$\|\Gamma_l\|_{\mathcal{L}(H)}\leq 1+C_G\Delta t_{l+1}\ \text{with}\ C_G:=b\nu(\chi),\quad\text{and}\ \|\mathcal{G}_l\|_{\mathcal{L}(H)}\leq 1+b\mathbf{1}_{\{p_{t_l}\neq 0\}}. \tag{18}$$

Set $N(T):=N((0,T]\times\chi)$. From Poisson distribution $\mathbb{P}(N(T)=m)=\frac{(T\nu(\chi))^m}{m!}e^{-T\nu(\chi)}$ and the estimate (18), it follows that for all $q\geq 1$,

$$\mathbb{E}\Big[\sup_{k\in\{0,1,\dots,n_T-1\}}\prod_{l=k+1}^{n_T-1}\|\Gamma_l\mathcal{G}_l\|^q_{\mathcal{L}(H)}\Big]\leq\mathbb{E}\Big[(1+b)^{qN(T)}\prod_{l=1}^{n_T}\big(1+C_G\Delta t_l\big)^q\Big]$$

$$\leq\mathbb{E}\Big[(1+b)^{qN(T)}e^{qC_G\sum_{l=1}^{n_T}\Delta t_l}\Big]\leq C_{q,T}\sum_{m=1}^{\infty}(1+b)^{qm}\frac{(T\nu(\chi))^m}{m!}e^{-T\nu(\chi)}<\infty. \tag{19}$$

Hence we arrive at $\mathcal{J}_1\leq C_{p,T}\int_0^T\|e_{\ell(\lfloor s\rfloor)-}\|_{L^p(\Omega,H)}ds$.

For the term $\mathcal{J}_2$, by Assumption 1, we have

$$\mathcal{J}_2=\Big\|\int_0^T\Big\{\prod_{l=\ell(\lfloor s\rfloor)+1}^{n_T-1}\Gamma_l\mathcal{G}_l\Big\}E(T-s)(-A)^{\frac{1+\delta}{4}}(-A)^{-\frac{1+\delta}{4}}\big(F_N(X^N(s))-F_N(X^N(\lfloor s\rfloor))\big)ds\Big\|_{L^p(\Omega,H)}$$

$$\leq C_\delta\int_0^T\Big\|\prod_{l=\ell(\lfloor s\rfloor)+1}^{n_T-1}\Gamma_l\mathcal{G}_l\Big\|_{L^{2p}(\Omega,\mathcal{L}(H))}(T-s)^{-\frac{1+\delta}{4}}\Big\|X^N(s)-X^N(\lfloor s\rfloor)\Big\|_{L^{4p}(\Omega,\dot H^{-\frac{1-\delta}{2}})}\times$$
$$\Big(1+\|X^N(s)\|_{L^{4p}(\Omega,\dot H^{\frac{1-\delta}{2}})}+\|X^N(\lfloor s\rfloor)\|_{L^{4p}(\Omega,\dot H^{\frac{1-\delta}{2}})}\Big)ds.$$

To estimate $\|X^N(s)-X^N(\lfloor s\rfloor)\|_{L^{2q}(\Omega,\dot H^{-\frac{1-\delta}{2}})}$ for $q\geq 1$ and $s\in[0,T]$, we note that

$$\Big\|\int_{\lfloor s\rfloor}^s(-A)^{-\frac{1-\delta}{4}}E(s-r)P_NdW(r)\Big\|^{2q}_{L^{2q}(\Omega,H)}\leq\mathbb{E}\Big[\mathbb{E}\Big[\Big\|\int_\zeta^s(-A)^{-\frac{1-\delta}{4}}E(s-r)P_NdW(r)\Big\|^{2q}\Big]\Big|_{\zeta=\lfloor s\rfloor}\Big]$$

$$\leq C_q\mathbb{E}\Big[\Big(\int_\zeta^s\|(-A)^{\frac{\delta}{2}}(-A)^{-\frac{1+\delta}{4}}E(s-r)\|^2_{\mathcal{L}_2(H)}dr\Big)^q\Big|_{\zeta=\lfloor s\rfloor}\Big]$$

$$\leq C_{q,\delta}\|(-A)^{-\frac{1+\delta}{4}}\|^{2q}_{\mathcal{L}_2(H)}\mathbb{E}\big[(s-\lfloor s\rfloor)^{q(1-\delta)}\big]\leq C_{q,\delta}(\Delta t)^{q(1-\delta)},$$

where we used the Burkholder–Davis–Gundy inequality (see e.g. [29, Theorem 4.36]) and the Hölder inequality. This, combining Proposition 1 shows

$$\Big\|X^N(s)-X^N(\lfloor s\rfloor)\Big\|_{L^{2q}(\Omega,\dot H^{-\frac{1-\delta}{2}})}$$

$$\leq\Big\|(-A)^{-\frac{1-\delta}{4}}(E(s-\lfloor s\rfloor)-\mathrm{I})X^N(\lfloor s\rfloor)\Big\|_{L^{2q}(\Omega,H)}+\int_{(s-\Delta t)\vee 0}^s\Big\|F_N(X^N(r))\Big\|_{L^{2q}(\Omega,H)}dr$$
$$+\Big\|\int_{\lfloor s\rfloor}^s(-A)^{-\frac{1-\delta}{4}}E(s-r)P_NdW(r)\Big\|_{L^{2q}(\Omega,H)}$$
$$+\Big\|\int_{\lfloor s\rfloor}^s\int_\chi(-A)^{-\frac{1-\delta}{4}}E(s-r)G_N(X^N(r),z)\nu(dz)dr\Big\|_{L^{2q}(\Omega,H)}$$

$$\leq C_{q,T,\delta}(\Delta t)^{\frac{1-\delta}{2}}+C_{q,T,\delta}\int_{(s-\Delta t)\vee 0}^s\int_\chi\Big(|g_1(z)|\Big\|X^N(r)\Big\|_{L^{2q}(\Omega,H)}+\|g(z)\|\Big)\nu(dz)dr\leq C_{q,T,\delta}(\Delta t)^{\frac{1-\delta}{2}}.$$

Hence, it follows from (19) that $\mathcal{J}_2\leq C_{p,T,\delta}(\Delta t)^{\frac{1-\delta}{2}}$.

For the term $\mathcal{J}_3$, by (19), similarly, we derive that

$$\mathcal{J}_3\leq C\int_0^T\Big\|\prod_{l=\ell(\lfloor s\rfloor)+1}^{n_T-1}\Gamma_l\mathcal{G}_l\Big\|_{L^{2p}(\Omega,\mathcal{L}(H))}(T-\lfloor s\rfloor)^{-\frac{1-\delta}{4}}\Big\|X^N(s)-X^N(\lfloor s\rfloor)\Big\|_{L^{2p}(\Omega,\dot H^{-\frac{1-\delta}{2}})}ds\leq C_{p,T,\delta}(\Delta t)^{\frac{1-\delta}{2}}.$$

The term $\mathcal{J}_4$ can be estimated as

$$\mathcal{J}_4 \le \int_0^T\int_\chi \Big\|\Big\{\prod_{l=\ell(\lfloor s\rfloor)+1}^{n_T-1}\Gamma_l\mathcal{G}_l\Big\}E(T-s)(-A)^{\frac12}(E(s-\lfloor s\rfloor)-\mathrm{I})(-A)^{-\frac12}G_N(X^N(s),z)\Big\|_{L^p(\Omega,H)}\nu(dz)ds$$
$$\le C(\Delta t)^{\frac12}\int_0^T\int_\chi (T-s)^{-\frac12}\big(|g_1(z)|\|X^N(s)\|_{L^{2p}(\Omega,H)}+\|g(z)\|\big)\nu(dz)ds \le C_{p,T}(\Delta t)^{\frac12}.$$

Combining estimates of terms $\mathcal{J}_1,\dots,\mathcal{J}_4$ above gives

$$\|e_{n_T-}\|_{L^p(\Omega,H)} \le C_{p,T}\int_0^T \|e_{\ell(\lfloor s\rfloor)-}\|_{L^p(\Omega,H)}ds + C_{p,T,\delta}(\Delta t)^{\frac{1-\delta}{2}},$$

which yields $\|e_{n_T-}\|_{L^p(\Omega,H)} \le C_{p,T,\delta}(\Delta t)^{\frac{1-\delta}{2}}$ by using the Grönwall inequality. According to (16) and $\|\mathcal{G}_l\|_{\mathcal{L}(H)} \le C$, we obtain

$$\|e_{n_T}\|_{L^p(\Omega,H)} \le C\|e_{n_T-}\|_{L^p(\Omega,H)} \le C_{p,T,\delta}(\Delta t)^{\frac{1-\delta}{2}}.$$

Combining **Steps 1–2** finishes the proof. □

4. Proof of Theorem 2

In this section, we present the proof of the L^p-strong convergence orders of the fully discrete scheme (13) for (1) driven by the additive Poisson noise with $\nu(\chi)\le\infty$. The proof relies on estimates of stochastic convolutions, the quantitative John–Nirenberg inequality and some *a priori* estimates on the nested conditional "L^2-norms" of both the stochastic convolution and the solution of the perturbed stochastic equation.

For $N\in\mathbb{N}_+$ and $t\in[0,T]$, denote stochastic convolutions $\mathcal{W}^N(t) := \int_0^t E(t-s)P_N dW(s)$ and $\mathcal{N}^N(t) := \int_0^t\int_\chi E(t-s)G_N(z)\tilde N(dz,ds)$ with $G_N := P_N G$. Let $Y^N(t) := X^N(t)-\mathcal{W}^N(t)-\mathcal{N}^N(t)$, which satisfies the perturbed stochastic equation:

$$dY^N(t) = AY^N(t)dt + F_N\big(Y^N(t)+\mathcal{W}^N(t)+\mathcal{N}^N(t)\big)dt,\quad t\in(0,T], \tag{20}$$

with $Y^N(0)=P_Nx_0$. It is known that Y^N satisfies

$$Y^N(t) = E(t)P_Nx_0 + \int_0^t E(t-\rho)F_N(Y^N(\rho)+\mathcal{W}^N(\rho)+\mathcal{N}^N(\rho))d\rho. \tag{21}$$

We list properties of the stochastic convolution with respect to the Wiener process as follows, whose proofs are given in the supplementary material [26].

Lemma 2. *(i) For any $k\ge1$, $\delta\in(0,1)$, and $0\le s<t\le T$, there exists a constant $C_{k,T,\delta}>0$ such that*

$$\sup_{N\in\mathbb{N}_+}\mathbb{E}\Big[\sup_{t\in(s,T]}\Big\|\int_s^t(-A)^{\frac{1-\delta}{4}}E(t-r)P_NdW(r)\Big\|^{2k}\Big] \le C_{k,T,\delta}. \tag{22}$$

(ii) For any $p\ge1$, $\delta\in(0,1)$, and $t\in[0,T]$, there exists a constant $C_{p,T,\delta}>0$ independent of Δt such that

$$\sup_{N\in\mathbb{N}_+}\Big\|(-A)^{-\frac{1-\delta}{4}}\big(\mathcal{W}^N(t)-\mathcal{W}^N(\lfloor t\rfloor)\big)\Big\|_{L^{2p}(\Omega,H)} \le C_{p,T,\delta}(\Delta t)^{\frac{1-\delta}{2}}. \tag{23}$$

We then introduce the quantitative John–Nirenberg inequality, which provides a useful way to bound the L^p-norm of a stochastic process by moment estimates of the nested conditional "L^1-norms". The conditional expectation given $\mathcal{F}_t$ is denoted by $\mathbb{E}^t$.

Lemma 3. *[23, Theorem 1.1] Let (E,d) be a metric space and $Z:[0,\infty)\times\Omega\to(E,\mathrm{d})$ be a right continuous with left limits and adapted integrable stochastic process. Then for every $\tau>0$ and $p\ge1$, there exists a constant $C_p>0$ such that*

$$\Big\|\sup_{t\in[0,\tau]}\mathrm{d}(Z_0,Z_t)\Big\|_{L^p(\Omega,\mathbb{R})} \le C_p\Big\|\sup_{t\in[0,\tau]}\mathbb{E}^t\Big[\sup_{s\in[t,\tau]}\mathbb{E}^s[\mathrm{d}(Z_{s-},Z_\tau)]\Big]\Big\|_{L^p(\Omega,\mathbb{R})}.$$

Let $[a,b]_<$ denote $\{(s,t)\in[a,b]^2:\ s<t\}$. For a random variable ϑ, a sub-σ-algebra $\mathcal{G}\subset\mathcal{F}$, and $p\ge1$, we set $\|\vartheta\|_{L^p(\Omega,H)|\mathcal{G}} := (\mathbb{E}[\|\vartheta\|^p|\mathcal{G}])^{1/p}$. For a measurable mapping $f:[0,T]\times\Omega\to H$, we set $\|f\|_{C_2^0|\mathcal{F}_s,[s,t]} := \sup_{r\in[s,t]}\|f(r)\|_{L^2(\Omega,H)|\mathcal{F}_s}$ for any $s,t\in[0,T]_<$. Let $s' := \lfloor s\rfloor+\Delta t$, that is, s' is the smallest grid point strictly bigger than s.

The following propositions give some *a priori* estimates including nested conditional "L^2-norms" of both stochastic convolutions and the solution of the perturbed stochastic equation. The proofs are based on the independent increments property of the compensated Poisson random measure and properties (4), (5) of semigroup $\{E(t)\}_{t\ge0}$, which are given in the supplementary material [26].

Proposition 2. *Let Assumptions 1, 3 and 4 hold. For each $N\in\mathbb{N}_+,\delta\in(0,1)$, and any time grid point t_i, $i=0,1,\dots,M$, there exists a constant $C_{T,\delta}>0$ such that*

$$\Big\|(-A)^{-\frac{1-\delta}{4}}\big(\mathcal{N}^N(\cdot)-\mathcal{N}^N(\lfloor\cdot\rfloor)\big)\Big\|_{C_2^0|\mathcal{F}_{t_i},[t_i,T]} \le C_{T,\delta}(\Delta t)^{\frac{1-\delta}{2}}\Big(1+\|\mathcal{N}^N(t_i)\|_{\dot H^{\frac{1-\delta}{2}}}\Big). \tag{24}$$

Proposition 3. *Let Assumptions 1, 3 and 4 hold. For any $p \geq 2$, there exists a constant $C_{p,T} > 0$ such that*

$$\sup_{N\in\mathbb{N}_+} \Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]} \sup_{r\in[0,T]} \mathbb{E}^s\big[\|\mathcal{N}^N(r)\|\big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})} \leq C_{p,T}, \tag{25}$$

and for any $p \geq 2$ and $\delta \in (0,1)$, there exists a constant $C_{p,T,\delta} > 0$ such that

$$\sup_{N\in\mathbb{N}_+} \Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]} \mathbb{E}^s\Big[\Big\|(-A)^{\frac{1-\delta}{4}}\mathcal{N}^N(s')\Big\|^2\Big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})} \leq C_{p,T,\delta}, \tag{26}$$

$$\sup_{N\in\mathbb{N}_+} \Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]} \mathbb{E}^s\Big[\Big\|(-A)^{\frac{1-\delta}{4}}\mathcal{N}^N(\cdot)\Big\|^2_{C_2^0|\mathcal{F}_{s'},[0,T]}\Big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})} \leq C_{p,T,\delta}, \tag{27}$$

$$\sup_{N\in\mathbb{N}_+} \Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]} \mathbb{E}^s\Big[\Big\|(-A)^{\frac{1-\delta}{4}}\mathcal{W}^N(\cdot)\Big\|^2_{C_2^0|\mathcal{F}_{s'},[0,T]}\Big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})} \leq C_{p,T,\delta}. \tag{28}$$

Proposition 4. *Let Assumptions 1, 3 and 4 hold.*

(i) For any $p \geq 2$, $\delta \in (0,1)$, and $t \in [0,T]$, there exists a constant $C_{p,T,\delta} > 0$ such that

$$\sup_{N\in\mathbb{N}_+} \Big\|(-A)^{-\frac{1-\delta}{4}}(Y^N(t) - Y^N(\lfloor t\rfloor))\Big\|_{L^p(\Omega,H)} \leq C_{p,T,\delta}(\Delta t)^{\frac{1-\delta}{2}}. \tag{29}$$

(ii) For any $p \geq 2$ and $\delta \in (0,1)$, there exists a constant $C_{p,T,\delta} > 0$ such that

$$\sup_{N\in\mathbb{N}_+} \Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]} \mathbb{E}^s\Big[\Big\|(-A)^{\frac{1-\delta}{4}}Y^N(\cdot)\Big\|^2_{C_2^0|\mathcal{F}_{s'},[s',T]}\Big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})} \leq C_{p,T,\delta}. \tag{30}$$

With these preliminaries, we present the proof of Theorem 2.

Proof. (**Proof of** Theorem 2) The error between the exact solution and the semi-discrete numerical solution (i.e., $\|X(T) - X^N(T)\|_{L^p(\Omega,H)} \leq C_{p,T,\delta}N^{-\frac{1-\delta}{2}}$) can be estimated similarly as in **Step 1** of the proof of Theorem 1 and thus is omitted. Hence it suffices to estimate the error $\|X^N(T) - X^N_{n_T}\|_{L^p(\Omega,H)}$.

It follows from the assumption on F, Assumption 4, (5), and Lemma 1 that

$$\begin{aligned}
\|X^N(T) - X^N_{n_T}\|_{L^p(\Omega,H)} &\leq \Big\| \int_0^T E(T-s)\big(F_N(X^N(s)) - F_N(X^N_{\ell(\lfloor s\rfloor)})\big)ds\Big\|_{L^p(\Omega,H)} \\
&+ \Big\| \int_0^T \int_\chi E(T-s)(\mathrm{I} - E(s-\lfloor s\rfloor))(-A)^{-\frac{1-\delta}{2}}(-A)^{\frac{1-\delta}{2}}G_N(z)\tilde{N}(dz,ds)\Big\|_{L^p(\Omega,H)} \\
&\leq I_1 + I_2 + I_3 + L_F\int_0^T \Big\|X^N(\lfloor s\rfloor) - X^N_{\ell(\lfloor s\rfloor)}\Big\|_{L^p(\Omega,H)}ds + C_{p,T}\Big(\int_0^T \Big\|(\mathrm{I} - E(s-\lfloor s\rfloor))(-A)^{-\frac{1-\delta}{2}}\Big\|^p_{\mathcal{L}(H)} \\
&\times \Big\{\int_\chi \Big\|(-A)^{\frac{1-\delta}{2}}G_N(z)\Big\|^p \nu(dz) + \Big(\int_\chi \Big\|(-A)^{\frac{1-\delta}{2}}G_N(z)\Big\|^2\nu(dz)\Big)^{\frac{p}{2}}\Big\}ds\Big)^{\frac{1}{p}} \\
&\leq C_{p,T,\delta}(\Delta t)^{\frac{1-\delta}{2}} + L_F\int_0^T \Big\|X^N(\lfloor s\rfloor) - X^N_{\ell(\lfloor s\rfloor)}\Big\|_{L^p(\Omega,H)}ds + I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &:= \Big\|\int_0^T E(T-s)\big(F_N(Y^N(s) + \mathcal{W}^N(\lfloor s\rfloor) + \mathcal{N}^N(\lfloor s\rfloor)) - F_N(X^N(\lfloor s\rfloor))\big)ds\Big\|_{L^p(\Omega,H)}, \\
I_2 &:= \Big\|\int_0^T E(T-s)\big(F_N(Y^N(s) + \mathcal{W}^N(s) + \mathcal{N}^N(\lfloor s\rfloor)) - F_N(Y^N(s) + \mathcal{W}^N(\lfloor s\rfloor) + \mathcal{N}^N(\lfloor s\rfloor))\big)ds\Big\|_{L^p(\Omega,H)}, \\
I_3 &:= \Big\|\int_0^T E(T-s)\big(F_N(X^N(s)) - F_N(Y^N(s) + \mathcal{W}^N(s) + \mathcal{N}^N(\lfloor s\rfloor))\big)ds\Big\|_{L^p(\Omega,H)}.
\end{aligned}$$

For the term I_1, by Assumption 1, we have

$$\begin{aligned}
I_1 &\leq C\int_0^T \Big\|E(T-s)(-A)^{\frac{1+\delta}{4}}\Big\|_{\mathcal{L}(H)}\Big\|(-A)^{-\frac{1-\delta}{4}}(Y^N(s) - Y^N(\lfloor s\rfloor))\Big\|_{L^{2p}(\Omega,H)}ds \times \\
&\Big(1 + \sup_{s\in[0,T]}\|Y^N(s)\|_{L^{2p}(\Omega,\dot{H}^{\frac{1-\delta}{2}})} + \sup_{s\in[0,T]}\|\mathcal{W}^N(s)\|_{L^{2p}(\Omega,\dot{H}^{\frac{1-\delta}{2}})} + \sup_{s\in[0,T]}\|\mathcal{N}^N(s)\|_{L^{2p}(\Omega,\dot{H}^{\frac{1-\delta}{2}})}\Big).
\end{aligned}$$

Combining (29) and the similar arguments as in the proof of Proposition 1 leads to

$$I_1 \leq C_{p,T,\delta}(\Delta t)^{\frac{1-\delta}{2}}\int_0^T (T-s)^{-\frac{1+\delta}{4}}ds \leq C_{p,T,\delta}(\Delta t)^{\frac{1-\delta}{2}}.$$

Similarly, by (23), we have

$$I_2 \leq C\int_0^T (T-s)^{-\frac{1+\delta}{4}}\Big\|(-A)^{-\frac{1-\delta}{4}}\big(\mathcal{W}^N(s) - \mathcal{W}^N(\lfloor s\rfloor)\big)\Big\|_{L^{2p}(\Omega,H)}ds \times \Big(1 + \sup_{s\in[0,T]}\|Y^N(s)\|_{L^{2p}(\Omega,\dot{H}^{\frac{1-\delta}{2}})}$$

$$+\sup_{s\in[0,T]}\|\mathcal{W}^N(s)\|_{L^{2p}(\Omega,\dot H^{\frac{1-\delta}{2}})}+\sup_{s\in[0,T]}\|\mathcal{N}^N(s)\|_{L^{2p}(\Omega,\dot H^{\frac{1-\delta}{2}})}\Big)\le C_{p,T,\delta}(\Delta t)^{\frac{1-\delta}{2}}.$$

For the term I_3, we claim that

$$I_3\le C_{p,T,\delta}(\Delta t)^{\frac{1-\delta}{2}}. \tag{31}$$

The estimate of term I_3 includes dealing with the temporal Hölder continuity of the stochastic convolution with respect to the compensated Poisson random measure to overcome the order barrier. To this end we aim to apply Lemma 3 to estimate the L^p-norm of I_3. For $(s,t)\in[0,T]_<$, define an integrable $\mathcal{F}_t$-adapted stochastic process:

$$\mathcal{A}^{N,\Delta}(t):=\int_0^t E(T-r)\big(F_N(X^N(r))-F_N(Y^N(r)+\mathcal{W}^N(r)+\mathcal{N}^N(\lfloor r\rfloor))\big)dr$$

with $\mathcal{A}^{N,\Delta}(0)=0$. By the Lipschitz condition (6) of F, we have

$$\begin{aligned}\mathbb{E}\big[\big\|\mathcal{A}^{N,\Delta}(t)-\mathcal{A}^{N,\Delta}(s)\big\|^2\big]&\le L_F|t-s|\int_s^t\mathbb{E}\big[\big\|\mathcal{N}^N(r)-\mathcal{N}^N(\lfloor r\rfloor)\big\|^2\big]dr\\&\le C|t-s|^2\sup_{r\in[0,T]}\mathbb{E}[\|\mathcal{N}^N(r)\|^2]\le C_T|t-s|^2.\end{aligned}$$

The Kolmogorov continuity theorem implies that $\mathcal{A}^{N,\Delta}$ is a.s. continuous. Thus by Lemma 3, we arrive at

$$\Big\|\sup_{t\in[0,T]}\|\mathcal{A}^{N,\Delta}(t)\|\Big\|_{L^p(\Omega,\mathbb{R})}\le C_p\Big\|\sup_{t\in[0,T]}\mathbb{E}^t\Big[\sup_{s\in[t,T]}\mathbb{E}^s\big[\big\|\mathcal{A}^{\mathcal{N},\Delta}(T)-\mathcal{A}^{\mathcal{N},\Delta}(s)\big\|\big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})},$$

which yields that

$$I_3=\|\mathcal{A}^{N,\Delta}(T)\|_{L^p(\Omega,H)}\le C_p\Big\|\sup_{t\in[0,T]}\mathbb{E}^t\Big[\sup_{s\in[t,T]}\mathbb{E}^s\big[\big\|\mathcal{A}^{\mathcal{N},\Delta}(T)-\mathcal{A}^{\mathcal{N},\Delta}(s)\big\|\big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})}. \tag{32}$$

Now we turn to estimating the term on the right hand side of the inequality in (32). We first show that, there exists a real-valued adapted stochastic process ξ such that

$$\mathbb{E}^s\big[\big\|\mathcal{A}^{N,\Delta}(T)-\mathcal{A}^{N,\Delta}(s)\big\|\big]\le\xi_s\quad a.s. \tag{33}$$

Indeed, using Assumption 1 and the conditional Hölder inequality gives

$$\begin{aligned}&\mathbb{E}^s\Big[\big\|\mathcal{A}^{N,\Delta}(T)-\mathcal{A}^{N,\Delta}(s)\big\|\Big]\le C\int_s^T(T-r)^{-\frac{1+\delta}{4}}\Big\|(-A)^{-\frac{1-\delta}{4}}\big(\mathcal{N}^N(r)-\mathcal{N}^N(\lfloor r\rfloor)\big)\Big\|_{L^2(\Omega,H)|\mathcal{F}_s}dr\\&\times\Big(1+\Big\|(-A)^{\frac{1-\delta}{4}}Y^N(\cdot)\Big\|_{C_2^0|\mathcal{F}_s,[s,T]}+\Big\|(-A)^{\frac{1-\delta}{4}}\mathcal{W}^N(\cdot)\Big\|_{C_2^0|\mathcal{F}_s,[s,T]}+\Big\|(-A)^{\frac{1-\delta}{4}}\mathcal{N}^N(\cdot)\Big\|_{C_2^0|\mathcal{F}_s,[0,T]}\Big)\\&\le C_T\Big\|(-A)^{-\frac{1-\delta}{4}}\big(\mathcal{N}^N(\cdot)-\mathcal{N}^N(\lfloor\cdot\rfloor)\big)\Big\|_{C_2^0|\mathcal{F}_s,[s,T]}\times\\&\quad\Big(1+\Big\|(-A)^{\frac{1-\delta}{4}}Y^N(\cdot)\Big\|_{C_2^0|\mathcal{F}_s,[s,T]}+\Big\|(-A)^{\frac{1-\delta}{4}}\mathcal{W}^N(\cdot)\Big\|_{C_2^0|\mathcal{F}_s,[s,T]}+\Big\|(-A)^{\frac{1-\delta}{4}}\mathcal{N}^N(\cdot)\Big\|_{C_2^0|\mathcal{F}_s,[0,T]}\Big).\end{aligned}$$

In the case of $s\in\{0,\Delta t,2\Delta t,\dots,T\}$, i.e., $\lfloor s\rfloor=s$, it follows from (24) and the Young inequality that $\mathbb{E}^s\big[\|\mathcal{A}^{N,\Delta}(T)-\mathcal{A}^{N,\Delta}(s)\|\big]\le\xi_s^1$ almost surely with

$$\begin{aligned}\xi_s^1:=&\,C_{T,\delta}(\Delta t)^{\frac{1-\delta}{2}}\times\Big(1+\Big\|(-A)^{\frac{1-\delta}{4}}\mathcal{N}^N(s)\Big\|^2+\Big\|(-A)^{\frac{1-\delta}{4}}Y^N(\cdot)\Big\|^2_{C_2^0|\mathcal{F}_s,[s,T]}\\&+\Big\|(-A)^{\frac{1-\delta}{4}}\mathcal{W}^N(\cdot)\Big\|^2_{C_2^0|\mathcal{F}_s,[s,T]}+\Big\|(-A)^{\frac{1-\delta}{4}}\mathcal{N}^N(\cdot)\Big\|^2_{C_2^0|\mathcal{F}_s,[0,T]}\Big).\end{aligned}$$

In the case that s is not a grid point, i.e., $\lfloor s\rfloor\neq s$, we note that when $T-s\le\Delta t$,

$$\mathbb{E}^s\big[\big\|\mathcal{A}^{N,\Delta}(T)-\mathcal{A}^{N,\Delta}(s)\big\|\big]\le C\Delta t\sup_{r\in[s,T]}\mathbb{E}^s\big[\big\|\mathcal{N}^N(r)-\mathcal{N}^N(\lfloor r\rfloor)\big\|\big]\le C\Delta t\sup_{r\in[s-\Delta t,T]}\mathbb{E}^s\big[\big\|\mathcal{N}^N(r)\big\|\big].$$

When $T-s>\Delta t$, we observe that $s'-s\le\Delta t$, $s'\le T$, and

$$\begin{aligned}\mathbb{E}^s\big[\big\|\mathcal{A}^{N,\Delta}(T)-\mathcal{A}^{N,\Delta}(s)\big\|\big]&\le\mathbb{E}^s\big[\mathbb{E}^{s'}\big[\|\mathcal{A}^{N,\Delta}(T)-\mathcal{A}^{N,\Delta}(s')\|\big]\big]+\mathbb{E}^s\big[\|\mathcal{A}^{N,\Delta}(s)-\mathcal{A}^{N,\Delta}(s')\|\big]\\&\le\mathbb{E}^s\big[\xi_{s'}^1\big]+\mathbb{E}^s\big[\|\mathcal{A}^{N,\Delta}(s)-\mathcal{A}^{N,\Delta}(s')\|\big].\end{aligned}$$

Hence, we derive that $\mathbb{E}^s\big[\|\mathcal{A}^{N,\Delta}(T)-\mathcal{A}^{N,\Delta}(s)\|\big]\le\xi_s^2$ almost surely with

$$\begin{aligned}\xi_s^2:=&\,C_{T,\delta}(\Delta t)^{\frac{1-\delta}{2}}\times\Big(1+\mathbb{E}^s\Big[\Big\|(-A)^{\frac{1-\delta}{4}}\mathcal{N}^N(s')\Big\|^2+\Big\|(-A)^{\frac{1-\delta}{4}}Y^N(\cdot)\Big\|^2_{C_2^0|\mathcal{F}_{s'},[s',T]}\\&+\Big\|(-A)^{\frac{1-\delta}{4}}\mathcal{W}^N(\cdot)\Big\|^2_{C_2^0|\mathcal{F}_{s'},[s',T]}+\Big\|(-A)^{\frac{1-\delta}{4}}\mathcal{N}^N(\cdot)\Big\|^2_{C_2^0|\mathcal{F}_{s'},[0,T]}\Big]+\sup_{r\in[s-\Delta t,T]}\mathbb{E}^s\big[\|\mathcal{N}^N(r)\|\big]\Big).\end{aligned}$$

Therefore (33) holds with $\xi_s := \xi_s^1 \mathbf{1}_{\{s=\lfloor s\rfloor\}} + \xi_s^2 \mathbf{1}_{\{s\neq\lfloor s\rfloor\}}$. To further estimate the term on the right hand side of (32), we use Propositions 2–4 to obtain

$$\begin{aligned}\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]} \xi_s^2\Big]\Big\|_{L^p(\Omega,\mathbb{R})} &\le C_{T,\delta}(\Delta t)^{\frac{1-\delta}{2}}\Big\{1+\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]} \mathbb{E}^s\Big[\Big\|(-A)^{\frac{1-\delta}{4}}\mathcal{N}^N(s')\Big\|^2\Big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&+\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]}\sup_{r\in[s-\Delta t,T]} \mathbb{E}^s\big[\|\mathcal{N}^N(r)\|\big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&+\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]} \mathbb{E}^s\Big[\Big\|(-A)^{\frac{1-\delta}{4}}\mathcal{N}^N(\cdot)\Big\|^2_{C_2^0|\mathcal{F}_{s'},[0,T]}\Big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&+\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]} \mathbb{E}^s\Big[\Big\|(-A)^{\frac{1-\delta}{4}}\mathcal{W}^N(\cdot)\Big\|^2_{C_2^0|\mathcal{F}_{s'},[s',T]}\Big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&+\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]} \mathbb{E}^s\Big[\Big\|(-A)^{\frac{1-\delta}{4}}Y^N(\cdot)\Big\|^2_{C_2^0|\mathcal{F}_{s'},[s',T]}\Big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\Big\} \le C_{p,T,\delta}(\Delta t)^{\frac{1-\delta}{2}}.\end{aligned}$$

Similarly, we can derive that $\big\|\sup_{t\in[0,T]}\mathbb{E}^t\big[\sup_{s\in[t,T]}\xi_s^1\big]\big\|_{L^p(\Omega,\mathbb{R})} \le C_{p,T,\delta}(\Delta t)^{\frac{1-\delta}{2}}$. As a result, (31) is proved.

Combining estimates of terms I_1, I_2, and I_3, and applying the Grönwall inequality complete the proof of the theorem. □

Remark 3. The convergence order of the scheme can be affected by the regularity of the jump coefficient. Specifically, in Theorem 2, if we replace Assumption 4 by

$$\int_\chi \big(\|(-A)^\beta G(z)\|^2 \vee \|(-A)^\beta G(z)\|^p\big)\nu(dz) \le C_p$$

for all $p\ge 2$ and some $\beta\in[\frac14,\frac12)$, then we can obtain the L^p-convergence orders $\frac{1-\delta}{2}$ in space and β in time.

5. Numerical experiments

In this section, we present some numerical experiments to illustrate the L^p-strong convergence orders of proposed schemes in both spatial and temporal directions. We also show the trajectories of the numerical solutions for SPDEs driven by Lévy noise.

5.1. Jump-adapted numerical scheme (12)

We consider the following SPDE with the linear multiplicative noise and Dirichlet Laplacian operator on the domain $\mathcal{O}=[0,1]$,

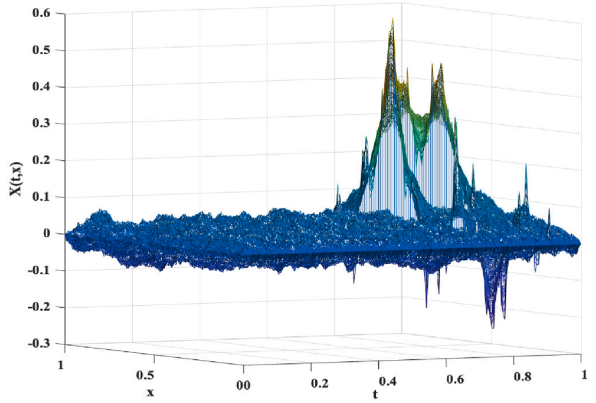
$$dX(t) = \Delta X(t)dt - \sin(X(t))dt + \gamma_1 dW(t) + \gamma_2 X(t)d\tilde N(t), \quad t>0, \quad X(0)=0, \tag{34}$$

where constants $\gamma_1,\gamma_2>0$. We take the Gaussian noise W as the space-time white noise, i.e., $Q=\mathrm{I}$. The Poisson counting process has a constant rate $\tilde\lambda$, which may vary across different numerical experiments.

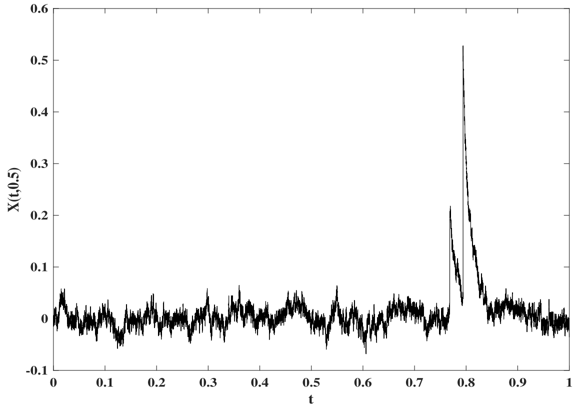
We first simulate a single path of the solution process X of (34) by the numerical method (12) in Fig. 1. In order to clearly show the jumping behavior, we take $\gamma_1=0.1,\gamma_2=10$. Take parameters $T=1$, $\tau=2^{-15}$, $N=100$, and different intensities $\tilde\lambda=2$ (in Fig. 1 (a)(b)) and $\tilde\lambda=5$ (in Fig. 1 (c)(d)). Pictures (a) and (c) illustrate one single path simulation of the solution process X, and pictures (b) and (d) show the contour slice of the solution at the fixed space point $x=0.5\in\mathcal{O}$. It is observed that the solution has more number of jumps when intensity $\tilde\lambda$ becomes larger.

Next, we verify the L^p-strong convergence orders of the method (12) in spatial and temporal directions respectively. The reference solutions are simulated by the same numerical method with a large spatial discretization parameter N or a small temporal discretization parameter τ. We use the Monte–Carlo method to simulate the expectations by computing averages of 1000 sample paths. Set $\gamma_1=\gamma_2=\tilde\lambda=1$. We first consider the spatial convergence order with respect to the space step sizes N^{-1}. Take $\tau=2^{-8}$ and $N=2^{12}$ for the reference solution. The numerical solutions are simulated with the same temporal step size $\tau=2^{-8}$, and smaller numbers $N=2^i$, $i=4,5,6,7,8$. It can be seen from Fig. 2 that the numerical method converges in the spatial direction in L^p $(p=2,5,10)$ with the same order close to $1/2$, which is consistent with Theorem 1.

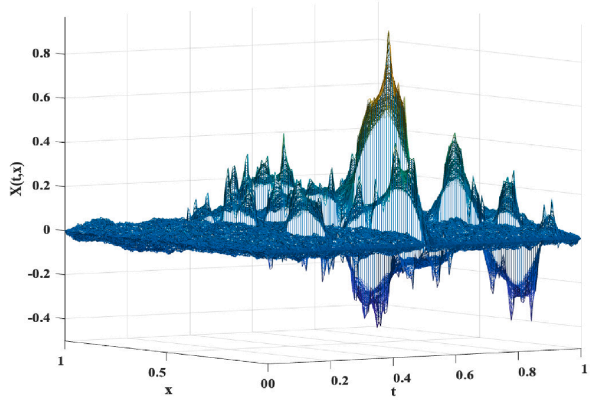
Then we consider the temporal convergence order. Set $N=100$. Take $\tau=2^{-15}$ for the reference solution, and the numerical solutions are simulated with large time step sizes $\tau=2^{-i}$, $i=6,7,8,9,10,11,12$. One can observe from Fig. 2 (b) that the numerical method converges in the temporal direction in L^p $(p=2,5,10)$ with the same order close to 1, which is greater than the theoretical result obtained in Theorem 1. To further explain this, we note that in the absence of Poisson noise (i.e., $\gamma_2=0$), our method (12) becomes the accelerated exponential Euler method for SPDEs driven by additive Gaussian space-time white noise, which is proved to be first-order strong convergence in time (see e.g. [30]). When the Poisson noise is involved, though theoretically the strong convergence order in time is shown to be $1/2$ in Theorem 1, the numerical experiment indicates that the jump-adapted strategy may retain first-order accuracy. We guess that this may be attributed to the reformulation of the stochastic convolution with respect to the Poisson random measure into an integral with respect to the intensity measure. A theoretical analysis for the optimal strong convergence order of the proposed scheme requires other skills and is left for future investigation.



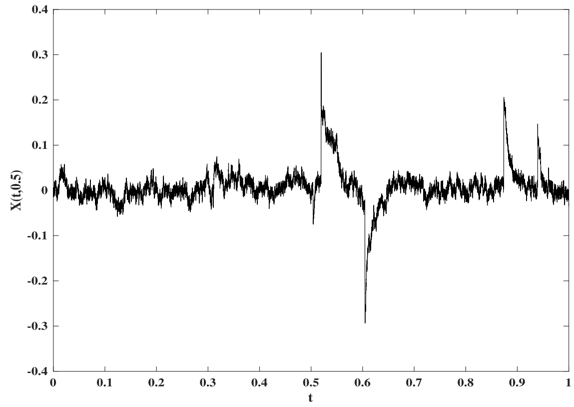
(a) Sample path of $X(\cdot,\cdot)$



(b) Sample trajectory of $X(\cdot,0.5)$

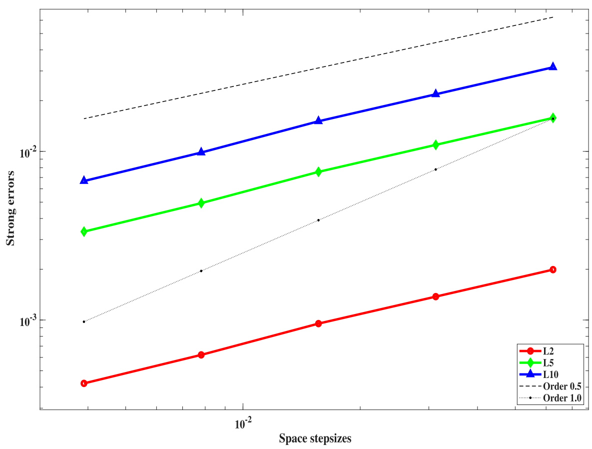


(c) Sample path of $X(\cdot,\cdot)$

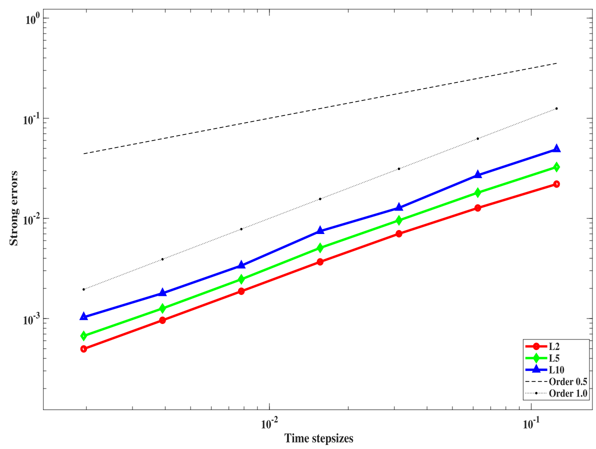


(d) Sample trajectory of $X(\cdot,0.5)$

Fig. 1. Sample paths of X, with (a)(b): $\tilde{\lambda} = 2$, and (c)(d): $\tilde{\lambda} = 5$.

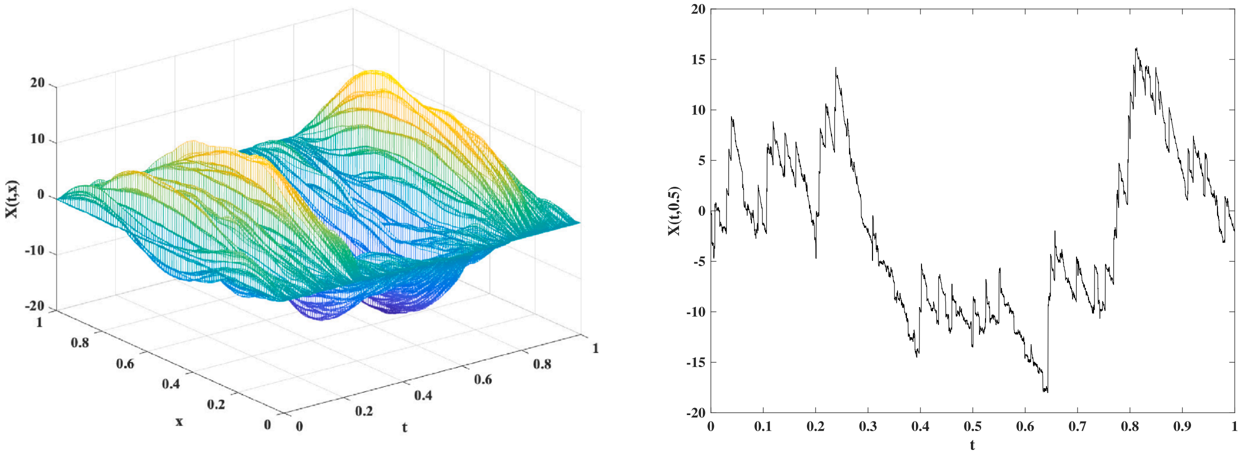


(a) Spatial convergence order



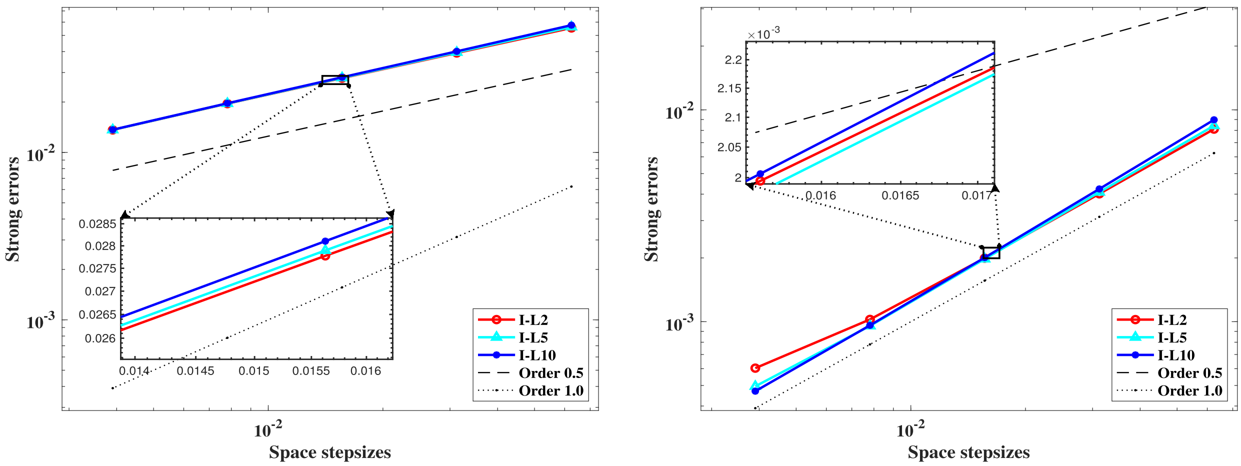
(b) Temporal convergence order

Fig. 2. L^p $(p = 2, 5, 10)$-strong convergence orders.



(a) Sample path of $X(\cdot,\cdot)$ (b) Sample trajectory $X(\cdot,0.5)$

Fig. 3. Sample path of X.



(a) $Q = \mathrm{I}$ (b) $\mathrm{Tr}(Q) < \infty$

Fig. 4. L^p ($p = 2, 5, 10$)-strong convergence orders in space.

5.2. Numerical scheme (13) with constant time step size

In this case, we consider the following SPDE with the additive Lévy noise on $\mathcal{O} = [0,1]$,

$$dX(t) = \Delta X(t)dt + (2X(t)+1)dt + dW(t) + g(z_0)d\tilde{N}(t), \quad t > 0, \quad X(0) = 0. \tag{35}$$

For the Poisson noise, we let $g(z_0) \in H = L^2(\mathcal{O})$ be given as $g(z_0) = \gamma_0 \sum_{i\in\mathbb{N}} i^{-1-\epsilon} e_i$ with $\gamma_0 > 0, \epsilon > 0$. We perform single sample path simulations of the process X for (35) when W is the space-time white noise in Fig. 3 (a). Take parameters $\tau = 2^{-12}$, $N = 100$, $T = 1$, and take a larger intensity $\tilde{\lambda} = 50$. The contour slice of the solution at the fixed space point $x = 0.5 \in \mathcal{O}$ is also presented in Fig. 3 (b).

Next, we consider the L^p-strong convergence orders both in space and in time for the method (13). Set $\gamma_0 = \pi^{-1}$. We use the Monte–Carlo method to simulate the expectations by computing averages of 1000 sample paths. When testing the spatial convergence order, we take parameters $N = 2^{12}$ and $\tau = 2^{-8}$ to simulate the reference solution. The numerical solution is simulated with $N = 2^i, i = 4,5,6,7,8$. Both the space-time white noise case $Q = \mathrm{I}$ and the trace class noise case $\mathrm{Tr}(Q) < \infty$ are considered, which are shown in Fig. 4 (a) and (b), respectively. For the trace class operator Q, we let its eigenvectors to be the same as $-\Delta$, and take the eigenvalues of Q as $q_i = i^{-1.1}$. We observe from Fig. 4 (a) that, in the space-time white noise case, the order of the L^p-strong convergence of (13) in space is close to $1/2$, as shown in Theorem 2. For the trace class noise case, due to the higher spatial regularity, it is shown in [16] that the scheme (13) exhibits first-order convergence in space, which can be observed from Fig. 4 (b).

Then we present the temporal convergence order of the scheme (13). For the noises, we take $Q = \mathrm{I}$ and $\tilde{\lambda} = 50$. The reference solution is approximated by scheme (13) with $N = 100$ and $\tau = 2^{-15}$. We take $N = 100, \tau = 2^{-i},\ i = 6,7,8,9,10,11,12$ for the numerical

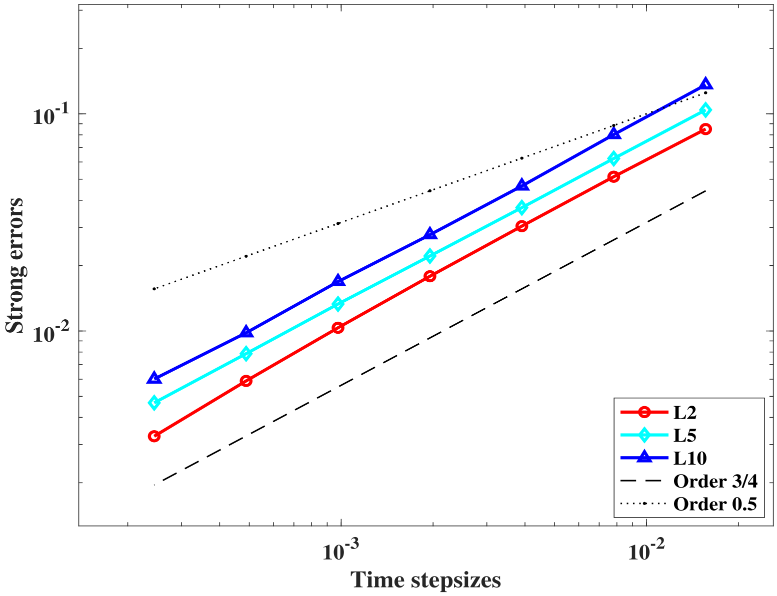
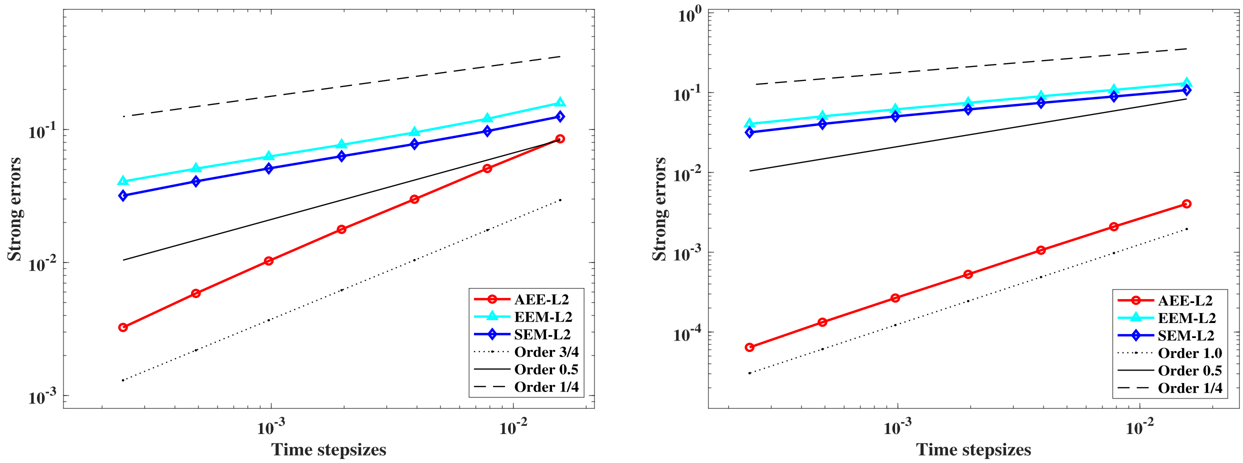


Fig. 5. L^p $(p = 2, 5, 10)$-strong convergence orders in time.



(a) $\tilde{\lambda} = 50$ (b) $\tilde{\lambda} = 0$

Fig. 6. Temporal strong convergence orders for various integrators.

solutions. From Fig. 5, one can observe that the scheme strongly converges in L^p, $p = 2, 5, 10$ with order close to $3/4$ in time, which is greater than the theoretical result $1/2$ order obtained in Theorem 2. We would like to mention that compared to the jump-adapted strategy (see Fig. 2), the temporal convergence order of scheme (13) with constant step size decreases from 1 to about $3/4$ numerically, which may be due to the inaccurate capture of jumps. However, the optimal strong convergence order for (13) with constant step size remains unknown, and is left for future research.

To further illustrate the numerical performance of proposed scheme, we compare the L^2-strong convergence orders of scheme (13) with the exponential Euler method (EEM) and the semi-implicit scheme (SEM), as introduced in [16]. It is worth noting that when there is no Poisson noise (i.e., $\gamma_0 = 0$), (13) reduces to the accelerated exponential Euler method, which exhibits high order convergence in time compared to EEM and SEM. To be specific, as shown in Fig. 6 (b), when $\tilde{\lambda} = 0$. scheme (13) achieves an L^2-strong convergence order close to 1 (see the red line labeled by "AEE"), which is consistent with the theoretical result in [30]. While EEM and SEM (see blue lines) are $1/4$ order convergence in time. When both Gaussian and Poisson noises are concerned (i.e., $\gamma_0 \neq 0$), our scheme (13) still performs better convergence order in time compared with EEM and SEM. As is shown in Fig. 6 (a), the L^2-strong convergence order of scheme (13) is approximately $3/4$ (see the red line still labeled by "AEE"), while both the EEM and SEM (see blue lines) remain $1/4$ order convergence.

CRediT authorship contribution statement

Chuchu Chen: Writing – original draft; **Tonghe Dang:** Writing – original draft; **Jialin Hong:** Writing – original draft; **Ziyi Lei:** Writing – original draft.

Data availability

Data will be made available on request.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Supplementary material

Supplementary material associated with this article can be found, in the online version at 10.1016/j.jcp.2025.114475

References

[1] S. Peszat, J. Zabczyk, Stochastic Partial Differential Equations with Lévy Noise: An Evolution Equation Approach, 113 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 2007. https://doi.org/10.1017/CBO9780511721373
[2] W.A. Woyczyński, Lévy processes in the physical sciences, in: Lévy Processes, Birkhäuser, Boston, MA, 2001. https://doi.org/10.1007/978-1-4612-0197-7.
[3] S. Duo, H. Wang, A fractional phase-field model using an infinitesimal generator of α stable Lévy process, J. Comput. Phys. 384 (2019) 253–269. https://doi.org/10.1016/j.jcp.2019.01.042.
[4] T. Dunst, E. Hausenblas, A. Prohl, Approximate Euler method for parabolic stochastic partial differential equations driven by space-time Lévy noise, SIAM J. Numer. Anal. 50 (6) (2012) 2873–2896. https://doi.org/10.1137/100818297.
[5] P. Przybyłowicz, M. Sobieraj, L. Stępień, Efficient approximation of SDEs driven by countably dimensional Wiener process and Poisson random measure, SIAM J. Numer. Anal. 60 (2) (2022) 824–855. https://doi.org/10.1137/21M1442747.
[6] O. Butkovsky, K. Dareiotis, M. Gerencsér, Strong rate of convergence of the Euler scheme for SDEs with irregular drift driven by Lévy noise, *Ann. Inst. Henri Poincaré Probab. Stat.*, to appear, 2024, https://arxiv.org/abs/2204.12926.
[7] C. Ling, Strong convergence of the Euler scheme for singular kinetic SDEs driven by α-stable processes, 2024, https://arxiv.org/abs/2412.05142.
[8] V. Schwarz, Stong order 1 adaptive approximation of jump-diffusion SDEs with discontinuous drift, 2025, https://arxiv.org/abs/2504.09452.
[9] C. Kelly, G.J. Lord, F. Sun, Strong convergence of a class of adaptive numerical methods for SDEs with jumps, Math. Comput. Simul. 227 (2025) 461–476. https://doi.org/10.1016/j.matcom.2024.08.020.
[10] S. Albeverio, E. Mastrogiacomo, B. Smii, Small noise asymptotic expansions for stochastic PDE's driven by dissipative nonlinearity and Lévy noise, Stochastic Process. Appl. 123 (6) (2013) 2084–2109. https://doi.org/10.1016/j.spa.2013.01.013.
[11] B. Xie, Uniqueness of invariant measures of infinite dimensional stochastic differential equations driven by Lévy noises, Potential Anal. 36 (1) (2012) 35–66. https://doi.org/10.1007/s11118-011-9220-6.
[12] S. Albeverio, V. Mandrekar, B. Rüdiger, Existence of mild solutions for stochastic differential equations and semilinear equations with non-Gaussian Lévy noise, Stochastic Process. Appl. 119 (3) (2009) 835–863. https://doi.org/10.1016/j.spa.2008.03.006.
[13] E. Hausenblas, SPDEs driven by Poisson random measure with non Lipschitz coefficients: existence results, Probab. Theory Related Fields 137 (2007) 161–200. https://doi.org/10.1007/s00440-006-0501-8.
[14] C. Chen, T. Dang, J. Hong, On structure preservation for fully discrete finite difference schemes of stochastic heat equation with Lévy space-time white noise, 2024, https://arxiv.org/abs/2409.14064.
[15] X. Yang, W. Zhao, Finite element methods and their error analysis for SPDEs driven by Gaussian and non-Gaussian noises, Appl. Math. Comput. 332 (2018) 58–75. https://doi.org/10.1016/j.amc.2018.03.039.
[16] J.-D. Mukam, A. Tambue, Optimal strong convergence rates of numerical methods for semilinear parabolic SPDE driven by Gaussian noise and Poisson random measure, Comput. Math. Appl. 77 (10) (2019) 2786–2803. https://doi.org/10.1016/j.camwa.2019.01.011.
[17] E. Hausenblas, I. Marchis, A numerical approximation of parabolic stochastic partial differential equations driven by a Poisson random measure, BIT Numer. Math. 46 (2006) 773–811. https://doi.org/10.1007/s10543-006-0099-3.
[18] E. Hausenblas, Finite element approximation of stochastic partial differential equations driven by Poisson random measures of jump type, SIAM J. Numer. Anal. 46 (1) (2007) 437–471. https://doi.org/10.1137/050654141.
[19] P. Przybyłowicz, V. Schwarz, M. Szölgyenyi, A higher-order approximation method for jump-diffusion SDEs with a discontinuous drift coefficient, J. Math. Anal. Appl. 538 (1) (2024) Paper No. 128319, 46. https://doi.org/10.1016/j.jmaa.2024.128319.
[20] Z. Lei, S. Gan, J. Liu, First order strong approximation of Ait–Sahalia-type interest rate model with Poisson jumps, Numer. Algorithms 94 (1) (2023) 93–130. https://doi.org/10.1007/s11075-022-01494-6.
[21] X. Yang, X. Wang, A transformed jump-adapted backward Euler method for jump-extended CIR and CEV models, Numer. Algorithms 74 (1) (2017) 39–57. https://doi.org/10.1007/s11075-016-0137-4.
[22] E. Platen, N.B. Liberati, Numerical Solution of Stochastic Differential Equations with Jumps in Finance, 64 of *Stochastic Modelling and Applied Probability*, Springer–Verlag, Berlin, 2010. https://doi.org/10.1007/978-3-642-13694-8.
[23] K. Lê, Maximal inequalities and weighted BMO processes, 2022, https://arxiv.org/abs/2211.15550.
[24] X. Wang, S. Gan, J. Tang, Higher order strong rate of convergence approximations of semilinear stochastic wave equation with additive space-time white noise, SIAM J. Sci. Comput. 36 (6) (2014) A2611–A2632. https://doi.org/10.1137/130937524.
[25] C. Marinelli, C. Prévôt, M. Röckner, Regular dependence on initial data for stochastic evolution equations with multiplicative Poisson noise, J. Funct. Anal. 258 (2) (2010) 616–649. https://doi.org/10.1016/j.jfa.2009.04.015.

[26] C. Chen, T. Dang, J. Hong, Z. Lei, Supplement to “Fully discrete schemes and L^p-strong convergence orders for the SPDE driven by Lévy noise" (2025).
[27] A. Barth, A. Stein, Approximation and simulation of infinite-dimensional Lévy processes, Stoch. Partial Differ. Equ. Anal. Comput. 6 (2) (2018) 286–334. https://doi.org/10.1007/s40072-017-0109-2.
[28] A. Barth, A. Lang, L^p and almost sure convergence of a Milstein scheme for stochastic partial differential equations, Stochastic Process. Appl. 123 (5) (2013) 1563–1587. https://doi.org/10.1016/j.spa.2013.01.003.
[29] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, 152 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 2nd edition, 2014. https://doi.org/10.1017/CBO9781107295513.
[30] A. Djurdjevac, M. Gerencsér, H. Kremp, Higher order approximation of nonlinear SPDEs with additive space-time white noise, 2024, https://arxiv.org/abs/2406.03058.