Supplement to “Fully discrete schemes and L^p-strong convergence orders for the SPDE driven by Lévy noise”

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Abstract

This file is the supplement to the main file titled “Fully discrete schemes and L^p-strong convergence orders for the SPDE driven by Lévy noise”.

Proof of Proposition 2.4. For brevity, we set $\mathcal{W}(t) := \int_0^t E(t-s)dW(s)$, and $Y_1(t) := X(t) - \mathcal{W}(t)$ for all $t \in [0,T]$. Then $\{Y_1(t)\}_{t\in[0,T]}$ satisfies

$$dY_1(t) = AY_1(t)dt + F\big(Y_1(t) + \mathcal{W}(t)\big)dt + \int_\chi G\big(Y_1(t) + \mathcal{W}(t), z\big)\tilde{N}(dz,dt), \quad t \in (0,T],$$

with initial value $Y_1(0) = x_0$. For all $p \geq 2$ and all $\alpha \in [0, \frac{1}{2})$, we note that there exists a constant $C_{p,T,\alpha} > 0$ such that $\sup_{t\in[0,T]} \|\mathcal{W}(t)\|_{L^p(\Omega,\dot{H}^\alpha)} \leq C_{p,T,\alpha}$. Using Assumption 1 and Lemma 2.3 leads to

$$\begin{aligned}
&\|(-A)^{\frac{\alpha}{2}}Y_1(t)\|_{L^p(\Omega,H)}\\
\leq\,& \|(-A)^{\frac{\alpha}{2}}E(t)x_0\|_{L^p(\Omega,H)} + \int_0^t \big\|(-A)^{\frac{\alpha}{2}}E(t-s)F\big(Y_1(s) + \mathcal{W}(s)\big)\big\|_{L^p(\Omega,H)}ds\\
&+ \Big\|\int_0^t \int_\chi (-A)^{\frac{\alpha}{2}}E(t-s)G\big(Y_1(s) + \mathcal{W}(s), z\big)\tilde{N}(dz,ds)\Big\|_{L^p(\Omega,H)}\\
\leq\,& \|x_0\|_{L^p(\Omega,\dot{H}^\alpha)} + C\int_0^t \|(-A)^{\frac{\alpha}{2}}E(t-s)\|_{\mathcal{L}(H)}\big(1 + \|Y_1(s)\|_{L^p(\Omega,H)} + \|\mathcal{W}(s)\|_{L^p(\Omega,H)}\big)ds\\
&+ C_{p,T}\Big(\int_0^t \mathbb{E}\Big[\int_\chi \big\|(-A)^{\frac{\alpha}{2}}G\big(Y_1(s) + \mathcal{W}(s), z\big)\big\|^p\nu(dz)\\
&+ \Big(\int_\chi \big\|(-A)^{\frac{\alpha}{2}}G\big(Y_1(s) + \mathcal{W}(s), z\big)\big\|^2\nu(dz)\Big)^{\frac{p}{2}}\Big]ds\Big)^{\frac{1}{p}}.
\end{aligned}$$

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When $\alpha = 0$, owing to the moment estimates of $\mathcal{W}$, Assumption 2, and the Hölder inequality, we arrive at

$$
\begin{aligned}
\|Y_1(t)\|^p_{L^p(\Omega,H)} \le\ & C_p\|x_0\|^p_{L^p(\Omega,H)} + C_{p,T}\int_0^t \big(1+\|Y_1(s)\|^p_{L^p(\Omega,H)}\big)ds \\
& + C_{p,T}\int_0^t \mathbb{E}\Big[\int_\chi \big(|g_1(z)|\big\|Y_1(s)+\mathcal{W}(s)\big\| + \|g(z)\|\big)^p\nu(dz) \\
& + \Big(\int_\chi \big(|g_1(z)|\big\|Y_1(s)+\mathcal{W}(s)\big\| + \|g(z)\|\big)^2\nu(dz)\Big)^{\frac{p}{2}}\Big]ds \\
\le\ & C_{p,T} + C_p\|x_0\|^p_{L^p(\Omega,H)} + C_{p,T}\int_0^t \|Y_1(s)\|^p_{L^p(\Omega,H)}ds.
\end{aligned}
$$

Applying the Grönwall inequality yields that $\|Y_1(t)\|_{L^p(\Omega,H)} \le C_{p,T}$.

Then for $\alpha \in (0,\frac{1}{2})$, we have

$$
\begin{aligned}
\|(-A)^{\frac{\alpha}{2}}Y_1(t)\|^p_{L^p(\Omega,H)} \le\ & C_p\|x_0\|^p_{L^p(\Omega,\dot{H}^\alpha)} + C_{p,T}\Big(\int_0^t (t-s)^{-\frac{\alpha}{2}}ds\Big)^p \sup_{s\in[0,t]}\big(1+\|Y_1(s)\|^p_{L^p(\Omega,H)}\big) \\
& + C_{p,T}\int_0^t \mathbb{E}\Big[\int_\chi \big\|g_1(z)(-A)^{\frac{\alpha}{2}}\big(Y_1(s)+\mathcal{W}(s)\big) + (-A)^{\frac{\alpha}{2}}g(z)\big\|^p\nu(dz) \\
& + \Big(\int_\chi \big\|g_1(z)(-A)^{\frac{\alpha}{2}}\big(Y_1(s)+\mathcal{W}(s)\big) + (-A)^{\frac{\alpha}{2}}g(z)\big\|^2\nu(dz)\Big)^{\frac{p}{2}}\Big]ds \\
\le\ & C_{p,T,\alpha} + C_p\|x_0\|^p_{L^p(\Omega,\dot{H}^\alpha)} + C_{p,T}\int_0^t \|(-A)^{\frac{\alpha}{2}}Y_1(s)\|^p_{L^p(\Omega,H)}ds.
\end{aligned}
$$

Applying the Grönwall inequality yields the desired assertion. $\square$

Proof of Lemma 4.1. (i) Denote $\mathcal{W}^N_{s,t} := \int_s^t E(t-r)P_N dW(r)$ for all $0 \le s \le t \le T$. Based on the factorization method, we have the expression $\mathcal{W}^N_{s,t} = \int_s^t (t-r)^{\alpha-1}e^{A(t-r)}\mathcal{Z}^N_{s,r}dr$ with $\mathcal{Z}^N_{s,r} = \frac{\sin(\pi\alpha)}{\pi}\int_s^r e^{A(r-\sigma)}(r-\sigma)^{-\alpha}P_N dW(\sigma)$ for $r \in [s,t]$, where $\alpha \in (0,1)$; see e.g. [1, Proposition 5.9, Theorem 5.10] for details. Note that for all $k \ge 1$ and $\delta \in (0,1)$,

$$
\begin{aligned}
& \mathbb{E}\Big[\sup_{t\in[s,T]}\|(-A)^{\frac{1-\delta}{4}}\mathcal{W}^N_{s,t}\|^{2k}\Big] \le \mathbb{E}\Big[\sup_{t\in[s,T]}\Big|\int_s^t (t-r)^{\alpha-1}e^{-\lambda_1(t-r)}\|(-A)^{\frac{1-\delta}{4}}\mathcal{Z}^N_{s,r}\|dr\Big|^{2k}\Big] \\
& \le C\mathbb{E}\Big[\sup_{t\in[s,T]}\Big(\int_s^t (t-r)^{p(\alpha-1)}e^{-p\lambda_1(t-r)}dr\Big)^{\frac{2k}{p}}\Big(\int_s^T \|(-A)^{\frac{1-\delta}{4}}\mathcal{Z}^N_{s,r}\|^q dr\Big)^{\frac{2k}{q}}\Big],
\end{aligned}
$$

where we utilized the Hölder inequality with $p,q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$ in the last step. When $2k \ge q$, by the Hölder inequality and the Burkholder–Davis–Gundy inequality, we derive

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\int_s^T \|(-A)^{\frac{1-\delta}{4}}\mathcal{Z}_{s,r}^N\|^q dr\Big)^{\frac{2k}{q}}\Big] \le C_T\mathbb{E}\Big[\int_s^T \|(-A)^{\frac{1-\delta}{4}}\mathcal{Z}_{s,r}^N\|^{2k}dr\Big]\\
&\le C_T\int_s^T \Big(\int_s^r (\frac{\sin(\pi\alpha)}{\pi})^2(r-\sigma)^{-2\alpha}\|(-A)^{\frac{2-\delta+\epsilon}{4}}e^{A(r-\sigma)}\|_{\mathcal{L}(H)}^2\|(-A)^{-\frac{1+\epsilon}{4}}\|_{\mathcal{L}_2(H)}^2 d\sigma\Big)^k dr\\
&\le C_T\int_s^T \Big(\int_s^r (r-\sigma)^{-2\alpha-1+\frac{\delta-\epsilon}{2}}d\sigma\Big)^k dr,
\end{aligned}
$$

which is finite if $-2\alpha+\frac{\delta-\epsilon}{2}>0$, i.e., $\alpha<\frac{\delta-\epsilon}{4}$. It means that

$$
\mathcal{Z}_{s,\cdot}^N \in L^{2k}(\Omega\times[s,T], D((-A)^{\frac{1-\delta}{4}})).
$$

Hence, when positive parameters α and p are chosen such that $\alpha<\frac{\delta-\epsilon}{4}$ and $p<\frac{1}{1-\alpha}$, we derive that $\sup_{t\in[s,T]}\Big(\int_s^t (t-r)^{p(\alpha-1)}e^{-p\lambda_1(t-r)}\mathrm{d}r\Big)^{\frac{2k}{p}} \le C_{k,p,T,\alpha}$. Therefore, for all $k\ge k_0$ with large number $k_0\ge\frac{q}{2}$, we obtain $\mathbb{E}\Big[\sup_{t\in[s,T]}\|(-A)^{\frac{1-\delta}{4}}\mathcal{W}_{s,t}^N\|^{2k}\Big]\le C_{k,T,\delta}$. For $k<k_0$ and $\delta\in(0,1)$, we have

$$
\mathbb{E}\Big[\sup_{t\in[s,T]}\|(-A)^{\frac{1-\delta}{4}}\mathcal{W}_{s,t}^N\|^{2k}\Big]\le \mathbb{E}\Big[\sup_{t\in[s,T]}\|(-A)^{\frac{1-\delta}{4}}\mathcal{W}_{s,t}^N\|^{2k_0}\Big]\le C_{k_0,T,\delta}.
$$

(ii) Recall that $\mathcal{W}^N(t)=\int_0^t E(t-s)P_N dW(s)$. For any $p\ge1$, $\delta\in(0,1)$, and $t\in[0,T]$, applying the Burkholder–Davis–Gundy inequality and properties (4)–(5) gives

$$
\begin{aligned}
&\big\|(-A)^{-\frac{1-\delta}{4}}\big(\mathcal{W}^N(t)-\mathcal{W}^N(\lfloor t\rfloor)\big)\big\|_{L^{2p}(\Omega,H)} \le \Big\|\int_{\lfloor t\rfloor}^t (-A)^{-\frac{1-\delta}{4}}E(t-r)P_N dW(r)\Big\|_{L^{2p}(\Omega,H)}\\
&+\Big\|\int_0^{\lfloor t\rfloor}(-A)^{-\frac{1-\delta}{4}}(E(t-\lfloor t\rfloor)-\mathrm{I})E(\lfloor t\rfloor-r)P_N dW(r)\Big\|_{L^{2p}(\Omega,H)}\\
&\le C_p\Big[\Big|\int_{\lfloor t\rfloor}^t (t-r)^{-\delta}\|(-A)^{-\frac{1+\delta}{4}}\|_{\mathcal{L}_2(H)}^2 dr\Big|^{\frac12}+(\Delta t)^{\frac{1-\delta}{2}}\Big|\int_0^{\lfloor t\rfloor}(\lfloor t\rfloor-r)^{-\frac{1-\delta}{2}}e^{-\lambda_1(\lfloor t\rfloor-r)}dr\Big|^{\frac12}\Big]\\
&\le C_{p,T,\delta}(\Delta t)^{\frac{1-\delta}{2}},
\end{aligned}
$$

where we used $\int_0^\infty x^{\eta-1}e^{-x}dx<\infty$ for $\eta>0$. The proof is completed. □

Note that the process $\int_t^\cdot\int_\chi E(\cdot-r)G_N(z)\tilde{N}(dz,dr)$ is progressively measurable on product space $(\Omega\times[t,T],\mathcal{F}_T\times\mathcal{B}([t,T]),\mathbb{P}\times dr)$. For any interval $[a,b]\in\mathcal{B}([t,T])$, due to the independent increments property of the compensated Poisson random measure, it holds that for $\beta\in[0,\frac14)$,

(P1) $\quad \sup_{s\in[a,b]}\Big\|\int_t^s\int_\chi E(s-r)(-A)^\beta G_N(z)\tilde{N}(dz,dr)\Big\|$ is $\mathcal{F}_t$-independent.

This property will be frequently utilized in the proofs of Propositions 4.3–4.5.

Proof of Proposition 4.3. Recall that $\mathcal{N}^N(t)=\int_0^t\int_\chi E(t-s)G_N(z)\tilde N(dz,ds)$. For any time grid point t_i, $i=0,1,\dots,M$, the property (P1) and Assumption 4 lead to

$$
\begin{aligned}
&\big\|(-A)^{-\frac{1-\delta}{4}}\big(\mathcal{N}^N(\cdot)-\mathcal{N}^N(\lfloor\cdot\rfloor)\big)\big\|_{\mathcal{C}_2^0|\mathcal{F}_{t_i},[t_i,T]}\\
\le&\sup_{s\in[t_i,T]}\Big\|\int_{\lfloor s\rfloor}^s\int_\chi(-A)^{-\frac{1-\delta}{4}}E(s-\rho)G_N(z)\tilde N(dz,d\rho)\Big\|_{L^2(\Omega,H)}\\
&+\sup_{s\in[t_i,T]}\Big\|\int_0^{\lfloor s\rfloor}\int_\chi(-A)^{-\frac{1-\delta}{4}}\big(E(s-\rho)-E(\lfloor s\rfloor-\rho)\big)G_N(z)\tilde N(dz,d\rho)\Big\|_{L^2(\Omega,H)|\mathcal{F}_{t_i}}\\
\le&C(\Delta t)^{\frac12}+\sup_{s\in[t_i,T]}\Big\|\big(E(s-\lfloor s\rfloor)-\mathrm{I}\big)(-A)^{-\frac{1-\delta}{2}}\Big\|_{\mathcal{L}(H)}\\
&\times\Big\|\int_0^{t_i}\int_\chi(-A)^{\frac{1-\delta}{4}}E(\lfloor s\rfloor-t_i)E(t_i-\rho)G_N(z)\tilde N(dz,d\rho)\Big\|+\sup_{s\in[t_i,T]}\Big\|\big(E(s-\lfloor s\rfloor)-\mathrm{I}\big)(-A)^{-\frac{1-\delta}{2}}\Big\|_{\mathcal{L}(H)}\\
&\times\Big\|\int_{t_i}^{\lfloor s\rfloor}\int_\chi(-A)^{\frac{1-\delta}{4}}E(\lfloor s\rfloor-t_i)E(t_i-\rho)G_N(z)\tilde N(dz,d\rho)\Big\|_{L^2(\Omega,H)}\\
\le&C_{T,\delta}(\Delta t)^{\frac{1-\delta}{2}}\Big(1+\Big\|\int_0^{t_i}\int_\chi E(t_i-\rho)G_N(z)\tilde N(dz,d\rho)\Big\|_{\dot H^{\frac{1-\delta}{2}}}\Big).
\end{aligned}
$$

The proof is completed. $\square$

Proof of Proposition 4.4. (i) *The proof of* (25). It follows from the property (P1) that

$$
\begin{aligned}
&\Big\|\sup_{t\in[0,T]}\mathbb{E}^t\Big[\sup_{s\in[t,T]}\sup_{r\in[0,T]}\mathbb{E}^s\big[\|\mathcal{N}^N(r)\|\big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
\le&\Big\|\sup_{t\in[0,T]}\mathbb{E}^t\Big[\sup_{s\in[t,T]}\sup_{r\in[s,T]}\mathbb{E}^s\Big[\Big\|\Big(\int_0^s+\int_s^r\Big)\int_\chi E(r-\rho)G_N(z)\tilde N(dz,d\rho)\Big\|\Big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&+\Big\|\sup_{t\in[0,T]}\mathbb{E}^t\Big[\sup_{s\in[t,T]}\sup_{r\in[0,s]}\mathbb{E}^s\Big[\Big\|\int_0^r\int_\chi E(r-\rho)G_N(z)\tilde N(dz,d\rho)\Big\|\Big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
\le&\Big\|\sup_{t\in[0,T]}\mathbb{E}^t\Big[\sup_{s\in[t,T]}\sup_{r\in[s,T]}\mathbb{E}\Big[\Big\|\int_s^r\int_\chi E(r-\rho)G_N(z)\tilde N(dz,d\rho)\Big\|\Big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&+\Big\|\sup_{t\in[0,T]}\mathbb{E}^t\Big[\sup_{s\in[t,T]}\Big\|\Big(\int_0^t+\int_t^s\Big)\int_\chi E(s-\rho)G_N(z)\tilde N(dz,d\rho)\Big\|\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&+\Big\|\sup_{t\in[0,T]}\mathbb{E}^t\Big[\sup_{r\in[0,T]}\Big\|\int_0^r\int_\chi E(r-\rho)G_N(z)\tilde N(dz,d\rho)\Big\|\Big]\Big\|_{L^p(\Omega,\mathbb{R})},
\end{aligned}
$$

and using (P1) again, we have

$$
\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\big[\sup_{s\in[t,T]}\sup_{r\in[0,T]}\mathbb{E}^s\big[\|\mathcal{N}^N(r)\|\big]\big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&\le C_{p,T} + \Big\|\sup_{t\in[0,T]}\Big\|\int_0^t\int_\chi E(t-\rho)G_N(z)\tilde{N}(dz,d\rho)\Big\|\Big\|_{L^p(\Omega,\mathbb{R})}\\
&\quad + \Big\|\sup_{t\in[0,T]}\mathbb{E}\Big[\sup_{s\in[t,T]}\Big\|\int_t^s\int_\chi E(s-\rho)G_N(z)\tilde{N}(dz,d\rho)\Big\|\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&\quad + \Big\|\sup_{t\in[0,T]}\mathbb{E}^t\Big[\sup_{r\in[0,t]}\Big\|\int_0^r\int_\chi E(r-\rho)G_N(z)\tilde{N}(dz,d\rho)\Big\|\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&\quad + \Big\|\sup_{t\in[0,T]}\mathbb{E}^t\Big[\sup_{r\in[t,T]}\Big\|\Big(\int_0^t+\int_t^r\Big)\int_\chi E(r-\rho)G_N(z)\tilde{N}(dz,d\rho)\Big\|\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&\le C_{p,T} + \Big\|\sup_{r\in[0,T]}\Big\|\int_0^r\int_\chi E(r-\rho)G_N(z)\tilde{N}(dz,d\rho)\Big\|\Big\|_{L^p(\Omega,\mathbb{R})}\\
&\quad + \Big\|\sup_{t\in[0,T]}\mathbb{E}\Big[\sup_{r\in[t,T]}\Big\|\int_t^r\int_\chi E(r-\rho)G_N(z)\tilde{N}(dz,d\rho)\Big\|\Big]\Big\|_{L^p(\Omega,\mathbb{R})} \le C_{p,T},
\end{aligned}
$$

where we also used Lemma 2.3 and Assumption 4.

(ii) *The proof of* (26). The proof is similar to that of (25). Recall that $s' := \lfloor s \rfloor + \Delta t > s$. Then we have

$$
\begin{aligned}
&\Big\|\sup_{t\in[0,T]}\mathbb{E}^t\Big[\sup_{s\in[t,T]}\mathbb{E}^s\big[\big\|(-A)^{\frac{1-\delta}{4}}\mathcal{N}^N(s')\big\|^2\big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&\le C\Big\|\sup_{t\in[0,T]}\mathbb{E}^t\Big[\sup_{s\in[t,T]}\mathbb{E}\Big[\Big\|\int_s^{s'}\int_\chi E(s'-r)(-A)^{\frac{1-\delta}{4}}G_N(z)\tilde{N}(dz,dr)\Big\|^2\Big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&\quad + C\Big\|\sup_{t\in[0,T]}\mathbb{E}^t\Big[\sup_{s\in[t,T]}\Big\|\int_0^s\int_\chi E(s'-r)(-A)^{\frac{1-\delta}{4}}G_N(z)\tilde{N}(dz,dr)\Big\|^2\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&\le C\Big\|\sup_{t\in[0,T]}\mathbb{E}^t\Big[\sup_{s\in[t,T]}\int_s^{s'}\int_\chi \big\|E(s'-r)(-A)^{\frac{1-\delta}{4}}G_N(z)\big\|^2\nu(dz)dr\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&\quad + C\Big\|\sup_{t\in[0,T]}\mathbb{E}^t\Big[\sup_{s\in[t,T]}\Big\|\Big(\int_t^s+\int_0^t\Big)\int_\chi E(s'-r)(-A)^{\frac{1-\delta}{4}}G_N(z)\tilde{N}(dz,dr)\Big\|^2\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&\le C_{p,T,\delta} + C\Big\|\sup_{t\in[0,T]}\mathbb{E}\Big[\sup_{s\in[t,T]}\Big\|\int_t^s\int_\chi E(s-r)(-A)^{\frac{1-\delta}{4}}G_N(z)\tilde{N}(dz,dr)\Big\|^2\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&\quad + C\Big\|\sup_{t\in[0,T]}\Big\|\int_0^t\int_\chi E(t-r)(-A)^{\frac{1-\delta}{4}}G_N(z)\tilde{N}(dz,dr)\Big\|^2\Big\|_{L^p(\Omega,\mathbb{R})} \le C_{p,T,\delta}.
\end{aligned}
$$

(iii) *The proof of* (27). Based on the property (P1), we have

$$
\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \mathbb{E}^t \Big[\sup_{s\in[t,T]} \mathbb{E}^s \big[\|(-A)^{\frac{1-\delta}{4}} \mathcal{N}^N(\cdot)\|^2_{\mathcal{C}_2^0|\mathcal{F}_{s'},[0,T]} \big] \Big] \Big\|_{L^p(\Omega,\mathbb{R})} \\
&\le C \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \Big[\sup_{s\in[t,T]} \mathbb{E}^s \Big[\sup_{r\in[0,s']} \Big\| \int_0^r \int_\chi E(r-\rho)(-A)^{\frac{1-\delta}{4}} G_N(z) \tilde{N}(dz,d\rho) \Big\|^2 \Big] \Big] \Big\|_{L^p(\Omega,\mathbb{R})} \\
&+ C \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \Big[\sup_{s\in[t,T]} \mathbb{E}^s \Big[\sup_{r\in[s',T]} \mathbb{E}^{s'} \Big[\Big\| \Big(\int_{s'}^r + \int_0^{s'} \Big) \int_\chi E(r-\rho)(-A)^{\frac{1-\delta}{4}} G_N(z) \tilde{N}(dz,d\rho) \Big\|^2 \Big] \Big] \Big] \Big\|_{L^p(\Omega,\mathbb{R})} \\
&\le C \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \Big[\sup_{s\in[t,T]} \mathbb{E}^s \Big[\sup_{r\in[0,s]} \Big\| \int_0^r \int_\chi E(r-\rho)(-A)^{\frac{1-\delta}{4}} G_N(z) \tilde{N}(dz,d\rho) \Big\|^2 \Big] \Big] \Big\|_{L^p(\Omega,\mathbb{R})} \\
&+ C \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \Big[\sup_{s\in[t,T]} \mathbb{E}^s \Big[\sup_{r\in[s,s']} \Big\| \Big(\int_0^s + \int_s^r \Big) \int_\chi E(r-\rho)(-A)^{\frac{1-\delta}{4}} G_N(z) \tilde{N}(dz,d\rho) \Big\|^2 \Big] \Big] \Big\|_{L^p(\Omega,\mathbb{R})} \\
&+ C \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \Big[\sup_{s\in[t,T]} \mathbb{E}^s \Big[\sup_{r\in[s',T]} \mathbb{E} \Big[\Big\| \int_{s'}^r \int_\chi E(r-\rho)(-A)^{\frac{1-\delta}{4}} G_N(z) \tilde{N}(dz,d\rho) \Big\|^2 \Big] \Big] \Big] \Big\|_{L^p(\Omega,\mathbb{R})} \\
&+ C \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \Big[\sup_{s\in[t,T]} \mathbb{E}^s \Big[\Big\| \Big(\int_0^s + \int_s^{s'} \Big) \int_\chi E(s'-\rho)(-A)^{\frac{1-\delta}{4}} G_N(z) \tilde{N}(dz,d\rho) \Big\|^2 \Big] \Big] \Big\|_{L^p(\Omega,\mathbb{R})}.
\end{aligned}
$$

Using (P1) and Lemma 2.3 yields that

$$
\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \mathbb{E}^t \Big[\sup_{s\in[t,T]} \mathbb{E}^s \Big[\sup_{r\in[s,s']} \Big\| \int_s^r \int_\chi E(r-\rho)(-A)^{\frac{1-\delta}{4}} G_N(z) \tilde{N}(dz,d\rho) \Big\|^2 \Big] \Big] \Big\|_{L^p(\Omega,\mathbb{R})} \\
&+ \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \Big[\sup_{s\in[t,T]} \mathbb{E}^s \Big[\sup_{r\in[s',T]} \mathbb{E} \Big[\Big\| \int_{s'}^r \int_\chi E(r-\rho)(-A)^{\frac{1-\delta}{4}} G_N(z) \tilde{N}(dz,d\rho) \Big\|^2 \Big] \Big] \Big] \Big\|_{L^p(\Omega,\mathbb{R})} \\
&+ \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \Big[\sup_{s\in[t,T]} \mathbb{E}^s \Big[\Big\| \int_s^{s'} \int_\chi E(s'-\rho)(-A)^{\frac{1-\delta}{4}} G_N(z) \tilde{N}(dz,d\rho) \Big\|^2 \Big] \Big] \Big\|_{L^p(\Omega,\mathbb{R})} \\
&\le \sup_{s\in[0,T]} \mathbb{E} \Big[\sup_{r\in[s,s']} \Big\| \int_s^r \int_\chi E(r-\rho)(-A)^{\frac{1-\delta}{4}} G_N(z) \tilde{N}(dz,d\rho) \Big\|^2 \Big] \\
&+ \sup_{s\in[0,T]} \mathbb{E} \Big[\sup_{r\in[s',T]} \mathbb{E} \Big[\Big\| \int_{s'}^r \int_\chi E(r-\rho)(-A)^{\frac{1-\delta}{4}} G_N(z) \tilde{N}(dz,d\rho) \Big\|^2 \Big] \Big] \\
&+ \sup_{s\in[0,T]} \mathbb{E} \Big[\Big\| \int_s^{s'} \int_\chi E(s'-\rho)(-A)^{\frac{1-\delta}{4}} G_N(z) \tilde{N}(dz,d\rho) \Big\|^2 \Big] \le C_{p,T,\delta}.
\end{aligned}
$$

As a result, we derive that

$$
\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]} \mathbb{E}^s\big[\|(-A)^{\frac{1-\delta}{4}}\mathcal{N}^N(\cdot)\|^2_{\mathcal{C}^0_2|\mathcal{F}_{s'},[0,T]}\big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&\le C_{p,T,\delta} + C\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{r\in[0,t]} \Big\| \int_0^r\int_\chi E(r-\rho)(-A)^{\frac{1-\delta}{4}} G_N(z)\tilde{N}(dz,d\rho)\Big\|^2\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&\quad + C\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]} \Big\|\Big(\int_0^t + \int_t^s\Big)\int_\chi E(r-\rho)(-A)^{\frac{1-\delta}{4}} G_N(z)\tilde{N}(dz,d\rho)\Big\|^2\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&\quad + C\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]} \Big\|\Big(\int_0^t + \int_t^s\Big)\int_\chi E(s-\rho)(-A)^{\frac{1-\delta}{4}} G_N(z)\tilde{N}(dz,d\rho)\Big\|^2\Big]\Big\|_{L^p(\Omega,\mathbb{R})}\\
&\le C_{p,T,\delta} + C\Big\| \sup_{t\in[0,T]} \Big\| \int_0^t\int_\chi E(t-\rho)(-A)^{\frac{1-\delta}{4}} G_N(z)\tilde{N}(dz,d\rho)\Big\|^2\Big\|_{L^p(\Omega,\mathbb{R})} \le C_{p,T,\delta},
\end{aligned}
$$

where in the second step, we repeatedly used (P1) and Lemma 2.3.

(iv) *The proof of* (28). By using (22) in Lemma 4.1 instead, the proof of (28) is similar to that of (27), which is omitted.

Combining (i)–(iv), the proof is completed. □

Proof of Proposition 4.5. (i) Owing to the linear growth condition of F, we derive that for any $t \in [0,T], \delta \in (0,1)$, and $p \ge 2$,

$$
\begin{aligned}
&\big\|(-A)^{-\frac{1-\delta}{4}}(Y^N(t) - Y^N(\lfloor t\rfloor))\big\|_{L^p(\Omega,H)}\\
&\le \int_{\lfloor t\rfloor}^t \big\|F_N(X^N(s))\big\|_{L^p(\Omega,H)}ds + \big\|(-A)^{-\frac{1-\delta}{2}}(E(t-\lfloor t\rfloor) - \mathrm{I})\big\|_{\mathcal{L}(H)}\times\\
&\quad \int_0^{\lfloor t\rfloor} \big\|(-A)^{\frac{1-\delta}{4}}E(\lfloor t\rfloor - s)\big\|_{\mathcal{L}(H)}\big\|F_N(X^N(s))\big\|_{L^p(\Omega,H)}ds \le C_{p,T,\delta}(\Delta t)^{\frac{1-\delta}{2}}.
\end{aligned}
$$

(ii) According to the estimates obtained in Propositions 4.3–4.4, we can derive that terms

$$
\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\big[\sup_{\zeta\in[0,T]} \big\|\mathcal{W}^N(\zeta)\big\|^2\big]\Big\|_{L^p(\Omega,\mathbb{R})}, \qquad \Big\| \sup_{t\in[0,T]} \mathbb{E}^t\big[\sup_{\zeta\in[0,T]} \big\|\mathcal{N}^N(\zeta)\big\|^2\big]\Big\|_{L^p(\Omega,\mathbb{R})},
$$

$$
\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\big[\sup_{s\in[t,T]} \mathbb{E}^s\big[\sup_{r\in[0,T]} \big\|\mathcal{W}^N(r)\big\|^2\big]\big]\Big\|_{L^p(\Omega,\mathbb{R})},
$$

$$
\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\big[\sup_{s\in[t,T]} \mathbb{E}^s\big[\sup_{r\in[0,T]} \big\|\mathcal{N}^N(r)\big\|^2\big]\big]\Big\|_{L^p(\Omega,\mathbb{R})},
$$

$$
\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\big[\sup_{s\in[t,T]} \mathbb{E}^s\big[\sup_{r\in[0,T]} \mathbb{E}^{s'}\big[\big\|\mathcal{W}^N(r)\big\|^2\big]\big]\big]\Big\|_{L^p(\Omega,\mathbb{R})},
$$

$$
\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\big[\sup_{s\in[t,T]} \mathbb{E}^s\big[\sup_{r\in[0,T]} \mathbb{E}^{s'}\big[\big\|\mathcal{N}^N(r)\big\|^2\big]\big]\big]\Big\|_{L^p(\Omega,\mathbb{R})}
$$

are all bounded for $p \ge 2$ with some positive constant $C_{p,T}$.

Recall that Y^N satisfies the perturbed stochastic equation (20). Using the linear growth condition of F, one has that for all $\zeta \in [0,T]$,

$$
\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \mathbb{E}^t \big[\|Y^N(\zeta)\|^2\big] \Big\|_{L^p(\Omega,\mathbb{R})} \\
&\le C\|x_0\|^2_{L^{2p}(\Omega,H)} + C_T \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \Big[\int_0^\zeta \big\|F_N(Y^N(\eta)+\mathcal{W}^N(\eta)+\mathcal{N}^N(\eta))\big\|^2 d\eta \Big] \Big\|_{L^p(\Omega,\mathbb{R})} \\
&\le C + C_T \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \Big[\int_0^\zeta \|Y^N(\eta)\|^2 d\eta \Big] \Big\|_{L^p(\Omega,\mathbb{R})} \\
&\quad + C_T \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \big[\sup_{\zeta\in[0,T]} \|\mathcal{W}^N(\zeta)\|^2 \big] \Big\|_{L^p(\Omega,\mathbb{R})} + C_T \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \big[\sup_{\zeta\in[0,T]} \|\mathcal{N}^N(\zeta)\|^2 \big] \Big\|_{L^p(\Omega,\mathbb{R})} \\
&\le C_{p,T} + C_T \int_0^\zeta \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \big[\|Y^N(\eta)\|^2\big] \Big\|_{L^p(\Omega,\mathbb{R})} d\eta.
\end{aligned}
$$

Applying the Grönwall inequality leads to $\sup_{\zeta\in[0,T]} \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \big[\big\|Y^N(\zeta)\big\|^2\big] \Big\|_{L^p(\Omega,\mathbb{R})} \le C_{p,T}$.

In addition, for all $\varrho \in [0,T]$,

$$
\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \mathbb{E}^t \big[\sup_{s\in[t,T]} \mathbb{E}^s \big[\|Y^N(\varrho)\|^2\big]\big] \Big\|_{L^p(\Omega,\mathbb{R})} \\
&\le C\|x_0\|^2_{L^{2p}(\Omega,H)} + C_T \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \Big[\sup_{s\in[t,T]} \mathbb{E}^s \Big[\int_0^\varrho \big\|F_N(Y^N(r)+\mathcal{W}^N(r)+\mathcal{N}^N(r))\big\|^2 dr \Big]\Big] \Big\|_{L^p(\Omega,\mathbb{R})} \\
&\le C_T + C_T \int_0^\varrho \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \big[\sup_{s\in[t,T]} \mathbb{E}^s \big[\|Y^N(r)\|^2\big]\big] \Big\|_{L^p(\Omega,\mathbb{R})} dr \\
&\quad + C_T \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \big[\sup_{s\in[t,T]} \mathbb{E}^s \big[\sup_{r\in[0,T]} \|\mathcal{W}^N(r)\|^2\big]\big] \Big\|_{L^p(\Omega,\mathbb{R})} \\
&\quad + C_T \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \big[\sup_{s\in[t,T]} \mathbb{E}^s \big[\sup_{r\in[0,T]} \|\mathcal{N}^N(r)\|^2\big]\big] \Big\|_{L^p(\Omega,\mathbb{R})} \\
&\le C_{p,T} + C_T \int_0^\varrho \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \big[\sup_{s\in[t,T]} \mathbb{E}^s \big[\|Y^N(r)\|^2\big]\big] \Big\|_{L^p(\Omega,\mathbb{R})} dr.
\end{aligned}
$$

The Grönwall inequality leads to $\sup_{\varrho\in[0,T]} \Big\| \sup_{t\in[0,T]} \mathbb{E}^t \big[\sup_{s\in[t,T]} \mathbb{E}^s \big[\|Y^N(\varrho)\|^2\big]\big] \Big\|_{L^p(\Omega,\mathbb{R})} \le C_{p,T}$.

Combining the above estimates, we obtain

$$
\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]} \mathbb{E}^s\big[\big\|(-A)^{\frac{1-\delta}{4}} Y^N(\cdot)\big\|^2_{\mathcal{C}_2^0|\mathcal{F}_{s'},[s',T]}\big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})} \\
&\le C\big\|(-A)^{\frac{1-\delta}{4}}x_0\big\|^2_{L^{2p}(\Omega,H)} + C_T\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]} \mathbb{E}^s\Big[\sup_{r\in[s',T]} \mathbb{E}^{s'}\Big[\int_0^r \big\|(-A)^{\frac{1-\delta}{4}} \\
&\quad \times E(r-\rho)F(Y^N(\rho)+\mathcal{W}^N(\rho)+\mathcal{N}^N(\rho))\big\|^2 d\rho\Big]\Big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})} \\
&\le C_T\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]} \mathbb{E}^s\Big[\sup_{r\in[s',T]} \sup_{\rho\in[0,r]} \mathbb{E}^{s'}\big[\|Y^N(\rho)\|^2\big]\int_0^r \|E(r-\eta)(-A)^{\frac{1-\delta}{4}}\|^2_{\mathcal{L}(H)}d\eta\Big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})} \\
&\quad + C_{p,T,\delta} + C_{T,\delta}\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\big[\sup_{s\in[t,T]} \mathbb{E}^s\big[\sup_{r\in[0,T]} \mathbb{E}^{s'}\big[\big\|\mathcal{W}^N(r)\big\|^2\big]\big]\big]\Big\|_{L^p(\Omega,\mathbb{R})} \\
&\quad + C_{T,\delta}\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\big[\sup_{s\in[t,T]} \mathbb{E}^s\big[\sup_{r\in[0,T]} \mathbb{E}^{s'}\big[\big\|\mathcal{N}^N(r)\big\|^2\big]\big]\big]\Big\|_{L^p(\Omega,\mathbb{R})},
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]} \mathbb{E}^s\big[\big\|(-A)^{\frac{1-\delta}{4}} Y^N(\cdot)\big\|^2_{\mathcal{C}_2^0|\mathcal{F}_{s'},[s',T]}\big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})} \\
&\le C_{p,T,\delta} + C_{T,\delta}\big\| \sup_{t\in[0,T]} \mathbb{E}^t\big[\sup_{s\in[t,T]} \mathbb{E}^s\big[\sup_{\rho\in[0,T]} \mathbb{E}^{s'}\big[\|Y^N(\rho)\|^2\big]\big]\big]\big\|_{L^p(\Omega,\mathbb{R})}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\big\| \sup_{t\in[0,T]} \mathbb{E}^t\big[\sup_{s\in[t,T]} \mathbb{E}^s\big[\sup_{\rho\in[0,T]} \mathbb{E}^{s'}\big[\|Y^N(\rho)\|^2\big]\big]\big]\big\|_{L^p(\Omega,\mathbb{R})} \\
&\le C_T\|x_0\|^2_{L^{2p}(\Omega,H)} \\
&\quad + C_T\Big\| \sup_{t\in[0,T]} \mathbb{E}^t\Big[\sup_{s\in[t,T]} \mathbb{E}^s\Big[\int_0^T \mathbb{E}^{s'}\big[\|Y^N(\vartheta)+\mathcal{W}^N(\vartheta)+\mathcal{N}^N(\vartheta)\|^2\big]d\vartheta\Big]\Big]\Big\|_{L^p(\Omega,\mathbb{R})} \\
&\le C_{p,T} + C_T\int_0^T \big\| \sup_{t\in[0,T]} \mathbb{E}^t\big[\sup_{s\in[t,T]} \mathbb{E}^s\big[\|Y^N(\vartheta)+\mathcal{W}^N(\vartheta)+\mathcal{N}^N(\vartheta)\|^2\big]\big]\big\|_{L^p(\Omega,\mathbb{R})}d\vartheta \\
&\le C_{p,T} + C_T\int_0^T \big\| \sup_{t\in[0,T]} \mathbb{E}^t\big[\sup_{s\in[t,T]} \mathbb{E}^s\big[\|Y^N(\vartheta)\|^2\big]\big]\big\|_{L^p(\Omega,\mathbb{R})}d\vartheta.
\end{aligned}
$$

Thus combining the above inequalities and then applying the Grönwall inequality yield the desired result. The proof is completed. □

[1] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, 2nd Edition, Vol. 152 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2014.
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