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An adaptive time-stepping fully discrete scheme for stochastic NLS equation: Strong convergence and numerical asymptotics ☆

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ABSTRACT

In this paper, we propose and analyze an adaptive time-stepping fully discrete scheme which possesses the optimal strong convergence order for the stochastic nonlinear Schrödinger equation with multiplicative noise. Based on the splitting skill and the adaptive strategy, the $\mathbb{H}^1$-exponential integrability of the numerical solution is obtained, which is a key ingredient to derive the strong convergence order. We show that the proposed scheme converges strongly with orders $\frac{1}{2}$ in time and 2 in space. To investigate the numerical asymptotic behavior, we establish the large deviation principle for the numerical solution. This is the first result on the study of the large deviation principle for the numerical scheme of stochastic partial differential equations with superlinearly growing drift. And as a byproduct, the error of the masses between the numerical and exact solutions is finally obtained.

1. Introduction

The stochastic nonlinear Schrödinger (NLS) equation is widely used to model the propagation of nonlinear dispersive waves in non-homogeneous or random media, and has important applications in various fields such as quantum physics, plasma physics, optical fiber communications and nonlinear optics (see e.g. [1,9,27] and references therein). In this paper, we focus on the numerical study of the following one-dimensional stochastic NLS equation driven by multiplicative noise of Stratonovich type

$$\mathrm{d}u = (\mathbf{i}\Delta u + \mathbf{i}\lambda|u|^2u)\mathrm{d}t - \mathbf{i}\sqrt{\epsilon}u\circ\mathrm{d}W(t), \quad \text{in } (0,T]\times\mathcal{O} \tag{1.1}$$

with the initial datum $u(0) = u_0 \in L^2(\mathcal{O};\mathbb{C}) =: \mathbb{H}$ and the homogeneous Dirichlet boundary condition, where $T > 0$, $\mathcal{O} = (0,1)$, $\epsilon > 0$ denotes the intensity of the noise, and $\lambda = 1$ or -1 corresponds to the focusing or defocusing case, respectively. Here, $\{W(t) : t \in [0,T]\}$ is a $L^2(\mathcal{O};\mathbb{R})$-valued Q-Wiener process on a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in[0,T]},\mathbb{P})$. There exists an orthonormal basis $\{e_k\}_{k\in\mathbb{N}_+}$ of $L^2(\mathcal{O};\mathbb{R})$ and a sequence of mutually independent, real-valued Brownian motions $\{\beta_k\}_{k\in\mathbb{N}_+}$ such that $W(t) = \sum_{k=1}^{\infty} Q^{\frac{1}{2}}e_k\beta_k(t),\ t \in [0,T]$.

Numerical analysis of the stochastic NLS equation (1.1) has been studied in recent decades, for instance, we refer to [10] for the θ-scheme, [17] for the Crank–Nicolson scheme, [15] for the splitting Crank–Nicolson scheme, [11] for the modified implicit Euler scheme, and [25] for the multi-symplectic scheme. These works are drift-implicit type schemes, while their implementation requires solving an algebraic equation at each iteration step, which needs additional computational efforts. In this regard, it is worth

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investigating explicit schemes, which are simple to implement and have lower complexity. However, the explicit, the exponential and the linear-implicit Euler type schemes with a uniform timestep fail to converge for a stochastic partial differential equation (SPDE) with superlinearly growing drift; see [13] for the stochastic NLS equation and [2] for the parabolic SPDE. To our knowledge, there are only a few works on the convergence analysis of the explicit scheme for the stochastic NLS equation (1.1). For instance, in [26], the author constructs an explicit splitting scheme in the temporal direction and obtains the convergence order in the probability sense. The author in [13] proposes a new kind of explicit splitting scheme, whose strong convergence orders are $\frac{1}{2}-$ and $s-$ in the temporal and spatial direction with $Q^{\frac{1}{2}} \in \mathcal{L}_2^s$ (in this case the solution has $\mathbb{H}^s$-regularity), respectively. In order to construct a drift-explicit scheme, whose strong convergence order is optimal, we apply the adaptive timestep skill to adapt the timestep size at each iteration. We refer to e.g. [7,8] for adaptive schemes for parabolic SPDEs with superlinearly growing drift. To our knowledge, there has been no work on the study of the adaptive time-stepping scheme for the stochastic NLS equation. The main purpose of this paper is twofold:

(i) Propose a drift-explicit, adaptive time-stepping fully discrete scheme for (1.1), whose strong convergence order is optimal.
(ii) Investigate the numerical asymptotic behavior of the proposed scheme as $\epsilon \to 0$ via the large deviation principle (LDP).

To be specific, in this work we propose an adaptive time-stepping fully discrete scheme, whose spatial direction is using the spectral Galerkin method, and temporal direction is based on the adaptive splitting exponential Euler scheme. A key ingredient to derive the strong convergence order is the $\mathbb{H}^1$-exponential integrability of both the exact and numerical solutions. It is studied in [15] that the exact solution and the drift-implicit type scheme of the stochastic NLS equation can have this exponential integrability due to the preservation of the mass of the solutions. The author in [13] uses the splitting skill to split the stochastic NLS equation into a Hamiltonian subsystem and a mass-decaying linear subsystem, so that the exponential integrability of the numerical solution is still possessed. We remark that this type of exponential integrability also has important applications in other problems, for instance the large deviation-type result (see e.g [14, Corollary 3.2]). To obtain the exponential integrability of the drift-explicit, adaptive time-stepping fully discrete scheme, we combine the splitting skill and the adaptive strategy for the proposed scheme to derive the a.s.-uniform boundedness of the mass of the numerical solution. Based on this $\mathbb{H}^1$-exponential integrability and the $\mathbb{H}^j$ $(j = 1, 2)$-regularity estimates, it is shown that this fully discrete scheme is convergent with strong orders $\frac{1}{2}$ in time and 2 in space, which are optimal in the sense that the orders coincide with the optimal temporal Hölder regularity and spatial Sobolev regularity, respectively.

Another aim of this paper is to further study the asymptotic behavior of the proposed adaptive time-stepping fully discrete scheme, by presenting the LDP for the numerical solution. The LDP for the SPDE with small noise is also called the Freidlin–Wentzell LDP, which characterizes the exponential decay probabilities that sample paths of the SPDE deviate from that of the corresponding deterministic equation as the intensity of the noise tends to zero. The study on the LDP and its relative topic for the stochastic NLS equation has received much attention in recent years (see e.g. [20,22–24,29,30]). To be specific, the LDP for the stochastic NLS equation has been well studied, see [22,24] for the additive type noise case and [20,23] for the multiplicative type noise case; in addition, the support result in the space of Hölder continuous functions is also derived (see [22,24] for the additive type noise case). For both the stochastic inhomogeneous dispersive equations and the stochastic NLS equation with variable coefficients, the author in [29] derives the LDP for the small noise asymptotics. Furthermore, the support result for the multiplicative type noise case is presented in [30], where the Stroock–Varadhan type theorem is obtained for the topological support of the probability distribution induced by global solutions in the Strichartz and local smoothing spaces. By contrast, less result is known for the case of numerical methods for the stochastic NLS equation.

A well-known approach proposed in [18] to establishing the LDP is the weak convergence method, which is by means of the equivalence to the Laplace principle. To apply this approach, the main difficulty lies in proving the compactness of solutions of the skeleton equation and the stochastic controlled equation in the space $C([0,T];\mathbb{H}_N)$. In this regard, by analyzing the conditional moment estimation of the solution of the stochastic controlled equation, we prove that the solution of the proposed fully discrete scheme satisfies the LDP on $C([0,T];\mathbb{H}_N)$ with the rate function given by the corresponding skeleton equation. To our knowledge, this is the first work on the study of the LDP for the numerical scheme of SPDEs with superlinearly growing drift. As a byproduct, the error of the masses between the numerical and exact solutions of (1.1) is finally obtained.

The outline of this paper is as follows. In the next section, we propose the adaptive time-stepping fully discrete scheme, and prove the a.s.-uniform boundedness of the mass, the $\mathbb{H}^j$ $(j = 1, 2)$-regularity estimates and the $\mathbb{H}^1$-exponential integrability of the numerical solution. In Section 3, we derive the optimal strong convergence order of the fully discrete scheme. Section 4 is devoted to establishing the LDP for the solution of the fully discrete scheme.

To close this section, we introduce some frequently used notations. The norm and the inner product of $\mathbb{H} = L^2(\mathcal{O};\mathbb{C})$ are denoted by $\|\cdot\|$ and $\langle u, v\rangle := \mathrm{Re}\big[\int_{\mathcal{O}} u(x)\bar{v}(x)\mathrm{d}x\big]$, respectively. Denote $L^p(\mathcal{O}) := L^p(\mathcal{O};\mathbb{C})$, $1 \le p \le \infty$, $H := L^2(\mathcal{O};\mathbb{R})$. Let $H^s := H^s(\mathcal{O})$ and $\mathbb{H}^s := \mathbb{H}^s(\mathcal{O})$, $s \in \mathbb{R}$ denote the real-valued and complex-valued Sobolev spaces, respectively. Then the domain of the Dirichlet Laplacian operator is $\mathbb{H}_0^1 \cap \mathbb{H}^2$. We denote the interpolation space of the Dirichlet negative Laplacian operator by $\dot{\mathbb{H}}^s$, $s \in \mathbb{R}$. It is known that $\mathbb{H}^s$ and $\dot{\mathbb{H}}^s$ are equivalent for $s = 1, 2$. Throughout the paper, we assume that the initial datum $u_0 \in \mathbb{H}_0^1 \cap \mathbb{H}^2$ is a deterministic function, and that the operator $Q^{\frac{1}{2}} \in \mathcal{L}_2^2 := \mathcal{L}_2(H;H^2)$, i.e., $\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2}^2 := \sum_{k=1}^{\infty} \|Q^{\frac{1}{2}} e_k\|_{H^2}^2 < \infty$. And hence $\|Q^{\frac{1}{2}}\|_{\mathcal{L}(H;H^2)} \le \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2} < \infty$. In sequel, C is a constant which may change from one line to another, and sometimes we write $C(a, b, c \dots)$ to emphasize the dependence on parameters $a, b, c, \dots$

2. The adaptive time-stepping fully discrete scheme

In this section, we first introduce the adaptive time-stepping fully discrete scheme of (1.1). Then we prove the a.s.-uniform boundedness of the mass, the $\mathbb{H}^j$ $(j = 1, 2)$-regularity estimates and the $\mathbb{H}^1$-exponential integrability of the numerical solution, which are important in the estimate of the strong convergence order of this fully discrete scheme. We remark that ϵ is a fixed positive parameter in this section and the next section, and we do not emphasize the dependence on ϵ of solutions of the stochastic NLS equation and its discretizations.

It is known that (1.1) has the following equivalent Itô formulation

$$\mathrm{d}u = \Big(\mathbf{i}\Delta u + \mathbf{i}\lambda|u|^2u - \frac{\epsilon}{2}F_Q u\Big)\mathrm{d}t - \mathbf{i}\sqrt{\epsilon}u\mathrm{d}W(t), \quad \text{in } (0,T]\times\mathcal{O}, \tag{2.1}$$

where $F_Q := \sum_{k=1}^{\infty}(Q^{\frac{1}{2}}e_k)^2$. The well-posedness and $\mathbb{H}^j$ $(j = 1, 2)$-regularity estimates for (2.1) have been studied; see e.g. [9,14,16,17,21].

It is known that the splitting skill can be used to construct convergent explicit numerical schemes for stochastic NLS equation; see e.g. [4,13,26]. Introduce a partition $0 = t_0 < t_1 < \cdots < t_m < \cdots < t_M = T$ with some $M \in \mathbb{N}_+$. As is shown in [15], one can split (2.1) in the time interval $T_m := [t_m, t_{m+1})$ into a deterministic NLS equation with random initial datum and a linear SPDE. Precisely, for $t \in T_m$,

$$\begin{cases}\mathrm{d}u_m^D(t) = \mathbf{i}\Delta u_m^D(t)\mathrm{d}t + \mathbf{i}\lambda|u_m^D(t)|^2u_m^D(t)\mathrm{d}t,\ u_m^D(t_m) = u_{m-1}^S(t_m), & \text{(a)}\\ \mathrm{d}u_m^S(t) = -\frac{\epsilon}{2}F_Q u_m^S(t)\mathrm{d}t - \mathbf{i}\sqrt{\epsilon}u_m^S(t)\mathrm{d}W(t), & \text{(b)}\\ \qquad u_m^S(t_m) = u_m^D(t_{m+1}); \end{cases} \tag{2.2}$$

especially, for $t \in T_0$, the initial datum of (2.2)(a) is $u_0^D(0) = u_0$.

Let $N \in \mathbb{N}_+$, and let $\mathbb{H}_N$ be the subspace of $\mathbb{H}$ consisting of the first N eigenvectors of the Dirichlet Laplacian operator. Denote by $P^N : \mathbb{H} \to \mathbb{H}_N$ the spectral Galerkin projection, which is defined by $\langle P^N u, v\rangle := \langle u, v\rangle$ for $u \in \mathbb{H}, v \in \mathbb{H}_N$. Applying the spectral Galerkin method to (2.2) in the spatial direction, we derive the semi-discrete scheme: For $t \in T_m$,

$$\begin{cases}\mathrm{d}u_m^{D,N}(t) = \mathbf{i}\Delta u_m^{D,N}(t)\mathrm{d}t + \mathbf{i}\lambda P^N|u_m^{D,N}(t)|^2u_m^{D,N}(t)\mathrm{d}t, & \text{(a)}\\ \qquad u_m^{D,N}(t_m) = u_{m-1}^{S,N}(t_m), & \\ \mathrm{d}u_m^{S,N}(t) = -\frac{\epsilon}{2}P^N F_Q u_m^{S,N}(t)\mathrm{d}t - \mathbf{i}\sqrt{\epsilon}P^N u_m^{S,N}(t)\mathrm{d}W(t), & \text{(b)}\\ \qquad u_m^{S,N}(t_m) = u_m^{D,N}(t_{m+1}), & \end{cases} \tag{2.3}$$

where the initial datum is $u_0^{D,N} = P^N u_0$.

To present the adaptive time-stepping scheme, the timestep at each iteration must be adapted with some adaptive timestep function $\tau : \mathbb{H} \to \mathbb{R}_+$ to control the numerical solution from divergence. Thus the partition $\{t_m : m = 0, \ldots, M\}$ of the split equation (2.2) and the semi-discrete scheme (2.3) is chosen the same as the one that will be used in the fully discrete scheme (2.4). In this case, to emphasize the dependence on T, we use M_T instead of M in the sequel. By further applying the adaptive exponential Euler scheme in the temporal direction of (2.3)(a), we obtain the fully discrete scheme, whose differential form reads as:

$$\begin{cases}\mathrm{d}u_{t,m}^{D,N} = \mathbf{i}\Delta u_{t,m}^{D,N}\mathrm{d}t + \mathbf{i}\lambda S^N(t-t_m)P^N|u_m^N|^2u_m^N\mathrm{d}t,\ u_{t_m,m}^{D,N} = u_m^N, & \text{(a)}\\ \mathrm{d}u_{t,m}^{S,N} = -\frac{\epsilon}{2}P^N F_Q u_{t,m}^{S,N}\mathrm{d}t - \mathbf{i}\sqrt{\epsilon}P^N u_{t,m}^{S,N}\mathrm{d}W(t), & \text{(b)}\\ \qquad u_{t_m,m}^{S,N} = u_{t_{m+1},m}^{D,N}, \end{cases} \tag{2.4}$$

where $t \in T_m$, and $t_{m+1} = t_m + \tau_m$ with $\tau_m := \tau(u_m^N)$. Here, $S^N(t) := P^N e^{\mathbf{i}t\Delta}$, $u_{m+1}^N := u_{t_{m+1},m}^{S,N}$, and the initial datum is $u_0^N = P^N u_0$. By (2.4)(a), we have the explicit one-step scheme for the deterministic part:

$$u_{t_{m+1},m}^{D,N} = S^N(\tau_m)(u_m^N + \mathbf{i}\lambda|u_m^N|^2u_m^N\tau_m). \tag{2.5}$$

If we denote the flows of (2.4)(a) and (2.4)(b) by $\Phi_{m,t-t_m}^{D,N}$ and $\Phi_{m,t-t_m}^{S,N}$, respectively for $t \in T_m$, then the solution of the fully discrete scheme (2.4) can be expressed as

$$u_m^N = u_{t_m,m-1}^{S,N} = \prod_{j=0}^{m-1}(\Phi_{j,\tau_j}^{S,N}\Phi_{j,\tau_j}^{D,N})u_0^N.$$

We remark that if the existing time span is longer than T after adding the last timestep, then we take a smaller timestep such that the existing time span just attains T after adding it. Namely, if $t_{M_T-1}+\tau_{M_T-1} > T$, then we enforce the last timestep $\tau_{M_T-1} := T - t_{M_T-1}$. In the sequel, we give some assumptions on the timestep function so that the numerical solution can attain T with finite many timesteps (see Remark 2.2). Without loss of generality, we take $1/0 = \infty$.

Assumption 1. Let τ_m satisfy

$$\tau_m \le \min\big\{L_1\|u_m^N\|^2\|u_m^N\|_{L^6(\mathcal{O})}^{-6}, T\delta\big\} \qquad a.s., \tag{2.6}$$

$$\tau_m \ge (\zeta\|u_m^N\|^{\beta} + \xi)^{-1}\delta \qquad a.s. \tag{2.7}$$

with constants $L_1, \zeta, \beta, \xi > 0$ and a small constant $\delta \in (0, 1)$.

Below, we give the estimate of the mass $\|u_m^N\|^2$ of the solution of (2.4). Hereafter, we use the notation $\underline{t} := \max\{m : t_m \le t\}$ to represent the maximal timestep number not exceeding t.

Lemma 2.1. *Under Assumption 1 (2.6), it holds that*

$$\sup_{t\in[0,T]} (\|u_{t,\underline{t}}^{D,N}\|^2 \vee \|u_{t,\underline{t}}^{S,N}\|^2) \le e^{L_1 T}\|u_0^N\|^2 \qquad a.s.$$

Proof. By the property $\|S^N(t)\|_{\mathcal{L}(\mathbb{H};\mathbb{H})} = 1$ and Assumption 1 (2.6), it follows from (2.5) that

$$\begin{aligned}\|u_{t_{m+1},m}^{D,N}\|^2 &= \|u_m^N + \mathbf{i}\lambda|u_m^N|^2 u_m^N \tau_m\|^2 = \|u_m^N\|^2 + \tau_m^2\|u_m^N\|_{L^6(\mathcal{O})}^6 \\ &\le (1+L_1\tau_m)\|u_m^N\|^2 \quad a.s.\end{aligned}$$

By the Itô formula, for $t \in T_m$,

$$\begin{aligned}&\|u_{t,m}^{S,N}\|^2 - \|u_{t_{m+1},m}^{D,N}\|^2 \\ &= 2\int_{t_m}^t \langle u_{r,m}^{S,N}, -\frac{\epsilon}{2}P^N F_Q u_{r,m}^{S,N}\rangle \mathrm{d}r + \epsilon\int_{t_m}^t \sum_{k=1}^\infty \|P^N u_{r,m}^{S,N} Q^{\frac12} e_k\|^2 \mathrm{d}r \\ &= -\epsilon\int_{t_m}^t \sum_{k=1}^\infty \|(\mathrm{Id}-P^N)u_{r,m}^{S,N}Q^{\frac12}e_k\|^2\mathrm{d}r \le 0 \quad a.s.\end{aligned} \tag{2.8}$$

Hence, combining the above two inequalities gives that

$$\begin{aligned}\|u_{m+1}^N\|^2 &\le \|u_{t_{m+1},m}^{D,N}\|^2 \le (1+L_1\tau_m)\|u_m^N\|^2 \le \prod_{j=0}^m (1+L_1\tau_j)\|u_0^N\|^2 \\ &\le e^{L_1 t_{m+1}}\|u_0^N\|^2 \le e^{L_1 T}\|u_0^N\|^2 \quad a.s.\end{aligned}$$

Moreover, we derive that for $t \in T_m$,

$$\begin{aligned}\|u_{t,m}^{D,N}\|^2 = \|u_m^N\|^2 + \|u_m^N\|_{L^6(\mathcal{O})}^6 (t-t_m)^2 &\le (1+L_1\tau_m)e^{L_1 t_m}\|u_0^N\|^2 \\ &\le e^{L_1 T}\|u_0^N\|^2 \quad a.s.\end{aligned}$$

and

$$\|u_{t,m}^{S,N}\|^2 \le \|u_{t_{m+1},m}^{D,N}\|^2 \le e^{L_1 T}\|u_0^N\|^2 \quad a.s.$$

The proof is finished. □

Remark 2.2. It follows from Lemma 2.1 and Assumption 1 (2.7) that

$$\tau_m \ge (\zeta\|u_m^N\|^\beta + \xi)^{-1}\delta \ge (\zeta e^{\frac12\beta L_1 T}\|u_0^N\|^\beta + \xi)^{-1}\delta =: \tau_{min}\delta,$$

which implies that under Assumption 1, the final time T is always attainable, i.e.,

$$M_T \le T(\inf_{t_m\in[0,T]} \tau_m)^{-1} \le T\tau_{min}^{-1}\delta^{-1} \quad \text{a.s.}$$

2.1. Regularity analysis

In this subsection, we give regularity analysis of the solution of the fully discrete scheme, including the $\mathbb{H}^j$ $(j = 1, 2)$-regularity estimates and the $\mathbb{H}^1$-exponential integrability. To this end, we make the following assumption on adaptive timesteps. Let the Hamiltonian be $\mathcal{H}(u) := \frac12\|\nabla u\|^2 - \frac{\lambda}{4}\|u\|_{L^4(\mathcal{O})}^4$, $u \in \mathbb{H}^1$.

Assumption 2. Let τ_m satisfy

$$\tau_m^{\frac12-\gamma}\lambda_N \le L_2 \quad a.s., \tag{2.9}$$

$$\tau_m^\gamma \mathcal{H}(u_m^N) \le L_3 \quad a.s. \tag{2.10}$$

for some $\gamma \in (0, \frac12)$ and constants $L_2, L_3 > 0$, where $\lambda_N = N^2\pi^2$ is the Nth eigenvalue of the Dirichlet negative Laplacian operator.

Remark 2.3. Note that the Gagliardo–Nirenberg inequality $\|u\|_{L^4(\mathcal{O})}^4 \le 2\|u\|^3\|\nabla u\|$, the inverse inequality $\|P^N u\|_{\dot{\mathbb{H}}^s} \le \lambda_N^{\frac{s}{2}}\|P^N u\|$, and Lemma 2.1 lead to $\mathcal{H}(u_m^N) \le C(\|\nabla u_m^N\|^2 + 1) \le C\lambda_N$. If both $\tau_m^{\frac12-\gamma}\lambda_N \le L_2$ and $\tau_m^\gamma\lambda_N \le L_3$ hold, then Assumption 2 is satisfied.

Proposition 2.4. *Under Assumptions 1 and 2, for $p \geq 2$, there exists a constant $C := C(p, \epsilon, T, \mathcal{H}(u_0^N)) > 0$ such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|u^{D,N}_{t,\underline{t}}\|^p_{\mathbb{H}^1}\Big] + \mathbb{E}\Big[\sup_{t\in[0,T]} \|u^{S,N}_{t,\underline{t}}\|^p_{\mathbb{H}^1}\Big] \leq C.$$

Proof. Direct calculation leads to

$$\begin{aligned}
&D\mathcal{H}(u)(v) = \langle \nabla u, \nabla v\rangle - \lambda\langle |u|^2 u, v\rangle,\\
&D^2\mathcal{H}(u)(v,w) = \langle \nabla w, \nabla v\rangle - \lambda\langle |u|^2 v, w\rangle - 2\lambda\langle u\mathrm{Re}(\bar{u}v), w\rangle.
\end{aligned}$$

It follows from the chain rule that

$$\begin{aligned}
&\mathcal{H}(u^{D,N}_{t_{m+1},m}) - \mathcal{H}(u^N_m) = \int_{t_m}^{t_{m+1}} D\mathcal{H}(u^{D,N}_{t,m})\mathrm{d}u^{D,N}_{t,m}\\
&= \int_{t_m}^{t_{m+1}} \Big\langle \nabla u^{D,N}_{t,m}, \mathbf{i}\lambda\nabla(S^N(t-t_m)|u^N_m|^2 u^N_m)\Big\rangle \mathrm{d}t\\
&- \lambda\int_{t_m}^{t_{m+1}} \Big\langle |u^{D,N}_{t,m}|^2 u^{D,N}_{t,m}, \mathbf{i}\Delta u^{D,N}_{t,m}\Big\rangle \mathrm{d}t\\
&- \lambda\int_{t_m}^{t_{m+1}} \Big\langle |u^{D,N}_{t,m}|^2 u^{D,N}_{t,m}, \mathbf{i}\lambda S^N(t-t_m)|u^N_m|^2 u^N_m\Big\rangle \mathrm{d}t\\
&= \int_{t_m}^{t_{m+1}} \Big\langle -\Delta u^{D,N}_{t,m}, \mathbf{i}\lambda(S^N(t-t_m) - \mathrm{Id})P^N|u^N_m|^2 u^N_m\\
&+ \mathbf{i}\lambda P^N(|u^N_m|^2 u^N_m - |u^{D,N}_{t,m}|^2 u^{D,N}_{t,m})\Big\rangle \mathrm{d}t\\
&- \int_{t_m}^{t_{m+1}} \Big\langle |u^{D,N}_{t,m}|^2 u^{D,N}_{t,m}, \mathbf{i}\lambda^2(S^N(t-t_m) - \mathrm{Id})P^N|u^N_m|^2 u^N_m\\
&+ \mathbf{i}\lambda^2 P^N(|u^N_m|^2 u^N_m - |u^{D,N}_{t,m}|^2 u^{D,N}_{t,m})\Big\rangle \mathrm{d}t.
\end{aligned}$$

By properties $\|(S(t)-\mathrm{Id})u\| \leq Ct^{\frac{1}{2}}\|u\|_{\dot{\mathbb{H}}^1}$, $\|S(t)\|_{\mathcal{L}(\mathbb{H};\mathbb{H})} = 1$ and the Gagliardo–Nirenberg inequality $\|u\|^3_{L^6(\mathcal{O})} \leq C\|\nabla u\|\|u\|^2$, we have that for $t \in T_m$,

$$\begin{aligned}
\|u^{D,N}_{t,m} - u^N_m\| &\leq \|(S(t-t_m)-\mathrm{Id})u^N_m\| + \|S(t-t_m)\mathbf{i}\lambda|u^N_m|^2 u^N_m(t-t_m)\|\\
&\leq (t-t_m)^{\frac{1}{2}}\|u^N_m\|_{\dot{\mathbb{H}}^1} + \|u^N_m\|^3_{L^6(\mathcal{O})}(t-t_m)\\
&\leq C(t-t_m)^{\frac{1}{2}}\|u^N_m\|_{\dot{\mathbb{H}}^1}.
\end{aligned} \tag{2.11}$$

Therefore, combining the cubic difference formula $|u|^2u - |v|^2v = (|u|^2+|v|^2)(u-v) + uv\overline{(u-v)}$ and the inverse inequality $\|P^N u\|_{\dot{\mathbb{H}}^s} \leq \lambda_N^{\frac{s}{2}}\|P^N u\|$, we obtain that

$$\begin{aligned}
&\mathcal{H}(u^{D,N}_{t_{m+1},m}) - \mathcal{H}(u^N_m)\\
&\leq \int_{t_m}^{t_{m+1}} \Big[\|\nabla u^{D,N}_{t,m}\|\tau_m^{\frac{1}{2}}\|P^N|u^N_m|^2u^N_m\|_{\dot{\mathbb{H}}^2} + \|\nabla u^{D,N}_{t,m}\|\Big\|P^N(|u^N_m|^2 + |u^{D,N}_{t,m}|^2)(u^N_m - u^{D,N}_{t,m})\\
&+ P^N u^N_m u^{D,N}_{t,m}\overline{(u^N_m - u^{D,N}_{t,m})}\Big\|_{\dot{\mathbb{H}}^1} + \|u^{D,N}_{t,m}\|^3_{L^6(\mathcal{O})}\tau_m^{\frac{1}{2}}\|P^N|u^N_m|^2u^N_m\|_{\dot{\mathbb{H}}^1}\\
&+ \|u^{D,N}_{t,m}\|^3_{L^6(\mathcal{O})}\Big\|P^N(|u^N_m|^2 + |u^{D,N}_{t,m}|^2)(u^N_m - u^{D,N}_{t,m}) + P^N u^N_m u^{D,N}_{t,m}\overline{(u^N_m - u^{D,N}_{t,m})}\Big\|\Big]\mathrm{d}t\\
&\leq C\int_{t_m}^{t_{m+1}} \Big[\|\nabla u^{D,N}_{t,m}\|\tau_m^{\frac{1}{2}}\lambda_N\|u^N_m\|^3_{L^6(\mathcal{O})} + \|\nabla u^{D,N}_{t,m}\|\lambda_N^{\frac{1}{2}}(\|u^N_m\|^2_{L^\infty(\mathcal{O})} + \|u^{D,N}_{t,m}\|^2_{L^\infty(\mathcal{O})})\times\\
&\|u^N_m - u^{D,N}_{t,m}\| + \|u^{D,N}_{t,m}\|^3_{L^6(\mathcal{O})}\tau_m^{\frac{1}{2}}\lambda_N^{\frac{1}{2}}\|u^N_m\|^3_{L^6(\mathcal{O})} + \|u^{D,N}_{t,m}\|^3_{L^6(\mathcal{O})}\times\\
&(\|u^N_m\|^2_{L^\infty(\mathcal{O})} + \|u^{D,N}_{t,m}\|^2_{L^\infty(\mathcal{O})})\|u^N_m - u^{D,N}_{t,m}\|\Big]\mathrm{d}t.
\end{aligned}$$

Applying the Gagliardo–Nirenberg inequalities $\|u\|^3_{L^6(\mathcal{O})} \leq C\|\nabla u\|\|u\|^2$, $\|u\|^2_{L^\infty(\mathcal{O})} \leq C\|\nabla u\|\|u\|$, the inverse inequality, and Lemma 2.1 yields

$$\begin{aligned}
&\mathcal{H}(u^{D,N}_{t_{m+1},m}) - \mathcal{H}(u^N_m)\\
&\leq C\int_{t_m}^{t_{m+1}} \Big[\|\nabla u^{D,N}_{t,m}\|\tau_m^{\frac{1}{2}}\lambda_N\|\nabla u^N_m\|\|u^N_m\|^2 + \|\nabla u^{D,N}_{t,m}\|\lambda_N^{\frac{1}{2}}\big(\|\nabla u^N_m\|\|u^N_m\|\\
&+ \|u^N_m - u^{D,N}_{t,m}\|_{\dot{\mathbb{H}}^1}\|u^N_m - u^{D,N}_{t,m}\|\big)\|u^N_m - u^{D,N}_{t,m}\|\\
&+ \|\nabla u^{D,N}_{t,m}\|\|u^{D,N}_{t,m}\|^2\tau_m^{\frac{1}{2}}\lambda_N^{\frac{1}{2}}\|\nabla u^N_m\|\|u^N_m\|^2 + \|\nabla u^{D,N}_{t,m}\|\|u^{D,N}_{t,m}\|^2\times
\end{aligned}$$

$$\left(\|\nabla u_m^N\|\|u_m^N\| + \|u_{t,m}^{D,N} - u_m^N\|_{\mathbb{H}^1}\|u_m^N - u_{t,m}^{D,N}\|\right)\|u_m^N - u_{t,m}^{D,N}\|\Big]\mathrm{d}t$$
$$\leq C\int_{t_m}^{t_{m+1}}\Big[\|\nabla u_{t,m}^{D,N}\|\|\nabla u_m^N\|\big(\tau_m^{\frac{1}{2}}\lambda_N + \tau_m^{\frac{1}{2}}\lambda_N^{\frac{1}{2}}\big) + \|\nabla u_{t,m}^{D,N}\|\|u_m^N - u_{t,m}^{D,N}\|\times$$
$$\left(\lambda_N^{\frac{1}{2}}\big(\|\nabla u_m^N\| + \lambda_N^{\frac{1}{2}}\|u_m^N - u_{t,m}^{D,N}\|^2\big)\right)\Big]\mathrm{d}t.$$

Noticing that $\|\nabla u_{t,m}^{D,N}\| \leq \|\nabla u_m^N\| + \|\nabla(u_{t,m}^{D,N} - u_m^N)\| \leq (1 + C\tau_m^{\frac{1}{2}}\lambda_N^{\frac{1}{2}})\|\nabla u_m^N\|$ due to (2.11) and the inverse inequality, we arrive at

$$\begin{aligned}
&\mathcal{H}(u_{t_{m+1},m}^{D,N}) - \mathcal{H}(u_m^N)\\
&\leq C\int_{t_m}^{t_{m+1}}\Big((1 + \tau_m^{\frac{1}{2}}\lambda_N^{\frac{1}{2}})\tau_m^{\frac{1}{2}}\lambda_N + (1 + \tau_m^{\frac{1}{2}}\lambda_N^{\frac{1}{2}})\tau_m^{\frac{1}{2}}(\lambda_N + \tau_m\lambda_N^2)\Big)\|\nabla u_m^N\|^2\mathrm{d}t\\
&\leq C\int_{t_m}^{t_{m+1}}(\tau_m^{\frac{1}{2}}\lambda_N + \tau_m\lambda_N^{\frac{3}{2}} + \tau_m^{\frac{3}{2}}\lambda_N^2 + \tau_m^2\lambda_N^{\frac{5}{2}})\|\nabla u_m^N\|^2\mathrm{d}t\\
&\leq CL_2\tau_m^{1+\gamma}\|\nabla u_m^N\|^2
\end{aligned}$$

under (2.9). Since the Gagliardo–Nirenberg inequality $\|u\|_{L^4(\mathcal{O})}^4 \leq 2\|\nabla u\|\|u\|^3$ and the Young inequality lead to $\mathcal{H}(u_m^N) \geq \frac{1}{4}(\|\nabla u_m^N\|^2 - \|u_m^N\|^6)$, which implies that $\|\nabla u_m^N\|^2 \leq C(\mathcal{H}(u_m^N) + \|u_0^N\|^6)$, we obtain

$$\mathcal{H}(u_{t_{m+1},m}^{D,N}) \leq \mathcal{H}(u_m^N) + C\tau_m^{1+\gamma}(\mathcal{H}(u_m^N) + 1). \tag{2.12}$$

Applying the Itô formula yields

$$\begin{aligned}
&\mathcal{H}(u_{t,m}^{S,N}) - \mathcal{H}(u_{t_{m+1},m}^{D,N})\\
&= \int_{t_m}^{t}\big\langle \nabla u_{r,m}^{S,N}, -\frac{\epsilon}{2}\nabla(F_Q u_{r,m}^{S,N})\big\rangle\mathrm{d}r - \int_{t_m}^{t}\big\langle \nabla u_{r,m}^{S,N}, \mathbf{i}\sqrt{\epsilon}u_{r,m}^{S,N}\nabla(\mathrm{d}W(r))\big\rangle\\
&\quad - \lambda\int_{t_m}^{t}\big\langle |u_{r,m}^{S,N}|^2 u_{r,m}^{S,N}, -\frac{\epsilon}{2}P^N F_Q u_{r,m}^{S,N}\mathrm{d}r - \mathbf{i}\sqrt{\epsilon}P^N u_{r,m}^{S,N}\mathrm{d}W(r)\big\rangle\\
&\quad + \frac{\epsilon}{2}\int_{t_m}^{t}\sum_{k=1}^{\infty}\|\nabla P^N u_{r,m}^{S,N}Q^{\frac{1}{2}}e_k\|^2\mathrm{d}r - \frac{\lambda\epsilon}{2}\int_{t_m}^{t}\sum_{k=1}^{\infty}\big\langle |u_{r,m}^{S,N}|^2(-\mathbf{i}P^N u_{r,m}^{S,N}Q^{\frac{1}{2}}e_k),\\
&\quad - \mathbf{i}P^N u_{r,m}^{S,N}Q^{\frac{1}{2}}e_k\big\rangle\mathrm{d}r - \lambda\epsilon\int_{t_m}^{t}\sum_{k=1}^{\infty}\big\langle u_{r,m}^{S,N}\mathrm{Re}(\overline{u_{r,m}^{S,N}}(-\mathbf{i}P^N u_{r,m}^{S,N}Q^{\frac{1}{2}}e_k)),\\
&\quad - \mathbf{i}P^N u_{r,m}^{S,N}Q^{\frac{1}{2}}e_k\big\rangle\mathrm{d}r.
\end{aligned}$$

Taking the expectation, using inequalities $\|u\|_{L^\infty} \leq C\|u\|_{\mathbb{H}^1}$ and $\|u\|_{L^6(\mathcal{O})}^3 \leq C\|\nabla u\|\|u\|^2$ for $u \in \mathbb{H}^1$, $(\|F_Q\|_{L^\infty(\mathcal{O})} \vee \|\nabla F_Q\|) \leq \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^1}^2$, and applying Lemma 2.1 lead to

$$\begin{aligned}
&\mathbb{E}[\mathcal{H}(u_{t,m}^{S,N})] - \mathbb{E}[\mathcal{H}(u_{t_{m+1},m}^{D,N})]\\
&\leq C\int_{t_m}^{t}\|\nabla u_{r,m}^{S,N}\|(\|\nabla F_Q\|\|u_{r,m}^{S,N}\|_{L^\infty(\mathcal{O})} + \|F_Q\|_{L^\infty(\mathcal{O})}\|\nabla u_{r,m}^{S,N}\|)\mathrm{d}r\\
&\quad + C\int_{t_m}^{t}\|u_{r,m}^{S,N}\|_{L^6(\mathcal{O})}^3\|F_Q\|_{L^\infty(\mathcal{O})}\|u_{r,m}^{S,N}\|\mathrm{d}r + C\int_{t_m}^{t}\|\nabla u_{r,m}^{S,N}\|^2\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^1}^2\mathrm{d}r\\
&\quad + C\int_{t_m}^{t}\|u_{r,m}^{S,N}\|_{L^\infty}^2\|u_{r,m}^{S,N}\|^2\sum_{k=1}^{\infty}\|Q^{\frac{1}{2}}e_k\|_{L^\infty(\mathcal{O})}^2\mathrm{d}r\\
&\leq C\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^1}^2\mathbb{E}\int_{t_m}^{t}(\|\nabla u_{r,m}^{S,N}\|^2 + 1)\mathrm{d}r,
\end{aligned}$$

which together with (2.12) and the assumption (2.10) gives that for $t \in T_m$,

$$\begin{aligned}
&\mathbb{E}[\mathcal{H}(u_{t,m}^{S,N})]\\
&\leq \mathbb{E}[\mathcal{H}(u_m^N)] + C\int_{t_m}^{t_{m+1}}\mathbb{E}[\tau_m^{\gamma}(\mathcal{H}(u_m^N) + 1)]\mathrm{d}r + C\int_{t_m}^{t}\mathbb{E}[\mathcal{H}(u_{r,m}^{S,N}) + 1]\mathrm{d}r\\
&\leq \mathbb{E}[\mathcal{H}(u_m^N)] + C\mathbb{E}\int_{t_m}^{t}(\mathcal{H}(u_{r,m}^{S,N}) + 1)\mathrm{d}r + C\mathbb{E}\tau_m.
\end{aligned}$$

By iteration, we have

$$\mathbb{E}[\mathcal{H}(u_{t,\underline{t}}^{S,N})] \leq \mathbb{E}[\mathcal{H}(u_0^N)] + C\int_0^t\mathbb{E}[\mathcal{H}(u_{r,\underline{r}}^{S,N})]\mathrm{d}r + CT,$$

which implies

$$\sup_{t\in[0,T]}\mathbb{E}[\mathcal{H}(u^{S,N}_{t,\underline{t}})]\le(\mathbb{E}[\mathcal{H}(u^N_0)]+C)e^{CT}$$

due to the Grönwall inequality. Hence, one derives $\sup_{t\in[0,T]}(\mathbb{E}[\|u^{D,N}_{t,\underline{t}}\|^2_{\mathbb{H}^1}]\vee\mathbb{E}[\|u^{S,N}_{t,\underline{t}}\|^2_{\mathbb{H}^1}])\le C$.

Moreover, by utilizing the Burkholder–Davis–Gundy inequality, we can also obtain the following supremum type inequality

$$\begin{aligned}&\mathbb{E}\Big[\Big|\sup_{t\in[0,T]}\int_0^t\Big\langle\nabla u^{S,N}_{s,\underline{s}},\mathbf{i}\sqrt{\epsilon}P^Nu^{S,N}_{s,\underline{s}}\nabla(\mathrm{d}W(s))\Big\rangle\Big|^2\Big]\\&+\mathbb{E}\Big[\Big|\sup_{t\in[0,T]}\int_0^t\big\langle|u^{S,N}_{r,m}|^2u^{S,N}_{r,m},\mathbf{i}\sqrt{\epsilon}P^Nu^{S,N}_{r,m}\mathrm{d}W(r)\big\rangle\Big|^2\Big]\\&\le C\mathbb{E}\Big[\int_0^T\|\nabla u^{S,N}_{s,\underline{s}}\|^2\|Q^{\frac12}\|^2_{\mathcal{L}^1_2}\mathrm{d}s\Big].\end{aligned}\tag{2.13}$$

Applying the above inequalities, one can finish the proof for the case of $p=2$. For the case of $p>2$, it can be proved similarly by means of the Itô formula, we omit the proof. □

Below, we prove the $\mathbb{H}^1$-exponential integrability for the solution of the fully discrete scheme. To this end, we first present a useful exponential integrability lemma, which is a variant of [14, Lemma 3.1] or [15, Lemma 2.1], and we refer to them for the proofs and more details.

Lemma 2.5. *Let X be an $\mathbb{H}$-valued adapted stochastic process with continuous sample paths satisfying $\int_{\underline{t}}^t\|\mu(X_t)\|+\|\sigma(X_t)\|^2\mathrm{d}t<\infty$ a.s. $\forall t\in[0,T]$, and $X_t=X_{\underline{t}}+\int_{\underline{t}}^t\mu(X_r)\mathrm{d}r+\int_{\underline{t}}^t\sigma(X_r)\mathrm{d}W(r)$. If there are two functionals $V,\overline{V}\in C^2(\mathbb{H};\mathbb{R})$ and a constant $\alpha>0$ such that*

$$\begin{aligned}&DV(X_s)\mu(X_s)+\frac12\mathrm{Tr}\big(D^2V(X_s)\sigma(X_s)\sigma(X_s)^*\big)+\frac{1}{2e^{\alpha(s-\underline{t})}}\|\sigma(X_s)^*DV(X_s)\|^2\\&+\overline{V}(X_s)\le\alpha V(X_s)\quad a.s.\quad\forall s\in[\underline{t},t),\end{aligned}$$

then for $t\in[0,T]$,

$$\mathbb{E}\Big[\exp\Big\{\frac{V(X_t)}{e^{\alpha(t-\underline{t})}}+\int_{\underline{t}}^t\frac{\overline{V}(X_r)}{e^{\alpha(r-\underline{t})}}\mathrm{d}r\Big\}\Big]\le\mathbb{E}\big[\exp\{V(X_{\underline{t}})\}\big].\tag{2.14}$$

Especially, when $\sigma\equiv0$,

$$\exp\Big\{\frac{V(X_t)}{e^{\alpha(t-\underline{t})}}+\int_{\underline{t}}^t\frac{\overline{V}(X_r)}{e^{\alpha(r-\underline{t})}}\mathrm{d}r\Big\}\le\exp\{V(X_{\underline{t}})\}\quad a.s.\tag{2.15}$$

Proposition 2.6. *Under Assumptions 1 and 2, there exist constants $\alpha_\lambda,C>0$ such that*

$$\sup_{t\in[0,T]}\mathbb{E}\Big[\exp\Big\{\frac{\mathcal{H}(u^{S,N}_{t,\underline{t}})}{e^{\alpha_\lambda t}}\Big\}\Big]\le C\mathbb{E}[\exp\{\mathcal{H}(u^N_0)\}].$$

Proof. Let $\mu_1(u^{D,N}_{t,m})=\mathbf{i}\Delta u^{D,N}_{t,m}+\mathbf{i}\lambda S^N(t-t_m)|u^N_m|^2u^N_m$ for $t\in T_m$. Similar to the proof of Proposition 2.4, we have

$$\begin{aligned}&D\mathcal{H}(u^{D,N}_{t,m})\mu_1(u^{D,N}_{t,m})\le C\tau_m^\gamma\|\nabla u^N_m\|^2\\&\le C\tau_m^\gamma(\|\nabla u^{D,N}_{t,m}\|^2+\lambda_N^2\tau_m\|u^N_m\|^2)\le C_0+C_1\tau_m^\gamma\mathcal{H}(u^{D,N}_{t,m}).\end{aligned}$$

Applying Lemma 2.5 (2.15) with $\underline{t}=t_m$, $\mu=\mu_1$, $\sigma\equiv0$, $V=\mathcal{H}$, $\overline{V}=-C_0$ and $\alpha=C_1\tau_m^\gamma$, letting $t=t_{m+1}^-$, and taking the limit, we obtain

$$\exp\Big\{\frac{\mathcal{H}(u^{D,N}_{t_{m+1},m})}{e^{\alpha\tau_m}}-C_0\int_{t_m}^{t_{m+1}}\frac{1}{e^{\alpha(r-t_m)}}\mathrm{d}r\Big\}\le\exp\{\mathcal{H}(u^N_m)\}.$$

Using the fact that $\int_{t_m}^{t_{m+1}}\frac{1}{e^{\alpha(r-t_m)}}\mathrm{d}r=\frac{1-e^{-\alpha\tau_m}}{\alpha}\le\tau_m$ yields

$$\exp\Big\{\frac{\mathcal{H}(u^{D,N}_{t_{m+1},m})}{e^{\alpha\tau_m}}\Big\}\le\exp\{\mathcal{H}(u^N_m)+C_0\tau_m\},$$

which gives

$$\begin{aligned}&\exp\{\mathcal{H}(u^{D,N}_{t_{m+1},m})\}\le\exp\{(\mathcal{H}(u^N_m)+C_0\tau_m)e^{\alpha\tau_m}\}\\&\le\exp\{(\mathcal{H}(u^N_m)+C_0\tau_m)(1+2C_1\tau_m^{1+\gamma})\}\end{aligned}\tag{2.16}$$

for $\tau_m\le T\delta$ with δ being small.

We claim that

$$\sup_{t\in T_m}\mathbb{E}\left[\exp\left\{\frac{\mathcal{H}(u_{t,m}^{S,N})}{e^{\alpha_\lambda(t-t_m)}}-\int_{t_m}^t\frac{\beta_\lambda}{e^{\alpha_\lambda(r-t_m)}}\mathrm{d}r\right\}\right]\le\mathbb{E}[\exp\{\mathcal{H}(u_{t_{m+1},m}^{D,N})\}], \tag{2.17}$$

where $\alpha_\lambda=C(e^{3L_1T}\|u_0^N\|^6+1)\|Q^{\frac12}\|^2_{\mathcal{L}_2^2}$ and $\beta_\lambda=C(e^{6L_1T}\|u_0^N\|^{12}+1)\|Q^{\frac12}\|^2_{\mathcal{L}_2^2}$. In fact, by letting $\mu_2(u)=-\frac{\epsilon}{2}P^NF_Qu$ and $\sigma_2(u)=-\mathbf{i}\sqrt{\epsilon}P^NuQ^{\frac12}$, we obtain

$$\begin{aligned}
&D\mathcal{H}(u_{t,m}^{S,N})\mu_2(u_{t,m}^{S,N})+\frac12\mathrm{Tr}\left[D^2\mathcal{H}(u_{t,m}^{S,N})\sigma_2(u_{t,m}^{S,N})\sigma_2(u_{t,m}^{S,N})^*\right]\\
&\quad+\frac{1}{2e^{\alpha_\lambda(t-t_m)}}\|\sigma_2(u_{t,m}^{S,N})^*D\mathcal{H}(u_{t,m}^{S,N})\|^2\\
&=\epsilon\left\langle\nabla u_{t,m}^{S,N},\nabla(-\frac12P^NF_Qu_{t,m}^{S,N})\right\rangle-\lambda\epsilon\left\langle|u_{t,m}^{S,N}|^2u_{t,m}^{S,N},-\frac12P^NF_Qu_{t,m}^{S,N}\right\rangle\\
&\quad+\frac{\epsilon}{2}\sum_{k=1}^\infty\|\nabla(-\mathbf{i}P^Nu_{t,m}^{S,N}Q^{\frac12}e_k)\|^2\\
&\quad-\frac{\lambda\epsilon}{2}\sum_{k=1}^\infty\left\langle|u_{t,m}^{S,N}|^2(P^Nu_{t,m}^{S,N}Q^{\frac12}e_k),P^Nu_{t,m}^{S,N}Q^{\frac12}e_k\right\rangle\\
&\quad-\lambda\epsilon\sum_{k=1}^\infty\left\langle u_{t,m}^{S,N}\mathrm{Re}(\overline{u_{t,m}^{S,N}}(-\mathbf{i}P^Nu_{t,m}^{S,N}Q^{\frac12}e_k)),-\mathbf{i}P^Nu_{t,m}^{S,N}Q^{\frac12}e_k\right\rangle+\frac{\epsilon}{2e^{\alpha_\lambda(t-t_m)}}\times\\
&\quad\sum_{k=1}^\infty\Big(\left\langle\nabla u_{t,m}^{S,N},-\mathbf{i}u_{t,m}^{S,N}(\nabla Q^{\frac12}e_k)\right\rangle-\lambda\left\langle|u_{t,m}^{S,N}|^2u_{t,m}^{S,N},-\mathbf{i}P^Nu_{t,m}^{S,N}Q^{\frac12}e_k\right\rangle\Big)^2\\
&\le C\|\nabla u_{t,m}^{S,N}\|^2\|Q^{\frac12}\|^2_{\mathcal{L}_2^1}+C\|u_{t,m}^{S,N}\|^3_{L^6(\mathcal{O})}\|u_{t,m}^{S,N}\|\|F_Q\|_{L^\infty(\mathcal{O})}+C\|\nabla u_{t,m}^{S,N}\|^2\times\\
&\quad\|Q^{\frac12}\|^2_{\mathcal{L}_2^1}(1+\|u_{t,m}^{S,N}\|^2)+\frac{C}{2e^{\alpha_\lambda(t-t_m)}}(\|\nabla u_{t,m}^{S,N}\|^2\|u_{t,m}^{S,N}\|^2\|Q^{\frac12}\|^2_{\mathcal{L}_2^2}\\
&\quad+\|u_{t,m}^{S,N}\|^6_{L^6(\mathcal{O})}\|u_{t,m}^{S,N}\|^2\|Q^{\frac12}\|^2_{\mathcal{L}_2^1})\\
&\le C\|\nabla u_{t,m}^{S,N}\|^2\|Q^{\frac12}\|^2_{\mathcal{L}_2^2}(\|u_{t,m}^{S,N}\|^6+1)+C\|u_{t,m}^{S,N}\|^6\|Q^{\frac12}\|^2_{\mathcal{L}_2^1}\\
&\le C(e^{3L_1T}\|u_0^N\|^6+1)\|Q^{\frac12}\|^2_{\mathcal{L}_2^2}\mathcal{H}(u_{t,m}^{S,N})+C(e^{6L_1T}\|u_0^N\|^{12}+1)\|Q^{\frac12}\|^2_{\mathcal{L}_2^2},
\end{aligned}$$

where the second inequality uses the Gagliardo–Nirenberg inequality $\|u\|^3_{L^6(\mathcal{O})}\le C\|\nabla u\|\|u\|^2$ for $u\in\mathbb{H}^1$, and in the last step we use the inequality $\|\nabla u_{t,m}^{S,N}\|^2\le 4\mathcal{H}(u_{t,m}^{S,N})+\|u_{t,m}^{S,N}\|^6$, $t\in T_m$ and Lemma 2.1. Applying Lemma 2.5 (2.14) with $\mu=\mu_2$, $\sigma=\sigma_2$, $V=\mathcal{H}$, $\overline{V}=-\beta_\lambda$ and $\alpha=\alpha_\lambda$ leads to (2.17).

Hence, it follows from (2.17) and the assumption $\tau_m\le T\delta$ that

$$\begin{aligned}
\mathbb{E}\left[\exp\left\{\frac{\mathcal{H}(u_{t,m}^{S,N})}{e^{\alpha_\lambda(t-t_m)}}\right\}\right]&\le\mathbb{E}\left[\exp\left\{\frac{\mathcal{H}(u_{t,m}^{S,N})}{e^{\alpha_\lambda(t-t_m)}}-\int_{t_m}^t\frac{\beta_\lambda}{e^{\alpha_\lambda(r-t_m)}}\mathrm{d}r\right\}e^{\beta_\lambda(t-t_m)}\right]\\
&\le\mathbb{E}[\exp\{\mathcal{H}(u_{t_{m+1},m}^{D,N})\}]e^{T\beta_\lambda\delta}\le\mathbb{E}\left[\exp\{(\mathcal{H}(u_m^N)+C_0\tau_m)(1+2C_1\tau_m^{1+\gamma})\}\right]e^{T\beta_\lambda\delta},
\end{aligned}$$

where in the last step we use (2.16). By considering $e^{-\alpha_\lambda t_m}\mathcal{H}$ instead of $\mathcal{H}$, we can obtain

$$\begin{aligned}
\mathbb{E}\left[\exp\left\{\frac{\mathcal{H}(u_{t,m}^{S,N})}{e^{\alpha_\lambda t}}\right\}\right]&\le\mathbb{E}\left[\exp\left\{\left(\frac{\mathcal{H}(u_m^N)}{e^{\alpha_\lambda t_m}}+C_0\tau_m\right)(1+2C_1\tau_m^{1+\gamma})\right\}\right]e^{T\beta_\lambda\delta}\\
&\le\mathbb{E}\left[\exp\left\{\frac{\mathcal{H}(u_m^N)}{e^{\alpha_\lambda t_m}}+2C_1\tau_m^{1+\gamma}\mathcal{H}(u_m^N)+C\tau_m\right\}\right]e^{T\beta_\lambda\delta}\\
&\le\mathbb{E}\left[\exp\left\{\frac{\mathcal{H}(u_m^N)}{e^{\alpha_\lambda t_m}}\right\}\right]e^{C\delta}
\end{aligned}$$

under the assumption that $\tau_m^\gamma\mathcal{H}(u_m^N)\le L_3$. By iteration and using Remark 2.2, we have

$$\mathbb{E}\left[\exp\left\{\frac{\mathcal{H}(u_{t,m}^{S,N})}{e^{\alpha_\lambda t}}\right\}\right]\le\mathbb{E}[\exp\{\mathcal{H}(u_0^N)\}]e^{C\delta(m+1)}\le\mathbb{E}[\exp\{\mathcal{H}(u_0^N)\}]e^{CT\tau_{min}^{-1}}.$$

The proof is finished. □

In order to derive the $\mathbb{H}^2$-regularity of the solution of the fully discrete scheme, we introduce the functional $f(u)=\|\Delta u\|^2+\lambda\langle\Delta u,|u|^2u\rangle$, $u\in\mathbb{H}^2$.

Proposition 2.7. *Under Assumptions 1 and 2, for $p\ge2$, there exists a constant $C:=C(p,\epsilon,T,f(u_0^N))>0$ such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|u_{t,\underline{t}}^{D,N}\|^p_{\mathbb{H}^2}\right]+\mathbb{E}\left[\sup_{t\in[0,T]}\|u_{t,\underline{t}}^{S,N}\|^p_{\mathbb{H}^2}\right]\le C.$$

Proof. Simple calculations give that

$$
\begin{aligned}
Df(u)(v) &= 2\langle \Delta u, \Delta v\rangle + 2\lambda\langle \Delta u, u\mathrm{Re}(\bar u v)\rangle + \lambda\langle \Delta u, |u|^2 v\rangle + \lambda\langle \Delta v, |u|^2 u\rangle,\\
D^2 f(u)(v,w) &= 2\langle \Delta v, \Delta w\rangle + 2\lambda\langle \Delta u, w\mathrm{Re}(\bar u v)\rangle + 2\lambda\langle \Delta w, u\mathrm{Re}(\bar u v)\rangle\\
&\quad + 2\lambda\langle \Delta u, u\mathrm{Re}(\bar v w)\rangle + 2\lambda\langle \Delta u, v\mathrm{Re}(\bar u w)\rangle + \lambda\langle \Delta w, |u|^2 v\rangle\\
&\quad + 2\lambda\langle \Delta v, u\mathrm{Re}(\bar u w)\rangle + \lambda\langle \Delta v, |u|^2 w\rangle.
\end{aligned}
$$

Step 1. By the chain rule, we obtain that for $t \in T_m$,

$$
\begin{aligned}
&f(u^{D,N}_{t_{m+1},m}) - f(u^N_m) = \int_{t_m}^{t_{m+1}} Df(u^{D,N}_{t,m})\,\mathrm{d}u^{D,N}_{t,m}\\
&= \int_{t_m}^{t_{m+1}} \Big[2\Big\langle \Delta u^{D,N}_{t,m}, \mathbf{i}\lambda\Delta\big(S^N(t-t_m)|u^N_m|^2 u^N_m\big)\Big\rangle\\
&\quad + 2\lambda\Big\langle \Delta u^{D,N}_{t,m}, u^{D,N}_{t,m}\mathrm{Re}\big[\overline{u^{D,N}_{t,m}}\big(\mathbf{i}\Delta u^{D,N}_{t,m} + \mathbf{i}\lambda S^N(t-t_m)|u^N_m|^2 u^N_m\big)\big]\Big\rangle\\
&\quad + \lambda\Big\langle \Delta u^{D,N}_{t,m}, |u^{D,N}_{t,m}|^2 \mathbf{i}\lambda S^N(t-t_m)|u^N_m|^2 u^N_m\Big\rangle\\
&\quad + \lambda\Big\langle |u^{D,N}_{t,m}|^2 u^{D,N}_{t,m}, \mathbf{i}\Delta^2 u^{D,N}_{t,m} + \mathbf{i}\lambda\Delta(S^N(t-t_m)|u^N_m|^2 u^N_m)\Big\rangle\Big]\,\mathrm{d}t.
\end{aligned}
$$

Utilizing the fact that $2\mathrm{Re}(\bar u v) = u\bar v + \bar u v$ yields

$$
\begin{aligned}
&f(u^{D,N}_{t_{m+1},m}) - f(u^N_m)\\
&= \int_{t_m}^{t_{m+1}} \Big[2\Big\langle \Delta u^{D,N}_{t,m}, \mathbf{i}\lambda\Delta[(S^N(t-t_m)-\mathrm{Id})|u^N_m|^2 u^N_m]\Big\rangle\\
&\quad + 2\Big\langle \Delta u^{D,N}_{t,m}, \mathbf{i}\lambda\Delta(|u^N_m|^2 u^N_m)\Big\rangle + \Big\langle \Delta u^{D,N}_{t,m}, \mathbf{i}\lambda^2|u^{D,N}_{t,m}|^2 S^N(t-t_m)|u^N_m|^2 u^N_m\Big\rangle\\
&\quad + \lambda\Big\langle \Delta u^{D,N}_{t,m}, -\mathbf{i}(u^{D,N}_{t,m})^2\overline{\Delta u^{D,N}_{t,m}}\Big\rangle + \lambda\Big\langle \Delta u^{D,N}_{t,m}, -\mathbf{i}\lambda(u^{D,N}_{t,m})^2 S^N(-(t-t_m))|u^N_m|^2\overline{u^N_m}\Big\rangle\\
&\quad + \lambda\Big\langle \Delta u^{D,N}_{t,m}, \mathbf{i}\lambda|u^{D,N}_{t,m}|^2 S^N(t-t_m)|u^N_m|^2 u^N_m\Big\rangle + \Big\langle \Delta u^{D,N}_{t,m}, -\mathbf{i}\lambda\Delta(|u^{D,N}_{t,m}|^2 u^{D,N}_{t,m})\Big\rangle\\
&\quad + \lambda\Big\langle \Delta(|u^{D,N}_{t,m}|^2 u^{D,N}_{t,m}), \mathbf{i}\lambda S^N(t-t_m)|u^N_m|^2 u^N_m\Big\rangle\Big]\mathrm{d}t\\
&=: \int_{t_m}^{t_{m+1}} \sum_{j=1}^{8} I_j\,\mathrm{d}t.
\end{aligned}
$$

Noticing that $\Delta(|u|^2 u) = 2|u|^2\Delta u + 4u|\nabla u|^2 + 2\bar u(\nabla u)^2 + u^2\Delta\bar u$, we arrive at

$$
\begin{aligned}
&I_2 + I_4 + I_7\\
&= \Big\langle \Delta u^{D,N}_{t,m}, \mathbf{i}\lambda\Delta(|u^N_m|^2 u^N_m - |u^{D,N}_{t,m}|^2 u^{D,N}_{t,m})\Big\rangle + \Big\langle \Delta u^{D,N}_{t,m}, \mathbf{i}\lambda(2|u^N_m|^2\Delta u^N_m + 4u^N_m|\nabla u^N_m|^2\\
&\quad + 2\overline{u^N_m}(\nabla u^N_m)^2 + (u^N_m)^2\overline{\Delta u^N_m})\Big\rangle - \Big\langle \Delta u^{D,N}_{t,m}, \mathbf{i}\lambda(u^{D,N}_{t,m})^2\overline{\Delta u^{D,N}_{t,m}}\Big\rangle.
\end{aligned}
$$

It follows from the inverse inequality, the Sobolev embedding inequality $\|u\|_{L^\infty(\mathcal{O})} \le C\|u\|_{\mathbb{H}^1}, u \in \mathbb{H}^1$, and the Young inequality that

$$
\begin{aligned}
&\Big\langle \Delta u^{D,N}_{t,m}, \mathbf{i}\lambda\Delta(|u^N_m|^2 u^N_m - |u^{D,N}_{t,m}|^2 u^{D,N}_{t,m})\Big\rangle\\
&\le C\|\Delta u^{D,N}_{t,m}\|\lambda_N(\|u^N_m\|^2_{L^\infty(\mathcal{O})} + \|u^{D,N}_{t,m}\|^2_{L^\infty(\mathcal{O})})\|u^N_m - u^{D,N}_{t,m}\|\\
&\le C\|\Delta u^{D,N}_{t,m}\|\lambda_N\tau_m^{\frac12}(\|u^N_m\|^2_{\mathbb{H}^1} + \|u^{D,N}_{t,m}\|^2_{\mathbb{H}^1})\|u^N_m\|_{\dot{\mathbb{H}}^1}\\
&\le C(\|\Delta u^{D,N}_{t,m}\|^2 + \|u^N_m\|^6_{\mathbb{H}^1} + \|u^{D,N}_{t,m}\|^6_{\mathbb{H}^1}),
\end{aligned}
$$

where we use the assumption (2.9) so that $\lambda_N\tau_m^{\frac12} < \infty$. Similar techniques, combining the fact that $\langle u, \mathbf{i}|v|^2 u\rangle = 0$ give

$$
\begin{aligned}
&2\Big\langle \Delta u^{D,N}_{t,m}, \mathbf{i}\lambda|u^N_m|^2\Delta u^N_m\Big\rangle = 2\Big\langle \Delta u^{D,N}_{t,m}, \mathbf{i}\lambda|u^N_m|^2(\Delta u^N_m - \Delta u^{D,N}_{t,m})\Big\rangle\\
&\le C\|\Delta u^{D,N}_{t,m}\|\|u^N_m\|^2_{L^\infty(\mathcal{O})}\lambda_N\|u^N_m - u^{D,N}_{t,m}\| \le C(\|\Delta u^{D,N}_{t,m}\|^2 + \|u^N_m\|^6_{\mathbb{H}^1}).
\end{aligned}
$$

And it can be shown that

$$
\begin{aligned}
&\Big\langle \Delta u^{D,N}_{t,m}, \mathbf{i}\lambda\big((u^N_m)^2\overline{\Delta u^N_m} - (u^{D,N}_{t,m})^2\overline{\Delta u^{D,N}_{t,m}}\big)\Big\rangle\\
&= \Big\langle \Delta u^{D,N}_{t,m}, \mathbf{i}\lambda\big((u^N_m)^2 - (u^{D,N}_{t,m})^2\big)\overline{\Delta u^N_m}\Big\rangle + \Big\langle \Delta u^{D,N}_{t,m}, \mathbf{i}\lambda(u^{D,N}_{t,m})^2\Delta\overline{(u^N_m - u^{D,N}_{t,m})}\Big\rangle\\
&\le \|\Delta u^{D,N}_{t,m}\|\|u^N_m - u^{D,N}_{t,m}\|(\|u^N_m\|_{\mathbb{H}^1} + \|u^{D,N}_{t,m}\|_{\mathbb{H}^1})\|\Delta u^{D,N}_{t,m}\|_{L^\infty(\mathcal{O})}\\
&\quad + \|\Delta u^{D,N}_{t,m}\|\|u^{D,N}_{t,m}\|^2_{\mathbb{H}^1}\|\Delta(u^N_m - u^{D,N}_{t,m})\|
\end{aligned}
$$

$$\le C\|\Delta u^{D,N}_{t,m}\|\lambda_N\tau_m^{\frac{1}{2}}\left[\|u^N_m\|_{\mathbb{H}^1}\|u^{D,N}_{t,m}\|_{\mathbb{H}^1}(\|u^N_m\|_{\mathbb{H}^1}+\|u^{D,N}_{t,m}\|_{\mathbb{H}^1})+\|u^N_m\|_{\mathbb{H}^1}\|u^{D,N}_{t,m}\|^2_{\mathbb{H}^1}\right]$$
$$\le C(\|\Delta u^{D,N}_{t,m}\|^2+\|u^N_m\|^6_{\mathbb{H}^1}+\|u^{D,N}_{t,m}\|^6_{\mathbb{H}^1})$$

and

$$\begin{aligned}&\left\langle \Delta u^{D,N}_{t,m},\mathbf{i}\lambda(4u^N_m|\nabla u^N_m|^2+2\overline{u^N_m}(\nabla u^N_m)^2)\right\rangle\\ &\le C\|\Delta u^{D,N}_{t,m}\|\|u^N_m\|_{\mathbb{H}^1}\|\nabla u^N_m\|^2_{L^4(\mathcal{O})}\\ &\le C\|\Delta u^{D,N}_{t,m}\|\|u^N_m\|_{\mathbb{H}^1}\|\Delta u^N_m\|^{\frac{1}{2}}\|\nabla u^N_m\|^{\frac{3}{2}}\\ &\le C\|\Delta u^{D,N}_{t,m}\|\|u^N_m\|^{\frac{5}{2}}_{\mathbb{H}^1}\left(\|\Delta u^{D,N}_{t,m}\|^{\frac{1}{2}}+\|\Delta(u^{D,N}_{t,m}-u^N_m)\|^{\frac{1}{2}}\right)\\ &\le C(\|\Delta u^{D,N}_{t,m}\|^2+\|u^N_m\|^{10}_{\mathbb{H}^1}+1).\end{aligned}$$

Moreover, the remaining terms can be estimated as follows:

$$I_1\le C\|\Delta u^{D,N}_{t,m}\|\lambda_N\tau_m^{\frac{1}{2}}\|u^N_m\|^3_{\mathbb{H}^1}\le C\|\Delta u^{D,N}_{t,m}\|^2+C\|u^N_m\|^6_{\mathbb{H}^1},$$

$$\begin{aligned}I_3+I_5+I_6&\le C\|\Delta u^{D,N}_{t,m}\|\|u^{D,N}_{t,m}\|^2_{\mathbb{H}^1}\|u^N_m\|^3_{L^6(\mathcal{O})}\\ &\le C\|\Delta u^{D,N}_{t,m}\|^2+C\|u^N_m\|^6_{\mathbb{H}^1}+C\|u^{D,N}_{t,m}\|^6_{\mathbb{H}^1}\end{aligned}$$

and

$$\begin{aligned}I_8&=-\lambda\left\langle\nabla(|u^{D,N}_{t,m}|^2u^{D,N}_{t,m}),\mathbf{i}\lambda S^N(t-t_m)\nabla(|u^N_m|^2u^N_m)\right\rangle\\ &\le C\|u^{D,N}_{t,m}\|^3_{\mathbb{H}^1}\|u^N_m\|^3_{\mathbb{H}^1}\le C\|u^{D,N}_{t,m}\|^6_{\mathbb{H}^1}+C\|u^N_m\|^6_{\mathbb{H}^1},\end{aligned}$$

where we use the integration by parts formula and the fact that $\mathbb{H}^s$ is an algebra for $s>\frac{1}{2}$, i.e., $\|uv\|_{\mathbb{H}^s}\le C\|u\|_{\mathbb{H}^s}\|v\|_{\mathbb{H}^s}$ for $u,v\in\mathbb{H}^s$.

Combining estimates of terms $I_i, i=1,\dots,8$, we derive

$$f(u^{D,N}_{t_{m+1},m})-f(u^N_m)\le C\int_{t_m}^{t_{m+1}}(\|\Delta u^{D,N}_{t,m}\|^2+\|u^N_m\|^{10}_{\mathbb{H}^1}+\|u^{D,N}_{t,m}\|^6_{\mathbb{H}^1}+1)\mathrm{d}t.$$

Since the Gagliardo–Nirenberg inequality and the Young inequality give

$$f(u)\ge\|\Delta u\|^2-\|\Delta u\|\|u\|^3_{L^6(\mathcal{O})}\ge\frac{1}{2}(\|\Delta u\|^2-\|u\|^6_{L^6(\mathcal{O})})\ge\frac{1}{2}\|\Delta u\|^2-C\|u\|^2_{\mathbb{H}^1}\|u\|^4,\tag{2.18}$$

we obtain

$$f(u^{D,N}_{t_{m+1},m})-f(u^N_m)\le C\int_{t_m}^{t_{m+1}}(f(u^{D,N}_{t,m})+\|u^N_m\|^{10}_{\mathbb{H}^1}+\|u^{D,N}_{t,m}\|^6_{\mathbb{H}^1}+1)\mathrm{d}t,$$

which implies

$$f(u^{D,N}_{t_{m+1},m})\le\left(f(u^N_m)+C\int_{t_m}^{t_{m+1}}(\|u^N_m\|^{10}_{\mathbb{H}^1}+\|u^{D,N}_{t,m}\|^6_{\mathbb{H}^1}+1)\mathrm{d}t\right)e^{C\tau_m}.\tag{2.19}$$

Step 2. Applying the Itô formula yields

$$\begin{aligned}f(u^{S,N}_{t,m})-f(u^{D,N}_{t_{m+1},m})&=\int_{t_m}^t 2\left\langle\Delta u^{S,N}_{r,m},\Delta(-\frac{\epsilon}{2}P^NF_Qu^{S,N}_{r,m}\mathrm{d}r-\mathbf{i}\sqrt{\epsilon}P^Nu^{S,N}_{r,m}\mathrm{d}W(r))\right\rangle\\
&+2\lambda\int_{t_m}^t\left\langle\Delta u^{S,N}_{r,m},u^{S,N}_{r,m}\mathrm{Re}(\overline{u^{S,N}_{r,m}}(-\frac{\epsilon}{2}P^NF_Qu^{S,N}_{r,m}\mathrm{d}r-\mathbf{i}\sqrt{\epsilon}P^Nu^{S,N}_{r,m}\mathrm{d}W(r)))\right\rangle\\
&+\lambda\int_{t_m}^t\left\langle\Delta u^{S,N}_{r,m},|u^{S,N}_{r,m}|^2(-\frac{\epsilon}{2}P^NF_Qu^{S,N}_{r,m}\mathrm{d}r-\mathbf{i}\sqrt{\epsilon}P^Nu^{S,N}_{r,m}\mathrm{d}W(r))\right\rangle\\
&+\lambda\int_{t_m}^t\left\langle|u^{S,N}_{r,m}|^2u^{S,N}_{r,m},\Delta(-\frac{\epsilon}{2}P^NF_Qu^{S,N}_{r,m}\mathrm{d}r-\mathbf{i}\sqrt{\epsilon}P^Nu^{S,N}_{r,m}\mathrm{d}W(r))\right\rangle\\
&+\lambda\epsilon\sum_{k=1}^\infty\int_{t_m}^t\left\langle\Delta P^N(\mathbf{i}u^{S,N}_{r,m}Q^{\frac{1}{2}}e_k),\mathbf{i}|u^{S,N}_{r,m}|^2P^Nu^{S,N}_{r,m}Q^{\frac{1}{2}}e_k\right\rangle\mathrm{d}r\\
&+\lambda\epsilon\int_{t_m}^t\sum_{k=1}^\infty\left\langle\Delta u^{S,N}_{r,m},u^{S,N}_{r,m}|P^Nu^{S,N}_{r,m}Q^{\frac{1}{2}}e_k|^2\right\rangle\mathrm{d}r+\epsilon\int_{t_m}^t\sum_{k=1}^\infty\|\Delta P^N(u^{S,N}_{r,m}Q^{\frac{1}{2}}e_k)\|^2\mathrm{d}r\\
&+2\lambda\epsilon\int_{t_m}^t\sum_{k=1}^\infty\Big[\left\langle\Delta u^{S,N}_{r,m},(-\mathbf{i}P^Nu^{S,N}_{r,m}Q^{\frac{1}{2}}e_k)\mathrm{Re}(\overline{u^{S,N}_{r,m}}(-\mathbf{i}P^Nu^{S,N}_{r,m}Q^{\frac{1}{2}}e_k))\right\rangle\\
&+\epsilon\left\langle\Delta(-\mathbf{i}P^Nu^{S,N}_{r,m}Q^{\frac{1}{2}}e_k),u^{S,N}_{r,m}\mathrm{Re}\left(\overline{u^{S,N}_{r,m}}(-\mathbf{i}P^Nu^{S,N}_{r,m}Q^{\frac{1}{2}}e_k)\right)\right\rangle\Big]\mathrm{d}r.\end{aligned}$$

By taking expectation and combining (2.19) and the fact that $e^{C\tau_m} \le 1 + 2C\tau_m$ for $\tau_m \le T\delta$ with δ being small, it follows from the fact that $\mathbb{H}^s$ is an algebra for $s > \frac{1}{2}$ that

$$\begin{aligned}
&\mathbb{E}[f(u^{S,N}_{t,m})] \\
&\le \mathbb{E}[f(u^N_m)(1+2C\tau_m)] + C\mathbb{E}\Big[(1+2C\tau_m)\Big(\int_{t_m}^{t_{m+1}} (\|u^N_m\|^{10}_{\mathbb{H}^1} + \|u^{D,N}_{r,m}\|^6_{\mathbb{H}^1} + 1)\mathrm{d}r\Big)\Big] \\
&\quad + C\mathbb{E}\Big[\int_{t_m}^{t} (\|\Delta u^{S,N}_{r,m}\|^2 + \|u^{S,N}_{r,m}\|^6_{\mathbb{H}^1} + 1)\mathrm{d}r\Big] \\
&\le \mathbb{E}[f(u^N_m)(1+2C\tau_m)] + C\mathbb{E}\Big[(1+2C\tau_m)\Big(\int_{t_m}^{t_{m+1}} (\|u^N_m\|^{10}_{\mathbb{H}^1} + \|u^{D,N}_{r,m}\|^6_{\mathbb{H}^1} + 1)\mathrm{d}r\Big)\Big] \\
&\quad + C\mathbb{E}\Big[\int_{t_m}^{t} (f(u^{S,N}_{r,m}) + \|u^{S,N}_{r,m}\|^6_{\mathbb{H}^1} + 1)\mathrm{d}r\Big],
\end{aligned}$$

where we use (2.18) in the last step. By iteration, we derive

$$\begin{aligned}
&\mathbb{E}[f(u^{S,N}_{t,m})] \\
&\le \mathbb{E}[f(u^N_0)] + C\mathbb{E}\Big[\sum_{k=0}^{m} f(u^N_k)\tau_k\Big] + C\mathbb{E}\Big[\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} f(u^{S,N}_{r,k})\mathrm{d}r\Big] \\
&\quad + C\mathbb{E}\Big[\int_{t_m}^{t} f(u^{S,N}_{r,m})\mathrm{d}r\Big] + C\mathbb{E}\Big[\sum_{k=0}^{m}\int_{t_k}^{t_{k+1}} (\|u^N_k\|^{10}_{\mathbb{H}^1} + \|u^{D,N}_{r,k}\|^6_{\mathbb{H}^1} + 1)\mathrm{d}r \\
&\quad + \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} (\|u^{S,N}_{r,k}\|^6_{\mathbb{H}^1} + 1)\mathrm{d}r\Big] + C\mathbb{E}\Big[\int_{t_m}^{t} (\|u^{S,N}_{r,m}\|^6_{\mathbb{H}^1} + 1)\mathrm{d}r\Big].
\end{aligned}$$

We claim that for $t \in T_m$,

$$\mathbb{E}\Big[\sum_{k=0}^{m} f(u^N_k)\tau_k\Big] \le C\Big(\int_0^t \mathbb{E}[f(u^{S,N}_{r,\underline{r}})]\mathrm{d}r + 1\Big). \tag{2.20}$$

In fact, noticing that for $k = 1, 2, \dots, m$,

$$\|u^N_k\|^2_{\mathbb{H}^2}\tau_{k-1} \le 2\int_{t_{k-1}}^{t_k} \|u^{S,N}_{r,k-1}\|^2_{\mathbb{H}^2}\mathrm{d}r + 2\int_{t_{k-1}}^{t_k} \|u^{S,N}_{r,k-1} - u^{S,N}_{t_k,k-1}\|^2_{\mathbb{H}^2}\mathrm{d}r$$

and $\frac{\tau_k}{\tau_{k-1}} \le \frac{T\delta}{\tau_{min}\delta} \le C$, we obtain

$$\begin{aligned}
&\sum_{k=0}^{m} \|u^N_k\|^2_{\mathbb{H}^2}\tau_k \le \|u^N_0\|^2_{\mathbb{H}^2}\tau_0 + C\sum_{k=1}^{m} \|u^N_k\|^2_{\mathbb{H}^2}\tau_{k-1} \\
&\le \|u^N_0\|^2_{\mathbb{H}^2}\tau_0 + C\sum_{k=1}^{m}\Big[\int_{t_{k-1}}^{t_k} \|u^{S,N}_{r,k-1}\|^2_{\mathbb{H}^2}\mathrm{d}r + \int_{t_{k-1}}^{t_k} \|u^{S,N}_{r,k-1} - u^{S,N}_{t_k,k-1}\|^2_{\mathbb{H}^2}\mathrm{d}r\Big].
\end{aligned} \tag{2.21}$$

Utilizing the property of the conditional expectation and the fact that $t_k = t_{k-1} + \tau_{k-1}$ is $\mathcal{F}_{t_{k-1}}$-measurable, one arrives at that for $r \in T_{k-1}$,

$$\begin{aligned}
&\mathbb{E}\Big[\Big\|\int_r^{t_k} P^N u^{S,N}_{s,k-1}\mathrm{d}W(s)\Big\|^2\Big] = \mathbb{E}\Big[\mathbb{E}\Big[\Big\|\int_r^{t_k} P^N u^{S,N}_{s,k-1}\mathrm{d}W(s)\Big\|^2\Big|\mathcal{F}_{t_{k-1}}\Big]\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\|\int_r^{y} P^N \widehat{u^{S,N}_{s,k-1}}\mathrm{d}W(s)\|^2\Big]\Big|_{y=t_k, z_0=u^{S,N}_{t_{k-1},k-1}}\Big] \\
&\le \mathbb{E}\Big[\mathbb{E}\Big[\int_r^{y}\sum_{j=1}^{\infty} \|P^N \widehat{u^{S,N}_{s,k-1}} Q^{\frac{1}{2}} e_j\|^2\mathrm{d}s\Big]\Big|_{y=t_k, z_0=u^{S,N}_{t_{k-1},k-1}}\Big] \\
&\le C\mathbb{E}[\tau_{k-1}],
\end{aligned}$$

where $\widehat{u^{S,N}_{s,k-1}}$, $s \in T_{k-1}$ is the solution of (2.4)(b) with initial datum z_0 at t_{k-1}. The above inequality yields

$$\begin{aligned}
&\mathbb{E}[\|u^{S,N}_{r,k-1} - u^{S,N}_{t_k,k-1}\|^2] \\
&\le C\mathbb{E}\Big[\Big\|\int_r^{t_k} P^N F_Q u^{S,N}_{s,k-1}\mathrm{d}s\Big\|^2\Big] + C\mathbb{E}\Big[\Big\|\int_r^{t_k} P^N u^{S,N}_{s,k-1}\mathrm{d}W(s)\Big\|^2\Big] \le C\mathbb{E}[\tau_{k-1}].
\end{aligned}$$

Thus, $\mathbb{E}\big[\|u^{S,N}_{r,k-1} - u^{S,N}_{t_k,k-1}\|^2_{\mathbb{H}^2}\big] \le C\lambda_N^2\mathbb{E}[\tau_{k-1}] \le C$ for $r \in T_{k-1}$, which together with (2.18) and (2.21) gives (2.20).

Hence, we derive

$$\mathbb{E}[f(u^{S,N}_{t,\underline{t}})] \le C\mathbb{E}[f(u^N_0)] + C\int_0^t \mathbb{E}[f(u^{S,N}_{r,\underline{r}})]\mathrm{d}r + \int_0^T \mathbb{E}\Big[\|u^N_{\underline{r}}\|^{10}_{\mathbb{H}^1}$$

$$+ \|u^{D,N}_{r,\underline{r}}\|^6_{\mathbb{H}^1} + \|u^{S,N}_{r,\underline{r}}\|^6_{\mathbb{H}^1} + 1\big]\mathrm{d}r,$$

which implies

$$\sup_{t\in[0,T]} \mathbb{E}[f(u^{S,N}_{t,\underline{t}})] \le C(\mathbb{E}[f(u^N_0)] + 1)e^{CT}$$

due to the Grönwall inequality.

Moreover, by utilizing the supremum type inequalities as in (2.13), one can finish the proof for the case of $p = 2$. For the case of $p > 2$, the proof is similar by the use of the Itô formula and thus is omitted. □

Remark 2.8. The conclusions in Propositions 2.6 and 2.7 still hold for the solution $\{u^D_{\underline{t}}(t), u^S_{\underline{t}}(t)\}$ of the split equation (2.2) and the solution $\{u^{D,N}_{\underline{t}}(t), u^{S,N}_{\underline{t}}(t)\}$ of the semi-discrete scheme (2.3) for $t \in [0,T]$, i.e.,

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\exp\Big\{\frac{\mathcal{H}(u^S_{\underline{t}}(t))}{e^{\alpha_\lambda t}}\Big\} + \exp\Big\{\frac{\mathcal{H}(u^{S,N}_{\underline{t}}(t))}{e^{\alpha_\lambda t}}\Big\}\Big] \le C,$$
$$\mathbb{E}\Big[\sup_{t\in[0,T]}\Big(\|u^D_{\underline{t}}(t)\|^p_{\mathbb{H}^2} + \|u^S_{\underline{t}}(t)\|^p_{\mathbb{H}^2} + \|u^{D,N}_{\underline{t}}(t)\|^p_{\mathbb{H}^2} + \|u^{S,N}_{\underline{t}}(t)\|^p_{\mathbb{H}^2}\Big)\Big] \le C.$$

The proofs are similar as before by considering $\mathcal{H}(u^D_{\underline{t}}(t))$, $\mathcal{H}(u^S_{\underline{t}}(t))$, $f(u^D_{\underline{t}}(t))$, $f(u^S_{\underline{t}}(t))$ and those of $u^{D,N}_{\underline{t}}(t)$, $u^{S,N}_{\underline{t}}(t)$ instead, and hence are omitted.

3. Optimal strong convergence order

In this section, based on the a.s.-uniform boundedness of the mass, the $\mathbb{H}^j$ $(j = 1,2)$-regularity estimates, and the $\mathbb{H}^1$-exponential integrability of the numerical solution given in Section 2, we show the optimal strong convergence order of the adaptive time-stepping scheme (2.4).

Theorem 3.1. *Under Assumptions 1 and 2, for $p \ge 2$, there exists a constant $C > 0$ such that*

$$\sup_{0\le m\le M_T} \|u(t_m) - u^N_m\|_{L^p(\Omega;\mathbb{H})} \le C(\delta^{\frac{1}{2}} + N^{-2}).$$

Proof. Noting that $u(t_m) - u^N_m = \big(u^{S,N}_{m-1}(t_m) - u^N_m\big) + \big(u^S_{m-1}(t_m) - u^{S,N}_{m-1}(t_m)\big) + \big(u(t_m) - u^S_{m-1}(t_m)\big)$, we split the estimate of the strong error into three steps.

Step 1. We first estimate the strong error between the semi-discrete scheme and the fully discrete scheme, i.e. $\Big\|u^{S,N}_{m-1}(t_m) - u^N_m\Big\|_{L^p(\Omega;\mathbb{H})} =: \|E_m\|_{L^p(\Omega;\mathbb{H})}$. Similar to the proof of (2.8), the differential form

$$\mathrm{d}(u^{S,N}_m(t) - u^{S,N}_{t,m})$$
$$= -\frac{\epsilon}{2}P^N F_Q(u^{S,N}_m(t) - u^{S,N}_{t,m})\mathrm{d}t - \mathbf{i}\sqrt{\epsilon}P^N(u^{S,N}_m(t) - u^{S,N}_{t,m})\mathrm{d}W(t),$$

combining the Itô formula yields that $\|u^{S,N}_m(t) - u^{S,N}_{t,m}\|^2 \le \|u^{D,N}_m(t_{m+1}) - u^{D,N}_{t_{m+1},m}\|^2$. Since

$$u^{D,N}_m(t) - u^{D,N}_{t,m} = E_m + \int_{t_m}^t \mathbf{i}\Delta(u^{D,N}_m(r) - u^{D,N}_{r,m})\mathrm{d}r$$
$$+ \int_{t_m}^t \mathbf{i}\lambda P^N\Big(|u^{D,N}_m(r)|^2 u^{D,N}_m(r) - S^N(r - t_m)|u^{D,N}_{t_m,m}|^2 u^{D,N}_{t_m,m}\Big)\mathrm{d}r, \tag{3.1}$$

we have

$$\|E_{m+1}\|^2$$
$$\le \|u^{D,N}_m(t_{m+1}) - u^{D,N}_{t_{m+1},m}\|^2 = \|E_m\|^2 + 2\Big\langle E_m, \int_{t_m}^{t_{m+1}} \mathbf{i}\Delta(u^{D,N}_m(t) - u^{D,N}_{t,m})\mathrm{d}t\Big\rangle$$
$$+ 2\Big\langle E_m, \mathbf{i}\lambda\int_{t_m}^{t_{m+1}} P^N(|u^{D,N}_m(t)|^2 u^{D,N}_m(t) - S^N(t - t_m)|u^{D,N}_{t_m,m}|^2 u^{D,N}_{t_m,m})\mathrm{d}t\Big\rangle$$
$$+ \Big\|\int_{t_m}^{t_{m+1}} \mathbf{i}\Delta(u^{D,N}_m(t) - u^{D,N}_{t,m})\mathrm{d}t$$
$$+ \mathbf{i}\lambda\int_{t_m}^{t_{m+1}} P^N\Big(|u^{D,N}_m(t)|^2 u^{D,N}_m(t) - S^N(t - t_m)|u^{D,N}_{t_m,m}|^2 u^{D,N}_{t_m,m}\Big)\mathrm{d}t\Big\|^2$$
$$=: \|E_m\|^2 + II_1 + II_2 + II_3.$$

For the term II_1, using (3.1) and the integration by parts formula, and combining the Gagliardo–Nirenberg inequality $\|u\|^3_{L^6(\mathcal{O})} \leq C\|\nabla u\|\|u\|^2$, $u \in \mathbb{H}$ give that

$$\begin{aligned}
II_1 &= 2\Big\langle \Delta E_m, -\int_{t_m}^{t_{m+1}}\int_{t_m}^{t} \Delta(u_m^{D,N}(r) - u_{r,m}^{D,N})\mathrm{d}r\mathrm{d}t \\
&\quad - \lambda \int_{t_m}^{t_{m+1}}\int_{t_m}^{t} P^N\Big(|u_m^{D,N}(r)|^2 u_m^{D,N}(r) - S^N(r-t_m)|u_{r,m}^{D,N}|^2 u_{r,m}^{D,N}\Big)\mathrm{d}r\mathrm{d}t\Big\rangle \\
&\leq C\|E_m\|_{\mathbb{H}^2}\tau_m^2\Big[\sup_{t\in T_m}\|u_m^{D,N}(t) - u_{t,m}^{D,N}\|_{\mathbb{H}^2} + \sup_{t\in T_m}\|u_m^{D,N}(t)\|^3_{L^6(\mathcal{O})} \\
&\quad + \sup_{t\in T_m}\|u_{t,m}^{D,N}\|^3_{L^6(\mathcal{O})}\Big] \\
&\leq C\tau_m^2\Big[\sup_{t\in T_m}\|u_m^{D,N}(t)\|^2_{\mathbb{H}^2} + \sup_{t\in T_m}\|u_{t,m}^{D,N}\|^2_{\mathbb{H}^2} + 1\Big].
\end{aligned}$$

For the term II_2, it follows from the property $\|(S(t) - \mathrm{Id})u\| \leq Ct^{\frac{1}{2}}\|u\|_{\dot{\mathbb{H}}^1}$ that

$$\begin{aligned}
II_2 &= 2\Big\langle E_m, \mathbf{i}\lambda\int_{t_m}^{t_{m+1}} P^N\Big((|u_m^{D,N}(t)|^2 + |u_{t_m,m}^{D,N}|^2)(u_m^{D,N}(t) - u_{t_m,m}^{D,N}) \\
&\quad + u_m^{D,N}(t)u_{t_m,m}^{D,N}\overline{(u_m^{D,N}(t) - u_{t_m,m}^{D,N})} + (\mathrm{Id} - S^N(t-t_m))|u_{t_m,m}^{D,N}|^2 u_{t_m,m}^{D,N}\Big)\mathrm{d}t\Big\rangle \\
&\leq C\|E_m\|\int_{t_m}^{t_{m+1}}\Big[(\|u_m^{D,N}(t)\|^2_{L^\infty(\mathcal{O})} + \|u_{t_m,m}^{D,N}\|^2_{L^\infty(\mathcal{O})})\|u_m^{D,N}(t) - u_{t_m,m}^{D,N}\| \\
&\quad + \tau_m^{\frac{1}{2}}\|u_{t_m,m}^{D,N}\|^3_{\mathbb{H}^1}\Big]\mathrm{d}t.
\end{aligned}$$

By the inverse inequality $\|P^N u\|_{\dot{\mathbb{H}}^s} \leq \lambda_N^{\frac{s}{2}}\|P^N u\|$, $u \in \mathbb{H}$, $\|u_m^{D,N}(t) - u_m^{D,N}(t_m)\| \leq \|(S^N(t-t_m) - \mathrm{Id})u_m^{D,N}(t_m)\| + \|\int_{t_m}^t S^N(t-s)P^N|u_m^{D,N}(s)|^2 u_m^{D,N}(s)\mathrm{d}s\| \leq C(t-t_m)^{\frac{1}{2}}\sup_{t\in T_m}\|u_m^{D,N}(t)\|_{\mathbb{H}^1}$, the Minkowskii inequality, and the Young inequality, we derive

$$\begin{aligned}
&II_2 \\
&\leq C\|E_m\|\int_{t_m}^{t_{m+1}}\big(\|u_m^{D,N}(t_m)\|^2_{L^\infty(\mathcal{O})} + \|u_{t_m,m}^{D,N}\|^2_{L^\infty(\mathcal{O})} + \tau_m\lambda_N^2\sup_{t\in T_m}\|u_m^{D,N}(t)\|^2\big) \\
&\quad \times(\tau_m^{\frac{1}{2}}\sup_{t\in T_m}\|u_m^{D,N}(t)\|_{\mathbb{H}^1} + \|E_m\|)\mathrm{d}t + C\|E_m\|\int_{t_m}^{t_{m+1}}\tau_m^{\frac{1}{2}}\|u_{t_m,m}^{D,N}\|^3_{\mathbb{H}^1}\mathrm{d}t \\
&\leq C\tau_m\|E_m\|^2(\|u_m^{D,N}(t_m)\|^2_{L^\infty(\mathcal{O})} + \|u_{t_m,m}^{D,N}\|^2_{L^\infty(\mathcal{O})} + 1) \\
&\quad + C\tau_m^2(\sup_{t\in T_m}\|u_m^{D,N}(t)\|^6_{\mathbb{H}^1} + \|u_{t_m,m}^{D,N}\|^6_{\mathbb{H}^1} + 1).
\end{aligned}$$

For the term II_3, it can be estimated as

$$II_3 \leq C\tau_m^2(\sup_{t\in T_m}\|u_m^{D,N}(t)\|^2_{\mathbb{H}^2} + \sup_{t\in T_m}\|u_{t,m}^{D,N}\|^2_{\mathbb{H}^2}).$$

Hence,

$$\begin{aligned}
\|E_{m+1}\|^2 &\leq \|E_m\|^2 + C\tau_m(1 + \|u_m^{D,N}(t_m)\|^2_{L^\infty(\mathcal{O})} + \|u_m^N\|^2_{L^\infty(\mathcal{O})})\|E_m\|^2 \\
&\quad + C\tau_m^2(\sup_{t\in T_m}\|u_m^{D,N}(t)\|^6_{\mathbb{H}^2} + \sup_{t\in T_m}\|u_{t,m}^{D,N}\|^6_{\mathbb{H}^2} + 1).
\end{aligned}$$

Applying the Grönwall inequality leads to

$$\begin{aligned}
\|E_{m+1}\|^2 &\leq C\sum_{j=0}^{m}\tau_j^2(\sup_{t\in T_j}\|u_j^{D,N}(t)\|^6_{\mathbb{H}^2} + \sup_{t\in T_j}\|u_{t,j}^{D,N}\|^6_{\mathbb{H}^2} + 1)\times \\
&\quad \exp\Big\{C\sum_{j=0}^{m}\tau_j\Big(1 + \|u_j^{D,N}(t_j)\|^2_{L^\infty(\mathcal{O})} + \|u_j^N\|^2_{L^\infty(\mathcal{O})}\Big)\Big\}.
\end{aligned} \tag{3.2}$$

Note that

$$\begin{aligned}
&\Big\|\exp\Big\{C\sum_{j=0}^{M_T-1}\tau_j\|\nabla u_j^N\|\Big\}\Big\|_{L^{4p}(\Omega)} \leq \Big\|\exp\Big\{\sum_{j=0}^{M_T-1}\tau_j\Big(\rho\|\nabla u_j^N\|^2 + C(\rho)\Big)\Big\}\Big\|_{L^{4p}(\Omega)} \\
&\leq \Big\|\frac{1}{T}\sum_{j=0}^{M_T-1}\tau_j\exp\Big\{T\Big(\rho\|\nabla u_j^N\|^2 + C(\rho)\Big)\Big\}\Big\|_{L^{4p}(\Omega)} \\
&\leq \frac{1}{T}\sum_{j=0}^{M_T-1}T\delta\Big(\mathbb{E}\Big[\exp\Big\{4pT\Big(\rho\|\nabla u_j^N\|^2 + C(\rho)\Big)\Big\}\Big]\Big)^{\frac{1}{4p}},
\end{aligned}$$

where in the second inequality we use the convexity of $e^a, a \in \mathbb{R}$, and in the last inequality we use the assumption $\tau_j \le T\delta$. Taking $\rho = \frac{1}{16pT\exp\{\alpha_\lambda T\}}$, and combining $\|\nabla u\|^2 \le 4\mathcal{H}(u) + \|u\|^6$, $u \in \mathbb{H}^1$ and Proposition 2.6 give

$$\mathbb{E}\Big[\exp\Big\{4pT\Big(\rho\|\nabla u_j^N\|^2 + C(\rho)\Big)\Big\}\Big] \le \mathbb{E}\Big[\exp\Big\{\frac{\mathcal{H}(u_j^N)}{e^{\alpha_\lambda t_j}} + C(\rho)\Big\}\Big] \le C,$$

which together with the Gagliardo–Nirenberg inequality $\|u\|^2_{L^\infty(\mathcal{O})} \le C\|\nabla u\|\|u\|$ for $u \in \mathbb{H}^1$ implies

$$\Big\|\exp\Big\{C\sum_{j=0}^m \tau_j \|u_j^N\|^2_{L^\infty(\mathcal{O})}\Big\}\Big\|_{L^{4p}(\Omega)} \le C. \tag{3.3}$$

Similarly, combining Remark 2.8, one can show that

$$\Big\|\exp\Big\{C\sum_{j=0}^m \tau_j \|u_j^{D,N}(t_j)\|^2_{L^\infty(\mathcal{O})}\Big\}\Big\|_{L^{4p}(\Omega)} \le C. \tag{3.4}$$

Hence, taking the pth power and expectation on both sides of (3.2) and using the assumption $\tau_m \le T\delta$ lead to

$$\begin{aligned}
&\mathbb{E}[\|E_{m+1}\|^{2p}]\\
&\le C(T\delta)^p\Big(\mathbb{E}\Big|\sum_{j=0}^m \tau_j(\sup_{t\in T_j}\|u_j^{D,N}(t)\|^6_{\mathbb{H}^2} + \sup_{t\in T_j}\|u_{t,j}^{D,N}\|^6_{\mathbb{H}^2} + 1)\Big|^{2p}\Big)^{\frac{1}{2}}\times\\
&\quad\Big\|\exp\Big\{C\sum_{j=0}^m \tau_j\|u_j^{D,N}(t_j)\|^2_{L^\infty(\mathcal{O})}\Big\}\Big\|^{\frac{p}{2}}_{L^{4p}(\Omega)}\Big\|\exp\Big\{C\sum_{j=0}^m \tau_j\|u_j^N\|^2_{L^\infty(\mathcal{O})}\Big\}\Big\|^{\frac{p}{2}}_{L^{4p}(\Omega)}\\
&\le C\delta^p.
\end{aligned}$$

Step 2. We estimate the strong error between the split equation (2.2) and the semi-discrete scheme, i.e., $\Big\|u_{m-1}^S(t_m) - u_{m-1}^{S,N}(t_m)\Big\|_{L^p(\Omega;\mathbb{H})} =: \|\hat{E}_m\|_{L^p(\Omega;\mathbb{H})}$. Applying the chain rule yields

$$\begin{aligned}
&\|u_m^D(t_{m+1}) - u_m^{D,N}(t_{m+1})\|^2 = \|u_m^D(t_m) - u_m^{D,N}(t_m)\|^2\\
&+ 2\int_{t_m}^{t_{m+1}} \Big\langle u_m^D(s) - u_m^{D,N}(s), \mathbf{i}\lambda\big(|u_m^D(s)|^2 u_m^D(s) - P^N|u_m^{D,N}(s)|^2 u_m^{D,N}(s)\big)\Big\rangle \mathrm{d}s,
\end{aligned}$$

and applying the Itô formula gives

$$\begin{aligned}
&\|u_m^S(t_{m+1}) - u_m^{S,N}(t_{m+1})\|^2 = \|u_m^S(t_m) - u_m^{S,N}(t_m)\|^2\\
&+ 2\epsilon\int_{t_m}^{t_{m+1}} \Big\langle u_m^S(s) - u_m^{S,N}(s), -\frac{1}{2}F_Q u_m^S(s) + \frac{1}{2}P^N F_Q u_m^{S,N}(s)\Big\rangle \mathrm{d}s\\
&+ 2\sqrt{\epsilon}\int_{t_m}^{t_{m+1}} \Big\langle u_m^S(s) - u_m^{S,N}(s), -\mathbf{i}(u_m^S(s) - P^N u_m^{S,N}(s))\mathrm{d}W(s)\Big\rangle\\
&+ \epsilon\int_{t_m}^{t_{m+1}} \sum_{k=1}^\infty \|(u_m^S(s) - P^N u_m^{S,N}(s))Q^{\frac{1}{2}}e_k\|^2 \mathrm{d}s.
\end{aligned}$$

Therefore, we derive

$$\begin{aligned}
&\|\hat{E}_{m+1}\|^2\\
&= \|\hat{E}_m\|^2 + 2\int_{t_m}^{t_{m+1}} \Big\langle u_m^D(s) - u_m^{D,N}(s), \mathbf{i}\lambda\big(|u_m^D(s)|^2 u_m^D(s)\\
&\quad - P^N|u_m^{D,N}(s)|^2 u_m^{D,N}(s)\big)\Big\rangle \mathrm{d}s\\
&\quad + 2\epsilon\int_{t_m}^{t_{m+1}} \Big\langle u_m^S(s) - u_m^{S,N}(s), -\frac{1}{2}F_Q u_m^S(s) + \frac{1}{2}P^N F_Q u_m^{S,N}(s)\Big\rangle \mathrm{d}s\\
&\quad + 2\sqrt{\epsilon}\int_{t_m}^{t_{m+1}} \Big\langle u_m^S(s) - u_m^{S,N}(s), -\mathbf{i}(u_m^S(s) - P^N u_m^{S,N}(s))\mathrm{d}W(s)\Big\rangle\\
&\quad + \epsilon\int_{t_m}^{t_{m+1}} \sum_{k=1}^\infty \|(u_m^S(s) - P^N u_m^{S,N}(s))Q^{\frac{1}{2}}e_k\|^2 \mathrm{d}s\\
&=: \|\hat{E}_m\|^2 + III_1 + III_2 + III_3 + III_4.
\end{aligned}$$

For the term III_1, combining the cubic difference formula gives

$$\begin{aligned}
&|III_1|\\
&\le C\int_{t_m}^{t_{m+1}} \|u_m^D(s) - u_m^{D,N}(s)\|\Big[\big(\|u_m^D(s)\|^2_{L^\infty(\mathcal{O})} + \|u_m^{D,N}(s)\|^2_{L^\infty(\mathcal{O})}\big)\times
\end{aligned}$$

$$\begin{aligned}&\|u_m^D(s)-u_m^{D,N}(s)\|+\|(\mathrm{Id}-P^N)|u_m^{D,N}(s)|^2u_m^{D,N}(s)\|\Big]\mathrm{d}s\\&\le C\int_{t_m}^{t_{m+1}}\|u_m^D(s)-u_m^{D,N}(s)\|^2\big(\|u_m^D(s)\|^2_{L^\infty(\mathcal{O})}+\|u_m^{D,N}(s)\|^2_{L^\infty(\mathcal{O})}+1\big)\mathrm{d}s\\&\quad+C\lambda_N^{-2}\int_{t_m}^{t_{m+1}}\|u_m^{D,N}(s)\|^6_{\mathbb{H}^2}\mathrm{d}s.\end{aligned}$$

By the properties $\|u_m^D(s)-u_m^D(t_m)\|^2\le C\tau_m\|u_m^D(t_m)\|^2_{\mathbb{H}^1}$, $\|u_m^D(s)-u_m^D(t_m)\|^2_{\mathbb{H}^1}\le C\tau_m\|u_m^D(t_m)\|^2_{\mathbb{H}^2}$ of $u_m^D(s)$, and those of $u_m^{D,N}(s)$ for $s\in T_m$, we arrive at

$$\begin{aligned}|III_1|&\le C\int_{t_m}^{t_{m+1}}(\|\hat{E}_m\|^2+\tau_m\|u_m^D(t_m)\|^2_{\mathbb{H}^1}+\tau_m\|u_m^{D,N}(t_m)\|^2_{\mathbb{H}^1})(\|u_m^D(t_m)\|^2_{L^\infty(\mathcal{O})}\\&\quad+\|u_m^{D,N}(t_m)\|^2_{L^\infty(\mathcal{O})}+\tau_m\|u_m^D(t_m)\|^2_{\mathbb{H}^2}+\tau_m\|u_m^{D,N}(t_m)\|^2_{\mathbb{H}^2}+1)\\&\quad+C\lambda_N^{-2}\int_{t_m}^{t_{m+1}}\|u_m^{D,N}(s)\|^6_{\mathbb{H}^2}\mathrm{d}s\\&\le C\tau_m\|\hat{E}_m\|^2(\|u_m^D(t_m)\|^2_{\mathbb{H}^1}+\|u_m^{D,N}(t_m)\|^2_{\mathbb{H}^1}+1)\\&\quad+C\int_{t_m}^{t_{m+1}}\tau_m(\|u_m^D(t_m)\|^4_{\mathbb{H}^1}+\|u_m^{D,N}(t_m)\|^4_{\mathbb{H}^1})+\lambda_N^{-2}\|u_m^{D,N}(s)\|^6_{\mathbb{H}^2}\mathrm{d}s.\end{aligned}$$

Terms III_2 and III_4 can be estimated respectively as

$$\begin{aligned}|III_2|&\le C\int_{t_m}^{t_{m+1}}\Big(\|u_m^S(s)-u_m^{S,N}(s)\|^2(\|F_Q\|_{L^\infty(\mathcal{O})}+1)\\&\quad+\lambda_N^{-2}\|F_Q\|^2_{H^2}\|u_m^{S,N}(s)\|^2_{\mathbb{H}^2}\Big)\mathrm{d}s\end{aligned}$$

and

$$|III_4|\le C\int_{t_m}^{t_{m+1}}\Big(\|u_m^S(s)-u_m^{S,N}(s)\|^2\|Q^{\frac12}\|^2_{\mathcal{L}_2^1}+\lambda_N^{-2}\|u_m^{S,N}(s)\|^2_{\mathbb{H}^2}\|Q^{\frac12}\|^2_{\mathcal{L}_2^2}\Big)\mathrm{d}s.$$

By the Hölder continuity and the triangle inequality, we obtain

$$\begin{aligned}&|III_2|+|III_4|\\&\le C\int_{t_m}^{t_{m+1}}\Big[\|\hat{E}_m\|^2+\|u_m^S(s)-u_m^S(t_m)\|^2+\|u_m^{S,N}(s)-u_m^{S,N}(t_m)\|^2\Big]\mathrm{d}s\\&\quad+C\int_{t_m}^{t_{m+1}}\lambda_N^{-2}\|u_m^{S,N}(s)\|^2_{\mathbb{H}^2}\mathrm{d}s.\end{aligned}$$

Combining estimates of terms III_j, $j=1,2,4$ yields that

$$\begin{aligned}&\|\hat{E}_{m+1}\|^2\\&\le\|\hat{E}_m\|^2+C\tau_m\|\hat{E}_m\|^2(\|u_m^D(t_m)\|^2_{\mathbb{H}^1}+\|u_m^{D,N}(t_m)\|^2_{\mathbb{H}^1}+1)\\&\quad+C\int_{t_m}^{t_{m+1}}\Big[\tau_m(\|u_m^D(t_m)\|^4_{\mathbb{H}^1}+\|u_m^{D,N}(t_m)\|^4_{\mathbb{H}^1})+\|u_m^S(s)-u_m^S(t_m)\|^2\\&\quad+\|u_m^{S,N}(s)-u_m^{S,N}(t_m)\|^2+\lambda_N^{-2}(\|u_m^{D,N}(s)\|^6_{\mathbb{H}^2}+\|u_m^{S,N}(s)\|^2_{\mathbb{H}^2})\Big]\mathrm{d}s+III_3.\end{aligned}$$

By iteration, we have

$$\begin{aligned}\|\hat{E}_{m+1}\|^2&\le\|\hat{E}_0\|^2+C\sum_{j=0}^m\tau_j\|\hat{E}_j\|^2(\|u_j^D(t_j)\|^2_{\mathbb{H}^1}+\|u_j^{D,N}(t_j)\|^2_{\mathbb{H}^1}+1)\\&\quad+C\int_0^{t_{m+1}}\Big[(\delta+\lambda_N^{-2})(\|u_{\underline{s}}^{D,N}(t_{\underline{s}})\|^6_{\mathbb{H}^2}+\|u_{\underline{s}}^D(t_{\underline{s}})\|^4_{\mathbb{H}^1}+\|u_{\underline{s}}^{S,N}(s)\|^2_{\mathbb{H}^2})\\&\quad+\|u_{\underline{s}}^S(s)-u_{\underline{s}}^S(t_{\underline{s}})\|^2+\|u_{\underline{s}}^{S,N}(s)-u_{\underline{s}}^{S,N}(t_{\underline{s}})\|^2\Big]\mathrm{d}s\\&\quad+2\int_0^{t_{m+1}}\Big\langle(\mathrm{Id}-P^N)(u_{\underline{s}}^S(s)-u_{\underline{s}}^{S,N}(s)),-\mathbf{i}(\mathrm{Id}-P^N)u_{\underline{s}}^{S,N}(s)\mathrm{d}W(s)\Big\rangle\\&=:\|\hat{E}_0\|^2+C\sum_{j=0}^m\tau_j\|\hat{E}_j\|^2(\|u_j^D(t_j)\|^2_{\mathbb{H}^1}+\|u_j^{D,N}(t_j)\|^2_{\mathbb{H}^1}+1)+J_1+J_2,\end{aligned}$$

which implies

$$\|\hat{E}_{m+1}\|^2\le(\lambda_N^{-2}\|u_0^N\|^2_{\mathbb{H}^2}+J_1+J_2)\exp\Big\{C\sum_{j=0}^m\tau_j(\|u_j^D(t_j)\|^2_{\mathbb{H}^1}+\|u_j^{D,N}(t_j)\|^2_{\mathbb{H}^1}+1)\Big\}.$$

Taking $L^p(\Omega)$-norm, and using the Hölder continuity of $u^S_{\underline{\cdot}}, u^{S,N}_{\underline{\cdot}}$ and the Burkholder–Davis–Gundy inequality give

$$\begin{aligned}&\|J_1+J_2\|_{L^{2p}(\Omega)}\\&\le C(\delta+\lambda_N^{-2})+C\int_0^T\left(\|u^S_{\underline{s}}(s)-u^S_{\underline{s}}(t_{\underline{s}})\|^2_{L^{4p}(\Omega;\mathbb{H})}+\|u^{S,N}_{\underline{s}}(s)-u^{S,N}_{\underline{s}}(t_{\underline{s}})\|^2_{L^{4p}(\Omega;\mathbb{H})}\right)\mathrm{d}s\\&\quad+C\mathbb{E}\Big[\Big(\int_0^T\lambda_N^{-4}(\|u^S_{\underline{s}}(s)\|^2_{\mathbb{H}^2}+\|u^{S,N}_{\underline{s}}(s)\|^2_{\mathbb{H}^2})\|u^{S,N}_{\underline{s}}(s)\|^2_{\mathbb{H}^2}\|Q^{\frac12}\|^2_{\mathcal{L}^1_2}\mathrm{d}s\Big)^p\Big]^{\frac{1}{2p}}\\&\le C(\delta+\lambda_N^{-2}).\end{aligned}$$

Moreover, one can show that

$$\Big\|\exp\Big\{C\sum_{j=0}^m\tau_j(\|u^D_j(t_j)\|^2_{\mathbb{H}^1}+\|u^{D,N}_j(t_j)\|^2_{\mathbb{H}^1}+1)\Big\|_{L^{2p}(\Omega)}\le C,$$

whose proof is similar to that of (3.3)–(3.4) and is omitted. Hence, we arrive at $\mathbb{E}[\|\hat{E}_{m+1}\|^{2p}]\le C(\lambda_N^{-2}+\delta)^p$.

Step 3. For the strong error between the original stochastic Schrödinger equation (2.1) and the split equation (2.2), i.e., $\Big\|u(t_m)-u^S_{m-1}(t_m)\Big\|_{L^p(\Omega;\mathbb{H})}$, it follows from [15, Theorem 2.2] that $\Big\|u(t_m)-u^S_{m-1}(t_m)\Big\|_{L^p(\Omega;\mathbb{H})}\le C\delta^{\frac12}$.

Combining *Steps* 1-3 finishes the proof. □

Remark 3.2. In practice, instead of verifying whether a timestep function satisfies the low bound in Assumption 1 (2.7), people usually introduce a backstop scheme with a uniform timestep and couple it with (2.4) to ensure that a simulation over the interval [0,T] can be completed in a finite number of timesteps; see e.g. [8] and references therein for more details.

4. Numerical asymptotics

In this section, we study the asymptotic behavior of the adaptive time-stepping fully discrete scheme (2.4) for the stochastic NLS equation (1.1) as the noise intensity ϵ tends to zero. Note that the dependence on ϵ of solutions is emphasized in this section, for example, solutions of (1.1) and (2.4) are denoted by $\{u^\epsilon(t) : t\in[0,T]\}$ and $\{u^{D,N,\epsilon}_{t,\underline{t}}, u^{S,N,\epsilon}_{t,\underline{t}} : t\in[0,T]\}$, respectively. The tool for this study is the theory of large deviation, which describes precisely the weak convergence of the law of the family $\{u^{S,N,\epsilon}_{\cdot,\underline{\cdot}}\}_{\epsilon\in(0,1)}$ towards the Dirac measure on the solution of the corresponding skeleton equation as $\epsilon\to0$.

Set $H_0 := Q^{\frac12}H$. Then H_0 is a Hilbert space with the inner product $\langle u,v\rangle_{H_0} := \langle Q^{-\frac12}u, Q^{-\frac12}v\rangle_H$ and the induced norm $\|\cdot\|^2_{H_0}=\langle\cdot,\cdot\rangle_{H_0}$, where $Q^{-\frac12}$ is the pseudo inverse of $Q^{\frac12}$. Denote $\mathcal{S}_M := \Big\{v\in L^2([0,T];H_0)\Big|\int_0^T\|v(s)\|^2_{H_0}\mathrm{d}s\le M\Big\}$ and $\mathcal{P}_M := \Big\{v : \Omega\times[0,T]\to H_0\Big| v \text{ is } \mathcal{F}_t\text{-predictable and } v\in\mathcal{S}_M \text{ a.s.}\Big\}$ for each $M\in(0,\infty)$. It can be checked that $\mathcal{S}_M$ is a compact Polish space endowed with the weak topology $d_1(g_1,g_2)=\sum_{k\ge1}\frac{1}{2^k}|\int_0^T\langle g_1(s)-g_2(s),\xi_k(s)\rangle_{H_0}\mathrm{d}s|$, where $\{\xi_k\}_{k\ge1}$ is an orthogonal basis of $L^2([0,T];H_0)$; see e.g. [6, Section 4] and [20, Section 2]. In the sequel, we denote by $\xrightarrow{d}$ the convergence in distribution.

In order to establish the LDP for the solution of (2.4), we consider the following stochastic controlled equation

$$\begin{cases}\mathrm{d}u^{D,N,\epsilon}_{v^\epsilon,m}(t)=\mathbf{i}\Delta u^{D,N,\epsilon}_{v^\epsilon,m}(t)\mathrm{d}t+\mathbf{i}\lambda S^N(t-t_m)P^N|u^{N,\epsilon}_{v^\epsilon,m}|^2u^{N,\epsilon}_{v^\epsilon,m}\mathrm{d}t,\\ \qquad u^{D,N,\epsilon}_{v^\epsilon,m}(t_m)=u^{N,\epsilon}_{v^\epsilon,m},\\ \mathrm{d}u^{S,N,\epsilon}_{v^\epsilon,m}(t)=-\frac{\epsilon}{2}P^NF_Qu^{S,N,\epsilon}_{v^\epsilon,m}(t)\mathrm{d}t-\mathbf{i}P^Nu^{S,N,\epsilon}_{v^\epsilon,m}(t)v^\epsilon(t)\mathrm{d}t\\ \qquad -\mathbf{i}\sqrt{\epsilon}P^Nu^{S,N,\epsilon}_{v^\epsilon,m}\mathrm{d}W(t),\quad u^{S,N,\epsilon}_{v^\epsilon,m}(t_m)=u^{D,N,\epsilon}_{v^\epsilon,m}(t_{m+1}),\end{cases}\tag{4.1}$$

and the skeleton equation

$$\begin{cases}\mathrm{d}w^{D,N}_{v,m}(t)=\mathbf{i}\Delta w^{D,N}_{v,m}(t)\mathrm{d}t+\mathbf{i}\lambda S^N(t-t_m)P^N|w^N_{v,m}|^2w^N_{v,m}\mathrm{d}t,\\ \qquad w^{D,N}_{v,m}(t_m)=w^N_{v,m},\\ \mathrm{d}w^{S,N}_{v,m}(t)=-\mathbf{i}P^Nw^{S,N}_{v,m}(t)v(t)\mathrm{d}t,\quad w^{S,N}_{v,m}(t_m)=w^{D,N}_{v,m}(t_{m+1})\end{cases}\tag{4.2}$$

for $t\in T_m$ with $v^\epsilon, v\in L^2([0,T];H_0)$. Here, the initial data both are u^N_0. Define measurable maps $\mathcal{G}^\epsilon,\mathcal{G}^0 : C([0,T];H)\to C([0,T];\mathbb{H}_N)$ by $\mathcal{G}^\epsilon\big(\sqrt{\epsilon}W+\int_0^\cdot v^\epsilon(s)\mathrm{d}s\big) := u^{S,N,\epsilon}_{v^\epsilon,\underline{\cdot}}(\cdot)$ and $\mathcal{G}^0\big(\int_0^\cdot v(s)\mathrm{d}s\big) := w^{S,N}_{v,\underline{\cdot}}(\cdot)$. And denote $u^{N,\epsilon}_{v^\epsilon,m+1} := u^{S,N,\epsilon}_{v^\epsilon,m}(t_{m+1})$ and $w^N_{v,m+1} := w^{S,N}_{v,m}(t_{m+1})$.

Similar assumptions to Assumptions 1 and 2 are given as follows.

Assumption 3. Let τ_m satisfy

$$\begin{aligned}&\tau_m\le\min\big\{L_1\|w^N_{v,m}\|^2\|w^N_{v,m}\|^{-6}_{L^6(\mathcal{O})},\ L_1\|u^{N,\epsilon}_{v^\epsilon,m}\|^2\|u^{N,\epsilon}_{v^\epsilon,m}\|^{-6}_{L^6(\mathcal{O})},\ T\delta\big\}\quad a.s.,\\&\tau_m\ge\max\big\{(\zeta\|u^{N,\epsilon}_{v^\epsilon,m}\|^\beta+\xi)^{-1}\delta,\ (\zeta\|w^N_{v,m}\|^\beta+\xi)^{-1}\delta\big\}\quad a.s.\end{aligned}$$

with constants $L_1,\zeta,\beta,\xi>0$ and small constant $\delta\in(0,1)$ independent of ϵ.

Assumption 4. Let τ_m satisfy

$$\tau_m^{\frac{1}{2}-\gamma}\lambda_N \le L_2 \quad a.s.,$$
$$\tau_m^{\gamma}\max\{\mathcal{H}(u^{N,\epsilon}_{v^\epsilon,m}),\mathcal{H}(w^N_{v,m})\} \le L_3 \quad a.s.$$

for some $\gamma \in (0,\frac{1}{2})$ and constants $L_2, L_3 > 0$ independent of ϵ.

The main result of this section is stated as follows.

Theorem 4.1. *Under Assumptions 3 and 4, the family* $\{u^{S,N,\epsilon}_{\cdot,\cdot}\}_{\epsilon\in(0,1)}$ *of solutions of (2.4) satisfies the LDP on* $C([0,T];\mathbb{H}_N)$*, i.e.,*

(i) for each closed subset F *of* $C([0,T];\mathbb{H}_N)$*,*

$$\limsup_{\epsilon\to 0}\epsilon\log\mathbb{P}(u^{S,N,\epsilon}_{\cdot,\cdot}\in F) \le -\inf_{x\in F} I(x);$$

(ii) for each open subset G *of* $C([0,T];\mathbb{H}_N)$*,*

$$\liminf_{\epsilon\to 0}\epsilon\log\mathbb{P}(u^{S,N,\epsilon}_{\cdot,\cdot}\in G) \ge -\inf_{x\in G} I(x),$$

where the good rate function $I : C([0,T];\mathbb{H}_N)\to[0,\infty]$ *is defined by*

$$I(f) = \inf_{\{v\in L^2([0,T];H_0):\, f=\mathcal{G}^0(\int_0^\cdot v(s)\mathrm{d}s)\}} \frac{1}{2}\int_0^T \|v(s)\|^2_{H_0}\mathrm{d}s.$$

Below we give the a.s.-uniform boundedness of the masses, and the $\mathbb{H}^1$-regularity estimates of solutions of (4.1) and (4.2), which are similar to those of the fully discrete scheme (2.4), i.e., Lemma 2.1 and Proposition 2.4.

Proposition 4.2. *Let* $M > 0$*, and let* $\{v^\epsilon\}_{\epsilon\in(0,1)}\subset\mathcal{P}_M$*. Under Assumptions 3 and 4,*

$$\sup_{\epsilon\in(0,1)}\sup_{t\in[0,T]}\big(\|u^{D,N,\epsilon}_{v^\epsilon,\underline{t}}\|^2 \vee \|u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}\|^2\big) \le e^{L_1 T}\|u^N_0\|^2 \quad a.s., \tag{4.3}$$

and for $p\ge 2$*,*

$$\sup_{\epsilon\in(0,1)}\mathbb{E}\Big[\sup_{t\in[0,T]}\|u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t)\|^p_{\mathbb{H}^1}\Big] \le C \quad a.s., \tag{4.4}$$

where the constant L_1 *is given in Assumption 3 and* $C := C(p,T,M,\mathcal{H}(u^N_0)) > 0$*.*

Proof. For the proof of (4.3), we note that $\langle u^{S,N,\epsilon}_{v^\epsilon,m}(s), -\mathbf{i}P^N u^{S,N,\epsilon}_{v^\epsilon,m}(s)v^\epsilon(s)\rangle = 0$. A similar proof to that of Lemma 2.1 leads to (4.3).

For the proof of (4.4), similar to the proof of (2.12), we have

$$\mathcal{H}(u^{D,N,\epsilon}_{v^\epsilon,m}(t_{m+1})) \le \mathcal{H}(u^{N,\epsilon}_{v^\epsilon,m}) + C\tau_m^{1+\gamma}(\mathcal{H}(u^{N,\epsilon}_{v^\epsilon,m})+1) \le \mathcal{H}(u^{N,\epsilon}_{v^\epsilon,m}) + C\tau_m$$

under Assumption 4. Applying the Itô formula to $\mathcal{H}(u^{S,N,\epsilon}_{v^\epsilon,\cdot}(\cdot))$, and noticing that

$$\big\langle \nabla u^{S,N,\epsilon}_{v^\epsilon,m}(s), \mathbf{i}\nabla\big(u^{S,N,\epsilon}_{v^\epsilon,m}(s)v^\epsilon(s)\big)\big\rangle = \big\langle \nabla u^{S,N,\epsilon}_{v^\epsilon,m}(s), \mathbf{i}u^{S,N,\epsilon}_{v^\epsilon,m}(s)\nabla v^\epsilon(s)\big\rangle,$$

we derive

$$\begin{aligned}
&\mathbb{E}[\mathcal{H}(u^{S,N,\epsilon}_{v^\epsilon,m}(t))] - \mathbb{E}[\mathcal{H}(u^{D,N,\epsilon}_{v^\epsilon,m}(t_{m+1}))]\\
&\le C\mathbb{E}\int_{t_m}^t \|\nabla u^{S,N,\epsilon}_{v^\epsilon,m}(s)\|\Big[\|\nabla u^{S,N,\epsilon}_{v^\epsilon,m}(s)\|\|F_Q\|_{L^\infty(\mathcal{O})} + \|u^{S,N,\epsilon}_{v^\epsilon,m}(s)\|_{L^\infty(\mathcal{O})}\|\nabla F_Q\|\\
&\quad + \|u^{S,N,\epsilon}_{v^\epsilon,m}(s)\|\|\nabla v^\epsilon(s)\|_{L^\infty(\mathcal{O})}\Big]\mathrm{d}s + C\mathbb{E}\int_{t_m}^t \|u^{S,N,\epsilon}_{v^\epsilon,m}(s)\|^3_{L^6(\mathcal{O})}\|u^{S,N,\epsilon}_{v^\epsilon,m}(s)\|\times\\
&\quad (\|F_Q\|_{L^\infty(\mathcal{O})} + \|v^\epsilon(s)\|_{L^\infty(\mathcal{O})})\mathrm{d}s + C\mathbb{E}\int_{t_m}^t \|u^{S,N,\epsilon}_{v^\epsilon,m}(s)\|^2_{\mathbb{H}^1}\mathrm{d}s\\
&\le C\mathbb{E}\int_{t_m}^t (\|\nabla u^{S,N,\epsilon}_{v^\epsilon,m}(s)\|^2+1)\mathrm{d}s + C\mathbb{E}\int_{t_m}^t \frac{1}{M}(\|\nabla v^\epsilon(s)\|^2_{L^\infty(\mathcal{O})} + \|v^\epsilon(s)\|^2_{L^\infty(\mathcal{O})})\mathrm{d}s,
\end{aligned} \tag{4.5}$$

where in the second inequality we use the Young inequality. By iteration and combining $\mathcal{H}(u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t)) \ge \frac{1}{4}(\|\nabla u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t)\|^2 - \|u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t)\|^6)$, we obtain

$$\begin{aligned}
\mathbb{E}[\mathcal{H}(u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t))] &\le \mathbb{E}[\mathcal{H}(u^N_0)] + C\mathbb{E}\int_0^t \mathcal{H}(u^{S,N,\epsilon}_{v^\epsilon,\underline{s}}(s))\mathrm{d}s + CT\\
&\quad + C\mathbb{E}\int_0^t \frac{1}{M}(\|\nabla v^\epsilon(s)\|^2_{L^\infty(\mathcal{O})} + \|v^\epsilon(s)\|^2_{L^\infty(\mathcal{O})})\mathrm{d}s.
\end{aligned}$$

It follows from $\|v\|_{H^2} \le \|Q^{\frac{1}{2}}\|_{\mathcal{L}(H,H^2)}\|Q^{-\frac{1}{2}}v\| \le \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2}\|v\|_{H_0}$ for $v \in H_0$ that

$$\int_0^T (\|\nabla v^\epsilon(s)\|^2_{L^\infty(\mathcal{O})} + \|v^\epsilon(s)\|^2_{L^\infty(\mathcal{O})})\mathrm{d}s$$
$$\le C\int_0^T \|v^\epsilon(s)\|^2_{H^2}\mathrm{d}s \le C\int_0^T \|v^\epsilon(s)\|^2_{H_0}\mathrm{d}s \le CM \quad a.s.$$

This leads to

$$\mathbb{E}[\mathcal{H}(u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t))] \le \mathbb{E}[\mathcal{H}(u^N_0)] + C\Big(\mathbb{E}\int_0^t \mathcal{H}(u^{S,N,\epsilon}_{v^\epsilon,\underline{s}}(s))\mathrm{d}s + 1\Big).$$

Hence,

$$\sup_{t\in[0,T]} \mathbb{E}[\mathcal{H}(u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t))] \le \Big(\mathbb{E}[\mathcal{H}(u^N_0)] + C\Big)e^{CT}.$$

The remaining proof is similar to that of Proposition 2.4 and hence is omitted. □

Proposition 4.3. *Let $M > 0$, and let $v \in \mathcal{S}_M$. Under Assumptions 3 and 4,*

$$\sup_{t\in[0,T]} \|w^N_{v,\underline{t}}\|^2 \le e^{L_1 T}\|w^N_0\|^2, \qquad \sup_{t\in[0,T]} \|w^{S,N}_{v,\underline{t}}(t)\|^2_{\mathbb{H}^1} \le C \quad a.s.,$$

where the constant L_1 is given in Assumption 3 and $C := C(T, M, \mathcal{H}(u^N_0)) > 0$.

Proof. It is clear that

$$\|w^{S,N}_{v,m}(t_{m+1})\|^2 = \|w^{S,N}_{v,m}(t_m)\|^2 = \|w^{D,N}_{v,m}(t_{m+1})\|^2,$$

which combining $\|w^{D,N}_{v,m}(t_{m+1})\|^2 \le (1 + L_1\tau_m)\|w^N_{v,m}\|^2$ implies that

$$\|w^N_{v,m}\|^2 \le e^{L_1 T}\|w^N_{v,0}\|^2.$$

Similar to the proof of (2.12), we have

$$\mathcal{H}(w^{D,N}_{v,m}(t_{m+1})) \le \mathcal{H}(w^N_{v,m}) + C\tau_m^{\gamma+1}(\mathcal{H}(w^N_{v,m}) + 1) \le \mathcal{H}(w^N_{v,m}) + C\tau_m$$

under Assumption 4. Applying the chain rule and the Young inequality gives

$$\begin{aligned}\mathcal{H}(w^{S,N}_{v,m}(t)) - \mathcal{H}(w^{S,N}_{v,m}(t_m)) &= \int_{t_m}^t \Big\langle \nabla w^{S,N}_{v,m}(s), \nabla\big(-\mathbf{i}P^N w^{S,N}_{v,m}(s)v(s)\big)\Big\rangle \mathrm{d}s \\ &\quad - \lambda\int_{t_m}^t \Big\langle |w^{S,N}_{v,m}(s)|^2 w^{S,N}_{v,m}(s), -\mathbf{i}P^N w^{S,N}_{v,m}(s)v(s)\Big\rangle \mathrm{d}s \\ &\le C\int_{t_m}^t \|\nabla w^{S,N}_{v,m}(s)\|\|v(s)\|_{H^2}\mathrm{d}s.\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{H}(w^{S,N}_{v,m}(t)) &\le \mathcal{H}(w^{D,N}_{v,m}(t_{m+1})) + C\int_{t_m}^t (\|\nabla w^{S,N}_{v,m}(s)\|^2 + \|v(s)\|^2_{H^2})\mathrm{d}s \\ &\le \mathcal{H}(w^N_{v,m}) + C\int_{t_m}^t (\mathcal{H}(w^{S,N}_{v,m}(s)) + \|v(s)\|^2_{H^2} + 1)\mathrm{d}s,\end{aligned}$$

which together with the iteration and the fact that

$$\int_0^T \|v^\epsilon(s)\|^2_{H^2}\mathrm{d}s \le C\int_0^T \|v^\epsilon(s)\|^2_{H_0}\mathrm{d}s \le CM$$

yields

$$\mathcal{H}(w^{S,N}_{v,m}(t)) \le \mathcal{H}(w^N_{v,0}) + C\int_0^t \mathcal{H}(w^{S,N}_{v,m}(s))\mathrm{d}s + CT + CM.$$

Applying the Grönwall inequality finishes the proof. □

Proposition 4.4. *Let $M > 0$. Under Assumptions 3 and 4, the set $K_M := \big\{\mathcal{G}^0(\int_0^\cdot v(s)\mathrm{d}s) : v \in \mathcal{S}_M\big\}$ is a compact subset in $\mathcal{C}([0,T];\mathbb{H}_N)$.*

Proof. It suffices to prove that K_M is sequentially compact in $\mathcal{C}([0,T];\mathbb{H}_N)$. Let $\{v^\epsilon, v\} \subset \mathcal{S}_M$ with $v^\epsilon \to v$ in $\mathcal{S}_M$. The property $\|S^N(t)\|_{\mathcal{L}(\mathbb{H};\mathbb{H})} = 1$ and Proposition 4.3 imply

$$\|w^{D,N}_{v^\epsilon,m}(t_{m+1}) - w^{D,N}_{v,m}(t_{m+1})\|^2$$

$$
\begin{aligned}
&= \|w^N_{\nu^\epsilon,m} - w^N_{\nu,m}\|^2 + 2\tau_m \left\langle w^N_{\nu^\epsilon,m} - w^N_{\nu,m}, \mathbf{i}\lambda\left(|w^N_{\nu^\epsilon,m}|^2 w^N_{\nu^\epsilon,m} - |w^N_{\nu,m}|^2 w^N_{\nu,m}\right)\right\rangle \\
&\quad + \tau_m^2 \left\| |w^N_{\nu^\epsilon,m}|^2 w^N_{\nu^\epsilon,m} - |w^N_{\nu,m}|^2 w^N_{\nu,m}\right\|^2 \\
&\le \|w^N_{\nu^\epsilon,m} - w^N_{\nu,m}\|^2 + C\tau_m \|w^N_{\nu^\epsilon,m} - w^N_{\nu,m}\|^2 \Big[\|w^N_{\nu^\epsilon,m}\|^2_{\mathbb{H}^1} + \|w^N_{\nu,m}\|^2_{\mathbb{H}^1} \\
&\quad + \tau_m (\|w^N_{\nu^\epsilon,m}\|^4_{\mathbb{H}^1} + w^N_{\nu,m}\Big\|^4_{\mathbb{H}^1}) \Big] \\
&\le \|w^N_{\nu^\epsilon,m} - w^N_{\nu,m}\|^2 + C\tau_m \|w^N_{\nu^\epsilon,m} - w^N_{\nu,m}\|^2 \quad a.s.
\end{aligned}
$$

Note that for $t \in T_m$,

$$
\frac{\mathrm{d}}{\mathrm{d}t}(w^{S,N}_{\nu^\epsilon,m}(t) - w^{S,N}_{\nu,m}(t)) = -\mathbf{i}P^N w^{S,N}_{\nu^\epsilon,m}(t)\nu^\epsilon(t) + \mathbf{i}P^N w^{S,N}_{\nu,m}(t)\nu(t).
$$

By the chain rule, we have that for $t \in T_m$,

$$
\begin{aligned}
&\|w^{S,N}_{\nu^\epsilon,m}(t) - w^{S,N}_{\nu,m}(t)\|^2 \\
&= \|w^{D,N}_{\nu^\epsilon,m}(t_{m+1}) - w^{D,N}_{\nu,m}(t_{m+1})\|^2 \\
&\quad + 2\int_{t_m}^t \Big\langle w^{S,N}_{\nu^\epsilon,m}(s) - w^{S,N}_{\nu,m}(s), -\mathbf{i}P^N(w^{S,N}_{\nu^\epsilon,m}(s)\nu^\epsilon(s) - w^{S,N}_{\nu,m}(s)\nu(s))\Big\rangle \mathrm{d}s \\
&\le \|w^N_{\nu^\epsilon,m} - w^N_{\nu,m}\|^2 + C\tau_m \|w^N_{\nu^\epsilon,m} - w^N_{\nu,m}\|^2 \\
&\quad + 2\int_{t_m}^t \Big\langle w^{S,N}_{\nu^\epsilon,m}(s) - w^{S,N}_{\nu,m}(s), -\mathbf{i}P^N w^{S,N}_{\nu^\epsilon,m}(s)(\nu^\epsilon(s) - \nu(s))\Big\rangle \mathrm{d}s,
\end{aligned}
$$

which together with the iteration yields that for $t \in T_m$,

$$
\begin{aligned}
&\|w^{S,N}_{\nu^\epsilon,m}(t) - w^{S,N}_{\nu,m}(t)\|^2 \le C \sup_{t\in[0,t_m]} \|w^N_{\nu^\epsilon,\underline{t}} - w^N_{\nu,\underline{t}}\|^2 \\
&+ 2\int_0^t \Big\langle w^{S,N}_{\nu^\epsilon,\underline{s}}(s) - w^{S,N}_{\nu,\underline{s}}(s), -\mathbf{i}P^N w^{S,N}_{\nu^\epsilon,\underline{s}}(s)(\nu^\epsilon(s) - \nu(s))\Big\rangle \mathrm{d}s.
\end{aligned}
$$

Denote $\psi_\epsilon(t) := \int_0^t -\mathbf{i}P^N w^{S,N}_{\nu^\epsilon,\underline{s}}(s)(\nu^\epsilon - \nu)(s)\mathrm{d}s$. Applying the integration by parts formula and combining Proposition 4.3 give that

$$
\begin{aligned}
&\int_0^t \Big\langle w^{S,N}_{\nu^\epsilon,\underline{s}}(s) - w^{S,N}_{\nu,\underline{s}}(s), -\mathbf{i}P^N w^{S,N}_{\nu^\epsilon,\underline{s}}(s)(\nu^\epsilon(s) - \nu(s))\Big\rangle \mathrm{d}s \\
&= \langle w^{S,N}_{\nu^\epsilon,\underline{t}}(t) - w^{S,N}_{\nu,\underline{t}}(t), \psi_\epsilon(t)\rangle \\
&\quad - \int_0^t \Big\langle -\mathbf{i}P^N w^{S,N}_{\nu^\epsilon,\underline{s}}(s)\nu^\epsilon(s) + \mathbf{i}P^N w^{S,N}_{\nu,\underline{s}}(s)\nu(s), \psi_\epsilon(s)\Big\rangle \mathrm{d}s \\
&\le \frac{1}{4}\|w^{S,N}_{\nu^\epsilon,\underline{t}}(t) - w^{S,N}_{\nu,\underline{t}}(t)\|^2 + C\sup_{s\in[0,t]}\|\psi_\epsilon(s)\|^2 + C\sup_{s\in[0,t]}\|\psi_\epsilon(s)\|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|w^{S,N}_{\nu^\epsilon,m}(t) - w^{S,N}_{\nu,m}(t)\|^2 \\
&\le C\sup_{t\in[0,t_m]}\|w^N_{\nu^\epsilon,\underline{t}} - w^N_{\nu,\underline{t}}\|^2 + C\sup_{s\in[0,T]}\|\psi_\epsilon(s)\|^2 + C\sup_{s\in[0,T]}\|\psi_\epsilon(s)\|.
\end{aligned}
$$

We use the induction method to prove the compactness. Suppose that $\sup_{t\in[0,t_m]}\|w^N_{\nu^\epsilon,\underline{t}} - w^N_{\nu,\underline{t}}\|^2 \to 0$ as $\epsilon \to 0$, then we show that $\sup_{t\in[0,t_{m+1}]}\|w^N_{\nu^\epsilon,\underline{t}} - w^N_{\nu,\underline{t}}\|^2 \to 0$ as $\epsilon \to 0$. Then it suffices to show that

$$
\sup_{s\in[0,T]}\|\psi_\epsilon(s)\|^2 + \sup_{s\in[0,T]}\|\psi_\epsilon(s)\| \to 0 \text{ as } \epsilon \to 0. \tag{4.6}
$$

In fact, for $h \in \mathcal{S}_M$, it follows from Proposition 4.3 that

$$
\begin{aligned}
\int_0^T \|\mathbf{i}Q\overline{w^{S,N}_{\nu^\epsilon_{\underline{s}}}}(s)P^N h(s)\|^2_{Q^{\frac{1}{2}}\mathbb{H}}\mathrm{d}s &\le C\int_0^T \|Q^{\frac{1}{2}}\|^2_{\mathcal{L}(H)}\|w^{S,N}_{\nu^\epsilon_{\underline{s}}}(s)\|^2_{\mathbb{H}^1}\|h(s)\|^2_H \mathrm{d}s \\
&\le C\int_0^T \|h(s)\|^2_{H_0}\mathrm{d}s < \infty,
\end{aligned}
$$

which together with $\nu^\epsilon \to \nu$ in $\mathcal{S}_M$ yields

$$
\lim_{\epsilon\to 0}\int_0^T \big\langle -\mathbf{i}P^N w^{S,N}_{\nu^\epsilon,\underline{s}}(s)(\nu^\epsilon(s) - \nu(s)), h(s)\big\rangle \mathrm{d}s = 0.
$$

This means that $-\mathbf{i}P^N w^{S,N}_{\nu^\epsilon,\underline{\cdot}}(\nu^\epsilon - \nu)(\cdot)$ converges to 0 as $\epsilon \to 0$ in $L^2([0,T];\mathbb{H})$ with respect to the weak topology. Moreover, one can show that the set $\{\psi_\epsilon\}_{\epsilon\in(0,1)}$ is a compact subset in $C([0,T];\mathbb{H}_N)$ by the Ascoli theorem (see [28, Theorem 47.1]). In fact, the equicontinuous of $\{\psi_\epsilon\}_{\epsilon\in(0,1)}$ can be deduced by

$$\begin{aligned}&\Big\|\int_{t_1}^{t_2} -\mathbf{i}P^N w^{S,N}_{v^\epsilon,\underline{s}}(s)(v^\epsilon - v)(s)\mathrm{d}s\Big\|\\ &\leq \sup_{s\in[0,T]} \|w^{S,N}_{v^\epsilon,\underline{s}}(s)\| \sqrt{|t_2-t_1|}\Big(\int_{t_1}^{t_2} \|v^\epsilon(s)-v(s)\|^2_{H^1}\mathrm{d}s\Big)^{\frac12}\\ &\leq \sup_{s\in[0,T]} \|w^{S,N}_{v^\epsilon,\underline{s}}(s)\| \sqrt{|t_2-t_1|}\Big(\int_{t_1}^{t_2} \|v^\epsilon(s)\|^2_{H^1}+\|v(s)\|^2_{H^1}\mathrm{d}s\Big)^{\frac12} \leq C\sqrt{|t_2-t_1|M}.\end{aligned}$$

Since

$$\begin{aligned}&\sup_{\epsilon\in(0,1)} \Big\|\int_0^t -\mathbf{i}P^N w^{S,N}_{v^\epsilon,\underline{s}}(s)(v^\epsilon - v)(s)\mathrm{d}s\Big\|_{\mathbb{H}^1}\\ &\leq C \sup_{s\in[0,T]} \|w^{S,N}_{v^\epsilon,\underline{s}}(s)\|_{\mathbb{H}^1} \sup_{\epsilon\in(0,1)} \int_0^T \|(v^\epsilon - v)(s)\|_{H^1}\mathrm{d}s \leq C,\end{aligned}$$

the compact Sobolev embedding $\mathbb{H}^1 \hookrightarrow \mathbb{H}$ implies that $\{\psi_\epsilon(t)\}_{\epsilon\in(0,1)}$ is compact in $\mathbb{H}$ for each fixed $t\geq 0$. Thus $\{\psi_\epsilon\}_{\epsilon\in(0,1)}$ is compact in $\mathcal{C}([0,T];\mathbb{H}_N)$, which combining [12, Proposition 3.3, Section VI] shows that $\psi_\epsilon \to 0$ in $\mathcal{C}([0,T];\mathbb{H}_N)$. Thus (4.6) is proved.

Note that (4.6) also implies that

$$\sup_{t\in[0,t_{m+1}]} \|w^{S,N}_{v^\epsilon,m}(t) - w^{S,N}_{v,m}(t)\|^2 \to 0 \text{ as } \epsilon\to 0 \quad a.s.$$

holds for the case of $m=0$. Combining the induction hypothesis, we finally obtain

$$\sup_{t\in[0,t_{m+1}]} \|w^{S,N}_{v^\epsilon,m}(t) - w^{S,N}_{v,m}(t)\|^2 \to 0 \text{ as } \epsilon\to 0 \quad a.s.,$$

and thus $\sup_{t\in[0,T]} \|w^{S,N}_{v^\epsilon,\underline{t}}(t) - w^{S,N}_{v,\underline{t}}(t)\|^2 \to 0$ as $\epsilon\to 0$ $a.s.$ The proof is finished. □

The following proposition shows that the solution of the stochastic controlled equation (4.1) converges to that of the skeleton equation (4.2) in distribution in $\mathcal{C}([0,T];\mathbb{H}_N)$ under certain conditions.

Proposition 4.5. *Let $M>0$, Assumptions 3 and 4 hold, and let $\{v^\epsilon\}_{\epsilon\in(0,1)}\subset\mathcal{P}_M$ satisfy that $v^\epsilon \xrightarrow[\epsilon\to0]{d} v$ as $\mathcal{S}_M$-valued random variables. Then $u^{S,N,\epsilon}_{v^\epsilon,\underline{\cdot}}(\cdot) \xrightarrow[\epsilon\to0]{d} w^{S,N}_{v,\underline{\cdot}}(\cdot)$ in $\mathcal{C}([0,T];\mathbb{H}_N)$.*

Proof. The proof is split into two steps.

Step 1: Show that $\{u^{S,N,\epsilon}_{v^\epsilon,\underline{\cdot}}(\cdot)\}_{\epsilon\in(0,1)}$ is weakly relatively compact in $\mathcal{C}([0,T];\mathbb{H}_N)$.

Following from [19, Theorem 8.6, Chapter 3], it suffices to prove that

(i) $\{u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t)\}_{\epsilon\in(0,1)}$ is tight for every $t\in[0,T]$;

(ii) There exists a family $\{\gamma_\epsilon(\theta,T):\theta,\epsilon\in(0,1)\}$ of nonnegative random variables satisfying

$$\begin{aligned}&\mathbb{E}\Big[1\wedge\|u^{S,N,\epsilon}_{v^\epsilon,\underline{t+\eta_1}}(t+\eta_1), u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t)\|^2\Big|\mathcal{F}_t\Big]\Big[1\wedge\|u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t), u^{S,N,\epsilon}_{v^\epsilon,\underline{t-\eta_2}}(t-\eta_2)\|^2\Big]\\ &\leq \mathbb{E}[\gamma_\epsilon(\theta,T)|\mathcal{F}_t]\end{aligned}$$

for $0\leq t\leq T, 0\leq\eta_1\leq\theta$, and $0\leq\eta_2\leq\theta\wedge t$; in addition,

$$\lim_{\theta\to0}\sup_{\epsilon\in(0,1)} \mathbb{E}[\gamma_\epsilon(\theta,T)] = 0 \tag{4.7}$$

and

$$\lim_{\theta\to0}\sup_{\epsilon\in(0,1)} \mathbb{E}\big[\|u^{S,N,\epsilon}_{v^\epsilon,\underline{\theta}}(\theta), u^{S,N,\epsilon}_{v^\epsilon,\underline{0}}(0)\|^2\big] = 0. \tag{4.8}$$

For the proof of (i), for arbitrary $\rho>0$ and $t\in[0,T]$, let $\Gamma_{\rho,t} := \big\{x\in\mathbb{H}_N : \|x\|_{\mathbb{H}^1}\leq R(\rho)\big\}$ with $R(\rho)$ being determined later. The compact Sobolev embedding $\mathbb{H}^1\hookrightarrow\mathbb{H}$ implies that $\Gamma_{\rho,t}$ is compact in $\mathbb{H}$. Since the Chebyshev inequality and (4.4) give that

$$\begin{aligned}&\mathbb{P}\Big(u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t)\in\Gamma_{\rho,t}\Big) = \mathbb{P}\Big(\|u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t)\|_{\mathbb{H}^1}\leq R(\rho)\Big)\\ &\geq 1-\frac{\sup_{\epsilon\in(0,1)}\sup_{t\in[0,T]}\mathbb{E}[\|u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t)\|_{\mathbb{H}^1}]}{R(\rho)}\\ &\geq 1-\frac{C}{R(\rho)} =: 1-\rho\end{aligned}$$

with $R(\rho)=\frac{C}{\rho}$, we obtain $\inf_{\epsilon\in(0,1)}\mathbb{P}\Big(u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t)\in\Gamma_{\rho,t}\Big)\geq 1-\rho$. Hence, $\{u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t)\}_{\epsilon\in(0,1)}$ is tight.

For the proof of (ii), noting that $a_1a_2 \le a_1\mathbb{I}_A + a_2\mathbb{I}_{A^c}$ for $0 < a_1, a_2 \le 1$ and a measurable set A, where $\mathbb{I}$ is the indicator function, we first prove the existence of $\{\gamma_\epsilon(\theta,T)\}_{\epsilon\in(0,1)}$ such that

$$\mathbb{E}\Big[\ \|u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t+\eta_1) - u^{S,N,\epsilon}_{v,\underline{t}}(t)\|^2\mathbb{I}_{\{\bar{t}-t\ge\frac{1}{2}\tau_{min}\}} + \|u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t) - u^{S,N,\epsilon}_{v,\underline{t}}(t-\eta_2)\|^2\mathbb{I}_{\{\bar{t}-t<\frac{1}{2}\tau_{min}\}}\Big|\mathcal{F}_t\Big] \le \mathbb{E}[\gamma_\epsilon(\theta,T)|\mathcal{F}_t]$$

for $0\le t\le T,\ 0\le\eta_1\le\theta<(1\wedge\frac{1}{2}\tau_{min})$ and $0\le\eta_2\le t\wedge\theta$, where $\bar{t} := \min\{t_m : t_m \ge t\}$. Recall that τ_{min} is given in Remark 2.2.

Note that for $t\in\{t\in[0,T] : \bar{t}-t\ge\frac{1}{2}\tau_{min}\}$,

$$\begin{aligned}
&\|u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t+\eta_1) - u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t)\|^2\\
&=\Big\|\int_t^{t+\eta_1} -\frac{\epsilon}{2}P^N F_Q u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(s) - \mathbf{i}P^N u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(s)v^\epsilon(s)\mathrm{d}s\\
&\quad-\int_t^{t+\eta_1}\mathbf{i}\sqrt{\epsilon}P^N u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(s)\mathrm{d}W(s)\Big\|^2\\
&\le C\theta\int_t^{t+\theta}\|u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(s)\|^2(\epsilon^2\|F_Q\|^2_{L^\infty(\mathcal{O})} + \|v^\epsilon(s)\|^2_{H^1})\mathrm{d}s\\
&\quad+C\epsilon\sup_{\eta_1\le\theta}\Big\|\int_t^{t+\eta_1}P^N u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(s)\mathrm{d}W(s)\Big\|^2 =: \mathcal{I}_1,
\end{aligned}$$

and for $t\in\{t\in[0,T] : \bar{t}-t<\frac{1}{2}\tau_{min}\}$,

$$\begin{aligned}
\|u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t) - u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(t-\eta_2)\|^2 &\le C\theta\int_{t-\theta}^t\|u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(s)\|^2(\epsilon^2\|F_Q\|^2_{L^\infty(\mathcal{O})} + \|v^\epsilon(s)\|^2_{H^1})\mathrm{d}s\\
&\quad+C\epsilon\sup_{\eta_2\le t\wedge\theta}\Big\|\int_{t-\eta_2}^t P^N u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(s)\mathrm{d}W(s)\Big\|^2 =: \mathcal{I}_2.
\end{aligned}$$

The random variable $\gamma_\epsilon(\theta,T)$ is chosen as $\gamma_\epsilon(\theta,T) = \mathcal{I}_1\mathbb{I}_{\{\bar{t}-t\ge\frac{1}{2}\tau_{min}\}} + \mathcal{I}_2\mathbb{I}_{\{\bar{t}-t<\frac{1}{2}\tau_{min}\}}$ for each $\epsilon\in(0,1)$ and $\theta<(1\wedge\frac{1}{2}\tau_{min})$. And we remark that if $\frac{1}{2}\tau_{min}\le\theta<1$, then we let $\gamma_\epsilon(\theta,T)\equiv 1$. Then it follows from the Burkholder–Davis–Gundy inequality and (4.3) that

$$\begin{aligned}
\sup_{\epsilon\in(0,1)}\mathbb{E}[\gamma_\epsilon(\theta,T)] &\le \sup_{\epsilon\in(0,1)}\Big\{C\theta^2\epsilon^2 + C\theta + C\epsilon\mathbb{E}\Big[\int_t^{t+\theta}\|u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(s)\|^2\|Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2^1}\mathrm{d}s\Big]\\
&\quad+C\epsilon\mathbb{E}\Big[\int_{t-\theta}^t\|u^{S,N,\epsilon}_{v^\epsilon,\underline{t}}(s)\|^2\|Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2^1}\mathrm{d}s\Big]\Big\}\\
&\le \sup_{\epsilon\in(0,1)}\Big[C\theta(\theta\epsilon^2+1)+C\epsilon\theta\Big] \le C\theta \to 0 \text{ as } \theta\to 0,
\end{aligned}$$

which proves (4.7). Finally, it is deduced from

$$\lim_{\theta\to0}\sup_{\epsilon\in(0,1)}\mathbb{E}\Big[\|u^{S,N,\epsilon}_{v^\epsilon,0}(\theta) - u^{S,N,\epsilon}_{v^\epsilon,0}(0)\|^2\Big] \le \lim_{\theta\to0}\sup_{\epsilon\in(0,1)}\Big[C\theta(\theta\epsilon^2+1)+C\epsilon\theta\Big] = 0$$

that (4.8) is satisfied, which finishes the proof that $\{u^{S,N,\epsilon}_{v^\epsilon,\cdot}(\cdot)\}_{\epsilon\in(0,1)}$ is weakly relatively compact in $C([0,T];\mathbb{H}_N)$.

Step 2: Show that $u^{S,N,\epsilon}_{v^\epsilon,\cdot}(\cdot)\xrightarrow[\epsilon\to0]{d} w^{S,N}_{v,\cdot}(\cdot)$ *if* $v^\epsilon\xrightarrow[\epsilon\to0]{d} v$.

Since $\{v^\epsilon\}$ is tight and $\mathcal{S}_M$ is a compact Polish space, $\{v^\epsilon\}$ is weakly relatively compact based on the Prohorov theorem (see e.g. [18, Theorem A.3.15]). Thus $\{(u^{S,N,\epsilon}_{v^\epsilon,\cdot}(\cdot), v^\epsilon)\}_{\epsilon\in(0,1)}$ is weakly relatively compact in $C([0,T];\mathbb{H}_N)\times\mathcal{S}_M$. Hence, there exists a subsequence $\epsilon_n\to0$ (as $n\to\infty$) such that $\{(u^{S,N,\epsilon_n}_{v^{\epsilon_n},\cdot}(\cdot), v^{\epsilon_n})\}_{\epsilon_n\in(0,1)}$ converges in distribution to an element taking values in $C([0,T];\mathbb{H}_N)\times\mathcal{S}_M$. It follows from the Skorohod representation theorem (see e.g. [18, Theorem A.3.9]) that there exists a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ on which a $C([0,T];\mathbb{H}_N)\times\mathcal{S}_M$-valued random variable $(\widetilde{u^{S,N}_{\cdot}}(\cdot),\tilde{v})$ is such that $\{(\widetilde{u^{S,N,\epsilon_n}_{v^{\epsilon_n},\cdot}}(\cdot), v^{\epsilon_n})\}_{\epsilon_n\in(0,1)}$ converges to $(\widetilde{u^{S,N}_{\cdot}}(\cdot),\tilde{v})$ in distribution. Denote by $\mathbb{E}_{\tilde{\mathbb{P}}}$ the expectation with respect to $\tilde{\mathbb{P}}$. We need to show that $\widetilde{u^{S,N}_{\cdot}}(\cdot)$ satisfies that for $t\in T_m$,

$$\begin{cases}
\mathrm{d}\widetilde{u^{D,N}_m}(t) = \mathbf{i}\Delta\widetilde{u^{D,N}_m}(t)\mathrm{d}t + \mathbf{i}\lambda S^N(t-t_m)P^N|\widetilde{u^N_m}|^2\widetilde{u^N_m}\mathrm{d}t,\ \widetilde{u^{D,N}_m}(t_m) = \widetilde{u^{S,N}_{m-1}}(t_m),\\
\mathrm{d}\widetilde{u^{S,N}_m}(t) = -\mathbf{i}P^N\widetilde{u^{S,N}_m}(t)\tilde{v}(t)\mathrm{d}t,\quad \widetilde{u^{S,N}_m}(t_m) = \widetilde{u^{D,N}_m}(t_{m+1}).
\end{cases} \tag{4.9}$$

To this end, for $t\in T_m$, define the map $\Upsilon_t : C([0,T];\mathbb{H}_N)\times\mathcal{S}_M\to[0,1]$ by

$$\begin{aligned}
&\Upsilon_t(f,\phi)\\
&=1\wedge\Big\|f(t) - S^N(\tau_m)\big(f(t_m) + \mathbf{i}\lambda P^N|f(t_m)|^2f(t_m)\tau_m\big) + \int_{t_m}^t\mathbf{i}P^Nf(s)\phi(s)\mathrm{d}s\Big\|.
\end{aligned}$$

We claim that Υ_t is continuous and bounded. In fact, noting that $C([0,T];\mathbb{H}^1)$ is dense in $C([0,T];\mathbb{H})$, we let $f_n\to f$ in $C([0,T];\mathbb{H})$ with $\sup_{n\in\mathbb{N}_+}\|f_n\|_{\mathbb{H}^1}\vee\|f\|_{\mathbb{H}^1}<\infty$ and let $\phi_n\to\phi$ in $\mathcal{S}_M$ with respect to the weak topology. By $\big|1\wedge\|x_1\| - 1\wedge\|x_2\|\big| \le 1\wedge\|x_1-x_2\| \le \|x_1-x_2\|$, we arrive at

$$|Y_t(f_n,\phi_n)-Y_t(f,\phi)| \\ \le C\|f_n-f\|_{C([0,T];\mathbb{H})}\Big(1+\int_{t_m}^t \|\phi_n(s)\|_{H^1}\mathrm{d}s\Big)+\Big\|\int_{t_m}^t \mathbf{i}P^N f(s)(\phi_n(s)-\phi(s))\mathrm{d}s\Big\|. \tag{4.10}$$

Similar to the proof of the convergence of $\{\psi_\epsilon\}_{\epsilon\in(0,1)}$ in Proposition 4.4, the last term in the right hand of (4.10) converges to 0 uniformly with respect to t.

Hence,

$$\lim_{n\to\infty}\mathbb{E}\Big[Y_t(u^{S,N,\epsilon_n}_{v^{\epsilon_n},\cdot}(\cdot),v^{\epsilon_n})\Big]=\mathbb{E}_{\tilde{\mathbb{P}}}[Y_t(\widetilde{u^{S,N}_{\cdot}}(\cdot),\tilde v)];$$

see e.g. [18, Page 375, Appendix A.3]. Since for $t\in T_m$,

$$\mathbb{E}\Big[Y_t(u^{S,N,\epsilon_n}_{v^{\epsilon_n},\cdot}(\cdot),v^{\epsilon_n})\Big] \\ =1\wedge\mathbb{E}\Big[\Big\|\epsilon_n\int_{t_m}^t\frac12 P^N F_Q u^{S,N,\epsilon_n}_{v^{\epsilon_n},m}(s)\mathrm{d}s+\sqrt{\epsilon_n}\int_{t_m}^t \mathbf{i}P^N u^{S,N,\epsilon_n}_{v^{\epsilon_n},m}(s)\mathrm{d}W(s)\Big\|\Big] \\ \le\frac{\epsilon_n}{2}\int_0^T\|F_Q\|_{L^\infty(\mathcal{O})}\|u^{S,N,\epsilon_n}_{v^{\epsilon_n},m}(s)\|\mathrm{d}s+\sqrt{\epsilon_n}\Big(\mathbb{E}\Big\|\int_{t_m}^t P^N u^{S,N,\epsilon_n}_{v^{\epsilon_n},m}(s)\mathrm{d}W(s)\Big\|^2\Big)^{\frac12} \\ \le C\epsilon_n\to 0 \text{ as } n\to\infty,$$

we obtain $\mathbb{E}_{\tilde{\mathbb{P}}}\Big[Y_t(\widetilde{u^{S,N}_{\cdot}}(\cdot),\tilde v)\Big]=0$. It follows from the definition of Y_t that

$$\widetilde{u^{S,N}_{\cdot}}(\cdot)=\mathcal{G}^0\Big(\int_0^{\cdot}\tilde v(s)\mathrm{d}s\Big)\quad \tilde{\mathbb{P}}\text{-a.s.}$$

Moreover, due to $(u^{S,N,\epsilon_n}_{v^{\epsilon_n},\cdot}(\cdot),v^{\epsilon_n})\xrightarrow[\epsilon_n\to0]{d}(\widetilde{u^{S,N}_{\cdot}}(\cdot),\tilde v)$, we have $v^{\epsilon_n}\xrightarrow[\epsilon_n\to0]{d}\tilde v$, which together with $v^\epsilon\xrightarrow[\epsilon\to0]{d}v$ yields that $v\overset{d}{=}\tilde v$ and consequently $w^{S,N}_{v,\cdot}(\cdot)\overset{d}{=}\widetilde{u^{S,N}_{\cdot}}(\cdot)$. Therefore,

$$(u^{S,N,\epsilon_n}_{v^{\epsilon_n},\cdot}(\cdot),v^{\epsilon_n})\xrightarrow[\epsilon_n\to0]{d}(w^{S,N}_{v,\cdot}(\cdot),v).$$

Repeating the above procedure, we derive that for any subsequence $\vartheta_n\to0$, there exists some subsubsequence $\vartheta_{n_k}\to0$, such that $(u^{S,N,\vartheta_{n_k}}_{v^{\vartheta_{n_k}},\cdot}(\cdot),v^{\vartheta_{n_k}})\xrightarrow[\vartheta_{n_k}\to0]{d}(w^{S,N}_{v,\cdot}(\cdot),v)$, which finally implies that $(u^{S,N,\epsilon}_{v^\epsilon,\cdot}(\cdot),v^\epsilon)\xrightarrow[\epsilon\to0]{d}(w^{S,N}_{v,\cdot}(\cdot),v)$; see e.g. [3, Theorem 2.6].

Combining *Steps* 1-2, we finish the proof. □

Proof of Theorem 4.1. Following [5, Theorem 4.4] or [6, Theorem 5], it suffices to prove that

(i) for any fixed $M<\infty$,

$$K_M=\Big\{\mathcal{G}^0\Big(\int_0^{\cdot}v(s)\mathrm{d}s\Big),v\in\mathcal{S}_M\Big\}$$

is a compact subset of $C([0,T];\mathbb{H}_N)$;

(ii) for $M<\infty$ and $\{v^\epsilon\}_{\epsilon\in(0,1)}\subset\mathcal{P}_M$ such that $v^\epsilon\xrightarrow[\epsilon\to0]{d}v$ as $\mathcal{S}_M$-valued random variables,

$$\mathcal{G}^\epsilon\Big(\sqrt{\epsilon}W+\int_0^{\cdot}v^\epsilon(s)\mathrm{d}s\Big)\xrightarrow[\epsilon\to0]{d}\mathcal{G}^0\Big(\int_0^{\cdot}v(s)\mathrm{d}s\Big),$$

which are given in Propositions 4.4 and 4.5, respectively. □

Recall that the mass conservation law $\|u(t)\|^2=\|u_0\|^2\ \forall t\in[0,T]$ holds for both the stochastic NLS equation (1.1) and the split equation (2.2). Even though the mass cannot be preserved exactly by the adaptive fully discrete scheme (2.4), the error of the masses between solutions of (1.1) and (2.4) can be given by means of the LDP for the numerical solution.

Corollary 4.6. *Under assumptions in Theorem 4.1, for any $\rho>0$, there is some $\epsilon_0>0$ such that for $\epsilon<\epsilon_0$,*

$$\exp\Big\{-\frac1\epsilon\inf_{x\in G^1_\rho}I(x)\Big\}+\exp\Big\{-\frac1\epsilon\inf_{x\in G^2_\rho}I(x)\Big\} \\ \le\mathbb{P}\Big(\Big|\sup_{t\in[0,T]}\|u^{S,N,\epsilon}_{t,\underline{t}}\|^2-\|u_0\|^2\Big|\ge\rho\Big) \\ \le\exp\Big\{-\frac1\epsilon\inf_{x\in F^1_\rho}I(x)\Big\}+\exp\Big\{-\frac1\epsilon\inf_{x\in F^2_\rho}I(x)\Big\},$$

where $G^1_\rho=\big\{x\in C([0,T];\mathbb{H}_N):\sup_{t\in[0,T]}\|x(t)\|^2>\|u_0\|^2+\rho+\hat\varepsilon\big\}$, $G^2_\rho=\big\{x\in C([0,T];\mathbb{H}_N):\sup_{t\in[0,T]}\|x(t)\|^2<\|u_0\|^2-\rho-\hat\varepsilon\big\}$ with $\hat\varepsilon>0$ being a small number, $F^1_\rho=\big\{x\in C([0,T];\mathbb{H}_N):\sup_{t\in[0,T]}\|x\|^2\ge\|u_0\|^2+\rho\big\}$, $F^2_\rho=\big\{x\in C([0,T];\mathbb{H}_N):\sup_{t\in[0,T]}\|x(t)\|^2\le\|u_0\|^2-\rho\big\}$, and I is given in Theorem 4.1.

Proof. It is straightforward that

$$\begin{aligned}&\mathbb{P}\Big(\Big|\sup_{t\in[0,T]}\|u^{S,N,\epsilon}_{t,\underline{t}}\|^2-\|u_0\|^2\Big|\geq\rho\Big)\\&=\mathbb{P}\Big(\sup_{t\in[0,T]}\|u^{S,N,\epsilon}_{t,\underline{t}}\|^2\geq\|u_0\|^2+\rho\Big)+\mathbb{P}\Big(\sup_{t\in[0,T]}\|u^{S,N,\epsilon}_{t,\underline{t}}\|^2\leq\|u_0\|^2-\rho\Big)\\&=:\mathcal{II}_1+\mathcal{II}_2.\end{aligned}$$

Note that $\{\omega : u^{S,N,\epsilon}_{\cdot,\underline{\cdot}}(\omega)\in G^1_\rho\}\subset\big\{\omega : \sup_{t\in[0,T]}\|u^{S,N,\epsilon}_{t,\underline{t}}(\omega)\|^2\geq\|u_0\|^2+\rho\big\}$ and $\{\omega : u^{S,N,\epsilon}_{\cdot,\underline{\cdot}}(\omega)\in G^2_\rho\}\subset\big\{\omega : \sup_{t\in[0,T]}\|u^{S,N,\epsilon}_{t,\underline{t}}(\omega)\|^2\leq\|u_0\|^2-\rho\big\}$. Terms $\mathcal{II}_j$ can be estimated by the LDP upper bound (resp. the LDP lower bound) with the closed subset F^j_ρ (resp. the open subset G^j_ρ) for $j=1,2$. □

Declaration of competing interest

None.

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