# On the accuracy of saddle point solvers

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# Saddle point problems

We consider a saddle point problem with the symmetric $2 \times 2$ block form

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

- $A$ is a square $n \times n$ nonsingular (symmetric positive definite) matrix,
- $B$ is a rectangular $n \times m$ matrix of (full column) rank $m$.

Applications: mixed finite element approximations, weighted least squares, constrained optimization etc. [Benzi,Golub, Liesen, 2005].
Numerous schemes: block diagonal preconditioners, block triangular preconditioners, constraint preconditioning, Hermitian/skew-Hermitian preconditioning and other splittings, combination preconditioning
References: [Bramble and Pasciak, 1988], [Silvester and Wathen, 1993, 1994], [Elman, Silvester and Wathen, 2002, 2005], [Kay, Loghin and Wathen, 2002], [Keller, Gould and Wathen 2000], [Perugia, Simoncini, Arioli, 1999], [Gould, Hribar and Nocedal, 2001], [Stoll, Wathen, 2008], ...

# Symmetric indefinite system, symmetric positive definite preconditioner

$$\mathcal{A} = \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \approx \mathcal{P} = \mathcal{R}^T \mathcal{R}$$

$\mathcal{A}$ symmetric indefinite, $\mathcal{P}$ positive definite ($\mathcal{R}$ nonsingular)

$$\left(\mathcal{R}^{-T} \mathcal{A} \mathcal{R}^{-1}\right) \mathcal{R} \begin{pmatrix} x \\ y \end{pmatrix} = \mathcal{R}^{-T} \begin{pmatrix} f \\ 0 \end{pmatrix}$$

$\mathcal{R}^{-T} \mathcal{A} \mathcal{R}^{-1}$ is symmetric indefinite!

# Iterative solution of preconditioned (symmetric indefinite) system

- Preconditioned MINRES is the MINRES on $\mathcal{R}^{-T}\mathcal{A}\mathcal{R}^{-1}$, minimizes the $\mathcal{P}^{-1} = \mathcal{R}^{-1}\mathcal{R}^{-T}$-norm of the residual on $K_n(\mathcal{P}^{-1}\mathcal{A}, \mathcal{P}^{-1}r_0)$
  $\equiv \mathcal{H}$-MINRES on $\mathcal{P}^{-1}\mathcal{A}$ with $\mathcal{H} = \mathcal{P}^{-1}$
- CG applied to indefinite system with $\mathcal{R}^{-T}\mathcal{A}\mathcal{R}^{-1}$:
  CG iterate exists at least at every second step (tridiagonal form $T_n$ is nonsingular at least at every second step)

  [Paige, Saunders, 1975]
- peak/plateau behavior:
  CG converges fast $\rightarrow$ MINRES is not much better than CG
  CG norm increases (peak) $\rightarrow$ MINRES stagnates (plateau)

  [Greenbaum, Cullum, 1996]

# Symmetric indefinite system, indefinite or nonsymmetric preconditioner

$\mathcal{P}$ symmetric indefinite or nonsymmetric

$$\mathcal{P}^{-1}\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} = \mathcal{P}^{-1}\begin{pmatrix} f \\ 0 \end{pmatrix}$$

$$\left(\mathcal{A}\mathcal{P}^{-1}\right)\mathcal{P}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

$\mathcal{P}^{-1}\mathcal{A}$ and $\mathcal{A}\mathcal{P}^{-1}$ are nonsymmetric!

# Iterative solution of preconditioned nonsymmetric system, positive definite inner product

- The existence of a short-term recurrence solution methods to solve the system with $\mathcal{P}^{-1}\mathcal{A}$ or $\mathcal{A}\mathcal{P}^{-1}$ for arbitrary right-hand side vector

  [Faber, Manteuffel 1984, Liesen, Strakoš, 2006]

- Matrices $\mathcal{P}^{-1}\mathcal{A}$ or $\mathcal{A}\mathcal{P}^{-1}$ can be symmetric (self-adjoint) in a given inner product induced by the **symmetric positive definite** $\mathcal{H}$. Then three term-recurrence method can be applied
  $$\mathcal{H}(\mathcal{P}^{-1}\mathcal{A}) = (\mathcal{P}^{-1}\mathcal{A})^T\mathcal{H} \iff (\mathcal{P}^{-T}\mathcal{H})^T\mathcal{A} = \mathcal{A}(\mathcal{P}^{-T}\mathcal{H})$$
  $$\mathcal{H}(\mathcal{A}\mathcal{P}^{-1}) = (\mathcal{A}\mathcal{P}^{-1})^T\mathcal{H} \iff \mathcal{H}\mathcal{A}\mathcal{P}^{-1} = \mathcal{P}^{-T}\mathcal{A}\mathcal{H}$$

- $\mathcal{H}(\mathcal{P}^{-1}\mathcal{A})$ **symmetric indefinite:** MINRES applied to $\mathcal{H}(\mathcal{P}^{-1}\mathcal{A})$ and preconditioned with $\mathcal{H}$
  $\equiv$ $\mathcal{H}$-MINRES on $\mathcal{P}^{-1}\mathcal{A}$

- $\mathcal{H}(\mathcal{P}^{-1}\mathcal{A})$ **positive definite:** CG applied to $\mathcal{H}(\mathcal{P}^{-1}\mathcal{A})$ and preconditioned with $\mathcal{H}$; works on $K_n(\mathcal{P}^{-1}\mathcal{A}, \mathcal{P}^{-1}r_0)$ and can be seen as the CG scheme applied to $\mathcal{P}^{-1}\mathcal{A}$ with a nonstandard inner product $\mathcal{H}$
  $\equiv$ $\mathcal{H}$-CG on $\mathcal{P}^{-1}\mathcal{A}$

# Iterative solution of preconditioned nonsymmetric system, symmetric bilinear form

- if there exists a **symmetric indefinite** $\mathcal{H}$ such that
  $\mathcal{H}(\mathcal{P}^{-1}\mathcal{A}) = (\mathcal{P}^{-1}\mathcal{A})^T\mathcal{H} = [\mathcal{H}(\mathcal{P}^{-1}\mathcal{A})]^T$
  $[(\mathcal{A}\mathcal{P}^{-1})^T\mathcal{H}]^T = \mathcal{H}(\mathcal{A}\mathcal{P}^{-1}) = (\mathcal{A}\mathcal{P}^{-1})^T\mathcal{H}$
  is **symmetric indefinite**

  MINRES method applied to $\mathcal{H}(\mathcal{P}^{-1}\mathcal{A})$ or $\mathcal{H}(\mathcal{A}\mathcal{P}^{-1})$

- **symmetric indefinite preconditioner** $\mathcal{H} = \mathcal{P}^{-1} = (\mathcal{P}^{-1})^T$ so that
  $(\mathcal{P}^{-1})^T(\mathcal{P}^{-1})\mathcal{A} = \mathcal{A}(\mathcal{P}^{-1})^T(\mathcal{P}^{-1})$
  $(\mathcal{P}^{-1})^T\mathcal{A}\mathcal{P}^{-1} = \mathcal{P}^{-1}\mathcal{A}\mathcal{P}^{-1}$
  right vs left preconditioning for symmetric $\mathcal{P}$
  $\mathcal{P}^{-1}K_n(\mathcal{A}\mathcal{P}^{-1}, r_0) = K_n(\mathcal{P}^{-1}\mathcal{A}, \mathcal{P}^{-1}r_0)$
  $(\mathcal{A}\mathcal{P}^{-1})^T = (\mathcal{P}^{-1})^T\mathcal{A} = \mathcal{P}^{-1}\mathcal{A}$

# Iterative solution of preconditioned nonsymmetric system, symmetric bilinear form

- $\mathcal{H}$-symmetric variant of the nonsymmetric Lanczos process:
  $\mathcal{A}\mathcal{P}^{-1}V_n = V_{n+1}T_{n+1,n}$, $(\mathcal{A}\mathcal{P}^{-1})^T W_n = W_{n+1}\tilde{T}_{n+1,n}$
  $W_n^T V_n = I \Longrightarrow W_n = \mathcal{H}V_n$
  
  [Freund, Nachtigal, 1995]

- $\mathcal{H}$-symmetric variant of Bi-CG
  $\mathcal{H}$-symmetric variant of QMR $\equiv$ ITFQMR
  
  [Freund, Nachtigal, 1995]

- QMR-from-BiCG:
  $\mathcal{H}$-symmetric Bi-CG + QMR-smoothing
  $\Longrightarrow$ $\mathcal{H}$-symmetric QMR
  
  [Freund, Nachtigal, 1995, Walker, Zhou 1994]

- peak/plateau behavior:
  QMR does not improve the convergence of Bi-CG (Bi-CG converges fast $\rightarrow$ QMR is not much better, Bi-CG norm increases $\rightarrow$ quasi-residual of QMR stagnates)
  
  [Greenbaum, Cullum, 1996]

# Simplified Bi-CG algorithm is a preconditioned CG algorithm

$\mathcal{H} = \mathcal{P}^{-1}$-symmetric variant of two-term Bi-CG on $\mathcal{A}\mathcal{P}^{-1}$ is the Hestenes-Stiefel CG algorithm on $\mathcal{A}$ preconditioned with $\mathcal{P}$

| $\mathcal{P}^{-1}$-symmetric Bi-CG($\mathcal{A}\mathcal{P}^{-1}$) | PCG($\mathcal{A}$) with $\mathcal{P}^{-1}$ |
|---|---|
| $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $r_0 = b - \mathcal{A}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ | |
| $\mathcal{P}^{-1}p_0 = \mathcal{P}^{-1}r_0$ , $\tilde{p}_0 = \tilde{r}_0 = \mathcal{P}^{-1}p_0$ | $z_0 = \mathcal{P}^{-1}r_0$ |
| $k = 0, 1, \ldots$ | |
| $\alpha_k = (r_k, \tilde{r}_k)/(\mathcal{A}\mathcal{P}^{-1}p_k, \tilde{p}_k)$ | $\alpha_k = (r_k, z_k)/(\mathcal{A}\mathcal{P}^{-1}p_k, \mathcal{P}^{-1}p_k)$ |
| $\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \alpha_k \mathcal{P}^{-1}p_k$ | |
| $r_{k+1} = r_k - \alpha_k \mathcal{A}\mathcal{P}^{-1}p_k$ | |
| $\tilde{r}_{k+1} = \mathcal{P}^{-1}r_{k+1}$ | $z_{k+1} = \mathcal{P}^{-1}r_{k+1}$ |
| $\beta_k = (r_{k+1}, \tilde{r}_{k+1})/(r_k, \tilde{r}_k)$ | $\beta_k = (r_{k+1}, z_{k+1})/(r_k, z_k)$ |
| $\mathcal{P}^{-1}p_{k+1} = \mathcal{P}^{-1}r_{k+1} + \beta_k \mathcal{P}^{-1}p_k$ | $\mathcal{P}^{-1}p_{k+1} = z_{k+1} + \beta_k \mathcal{P}^{-1}p_k$ |
| $\tilde{p}_{k+1} = \mathcal{P}^{-1}p_{k+1}$ | |

# Saddle point problem and indefinite constraint preconditioner

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

$$\mathcal{P} = \begin{pmatrix} I & B \\ B^T & 0 \end{pmatrix}, \quad \mathcal{H} = \mathcal{P}^{-1}$$

PCG applied to indefinite system with indefinite preconditioner; will not work for arbitrary right-hand side, particular right-hand side or initial guess: $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, r_0 = \begin{pmatrix} s_0 \\ 0 \end{pmatrix}$, here $g = 0$ and $x_0 = y_0 = 0$

[Lukšan, Vlček, 1998], [Gould, Keller, Wathen 2000]
[Perugia, Simoncini, Arioli, 1999], [R, Simoncini, 2002]

# Saddle point problem and indefinite constraint preconditioner - preconditioned system

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} I & B \\ B^T & 0 \end{pmatrix}$$

$$\mathcal{A}\mathcal{P}^{-1} = \begin{pmatrix} A(I - \Pi) + \Pi & (A - I)B(B^T B)^{-1} \\ 0 & I \end{pmatrix}$$

$\Pi = B(B^T B)^{-1} B^T$ - orth. projector onto $span(B)$

# Indefinite constraint preconditioner: spectral properties of preconditioned system

$\mathcal{A}\mathcal{P}^{-1}$ **nonsymmetric** and **non-diagonalizable**!
but it has a 'nice' spectrum:

$$
\begin{aligned}
\sigma(\mathcal{A}\mathcal{P}^{-1}) &\subset \{1\} \cup \sigma(A(I-\Pi)+\Pi) \\
&\subset \{1\} \cup \sigma((I-\Pi)A(I-\Pi)) - \{0\}
\end{aligned}
$$

and only 2 by 2 Jordan blocks!

[Lukšan, Vlček 1998], [Gould, Wathen, Keller, 1999], [Perugia, Simoncini 1999]

# Basic properties of any Krylov method with the constraint preconditioner

$$e_{k+1} = \begin{pmatrix} x - x_{k+1} \\ y - y_{k+1} \end{pmatrix}$$

$$r_{k+1} = \begin{pmatrix} f \\ 0 \end{pmatrix} - \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix}$$

$$r_0 = \begin{pmatrix} s_0 \\ 0 \end{pmatrix} \Rightarrow r_{k+1} = \begin{pmatrix} s_{k+1} \\ 0 \end{pmatrix}$$
$$\Rightarrow B^T(x - x_{k+1}) = 0$$
$$\Rightarrow x_{k+1} \in Null(B^T)!$$

# The energy-norm of the error in the preconditioned CG method

$$r_{k+1}^T \mathcal{P}^{-1} r_j = 0, \; j = 0, \dots, k$$

$x_{k+1}$ is an iterate from CG applied to

$$(I - \Pi)A(I - \Pi)x = (I - \Pi)f!$$

satisfying

$$\|x - x_{k+1}\|_A = \min_{u \in x_0 + span\{(I-\Pi)s_j\}} \|x - u\|_A$$

[Lukšan, Vlček 1998], [Gould, Wathen, Keller, 1999]

# The residual norm in the preconditioned CG method

$$\|x_{k+1} - x\| \to 0$$

but in general

$$y_{k+1} \not\to y$$

which is reflected in

$$\|r_{k+1}\| = \left\| \begin{pmatrix} s_{k+1} \\ 0 \end{pmatrix} \right\| \not\to 0!$$

but under appropriate scaling yes!

# The residual norm in the preconditioned CG method

$$x_{k+1} \rightarrow x$$

$$x - x_{k+1} = \phi_{k+1}((I - \Pi)A(I - \Pi))(x - x_0)$$

$$r_{k+1} = \phi_{k+1}(A(I - \Pi) + \Pi)s_0$$

$$\sigma((I - \Pi)A(I - \Pi)) \subset \sigma(A(I - \Pi) + \Pi)$$

$$\{1\} \in \sigma((I - \Pi)\alpha A(I - \Pi)) - \{0\}$$
$$\Rightarrow \|r_{k+1}\| = \left\| \begin{pmatrix} s_{k+1} \\ 0 \end{pmatrix} \right\| \rightarrow 0!$$

# How to avoid the misconvergence of the scheme

- Scaling by a constant $\alpha > 0$ such that

$$\{1\} \in conv(\sigma((I-\Pi)\alpha A(I-\Pi)) - \{0\})$$

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad \begin{pmatrix} \alpha A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ \alpha y \end{pmatrix} = \begin{pmatrix} \alpha f \\ 0 \end{pmatrix}$$

$$v: \quad \|(I-\Pi)v\| \neq 0, \quad \alpha = \frac{1}{((I-\Pi)v, A(I-\Pi)v)}!$$

- Scaling by a diagonal $A \rightarrow (diag(A))^{-1/2} A (diag(A))^{-1/2}$ often gives what we want!
- Different direction vector so that $\|r_{k+1}\| = \|s_{k+1}\|$ is locally minimized!

$$y_{k+1} = y_k + (B^T B)^{-1} B^T s_k$$

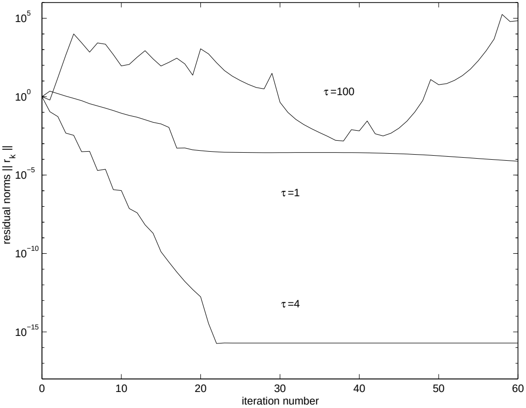
[Braess, Deuflhard,Lipikov 1999], [Hribar, Gould, Nocedal, 1999]
[Jiránek, R, 2008]

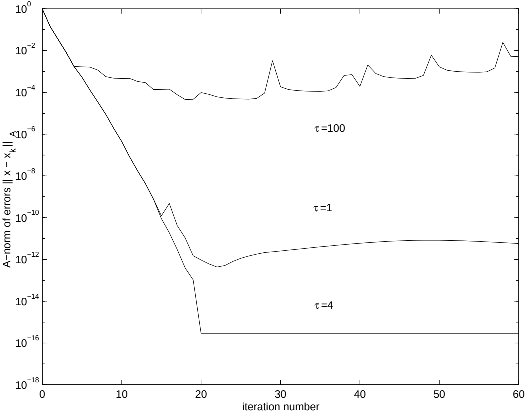
# Numerical example

$$A = tridiag(1,4,1) \in \mathsf{R}^{25,25},\ B = rand(25,5) \in \mathsf{R}^{25,5}$$
$$f = rand(25,1) \in \mathsf{R}^{25}$$

$$\sigma(A) \subset [2.0146, 5.9854]$$

| $\alpha = 1/\tau$ | $\sigma(\begin{pmatrix} \alpha A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} I & B \\ B^T & 0 \end{pmatrix}^{-1})$ |
|---|---|
| $1/100$ | $[0.0207, 0.0586] \cup \{1\}$ |
| $1/10$ | $[0.2067, 0.5856] \cup \{1\}$ |
| $\mathbf{1/4}$ | $[\mathbf{0.5170}, \mathbf{1.4641}]$ |
| $1$ | $\{1\} \cup [2.0678, 5.8563]$ |
| $4$ | $\{1\} \cup [8.2712, 23.4252]$ |


10^5
10^0
10^−5
10^−10
10^−15
residual norms || r_k ||
τ =100
τ =1
τ =4
0
10
20
30
40
50
60
iteration number



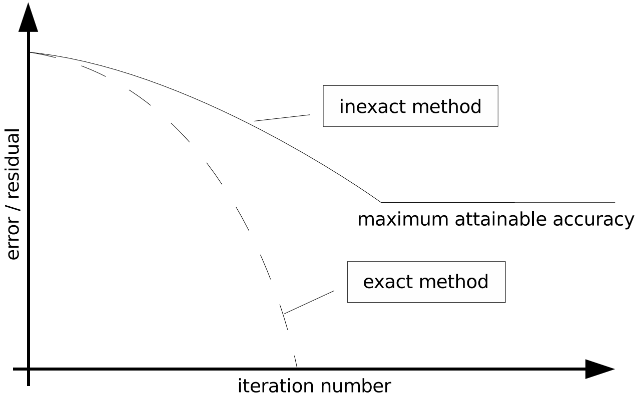
A−norm of errors || x − x_k ||_A
iteration number
τ =100
τ =1
τ =4
10^0
10^−2
10^−4
10^−6
10^−8
10^−10
10^−12
10^−14
10^−16
10^−18
0
10
20
30
40
50
60

# Inexact saddle point solvers

1. **exact method**: exact constraint preconditioning, exact arithmetic : outer iteration for solving the preconditioned system;
2. **inexact method** with approximate or incomplete factorization scheme to solve inner problems with $(B^T B)^{-1}$: structure-based or with appropriate dropping criterion; inner iteration method
3. **the rounding errors**: finite precision arithmetic.

References: [Gould, Hribar and Nocedal, 2001], [R, Simoncini, 2002] with the use of [Greenbaum 1994,1997], [Sleijpen, et al. 1994]

# Delay of convergence and limit on the final accuracy



# Preconditioned CG in finite precision arithmetic

$$\begin{pmatrix} \bar{x}_{k+1} \\ \bar{y}_{k+1} \end{pmatrix}, \quad \bar{r}_{k+1} = \begin{pmatrix} \bar{s}_{k+1}^{(1)} \\ \bar{s}_{k+1}^{(2)} \end{pmatrix}$$

$$\|x - \bar{x}_{k+1}\|_A \leq \gamma_1 \|\Pi(x - \bar{x}_{k+1})\| + \gamma_2 \|(I - \Pi)A(I - \Pi)(x - \bar{x}_{k+1})\|$$

**Exact arithmetic:**

$$\|\Pi(x - x_{k+1})\| = 0$$

$$\|(I - \Pi)A(I - \Pi)(x - x_{k+1})\| \to 0$$

# Forward error of computed approximate solution: departure from the null-space of $B^T$ + projection of the residual onto it

$$\|x - \bar{x}_{k+1}\|_A \leq \gamma_3 \|B^T(x - \bar{x}_{k+1})\| + \gamma_2 \|(I - \Pi)(f - A\bar{x}_{k+1} - B\bar{y}_{k+1})\|$$

**can be monitored by easily computable quantities:**

$$B^T(x - \bar{x}_{k+1}) \sim \bar{s}_{k+1}^{(2)}$$

$$(I - \Pi)(f - A\bar{x}_{k+1} - B\bar{y}_{k+1}) \sim (I - \Pi)\bar{s}_{k+1}^{(1)}$$

# Maximum attainable accuracy of the scheme

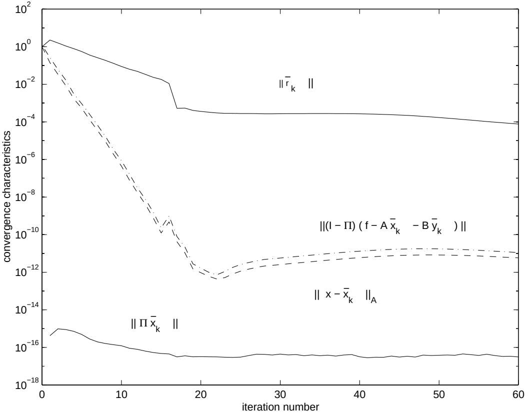
$$\|(f - A\bar{x}_{k+1} - B\bar{y}_{k+1}) - \bar{s}^{(1)}_{k+1}\|,$$
$$\|B^T(x - \bar{x}_{k+1}) - \bar{s}^{(2)}_{k+1}\| \leq$$
$$\leq \| \begin{pmatrix} f \\ 0 \end{pmatrix} - \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_{k+1} \\ \bar{y}_{k+1} \end{pmatrix} - \begin{pmatrix} \bar{s}^{(1)}_{k+1} \\ \bar{s}^{(2)}_{k+1} \end{pmatrix} \|$$

$$\leq c_1 \varepsilon \kappa(\mathcal{A}) \max_{j=0,\dots,k+1} \|\bar{r}_j\|$$

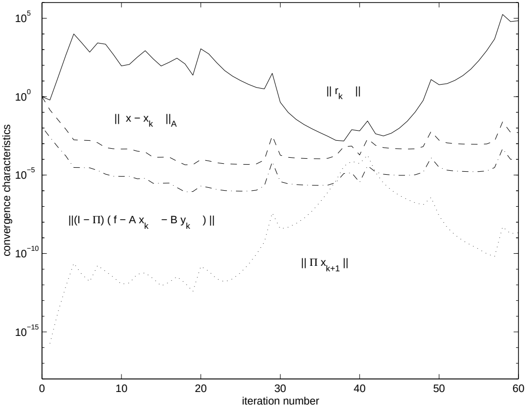
[Greenbaum 1994,1997], [Sleijpen, et al. 1994]

good scaling: $\|\bar{r}_j\| \to 0$ nearly monotonically

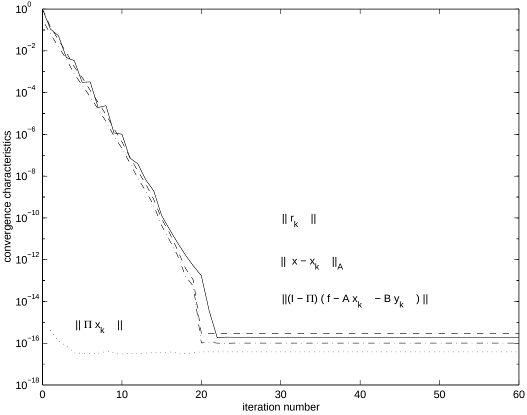
$\|\bar{r}_0\| \sim \max_{j=0,\dots,k+1} \|\bar{r}_j\|$



$10^{2}$
$10^{0}$
$10^{-2}$
$10^{-4}$
$10^{-6}$
$10^{-8}$
$10^{-10}$
$10^{-12}$
$10^{-14}$
$10^{-16}$
$10^{-18}$
convergence characteristics
0
10
20
30
40
50
60
iteration number
$\|\bar{r}_k\|$
$\|(I-\Pi)(f - A\bar{x}_k - B\bar{y}_k)\|$
$\|x-\bar{x}_k\|_A$
$\|\Pi\bar{x}_k\|$



convergence characteristics
iteration number
$10^{5}$
$10^{0}$
$10^{-5}$
$10^{-10}$
$10^{-15}$
0
10
20
30
40
50
60
$\| r_k \|$
$\| x - x_k \|_A$
$\|(I - \Pi)(f - A x_k - B y_k)\|$
$\| \Pi x_{k+1} \|$



convergence characteristics
iteration number
$10^{0}$
$10^{-2}$
$10^{-4}$
$10^{-6}$
$10^{-8}$
$10^{-10}$
$10^{-12}$
$10^{-14}$
$10^{-16}$
$10^{-18}$
0
10
20
30
40
50
60
$\| r_k \|$
$\| x - x_k \|_A$
$\|(I - \Pi)(f - A x_k - B y_k)\|$
$\| \Pi x_k \|$

# Conclusions

- Short-term recurrence methods are applicable for saddle point problems with indefinite preconditioning at a cost comparable to that of symmetric solvers. There is a tight connection between the simplified Bi-CG algorithm and the classical CG.
- The convergence of CG applied to saddle point problem with indefinite preconditioner for all right-hand side vectors is not guaranteed. For a particular set of right-hand sides the convergence can be achieved by the appropriate scaling of the saddle point problem or by a different back-substitution formula for dual unknowns.
- Since the numerical behavior of CG in finite precision arithmetic depends heavily on the size of computed residuals, a good scaling of the problems leads to approximate solutions satisfying both two block equations to the working accuracy.

## Thank you for your attention.

http://www.cs.cas.cz/∼miro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# Null-space projection method

- compute $x \in N(B^T)$ as a solution of the projected system

$$(I - \Pi)A(I - \Pi)x = (I - \Pi)f,$$

- compute $y$ as a solution of the least squares problem

$$By \approx f - Ax,$$

$\Pi = B(B^TB)^{-1}B^T$ is the orthogonal projector onto $R(B)$.

Results for schemes, where the least squares with $B$ are solved inexactly. Every computed approximate solution $\bar{v}$ of a least squares problem $Bv \approx c$ is interpreted as an exact solution of a perturbed least squares

$$(B + \Delta B)\bar{v} \approx c + \Delta c, \ \|\Delta B\| \leq \tau\|B\|, \ \|\Delta c\| \leq \tau\|c\|, \ \tau\kappa(B) \ll 1.$$

# Null-space projection method

choose $x_0$, solve $By_0 \approx f - Ax_0$

compute $\alpha_k$ and $p_k^{(x)} \in N(B^T)$

$x_{k+1} = x_k + \alpha_k p_k^{(x)}$

solve $Bp_k^{(y)} \approx r_k^{(x)} - \alpha_k Ap_k^{(x)}$

back-substitution:

**A:** $y_{k+1} = y_k + p_k^{(y)}$,

**B:** solve $By_{k+1} \approx f - Ax_{k+1}$,

**C:** solve $Bv_k \approx f - Ax_{k+1} - By_k$,

$y_{k+1} = y_k + v_k$.

inner iteration

$r_{k+1}^{(x)} = r_k^{(x)} - \alpha_k Ap_k^{(x)} - Bp_k^{(y)}$

outer iteration

# Accuracy in the saddle point system

$$\|f - Ax_k - By_k - r_k^{(x)}\| \leq \frac{O(\alpha_3)\kappa(B)}{1 - \tau\kappa(B)}(\|f\| + \|A\|X_k),$$

$$\|-B^T x_k\| \leq \frac{O(\tau)\kappa(B)}{1 - \tau\kappa(B)}\|B\|X_k,$$

$$X_k \equiv \max\{\|x_i\| \,|\, i = 0, 1, \ldots, k\}.$$

| Back-substitution scheme | | $\alpha_3$ |
|---|---|---|
| **A**: | Generic update<br>$y_{k+1} = y_k + p_k^{(y)}$ | $u$ |
| **B**: | Direct substitution<br>$y_{k+1} = B^\dagger(f - Ax_{k+1})$ | $\tau$ |
| **C**: | Corrected dir. subst.<br>$y_{k+1} = y_k + B^\dagger(f - Ax_{k+1} - By_k)$ | $u$ |

additional least square with B

# Maximum attainable accuracy of inexact null-space projection schemes

The limiting (maximum attainable) accuracy is measured by the ultimate (asymptotic) values of:

1. **the true projected residual**: $(I - \Pi)f - (I - \Pi)A(I - \Pi)x_k$;
2. **the residuals in the saddle point system**: $f - Ax_k - By_k$ and $-B^T x_k$;
3. **the forward errors**: $x - x_k$ and $y - y_k$.

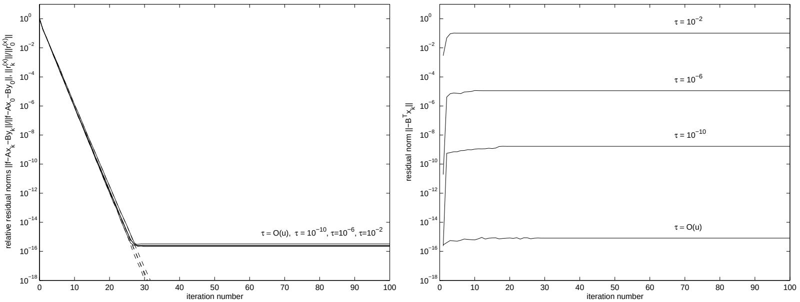
**Numerical experiments: a small model example**

$$A = \mathrm{tridiag}(1, 4, 1) \in \mathbb{R}^{100\times 100},\ B = \mathrm{rand}(100, 20),\ f = \mathrm{rand}(100, 1),$$

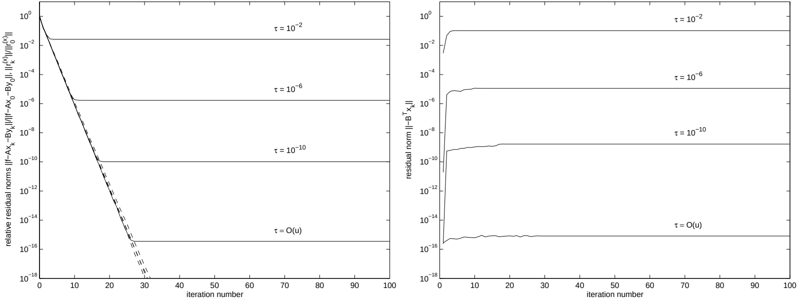
$$\kappa(A) = \|A\| \cdot \|A^{-1}\| = 7.1695 \cdot 0.4603 \approx 3.3001,$$

$$\kappa(B) = \|B\| \cdot \|B^{\dagger}\| = 5.9990 \cdot 0.4998 \approx 2.9983.$$

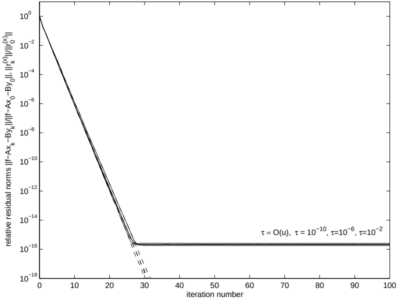
# Generic update: $y_{k+1} = y_k + p_k^{(y)}$

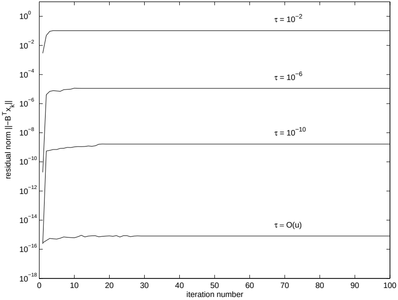


# Direct substitution: $y_{k+1} = B^{\dagger}(f - Ax_{k+1})$



## Corrected direct substitution: $y_{k+1} = y_k + B^\dagger(f - Ax_{k+1} - By_k)$



## Schur complement reduction method

- Compute $y$ as a solution of the Schur complement system

$$B^T A^{-1} B y = B^T A^{-1} f,$$

- compute $x$ as a solution of

$$Ax = f - By.$$

- inexact solution of systems with $A$: **every computed solution $\hat{u}$ of $Au = b$ is interpreted an exact solution of a perturbed system**

$$(A + \Delta A)\hat{u} = b + \Delta b, \ \|\Delta A\| \leq \tau \|A\|, \ \|\Delta b\| \leq \tau \|b\|, \ \tau\kappa(A) \ll 1.$$

# Iterative solution of the Schur complement system

$$
\left.
\begin{array}{l}
\text{choose } y_0,\ \text{solve } Ax_0 = f - By_0 \\
\text{compute } \alpha_k \text{ and } p_k^{(y)} \\
y_{k+1} = y_k + \alpha_k p_k^{(y)} \\
\quad \left|
\begin{array}{l}
\text{solve } Ap_k^{(x)} = -Bp_k^{(y)} \\
\color{red}{\text{back-substitution:}} \\
\textbf{A: } x_{k+1} = x_k + \alpha_k p_k^{(x)}, \\
\textbf{B: } \text{solve } Ax_{k+1} = f - By_{k+1}, \\
\textbf{C: } \text{solve } Au_k = f - Ax_k - By_{k+1}, \\
\qquad x_{k+1} = x_k + u_k.
\end{array}
\right\} \begin{array}{l}\text{inner}\\ \text{iteration}\end{array} \\
r_{k+1}^{(y)} = r_k^{(y)} - \alpha_k B^T p_k^{(x)}
\end{array}
\right\} \begin{array}{l}\text{outer}\\ \text{iteration}\end{array}
$$

# Maximum attainable accuracy of inexact Schur complement schemes

The limiting (maximum attainable) accuracy is measured by the ultimate (asymptotic) values of:

1. **the Schur complement residual**: $B^TA^{-1}f - B^TA^{-1}By_k$;
2. **the residuals in the saddle point system**: $f - Ax_k - By_k$ and $-B^Tx_k$;
3. **the forward errors**: $x - x_k$ and $y - y_k$.

**Numerical experiments: a small model example**

$$A = \mathrm{tridiag}(1, 4, 1) \in \mathbb{R}^{100\times 100},\ B = \mathrm{rand}(100, 20),\ f = \mathrm{rand}(100, 1),$$

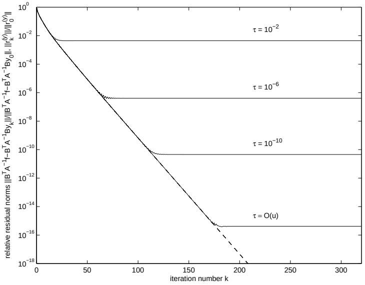
$$\kappa(A) = \|A\| \cdot \|A^{-1}\| = 7.1695 \cdot 0.4603 \approx 3.3001,$$

$$\kappa(B) = \|B\| \cdot \|B^{\dagger}\| = 5.9990 \cdot 0.4998 \approx 2.9983.$$

# Accuracy in the outer iteration process

$$\| - B^T A^{-1} f + B^T A^{-1} B y_k - r_k^{(y)} \| \le \frac{O(\tau)\kappa(A)}{1 - \tau\kappa(A)} \|A^{-1}\| \|B\| (\|f\| + \|B\| Y_k).$$

$$Y_k \equiv \max\{\|y_i\| \,|\, i = 0, 1, \ldots, k\}.$$




$$B^T (A + \Delta A)^{-1} B \hat{y} = B^T (A + \Delta A)^{-1} f,$$

$$\|B^T A^{-1} f - B^T A^{-1} B \hat{y}\| \le \frac{\tau\kappa(A)}{1 - \tau\kappa(A)} \|A^{-1}\| \|B\|^2 \|\hat{y}\|.$$

# Accuracy in the saddle point system

$$\|f - Ax_k - By_k\| \leq \frac{O(\alpha_1)\kappa(A)}{1 - \tau\kappa(A)}(\|f\| + \|B\|Y_k),$$

$$\| - B^T x_k - r_k^{(y)}\| \leq \frac{O(\alpha_2)\kappa(A)}{1 - \tau\kappa(A)}\|A^{-1}\|\|B\|(\|f\| + \|B\|Y_k),$$

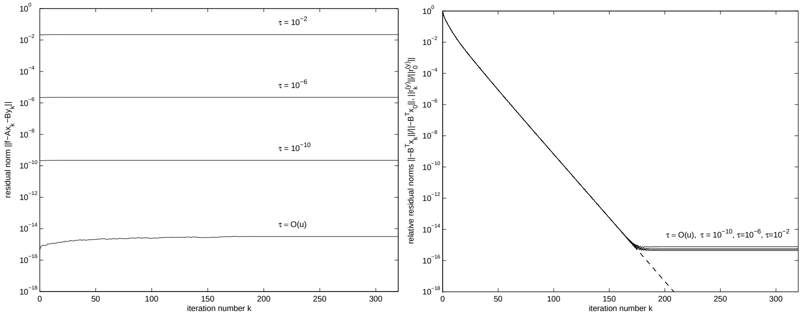
$$Y_k \equiv \max\{\|y_i\| \,|\, i = 0, 1, \ldots, k\}.$$

| Back-substitution scheme | $\alpha_1$ | $\alpha_2$ |
|---|---|---|
| **A**: Generic update<br>$x_{k+1} = x_k + \alpha_k p_k^{(x)}$ | $\tau$ | $u$ |
| **B**: Direct substitution<br>$x_{k+1} = A^{-1}(f - By_{k+1})$ | $\tau$ | $\tau$ |
| **C**: Corrected dir. subst.<br>$x_{k+1} = x_k + A^{-1}(f - Ax_k - By_{k+1})$ | $u$ | $\tau$ |

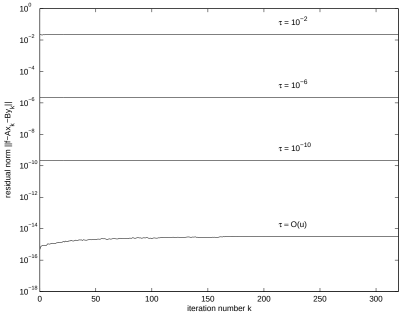
additional system with A

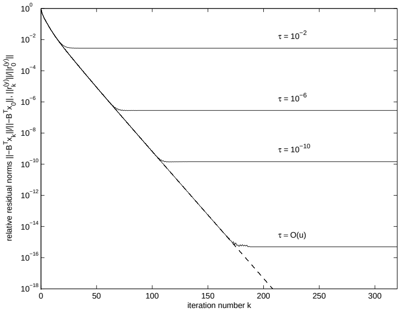
$$-B^T A^{-1} f + B^T A^{-1} B y_k = -B^T x_k - B^T A^{-1}(f - Ax_k - By_k)$$

## Generic update: $x_{k+1} = x_k + \alpha_k p_k^{(x)}$

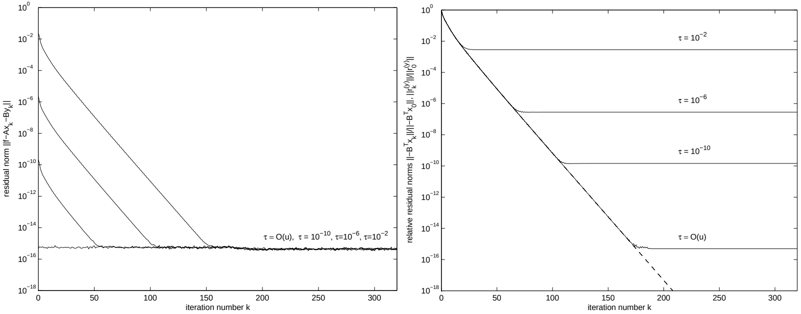


# Direct substitution: $x_{k+1} = A^{-1}(f - By_{k+1})$



# Corrected direct substitution: $x_{k+1} = x_k + A^{-1}(f - Ax_k - By_{k+1})$



# Related results in the context of saddle-point problems and Krylov subspace methods

- General framework of inexact Krylov subspace methods: in exact arithmetic the effects of relaxation in matrix-vector multiplication on the ultimate accuracy of several solvers [?], [?].
- The effects of rounding errors in the Schur complement reduction (block LU decomposition) method and the null-space method [?], [Arioli, 2000], the maximum attainable accuracy studied in terms of the user tolerance specified in the outer iteration [?], [?].
- Error analysis in computing the projections into the null-space and constraint preconditioning, limiting accuracy of the preconditioned CG , residual update strategy when solving constrained quadratic programming problems [?], or in cascadic multigrid method for elliptic problems [?].
- Theory for a general class of iterative methods based on coupled two-term recursions, all bounds of the limiting accuracy depend on the maximum norm of computed iterates, fixed matrix-vector multiplication, cf. [Greenbaum, 1997].

# General comments and considerations, future work

**"new_value = old_value + small_correction"**

- Fixed-precision iterative refinement for improving the computed solution $x_{\text{old}}$ to a system $Ax = b$: solving update equations $Az_{\text{corr}} = r$ that have residual $r = b - Ay_{\text{old}}$ as a right-hand side to obtain $x_{\text{new}} = x_{\text{old}} + z_{\text{corr}}$, see [?], [?].
- Stationary iterative methods for $Ax = b$ and their maximum attainable accuracy [?]: assuming splitting $A = M - N$ and inexact solution of systems with $M$, use $x_{\text{new}} = x_{\text{old}} + M^{-1}(b - Ax_{\text{old}})$ rather than $x_{\text{new}} = M^{-1}(Nx_{\text{old}} + b)$, [?].
- Two-step splitting iteration framework: $A = M_1 - N_1 = M_2 - N_2$ assuming inexact solution of systems with $M_1$ and $M_2$, reformulation of $M_1 x_{1/2} = N_1 x_{\text{old}} + b$, $M_2 x_{\text{new}} = N_2 x_{1/2} + b$, Hermitian/skew-Hermitian splitting (HSS) iteration [Bai, Golub, and Ng, 2003].
- Inexact preconditioners for saddle point problems: SIMPLE and SIMPLE(R) type algorithms [Vuik and Saghir, 2002] and constraint preconditioners [?].